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On the CLT for Linear Eigenvalue Statistics of a Tensor Model of Sample Covariance Matrices

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In [18], there was proved the CLT for linear eigenvalue statistics $\mathrm{Tr}\,\varphi(M_n)$ of sample covariance matrices of the form $M_n = \sum_{\alpha=1}^m \mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)} (\mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)})^T$, where $(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})_\alpha$ are iid copies of $\mathbf{y} \in \mathbb{R}^n$ satisfying $\mathbf{E}\mathbf{y}\mathbf{y}^T = n^{-1} I_n$, $\mathbf{E} y_i^2 y_j^2 = (1+\delta_{ij} d) n^{-2} + a(1+\delta_{ij} d_1) n^{-3} + O(n^{-4})$ for some $a, d, d_1 \in \mathbb{R}$. It was shown that given a smooth enough test function φ, $\mathbf{Var}\,\mathrm{Tr}\,\varphi(M_n) = O(n)$ as $m, n \to \infty$, $m/n^2 \to c > 0$, and $(\mathrm{Tr}\,\varphi(M_n) - \mathbf{E}\,\mathrm{Tr}\,\varphi(M_n))/\sqrt{n}$ converges in distribution to a Gaussian mean zero random variable with variance $V[\varphi]$ proportional to $a+d$. It was noticed that if $\mathbf{y}$ is uniformly distributed on the unit sphere then $a + d = 0$ and $V[\varphi]$ vanishes. In this note we show that in this case $\mathbf{Var}\,\mathrm{Tr}(M_n - zI_n)^{-1} = O(1)$, so that the CLT should be valid for linear eigenvalue statistics themselves without a normalising factor in front (in contrast to the Gaussian case.)

Key words: sample covariance matrices, CLT, linear eigenvalue statistics

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1. Introduction: model and main results

Consider the following model. Let $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ be a random vector having an unconditional distribution (which means that $(y_i)_i$ and $(\pm y_i)_i$ have the same distribution for any choice of signs) and satisfying the following moment conditions as $n \to \infty$:

$$\begin{aligned} &\mathbf{E} y_i = 0, \quad \mathbf{E} y_i y_j = n^{-1}\delta_{ij}, \quad i, j \le n, \\ &a_{2,2} := \mathbf{E} y_i^2 y_j^2 = n^{-2} + a n^{-3} + O\big(n^{-4}\big), \quad \forall i \ne j, \\ &\mathbf{E} y_j^4 - 3a_{2,2} = b n^{-2} + O\big(n^{-3}\big) \end{aligned} \tag{1.1}$$

for some $a, b \in \mathbb{R}$. Note that a vector $\mathbf{y} \sim U(S^{n-1})$ uniformly distributed on the unit sphere satisfies these conditions (with $a = -2$, $b = 0$) as well as a normalised standard Gaussian vector $\mathbf{y} \sim \mathcal{N}(0, n^{-1} I_n)$ (with $a = b = 0$). Given $m = m(n) \in \mathbb{N}$, let $(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})_{\alpha=1}^m$ be independent copies of $\mathbf{y}$ and let $\{Y_1, \dots, Y_m\}$ be a multivariate sample of tensor products of pairs $\{\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)}\}$:

$$Y_\alpha = \mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)} = (\mathbf{y}_{\alpha i}^{(1)} \mathbf{y}_{\alpha j}^{(2)})_{i,j} \in \mathbb{R}^{n^2}, \quad \alpha = 1, \dots, m. \tag{1.2}$$

Consider an $n^2 \times n^2$ sample covariance matrix of the form

$$M_n = \sum_{\alpha=1}^{m} Y_\alpha Y_\alpha^T. \tag{1.3}$$

In [18], there was studied the asymptotic behaviour of linear eigenvalue statistics $\mathrm{Tr}\,\varphi(M_n)$ as $m, n \to \infty$, $m/n^2 \to c > 0$. In particular, it was shown that for a smooth enough test function φ, the variance of $\mathrm{Tr}\,\varphi(M_n)$ grows to infinity not faster than n,

$$\mathbf{Var}\,\mathrm{Tr}\,\varphi(M_n) = O(n), \tag{1.4}$$

and $(\mathrm{Tr}\,\varphi(M_n) - \mathbf{E}\,\mathrm{Tr}\,\varphi(M_n))/\sqrt{n}$ converges in distribution to a Gaussian random variable with zero mean and variance

$$V^{(2)}[\varphi] = \frac{(a+b+2)}{2c\pi^2} \left(\int_{a_-}^{a_+} \varphi(\mu) \frac{\mu - a_m}{\sqrt{(a_+ - \mu)(\mu - a_-)}} d\mu \right)^2, \tag{1.5}$$

where

$$a_\pm = (1 \pm \sqrt{c})^2, \quad a_m = (a_+ + a_-)/2 = 1 + c$$

(see Theorem 1.9 and Remark 1.10 of [18] for the details.)

If $a + b + 2 = 0$ then $V[\varphi]$ vanishes and the limit of $(\mathrm{Tr}\,\varphi(M_n) - \mathbf{E}\,\mathrm{Tr}\,\varphi(M_n))/\sqrt{n}$ is trivial. This is precisely the situation we have when vectors $(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})_\alpha$ in the definition of M_n are uniformly distributed on the unit sphere. In this case in order to describe fluctuations of linear eigenvalue statistics of corresponding matrix M_n, we need to refine (1.4) finding the correct order of $\mathbf{Var}\,\mathrm{Tr}\,\varphi(M_n)$ and then after proper normalization find the corresponding limiting variance. This is the most important step while proving the CLT for the linear eigenvalue statistics and this is the main aim of the present note.

The question of the validity of CLT for linear eigenvalue statistics of random matrices, in particular sample covariance matrices, has been a subject of extensive research with numerous significant findings. It dates back to the investigation of fluctuations of the traces of matrix powers [2, 15] and matrix resolvents [11]. The first CLTs for the traces of arbitrary smooth enough test functions $\mathrm{Tr}\,\varphi(M_n)$ were obtained in [9, 13, 14] for the case of Gaussian matrix entries and in [1, 4, 19] for more general models. Further study of fluctuations of linear spectral statistics has been caring out mostly in three main directions: relaxing regularity conditions for the test functions, relaxing moment conditions, and exploring more complex matrix structures (sparsification, considering different types of dependence of matrix elements etc.) We refer the Reader to [6, 12, 16–18, 21, 23–26] and the references therein. Based on our knowledge it is expected that, up to certain moment and regularity conditions, the asymptotic behavior of eigenvalue statistics of random matrices depends mostly on the structure of the matrix being considered and not on the precise distribution of matrix elements. Thus, in the case of basic models for Wigner and sample covariance random matrices, the CLTs for linear eigenvalue statistics depend only on the first four moments of the corresponding matrix and sample entries, provided that the test functions are smooth enough. We also expect that the standard Gaussian vectors and vectors uniformly distributed on the unit sphere result in similar asymptotic behavior of samples dependent on these vectors, and this is precisely what we have in the case of CLTs for linear eigenvalue statistics for matrices of the form $\sum_{\alpha=1}^m \tau_\alpha \mathbf{y}_\alpha \mathbf{y}_\alpha^T$, where up to a constant the limiting variance is the same for $(\mathbf{y}_\alpha)_\alpha \sim_{\text{iid}} U(S^{n-1})$ and $(\mathbf{y}_\alpha)_\alpha \sim_{\text{iid}} \mathcal{N}(0, n^{-1}I_n)$. What makes our model (1.2)–(1.3) special and interesting for us is that not only the limiting expression of the variance but the order of the variance of linear eigenvalue statistic depends on the distributions of $(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})_\alpha$ in the definition of M_n (compare (1.5) and (1.8) below.)

Before stating our result, we present some of the results proven in [12]. Consider the sample covariance matrix of the form $M_n^{(1)} = \sum_{\alpha=1}^m \mathbf{y}_\alpha \mathbf{y}_\alpha^T$, where $(\mathbf{y}_\alpha)_\alpha$ are independent and identically distributed copies of a vector $\mathbf{y}$ with an unconditional distribution satisfying $\mathbf{E}\mathbf{y}\mathbf{y}^T = n^{-1}I_n$ and such that for any deterministic $n \times n$ matrix A_n with $\|A_n\|_{op} = 1$, $\mathbf{Var}(A_n\mathbf{y}, \mathbf{y}) = o(1)$, as $n \to \infty$. The almost sure convergence of the empirical spectral distribution of these matrices to the Marchenko–Pastur law [20] was proved in [22] (see also [5].) In [12] it was shown that if additionally $\mathbf{y}$ satisfies (1.1) and $\mathbf{E}|(A_n\mathbf{y}, \mathbf{y}) - \mathbf{E}(A_n\mathbf{y}, \mathbf{y})|^4 = O(n^{-2})$, then given a test function $\varphi \in H_{2+\delta}$, $\delta > 0$, the centered linear eigenvalue statistic $\operatorname{Tr}\varphi(M_n^{(1)}) - \mathbf{E}\operatorname{Tr}\varphi(M_n^{(1)})$ converges in distribution to a Gaussian random variable with zero mean and variance

$$V^{(1)}[\varphi] = \frac{1}{2\pi^2}\int_{a_-}^{a_+}\int_{a_-}^{a_+}\left(\frac{\triangle\varphi}{\triangle\lambda}\right)^2 \frac{(4c - (\lambda_1 - a_m)(\lambda_2 - a_m))d\lambda_1 d\lambda_2}{\sqrt{(a_+ - \lambda_1)(\lambda_1 - a_-)}\sqrt{(a_+ - \lambda_2)(\lambda_2 - a_-)}}$$
$$+ \frac{a+b}{4c\pi^2}\left(\int_{a_-}^{a_+}\varphi(\mu)\frac{\mu - a_m}{\sqrt{(a_+ - \mu)(\mu - a_-)}}d\mu\right)^2, \tag{1.6}$$

where $\triangle\varphi/\triangle\lambda = \varphi(\lambda_1) - \varphi(\lambda_2)/(\lambda_1 - \lambda_2)$ (see Theorem 1.8 of [12] for the details.)

The proofs of (1.5) and (1.6) follow the method of Stieltjes transform used by many authors before (see [4, 11, 20, 25] and references therein.) One of the main steps of these proofs is to find the order of fluctuations of the traces $\gamma_n^{(1)}(z) = \operatorname{Tr} G(z)$ of the resolvent $G(z) = (M_n^{(1)} - zI_n)^{-1}$, $\Im z \neq 0$, and to find the limit of the properly normalized covariance of the resolvent traces, which allows then to get the limiting variance of the linear eigenvalue statistics. Thus, in [12] in order to get (1.6) it was shown that $\mathbf{Var}\,\gamma_n^{(1)}(z) \le C|\Im z|^{-6}$, $\Im z \neq 0$, and

$$C^{(1)}(z_1, z_2) := \lim_{n\to\infty} \mathbf{Cov}\left\{\gamma_n^{(1)}(z_1), \gamma_n^{(1)}(z_2)\right\}$$
$$= \frac{\partial^2}{\partial z_1 \partial z_2}\left(2\log\frac{\Delta f}{\Delta z} + \frac{c(a+b)}{(1+f(z_1))(1+f(z_2))}\right),$$

where f is the Stieltjes transform of the Marchenko–Pastur law.

In our work we focus mainly on this step and study the fluctuations of the resolvent traces. Our main result is the following theorem.

Theorem 1.1. *Given $n, m \in \mathbb{N}$, let $M_n = \sum_{\alpha=1}^m \mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)}(\mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)})^T$, where $(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})_\alpha$ are independent and identically distributed vectors uniformly* *distributed on the unit sphere. Suppose that* $m = m(n) \to \infty$ *and* $m/n^2 \to c > 0$ *as* $n \to \infty$*. Let* $G(z) := (M_n - zI_n)^{-1}$*,* $\gamma_n(z) := \operatorname{Tr} G(z)$*,* $\Im z \neq 0$*. Then* $\mathbf{Var}\,\gamma_n(z) = O(1)$*,* $\Im z \neq 0$*, and there exists* $\eta_0 \in (0,\infty)$ *such that*

$$\begin{aligned} C(z_1,z_2) : &= \lim_{n\to\infty} \mathbf{Cov}\{\gamma_n(z_1),\gamma_n(z_2)\} \\ &= C^{(1)}(z_1,z_2)\Big|_{a+b=-2} + 3\left(\frac{cf^2(z_1)}{(1+f(z_1))^2}\right)'\left(\frac{cf^2(z_2)}{(1+f(z_2))^2}\right)', \end{aligned} \tag{1.7}$$

where $z_1, z_2 \in \{z \in \mathbb{C} : \Im z \geq \eta_0\}$ *and* f *is the Stieltjes transform of the Marchenko–Pastur law.*

Remark 1.1.

1. In Section 2, for the Readers convenience we gather some useful properties of f and give the explicit form of $C(z_1, z_2)$.
2. In what follows for the sake of simplicity we suppose that $m = cn^2$. Our main results, Theorems 3.1 and 1.1, remain valid in general case $m/n^2 \to c > 0$, $m, n \to \infty$ (see Remark 3.1.)

Having proved Theorem 1.1 and following step by step the scheme proposed in [25] (used in [12, 18]), one can prove the corresponding CLT for the linear eigenvalue statistics and show that under conditions of Theorem 3.1, given a smooth enough test function φ ($\varphi \in H^s$ for some $s > 2$), the centered linear eigenvalue statistic $\operatorname{Tr}\varphi(M_n) - \mathbf{E}\operatorname{Tr}\varphi(M_n)$ converges in distribution to a Gaussian random variable with zero mean and variance

$$V[\varphi] = V^{(1)}[\varphi]\Big|_{a+b=-2} + \frac{3}{(c\pi)^2}\left(\int_{a_-}^{a_+}\varphi(\mu)\frac{2c-(\mu-a_m)^2}{\sqrt{(a_+-\mu)(\mu-a_-)}}d\mu\right)^2. \tag{1.8}$$

To find a regularity class for the test functions, one can get an analog of Lemma 3.2 of [12] and then apply Proposition 1 of [25]. (We also refer the Reader to [17,26] as to some optimal results on regularity classes for the CLTs for Wigner and sample covariance matrices.)

Though the scheme of the proof of Theorem 1.1 and similar results is quite standard nowadays, there are some difficulties in the details, namely in the estimating of the error terms, such as variances of various resolvent statistics, and in showing that what we expect to be small is indeed small enough. In addition to a bit complicate structure of the sample, we need to take into account some delicate cancellations we have due to the moment conditions specific to the vectors uniformly distributed on a sphere. As in many other papers dealing with random matrices without independence structures in columns (see, e.g., [22] and [5]), our research is based on the asymptotic analysis of the bilinear forms (AY, Y). In Section 3, using a bootstrapping argument we get the order and find the limit of the properly normalised covariance of bilinear forms $(G(z)Y, Y)$. Section 4 contains the proof of Theorem 1.1. In Section 5 we gather auxiliary technical results.

2. Notations and preliminary results

Moments of $\mathbf{y} \sim U(S^{n-1})$ and $Y = \mathbf{y}^{(1)} \otimes \mathbf{y}^{(2)}$. Let $\mathbf{y} = (y_1, \ldots, y_n) \in S^{n-1}$ be a random vector uniformly distributed on the unit sphere in $\mathbb{R}^n$. It is easy to see that this distribution is unconditional, and that

$$\mathbf{E} y_i = 0, \quad \mathbf{E} y_i y_j = n^{-1}\delta_{ij}, \quad i, j \le n.$$

Also it can be shown that

$$\begin{aligned} &a_{2,2} = \mathbf{E} y_i^2 y_j^2 = \frac{1}{n(n+2)} = \frac{1}{n^2} - \frac{2}{n^3} + O(n^{-4}), \quad i \neq j, \\ &\mathbf{E} y_j^4 - 3a_{2,2} = 0, \quad \mathbf{E} y_j^4 - 3\big(\mathbf{E} y_j^2\big)^2 = -\frac{6}{n^2(n+2)}, \\ &\mathbf{E} y_j y_s y_p y_q = a_{2,2}(\delta_{js}\delta_{pq} + \delta_{jp}\delta_{sq} + \delta_{jq}\delta_{sp}). \end{aligned}$$

This shows that in (1.1) $a = -2$ and $b = 0$, so that $V[\varphi] = 0$.

Now let $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ be independent and identically distributed copies of $\mathbf{y}$ and let $Y = \mathbf{y}^{(1)} \otimes \mathbf{y}^{(2)} = \big(y_i^{(1)} y_j^{(2)}\big)_{i,j=1}^n$. We have

$$\begin{aligned} &\mathbf{E} Y_{ij} = 0, \quad \mathbf{E} Y_{ij} Y_{pq} = n^{-2}\delta_{ij}\delta_{pq}, \\ &\mathbf{E} Y_{ij}^2 Y_{pq}^2 = a_{2,2}^2 = \frac{1}{n^2(n+2)^2} = \frac{1}{n^4} - \frac{4}{n^5} + \frac{12}{n^6} + O\big(n^{-7}\big), \\ &\mathbf{E} Y_{jj'} Y_{ss'} Y_{pp'} Y_{qq'} = a_{2,2}^2(\delta_{js}\delta_{pq} + \delta_{jp}\delta_{sq} + \delta_{jq}\delta_{sp})(\delta_{j's'}\delta_{p'q'} + \delta_{j'p'}\delta_{s'q'} + \delta_{j'q'}\delta_{s'p'}). \end{aligned} \tag{2.1}$$

Note that $\|Y\| = 1$, though the distribution of Y is not uniform on S^{n^2-1}.

Here and in what follows given a vector X we use notation $\|X\|$ for the Euclidian norm of X. Also given a matrix M, we use $\|M\|_{op}$ for its operator norm, $\|M\|_{op} = \sup_{X:\,\|X\|=1} \|MX\|$, and $\|M\|_{HS}$ for its Hilbert–Schmidt norm, $\|M\|_{HS} = (\sum_{i,j} M_{ij}^2)^{1/2}$.

The Stieltjes transform of the Marchenko–Pastur law. Here we gather some simple facts on the Stieltjes transform $f(z)$ of the Marchenko–Pastur law (see [20]) that we need in what follows. We have

$$zf^2 + f(z+1-c) + 1 = 0, \; f = \frac{1}{2z}\left[-(z+1-c) + \sqrt{(z+1-c)^2 - 4z}\right], \tag{2.2}$$

$$\begin{aligned} &(z - c/(1+f))^{-1} = -f, \quad \big(z - c/(1+f)^2\big)^{-1} = -f'/f, \\ &\frac{\Delta z}{\Delta f} = \frac{1}{f(z_1)f(z_2)} - \frac{c}{(1+f(z_1))(1+f(z_2))}, \quad \frac{f'}{f(f+1)} = -\frac{1}{sq(z)}, \\ &\left(\frac{1}{f+1}\right)' = -\frac{1}{2}\left[1 + \frac{z-(1+c)}{sq(z)}\right], \quad \left(\frac{1}{f}\right)' = -\frac{1}{2c}\left[1 + \frac{z-(1+c)}{sq(z)}\right], \end{aligned}$$

where $sq(z) := \sqrt{(z+1-c)^2 - 4z}$. This allows to get the explicit expression in (1.7):

$$C(z_1, z_2) = -\frac{1}{(\Delta z)^2}\left[1 + \frac{4c - (z_1 - a_m)(z_2 - a_m)}{sq(z_1)sq(z_2)}\right] - \frac{1}{2c}\prod_{i=1,2}\left[\frac{z_i - a_m}{sq(z_i)} - 1\right]$$

$$+\frac{3}{c^2}\prod_{i=1,2}\left[z_i - a_m + \frac{2c-(z_i-a_m)^2}{sq(z_i)}\right],$$

where $a_m = 1+c$. Note that compared with the previously known results for the $n\times n$ sample covariance matrices $M_n^{(1)}$, here we have an additional term (the last one, see [12, 24].)

The resolvent and some related spectral statistics. Let $G(z) = (M_n - z)^{-1}$, $z \in \mathbb{C}$, be the resolvent of M_n. We have $\|G(z)\|_{op} \le 1/|\Im z|$, $\|G(z)\|_{HS} \le n^{1/2}/|\Im z|$. Introduce $n\times n$ matrices $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ by the formulas

$$\mathcal{G} = \Big(\sum_q G_{jq,kq}\Big)_{j,k}, \quad \widetilde{\mathcal{G}} = \Big(\sum_q G_{qj,qk}\Big)_{j,k}.$$

It is easy to check that $\|\mathcal{G}(z)\|_{op} \le n/|\Im z|$, $\|\widetilde{\mathcal{G}}(z)\|_{op} \le n/|\Im z|$. Introduce also

$$\gamma_n(z) = \operatorname{Tr} G(z), \quad g_n = n^{-2}\gamma_n, \tag{2.3}$$

$$g_n^{(1)}(z_1,z_2) = \frac{1}{n^3}\sum_{j,s,p,q} G_{js,ps}(z_1)G_{jq,pq}(z_2) = n^{-3}\operatorname{Tr}\mathcal{G}(z_1)\mathcal{G}(z_2), \tag{2.4}$$

$$\widetilde{g}_n^{(1)}(z_1,z_2) = n^{-3}\operatorname{Tr}\widetilde{\mathcal{G}}(z_1)\widetilde{\mathcal{G}}(z_2),$$

$$g_n^{(2)}(z_1,z_2) = \frac{1}{n^2}\sum_{j,s,p,q} G_{js,pq}(z_1)G_{ps,jq}(z_2). \tag{2.5}$$

Here and in what follows the summations over the Latin indices are from 1 to n and over the Greek indices are from 1 to m. Let $f_n = \mathbf{E}g_n$, $f_n^{(i)} = \mathbf{E}g_n^{(i)}$, $i = 1,2$. We normalize the introduced functions so that they are uniformly bounded in n.

The following statement was proved in [18, Lemmas 6.1,7.2]:

Lemma 2.1 ([18])**.** *Let γ_n be defined in (2.3). Given a compact set $K \subset \mathbb{C}\setminus\mathbb{R}$, we have uniformly in $z, z_1, z_2 \in K$ as $n\to\infty$:*

(i) $\mathbf{Var}\,\gamma_n(z) = O(n)$, $\mathbf{Var}\,g_n = O\big(n^{-3}\big)$,

(ii) $\mathbf{Var}\,g_n^{(1)}$, $\mathbf{Var}\,\widetilde{g}_n^{(1)} = O\big(n^{-2}\big)$.

Here and in what follows the constants hidden in $O\big(n^{-\ell}\big)$ may depend only on K. Let

$$M_n^\alpha = M_n - Y_\alpha Y_\alpha^T = \sum_{\beta\ne\alpha,\beta=1}^{m} Y_\alpha Y_\alpha^T.$$

In what follows we use the upper index α for the quantities which does not depend on Y_α:

$$G^\alpha(z) := (M_n^\alpha - z)^{-1}, \quad g_n^\alpha := n^{-2}\operatorname{Tr} G^\alpha, \quad f_n^\alpha := \mathbf{E}g_n^\alpha,$$

and so on. Let $\mathbf{E}_\alpha = \mathbf{E}_{Y_\alpha}$ denote expectation with respect to Y_α. We have by (2.1),

$$\mathbf{E}_\alpha(G^\alpha Y_\alpha, Y_\alpha) = g_n^\alpha, \quad \mathbf{E}(G^\alpha Y_\alpha, Y_\alpha) = f_n^\alpha.$$

Also given $\xi = \xi(Y_1, \dots, Y_m)$, we put

$$\xi^\circ = \xi - \mathbf{E}\xi, \quad (\xi)^\circ_\alpha = \xi - \mathbf{E}_\alpha \xi,$$

so that $\mathbf{Var}\,\xi = \mathbf{E}|\xi^\circ|^2$, $\mathbf{Var}_\alpha\,\xi = \mathbf{E}_\alpha|(\xi)^\circ_\alpha|^2$, $\mathbf{Cov}\{\xi_1, \xi_2\} = \mathbf{E}\,\xi_1\overline{\xi_2}^\circ$.

3. Covariance and central moments of bilinear forms $(G(z)Y, Y)$

In this section we establish some auxiliary results concerning the asymptotic properties of the bilinear forms $(G^\alpha Y_\alpha, Y_\alpha)$, $\alpha \le m$. Let Y be defined in (1.2). Take any two matrices F, H which do not depend on Y and have the operator norms uniformly bounded in n. With the help of (2.1) one can get

$$\begin{aligned}\mathbf{E}_Y(FY,Y)(HY,Y)^\circ_Y &= \big(a_{2,2}^2 - n^{-4}\big)\operatorname{Tr} F \operatorname{Tr} H + 2a_{2,2}^2 \operatorname{Tr} FH \\ &+ 2a_{2,2}^2 \sum_{j,p,s,q} (F_{js,ps}H_{jq,pq} + F_{js,jq}H_{ps,pq} + F_{js,pq}H_{ps,jq}),\end{aligned} \tag{3.1}$$

where $|\operatorname{Tr} FH| \le n^2\|F\|_{op}\|H\|_{op}$ and by the Cauchy–Schwarz inequality,

$$\Big|\sum_{j,p,s,q} F_{js,pq}H_{ps,jq}\Big| \le n^2\|F\|_{op}\|H\|_{op} \quad\text{and}\quad \Big|\sum_{j,p,s,q} F_{js,ps}H_{jq,pq}\Big| \le n^3\|F\|_{op}\|H\|_{op}.$$

This, (2.1), and (3.1) yield

$$|\mathbf{E}_Y(FY,Y)(HY,Y)^\circ_Y| \le Cn^{-1}\|F\|_{op}\|H\|_{op}, \tag{3.2}$$

where C is an absolute constant. Let now $F = G(z_1)$ and $H = G(z_2)$, z_1, $z_2 \in \mathbb{C} \setminus \mathbb{R}$, where $G(z) = (M_n - z)^{-1}$ is the resolvent of M_n. Suppose that Y and $Y_1, \dots, Y_m$ in the definition of M_n are mutually independent. Let

$$D_n(z_1, z_2) := \mathbf{Cov}\{(G(z_1)Y,Y),(G(z_2)Y,Y)\} = \mathbf{E}(G(z_1)Y,Y)(G(z_2)Y,Y)^\circ.$$

Since $\|G(z)\|_{op} \le |\Im z|^{-1}$, we have by (3.2)

$$\mathbf{E}_Y(G(z_1)Y,Y)(G(z_2)Y,Y)^\circ_Y = O\big(n^{-1}\big). \tag{3.3}$$

Also we have

$$\begin{aligned}D_n(z_1,z_2) = \mathbf{E}_{Y_1,\dots,Y_m}&\big(\mathbf{E}_Y(G(z_1)Y,Y))(G(z_2)Y,Y)^\circ_Y\big) \\ &+ \mathbf{Cov}_{Y_1,\dots,Y_m}\{g_n(z_1), g_n(z_2)\},\end{aligned} \tag{3.4}$$

where by Lemma 2.1(i) the second term is of order $O\big(n^{-3}\big)$. Hence

$$D_n(z_1,z_2) = O\big(n^{-1}\big). \tag{3.5}$$

The main purpose of this section is to show that $D_n(z_1,z_2) = O\big(n^{-2}\big)$ (see Lemma 3.1 below) and then to find the limit of $n^2 D_n(z_1,z_2)$ as $n \to \infty$ (see Theorem 1.1).

Lemma 3.1. *Let $f_n = \mathbf{E}g_n$ and $f_n^{(i)} = \mathbf{E}g_n^{(i)}$ (see (2.3)–(2.5).) Given a compact set $K \subset \mathbb{C} \setminus \mathbb{R}$, we have uniformly in $z, z_1, z_2 \in K$ as $n \to \infty$, $m = cn^2$:*

(i) $D_n(z_1, z_2) = O\big(n^{-2}\big)$,

(ii) $f_n(z) = f(z) + O\big(n^{-2}\big)$,

(iii) $f_n^{(1)}(z_1, z_2) = f(z_1)f(z_2) + \dfrac{1}{n}\dfrac{cf^2(z_1)f^2(z_2)}{(1+f(z_1))(1+f(z_2))} + O\big(n^{-3/2}\big)$,

(iv) $f_n^{(2)}(z_1, z_2) = f(z_1)f(z_2) + \dfrac{cf^2(z_1)f^2(z_2)}{(1+f(z_1))(1+f(z_2))} + O\big(n^{-1/2}\big)$.

Proof. To prove (i)–(iv) we use a bootstrap argument: first we prove a weaker statement and show that

$$f_n(z) = f(z) + O\big(n^{-1}\big) \quad \text{and} \quad f_n^{(1)}(z_1, z_2) = f(z_1)f(z_2) + O\big(n^{-1}\big) \tag{3.6}$$

and then repeating the argument and using (3.5) and (3.6) we get (i)–(iii).

We start with the first equality in (3.6). The rank one perturbation formula

$$G - G^\alpha = -\frac{G^\alpha Y_\alpha Y_\alpha^T G^\alpha}{1 + (G^\alpha Y_\alpha, Y_\alpha)} \tag{3.7}$$

implies that

$$\operatorname{Tr} G - \operatorname{Tr} G^\alpha = -\frac{(G^{\alpha 2} Y_\alpha, Y_\alpha)}{1 + (G^\alpha Y_\alpha, Y_\alpha)} := -\frac{B_\alpha}{A_\alpha}, \tag{3.8}$$

$$(GY_\alpha, Y_\alpha) = (G^\alpha Y_\alpha, Y_\alpha) - \frac{(G^\alpha Y_\alpha, Y_\alpha)^2}{A_\alpha} = \frac{(G^\alpha Y_\alpha, Y_\alpha)}{A_\alpha} = 1 - \frac{1}{A_\alpha}. \tag{3.9}$$

Taking into account that $\|Y_\alpha\|_2 = 1$ and $|(G^\alpha Y_\alpha, Y_\alpha)^k| \le |\Im z|^{-k}$, we get

$$|A_\alpha|,\ |A_\alpha^{-1}| \le 1 + |\Im z|^{-1}. \tag{3.10}$$

This and (3.8) lead to

$$|f_n - f_n^\alpha| = O(n^{-2}), \tag{3.11}$$

hence, $\mathbf{E}A_\alpha(z) = 1 + f_n^\alpha(z) = 1 + f_n(z) + O\big(n^{-2}\big)$. We also have

$$|(\mathbf{E}A_\alpha)^{-1}| = \max\{2, 4/|\Im z|\} \tag{3.12}$$

(see (4.9) in [18].) Applying the resolvent identity and (3.7) we get

$$zG(z) = -I + M_n G(z) = -I + \sum_\alpha Y_\alpha Y_\alpha^T G = -I + \sum_\alpha \frac{Y_\alpha Y_\alpha^T G^\alpha}{A_\alpha}, \tag{3.13}$$

so that taking the trace and applying (3.9) and equality $m = cn^2$ we get

$$z\frac{1}{n^2}\operatorname{Tr} G(z) = -1 + \frac{1}{n^2}\sum_\alpha \frac{(G^\alpha Y_\alpha, Y_\alpha)}{A_\alpha} = -1 + c - \frac{c}{m}\sum_\alpha \frac{1}{A_\alpha}. \tag{3.14}$$

Note that for every $k \in \mathbb{N}$ we have

$$\frac{1}{A_\alpha} = \frac{1}{\mathbf{E}A_\alpha} - \frac{A_\alpha^\circ}{(\mathbf{E}A_\alpha)^2} + \frac{(A_\alpha^\circ)^2}{(\mathbf{E}A_\alpha)^3} + \cdots + \frac{(-1)^k (A_\alpha^\circ)^k}{A_\alpha (\mathbf{E}A_\alpha)^{k-1}}, \tag{3.15}$$

and in particular,

$$\frac{1}{A_\alpha} = \frac{1}{1+f_n} - \frac{A_\alpha^\circ}{(1+f_n)^2} + \frac{(A_\alpha^\circ)^2}{A_\alpha(1+f_n)^2} + O\big(n^{-2}\big),$$

where we also used (3.10)–(3.12). This and (3.14) leads to

$$z f_n(z) = -1 + c - \frac{c}{1+f_n} + \frac{c}{m(1+f_n)^2} \sum_\alpha \mathbf{E} \frac{A_\alpha^{\circ 2}}{A_\alpha} + O\big(n^{-2}\big).$$

Since $\mathbf{E}_\alpha A_\alpha(z) = 1 + g_n^\alpha(z)$ and

$$A_\alpha^\circ = (A_\alpha)_\alpha^\circ + g_n^{\alpha\circ}, \tag{3.16}$$

we get

$$\mathbf{Var}\, A_\alpha \le 2\big(\mathbf{E}\mathbf{E}_\alpha |(A_\alpha)_\alpha^\circ|^2 + \mathbf{Var}\{g_n^\alpha\}\big) = O\big(n^{-1}\big), \tag{3.17}$$

where we have applied Lemma 2.1(i) and used that by (3.3), $\mathbf{E}_\alpha |(A_\alpha)_\alpha^\circ|^2 = O\big(n^{-1}\big)$. Thus

$$z f_n(z) = -1 + c - \frac{c}{1+f_n} + O\big(n^{-1}\big). \tag{3.18}$$

On the other hand,

$$z f(z) = -1 + c - \frac{c}{1+f}$$

(see (2.2).) This implies

$$f_n(z) - f(z) = \big(z - c(1+f)^{-1}(1+f_n)^{-1}\big)^{-1} O\big(n^{-1}\big), \tag{3.19}$$

and the first statement of (3.6) follows. Now we turn to the second part of (3.6). By (3.7),

$$G_{\mathbf{jp}} = G_{\mathbf{jp}}^\alpha - \frac{(G^\alpha Y_\alpha)_{\mathbf{j}} (G^\alpha Y_\alpha)_{\mathbf{p}}}{A_\alpha}, \quad (M_n G)_{\mathbf{jp}} = \sum_\alpha \frac{Y_{\alpha \mathbf{j}} (G^\alpha Y_\alpha)_{\mathbf{p}}}{A_\alpha}.$$

This and (3.13) allows to get (cf (3.14))

$$\begin{aligned} z_1 g_n^{(1)}(z_1, z_2) &= -g_n(z_2) + n^{-3} \sum_{j,p,s,q} \sum_\alpha \frac{(G^\alpha(z_1) Y_\alpha)_{ps} Y_{\alpha\, js}}{A_\alpha(z_1)} G^\alpha_{jq,pq}(z_2) \\ &\quad - n^{-3} \sum_{j,p,s,q} \sum_\alpha \frac{Y_{\alpha\, js} (G^\alpha(z_1) Y_\alpha)_{ps}}{A_\alpha(z_1)} \frac{(G^\alpha(z_2) Y_\alpha)_{jq} (G^\alpha(z_2) Y_\alpha)_{pq}}{A_\alpha(z_2)} \\ &= -g_n(z_2) + T_1 - T_2, \end{aligned} \tag{3.20}$$

where

$$T_1 = \frac{1}{n^2}\sum_{\alpha}\frac{h_{n\alpha}(z_1,z_2)}{A_\alpha(z_1)},$$

$$h_{n\alpha} = \frac{1}{n}\sum_{j,s,p}(G^\alpha(z_1)Y_\alpha)_{ps}Y_{\alpha\,js}\mathcal{G}^\alpha_{jp}(z_2), \tag{3.21}$$

$$T_2 = \frac{1}{n^3}\sum_{\alpha}\frac{H_{n\alpha}(z_1,z_2)}{A_\alpha(z_1)A_\alpha(z_2)},$$

$$H_{n\alpha} = \sum_{j,s,p,q} Y_{\alpha\,js}(G^\alpha(z_1)Y_\alpha)_{ps}(G^\alpha(z_2)Y_\alpha)_{jq}(G^\alpha(z_2)Y_\alpha)_{pq}. \tag{3.22}$$

Applying the Cauchy–Schwarz inequality we get

$$\Big|\sum_{j,p,s,q} Y_{\alpha\,js}(G^\alpha Y_\alpha)_{ps}(G^\alpha Y_\alpha)_{jq}(G^\alpha Y_\alpha)_{pq}\Big| \le \|Y_\alpha\|\,\|G^\alpha Y_\alpha\|\,\|G^\alpha Y_\alpha\|^2$$
$$= O(1). \tag{3.23}$$

This and (3.10) yield $T_2 = O\big(n^{-1}\big)$. It is easy to check that $\mathbf{E}h_{n\alpha}(z_1,z_2) = f_n^{(1)\alpha}(z_1,z_2)$. Also it can be shown that

$$h_{n\alpha} = O(1) \quad \text{and} \quad \mathbf{Var}\, h_{n\alpha} = O\big(n^{-1}\big)$$

(see Lemma 5.1.) This, (3.15), and (3.17) allow to get

$$\mathbf{E}T_1 = n^{-2}\sum_{\alpha}\mathbf{E}\frac{h_{n\alpha}(z_1,z_2)}{A_\alpha(z_1)}$$
$$= n^{-2}\sum_{\alpha}\frac{f_n^{(1)\alpha}(z_1,z_2)}{1+f_n^\alpha(z_1)} - n^{-2}\sum_{\alpha}\frac{\mathbf{E}A_\alpha^\circ(z_1)h_{n\alpha}(z_1,z_2)A_\alpha^{-1}(z_1)}{1+f_n^\alpha(z_1)},$$

where

$$\mathbf{E}A_\alpha^\circ h_{n\alpha}A_\alpha^{-1} = \frac{\mathbf{E}A_\alpha^\circ h_{n\alpha} + \mathbf{E}A_\alpha^{\circ 2}h_{n\alpha}A_\alpha^{-1}}{1+f_n^\alpha} = O\big(n^{-1}\big). \tag{3.24}$$

Since replacing f_n^α and $f_n^{(1)\alpha}$ with f_n and $f_n^{(1)}$ results in terms of order $O\big(n^{-2}\big)$, this leads to $\mathbf{E}T_1 = cf_n^{(1)}(z_1,z_2)/(1+f_n(z_1)) + O\big(n^{-1}\big)$ and we finally get from (3.20)

$$z_1 f_n^{(1)}(z_1,z_2) = -f_n(z_2) + \frac{cf_n^{(1)}(z_1,z_2)}{1+f_n(z_1)} + O\big(n^{-1}\big).$$

This and (3.19) yield the second equality in (3.6). In the second round of the proof we repeat the schemes proposed above and equipped with (3.6) get (i)–(iii).

(i) It follows from (3.1) with $F = G(z_1)$ and $H = G(z_2)$ and (2.1) that

$$n\mathbf{E}_Y(G(z_1)Y,Y)(G(z_2)Y,Y)_Y^\circ = \frac{-4n^2-4n}{(n+2)^2}g_n(z_1)g_n(z_2) + \frac{2\,\mathrm{Tr}\,G(z_1)G(z_2)}{n(n+2)^2}$$

$$+ \frac{2n}{(n+2)^2}\Big(g_n^{(1)}(z_1,z_2) + \widetilde{g}_n^{(1)}(z_1,z_2) + n^{-1}g_n^{(2)}(z_1,z_2)\Big). \quad (3.25)$$

Since $G(z_1)G(z_2) = (G(z_1)-G(z_2))/(z_1-z_2)$ and $\mathbf{E}g_n(z_1)g_n(z_2) = f_n(z_1)f_n(z_2)+ O\big(n^{-3}\big)$ (see Lemma 2.1(i)), this together with (3.4) lead to

$$nD_n(z_1,z_2) = -(4-12n^{-1})f_n(z_1)f_n(z_2) + \frac{2}{n}\frac{f_n(z_1)-f_n(z_2)}{z_1-z_2}$$
$$+ (2-8n^{-1})\Big(f_n^{(1)}(z_1,z_2) + \widetilde{f}_n^{(1)}(z_1,z_2) + n^{-1}f_n^{(2)}(z_1,z_2)\Big) + O\big(n^{-2}\big). \quad (3.26)$$

Substituting here (3.6), we get that the right-hand side is of order $O\big(n^{-1}\big)$, and so (i) follows.

(ii) The proof of (ii) repeats that one of (3.6) except that now by (i) we have

$$\mathbf{Var}\, A_\alpha = O\big(n^{-2}\big), \quad (3.27)$$

which leads to $O\big(n^{-2}\big)$ in the right-hand sides of (3.17)–(3.18).

(iii) We already have (3.20)–(3.24), where now applying (3.27), we get $O\big(n^{-3/2}\big)$ in the right-hand side in (3.24) so that

$$\mathbf{E}T_1 = cf_n^{(1)}(z_1,z_2)/(1+f_n(z_1)) + O\big(n^{-3/2}\big). \quad (3.28)$$

It follows from (3.22) that

$$\mathbf{E}T_2 = n^{-3}\sum_\alpha \frac{\mathbf{E}H_{n\alpha}(z_1,z_2)}{\mathbf{E}(A_\alpha(z_1)A_\alpha(z_2))} - n^{-3}\sum_\alpha \mathbf{E}\frac{H_{n\alpha}(z_1,z_2)(A_\alpha(z_1)A_\alpha(z_2))^\circ}{A_\alpha(z_1)A_\alpha(z_2)\mathbf{E}(A_\alpha(z_1)A_\alpha(z_2))}.$$

By (3.23), $H_{n\alpha} = O(1)$. Also it can be shown that $\mathbf{E}H_{n\alpha} = f(z_1)f^2(z_2)+O\big(n^{-1}\big)$, $n\to\infty$ (see Lemma 5.2.) This and (3.27) allow to get

$$\mathbf{E}T_2 = \frac{1}{n}\frac{cf(z_1)f^2(z_2)}{(1+f(z_1))(1+f(z_2))} + O\big(n^{-3/2}\big). \quad (3.29)$$

Finally, applying (3.20) and (3.28) we get

$$z_1f_n^{(1)}(z_1,z_2) = -f(z_2) + \frac{cf_n^{(1)}(z_1,z_2)}{1+f(z_1)} - \frac{1}{n}\frac{cf(z_1)f^2(z_2)}{(1+f(z_1))(1+f(z_2))} + O\big(n^{-3/2}\big).$$

This leads to (iii).

(iv) Similar to (3.20) we have

$$z_1f_n^{(2)}(z_1,z_2) = -f_n(z_2) + \mathbf{E}T_1' - \mathbf{E}T_2', \quad (3.30)$$

where

$$T_1' = n^{-2}\sum_\alpha \frac{b_{n\alpha}(z_1,z_2)}{A_\alpha(z_1)},$$
$$b_{n\alpha}(z_1,z_2) = \sum_{j,p,s,q}(G^\alpha(z_1)Y_\alpha)_{pq}Y_{\alpha\, js}G^\alpha_{ps,jq}(z_2),$$

$$T_2' = \frac{1}{n^2}\sum_\alpha \frac{H_{n\alpha}'(z_1,z_2)}{A_\alpha(z_1)A_\alpha(z_2)},$$
$$H_{n\alpha}' = \sum_{j,s,p,q} Y_{\alpha\, js}(G^\alpha(z_2)Y_\alpha)_{ps}(G^\alpha(z_2)Y_\alpha)_{jq}(G^\alpha(z_1)Y_\alpha)_{pq}.$$

It can be shown that $|b_{n\alpha}| = O(n)$, $\mathbf{E}b_{n\alpha} = f_n^{(2)\alpha} = f_n^{(2)} + O(n^{-1})$, and $\mathbf{E}|b_{n\alpha}|^2 = O(n)$. This together with (3.15) and (3.27) leads to (cf (3.28))

$$\begin{aligned}\mathbf{E}T_1' &= n^{-2}\sum_\alpha \frac{f_n^{(2)\alpha}(z_1,z_2)}{1+f_n^\alpha(z_1)} - n^{-2}\sum_\alpha \frac{\mathbf{E}A_\alpha^\circ(z_1)b_{n\alpha}(z_1,z_2)A_\alpha^{-1}(z_1)}{1+f_n^\alpha(z_1)}\\ &= \frac{f_n^{(2)}(z_1,z_2)}{1+f_n(z_1)} + O\big(n^{-1/2}\big).\end{aligned} \tag{3.31}$$

Also comparing T_2 and T_2' one can see that $\mathbf{E}T_2' = n\mathbf{E}T_2$, where $\mathbf{E}T_2$ is given by (3.29). Hence

$$z_1 f_n^{(2)}(z_1,z_2) = -f(z_2) + \frac{cf_n^{(2)}(z_1,z_2)}{1+f(z_1)} - \frac{cf(z_1)f^2(z_2)}{(1+f(z_1))(1+f(z_2))} + O\big(n^{-1/2}\big).$$

This implies (iv) and finishes the proof of the lemma. □

Remark 3.1. Note that in general case when m is not identically equal to cn^2 but $c_n := m/n^2 \to c$ as $n\to\infty$, the right-hand sides of (ii)–(iv) can have additional terms of order bigger than $O\big(n^{-3/2}\big)$, though item (i) as well as Theorem 3.1 below remain valid. For example, if $c_n = c + n^{-\gamma}c_1$ for some $c > 0$, $c_1 \neq 0$, $\gamma\in(0,1)$, then as it follows from the proof above that $f_n = f + c_1 n^{-\gamma} f'(1+f)^{-1}$, but still $D_n(z_1,z_2) = O\big(n^{-2}\big)$.

Now we are ready to prove the main result of this section.

Theorem 3.1. *Given z_1, $z_2\in\mathbb{C}\setminus\mathbb{R}$, let $D_n(z_1,z_2)$ be defined in (3.4). Then*

$$D(z_1,z_2) := \lim_{n\to\infty} n^2 D_n(z_1,z_2) = 2\frac{\Delta f}{\Delta z} - 2f(z_1)f(z_2) + \frac{6cf^2(z_1)f^2(z_2)}{(1+f(z_1))(1+f(z_2))},$$

where $\Delta f/\Delta z = (f(z_1)-f(z_2))/(z_1-z_2)$.

Proof. Let $F := cf^2(z_1)f^2(z_2)/(1+f(z_1))(1+f(z_2))$. It follows from (3.26) and Lemma 3.1 that

$$\begin{aligned}nD_n(z_1,z_2) =&(-4+12n^{-1})\left(f(z_1)+O\big(n^{-2}\big)\right)\left(f(z_2)+O\big(n^{-2}\big)\right) + \frac{2}{n}\frac{\Delta f}{\Delta z}\\ &+(2-8n^{-1})\big(2f(z_1)f(z_2) + 2n^{-1}F + n^{-1}(f(z_1)f(z_2)+F)\big) + O\big(n^{-2}\big)\\ =&\frac{1}{n}\left(-2f(z_1)f(z_2) + 2\frac{\Delta f}{\Delta z} + 6F\right) + O\big(n^{-2}\big),\end{aligned}$$

and the theorem follows. □

Lemma 3.1 allows also to refine the results of Lemma 2.1 and, in particular, to show that the variance of the resolvent's trace is uniformly bounded in n. Namely, we have

Lemma 3.2. *Let γ_n and A_α be defined in (2.3) and (3.8), respectively. Given a compact set $K \subset \mathbb{C} \setminus \mathbb{R}$, we have uniformly in $z \in K$ as $n \to \infty$:*

(i) $\mathbf{E}_\alpha\{|(A_\alpha)^\circ_\alpha|^p\} = O\big(n^{-p}\big)$,
(ii) $\mathbf{E}\{|\gamma_n^\circ|^p\} = O(1)$,
(iii) $\mathbf{E}\{|A_\alpha^\circ|^p\} = O\big(n^{-p}\big)$.

Proof. (i) The proof of the first part is similar to the proof of [12, Lemma 2.1]. It is based on (3.27) and the dimension free Khinchine–Kahane-type inequality by Bourgain [8] (see also [7]), which says that if P_d is a polynomial of degree d, and $\mathbf{y} \in \mathbb{R}^n$ has a log-concave distribution, then

$$\mathbf{E}\{|P_d(\mathbf{y})|^p\} \le C(d,p)\mathbf{E}\{|P_d(\mathbf{y})|\}^p, \tag{3.32}$$

where $C(d,q)$ depends only on d and p and does not depend on n. Now since Y defined in (1.2) has a log-concave distribution, substituting $P_4(Y) = |(G^\alpha Y, Y)^\circ|^2$ in (3.32) and applying (3.27) we get (i).

(ii) The proof of the second part repeats the proof of [25, Proposition 2] (see also [12, Lemma 3.2]) combined with (i). For the Reader's convenience we provide here the main steps. It follows from [10] that for every $p \ge 2$ there exists $C_p > 0$ such that

$$\mathbf{E}\{|\gamma_n^\circ|^p\} \le C_p n^{p-2} \sum_\alpha \mathbf{E}\{|(\gamma_n)^\circ_\alpha|^p\}, \tag{3.33}$$

where by (3.8), (3.10), and (3.12) we have

$$\begin{aligned}
\mathbf{E}\{|(\gamma_n)^\circ_\alpha|^p\} &= \mathbf{E}\{|\gamma_n - \gamma_n^\alpha - \mathbf{E}_\alpha\{\gamma_n - \gamma_n^\alpha\}|^p\} \\
&\le C\mathbf{E}\Big\{\Big|\frac{B_\alpha}{A_\alpha} - \frac{\mathbf{E}_\alpha\{B_\alpha\}}{\mathbf{E}_\alpha\{A_\alpha\}}\Big|^p\Big\} = C\mathbf{E}\Big\{\Big|\frac{(B_\alpha)^\circ_\alpha}{\mathbf{E}_\alpha\{A_\alpha\}} - \frac{B_\alpha}{A_\alpha} \cdot \frac{(A_\alpha)^\circ_\alpha}{\mathbf{E}_\alpha\{A_\alpha\}}\Big|^p\Big\} \\
&\le C'\mathbf{E}\big\{\mathbf{E}_\alpha\{|(B_\alpha)^\circ_\alpha|^p\} + \mathbf{E}_\alpha\{|(A_\alpha)^\circ_\alpha|^p\}\big\}.
\end{aligned}$$

Here C, C' depends only on $z \in K$ and p. Since both A_α and B_α satisfy (i), this and (3.33) imply (ii).

It remains to note, that now (iii) follows from (3.16) and (ii). ◻

4. Proof of Theorem 1.1

We start with a technical lemma.

Lemma 4.1. *Let A_α be defined in (3.8). As $n \to \infty$, we have*

$$\mathbf{Var}\,\mathbf{E}_\alpha\{(A_\alpha^\circ)^2\} = O\big(n^{-9/2}\big)$$

uniformly in $z_1, z_2 \in \{z \in \mathbb{C} : \Im z \ge \eta_0 > 0\}$.

Proof. It follows from (3.25) and Lemma 2.1 that

$$\begin{aligned}\mathbf{Var}\,\mathbf{E}_\alpha\{(A_\alpha^\circ)^2\} &\le Cn^{-2}\left(\mathbf{Var}\, g_n^{\alpha 2} + n^{-2}\,\mathbf{Var}\,\partial g_n^\alpha/\partial z\right.\\ &\qquad \left.+\,2\,\mathbf{Var}\, g_n^{(1)\alpha} + n^{-2}\,\mathbf{Var}\, g_n^{(2)\alpha}\right)\\ &\le Cn^{-2}\,\mathbf{Var}\, g_n^{(1)\alpha} + Cn^{-4}\,\mathbf{Var}\, g_n^{(2)\alpha} + O(n^{-5}),\end{aligned}$$

where we do not distinguish between $g_n^{(1)\alpha}$ and $\widetilde{g_n}^{(1)\alpha}$. It can be shown that $\mathbf{Var}\, g_n^{(1)} = O(n^{-5/2})$ (cf Lemma 2.1(ii)) and $\mathbf{Var}\, g_n^{(2)} = O(n^{-1/2})$. We postpone the proof of this bound to Section 5 (see Lemma 5.3). This finishes the proof of the lemma. □

Let $C_n(z_1, z_2) := \mathbf{E}\gamma_n(z_1)\gamma_n^\circ(z_2)$. In order to prove Theorem 1.1 we need to show that the limit of every converging subsequence of $\{C_n(z_1, z_2)\}_n$ is given by

$$\begin{aligned}C(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2}\Bigg[2\log\frac{\Delta f}{\Delta z} &- \frac{2c}{(1+f(z_1))(1+f(z_2))}\\ &+ \frac{3c^2 f^2(z_1) f^2(z_2)}{(1+f(z_1))^2(1+f(z_2))^2}\Bigg],\end{aligned} \tag{4.1}$$

$z_1, z_2 \in \{z \in \mathbb{C} : \Im z \ge \eta_0\}$ for some $\eta_0 \in (0, \infty)$. By (3.14), we have

$$\begin{aligned}z_1 C_n(z_1, z_2) &= -\sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)\gamma_n^{\alpha\circ}(z_2)\} - \sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)(\gamma_n - \gamma_n^\alpha)^\circ(z_2)\}\\ &=: T_n^{(1)} + T_n^{(2)}.\end{aligned} \tag{4.2}$$

It follows from (3.15) with $k = 3$ that

$$\begin{aligned}T_n^{(1)} &= \sum_\alpha \left[\frac{\mathbf{E}\{A_\alpha(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{(1+f_n^\alpha(z_1))^2} - \frac{\mathbf{E}\{A_\alpha^{\circ 2}(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{(1+f_n^\alpha(z_1))^3} + \frac{\mathbf{E}\{A_\alpha^{-1}A_\alpha^{\circ 3}(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{(1+f_n^\alpha(z_1))^3}\right]\\ &= \frac{1}{n^2}\sum_\alpha \frac{\mathbf{E}\{\gamma_n^\alpha(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{(1+f_n^\alpha(z_1))^2} + O(n^{-1/4}),\end{aligned}$$

where we used that by the Cauchy–Schwarz inequality and Lemmas 3.2 and 4.1,

$$\begin{aligned}&\left|\mathbf{E}\{A_\alpha^{\circ 2}(z_1)\gamma_n^{\alpha\circ}(z_2)\}\right| \le \left(\mathbf{Var}\,\mathbf{E}_\alpha\{(A_\alpha^\circ)^2\}\,\mathbf{Var}\,\gamma_n^\alpha\right)^{1/2} = O(n^{-9/4}), \quad \text{and}\\ &\left|\mathbf{E}\{A_\alpha^{-1}A_\alpha^{\circ 3}(z_1)\gamma_n^{\alpha\circ}(z_2)\}\right| \le C\left(\mathbf{E}\{|A_\alpha^\circ|^6\}\,\mathbf{Var}\,\gamma_n^\alpha\right)^{1/2} = O(n^{-3}).\end{aligned}$$

Applying (3.8), (3.15), the Cauchy–Schwarz inequality, and Lemma 3.2 we also get

$$\begin{aligned}\left|\mathbf{E}\{(\gamma_n^\alpha - \gamma_n)^\circ(z_1)\gamma_n^{\alpha\circ}(z_2)\}\right| &= \left|\mathbf{E}\{(B_\alpha/A_\alpha)^\circ(z_1)\gamma_n^{\alpha\circ}(z_2)\}\right|\\ &\le \left(\mathbf{Var}\{B_\alpha/A_\alpha\}\,\mathbf{Var}\,\gamma_n^\alpha\right)^{1/2} = O(n^{-1}).\end{aligned}$$

Hence,

$$T_n^{(1)} = \frac{cC_n(z_1, z_2)}{(1+f(z_1))^2} + O(n^{-1/4}). \tag{4.3}$$

Consider now $T_n^{(2)}$ of (4.2). By (3.8),

$$T_n^{(2)} = \sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)(B_\alpha/A_\alpha)^\circ(z_2)\}.$$

Denote for the moment $A_i := A_\alpha(z_i)$, $i = 1, 2$, $B_2 := B_\alpha(z_2)$. Applying (3.15) with $k = 2$ with respect to A_1 and A_2 and Lemma 3.2(iii) with $p = 3$ to estimate the reminder term, we get

$$\begin{aligned}\mathbf{E}\{(1/A_1)^\circ(B_2/A_2)^\circ\} &= \frac{\mathbf{E}\{(-A_1^\circ + A_1^{-1}A_1^{\circ 2})(B_2\mathbf{E}\{A_2\} - B_2A_2^\circ + B_2A_2^{-1}A_2^{\circ 2})^\circ\}}{\mathbf{E}\{A_1\}^2\mathbf{E}\{A_2\}^2} \\ &= \frac{-\mathbf{E}\{A_1^\circ B_2\}\mathbf{E}\{A_2\} + \mathbf{E}\{B_2\}\mathbf{E}\{A_1^\circ A_2\}}{\mathbf{E}\{A_1\}^2\mathbf{E}\{A_2\}^2} + O\big(n^{-3}\big).\end{aligned}$$

Taking into account that $B_\alpha(z) = \partial A_\alpha(z)/\partial z$, and applying Theorem 3.1 we get

$$\begin{aligned}T_n^{(2)} &= -\sum_\alpha \frac{1}{\mathbf{E}\{A_\alpha(z_1)\}^2}\frac{\partial}{\partial z_2}\frac{\mathbf{E}\{A_\alpha(z_1)A_\alpha^\circ(z_2)\}}{\mathbf{E}\{A_\alpha(z_2)\}} + O\big(n^{-1}\big) \\ &= -\frac{c}{(1+f(z_1))^2}\frac{\partial}{\partial z_2}\frac{D(z_1,z_2)}{1+f(z_2)} + O\big(n^{-1}\big).\end{aligned}$$

This, (4.2)–(4.3), and (2.2) lead to

$$C(z_1,z_2) = -\frac{c}{f_1}\left(\frac{\partial}{\partial z_1}\frac{1}{1+f_1}\right)\frac{\partial}{\partial z_2}\frac{D(z_1,z_2)}{1+f_2},$$

where for shortness we use notations $f_i = f(z_i)$. Substituting the expression for D, we get

$$\begin{aligned}C(z_1,z_2) = &-2\frac{c}{f_1}\left(\frac{\partial}{\partial z_1}\frac{1}{1+f_1}\right)\frac{\partial}{\partial z_2}\frac{1}{1+f_2}\frac{\Delta f}{\Delta z} \\ &+ \frac{\partial^2}{\partial z_1\partial z_2}\left[-\frac{2c}{(1+f_1)(1+f_2)} + \frac{3c^2f_1^2f_2^2}{(1+f_1)^2(1+f_2)^2}\right].\end{aligned}$$

By (2.2), we have for the first term on the right-hand side

$$\begin{aligned}-\frac{c}{f_1}\frac{\partial}{\partial z_1}\left(\frac{1}{1+f_1}\right)\frac{\partial}{\partial z_2}\frac{1}{1+f_2}\frac{\Delta f}{\Delta z} &= \frac{\partial}{\partial z_2}\frac{1}{f_1f_2}\left(\frac{\partial}{\partial z_1}\frac{cf_1f_2}{(1+f_1)(1+f_2)}\right)\frac{\Delta f}{\Delta z} \\ &= -\frac{\partial^2}{\partial z_1\partial z_2}\log f_1f_2\frac{\Delta z}{\Delta f} = \frac{\partial^2}{\partial z_1\partial z_2}\log\frac{\Delta f}{\Delta z}.\end{aligned}$$

This leads to (4.1) and finishes the proof.

5. Auxiliary results

Lemma 5.1. *Let $Y, Y_1, \ldots, Y_m$ be mutually independent identically distributed vectors defined in (1.2), $M_n = \sum_{\alpha=1}^m Y_\alpha Y_\alpha^T$, $m = cn^2$, $G(z) = (M_n - zI_n)^{-1}$, $\mathcal{G} = \big(\sum_q G_{jq,kq}\big)_{j,k}$, and*

$$h_n(z_1,z_2) = n^{-1}\sum_{j,s,p}(G(z_1)Y)_{ps}Y_{js}\mathcal{G}_{jp}(z_2).$$

Then $h_n = O(1)$ and $\mathbf{Var}\, h_n(z_1,z_2) = O\big(n^{-1}\big)$.

Proof. Since $\|\mathcal{G}\|_{op} = O(n)$, we have $|h_n| \le \|\mathcal{G}(z_2)\|_{op}\|Y\|\|G(z_1)Y\| = O(1)$. Applying (2.1) we get

$$\begin{aligned}
\mathbf{Var}_Y\, h_n &= \frac{1}{n^2} \sum_{j,p,s,t,u,v} \mathcal{G}_{jp}\overline{\mathcal{G}}_{uv}\mathbf{E}_Y(GY)_{ps}Y_{js}(\overline{G}Y)_{ut}Y_{ut} \\
&\quad - \frac{1}{n^2}\Big|\sum_{j,p,s}\mathcal{G}_{jp}\mathbf{E}_Y(GY)_{ps}Y_{js}\Big|^2 \\
&= \frac{a_{2,2}^2}{n^2}\sum_{j,p,s,d,u,v}\mathcal{G}_{jp}\overline{\mathcal{G}}_{uv}\Big[G_{ps,jd}\overline{G}_{us,vd} + G_{ps,jd}\overline{G}_{ud,vs} \\
&\quad + G_{ps,vd}\overline{G}_{us,jd} + G_{ps,vd}\overline{G}_{ud,js}\Big] \\
&\quad + \frac{2a_{2,2}^2}{n^2}\operatorname{Tr}\mathcal{G}^2\overline{\mathcal{G}}^2 + \frac{1}{n^6}\sum_{j,p,s}(\mathcal{G}\overline{\mathcal{G}})_{up}(G\overline{G})_{us,ps} \\
&\quad + \frac{a_{2,2}^2}{n^2}\sum_{j,p,s,d}(\mathcal{G}\overline{\mathcal{G}})_{up}G_{ps,jd}\overline{G}_{ud,js} + \big(a_{2,2}^2 n^{-2} - n^{-6}\big)(\operatorname{Tr}\mathcal{G}\overline{\mathcal{G}})^2.
\end{aligned}$$

To estimate the terms on the right-hand side we use the Cauchy–Schwartz inequality and bounds $\|G\|_{op} = O(1)$, $\|\mathcal{G}\|_{op} = O(n)$, $a_{2,2}^2 = n^{-4} + O(n^{-5})$. Thus for the first term on the right-hand side we have

$$\begin{aligned}
&\frac{1}{n^6}\Big|\sum_{j,p,s,d,u,v}\mathcal{G}_{jp}\overline{\mathcal{G}}_{uv}G_{ps,jd}\overline{G}_{us,vd}\Big| \\
&\qquad\le \frac{1}{n^6}\Big(\sum_{j,p,u,v}|\mathcal{G}_{jp}\overline{\mathcal{G}}_{uv}|^2\sum_{j,p,u,v}\Big|\sum_{s,d}G_{ps,jd}\overline{G}_{us,vd}\Big|^2\Big)^{1/2} \\
&\qquad\le \frac{1}{n^6}\operatorname{Tr}\mathcal{G}\overline{\mathcal{G}}\sum_{j,p,s,d}|G_{ps,jd}|^2 = O\big(n^{-1}\big).
\end{aligned}$$

Similarly, it can be shown that the remaining terms also have order at most $O(n^{-1})$. Hence $\mathbf{Var}_Y\, h_n = O(n^{-1})$. Since $\mathbf{E}\, h_n = \mathbf{E} g_n^{(1)}$ and by Lemma 2.1 $\mathbf{Var}\, g_n^{(1)} = O(n^{-2})$, we have $\mathbf{Var}\, h_n = O(n^{-1})$. This finishes the proof of the lemma. □

Lemma 5.2. *Let $Y, Y_1, \dots, Y_m$ be mutually independent identically distributed unit vectors satisfying (2.1), $M_n = \sum_{\alpha=1}^m Y_\alpha Y_\alpha^T$, $m = cn^2$, $G(z) = (M_n - zI_n)^{-1}$, $\mathcal{G} = \big(\sum_q G_{jq,kq}\big)_{j,k}$, and*

$$H_n(z_1,z_2) = \sum_{j,s,p,q} Y_{js}(G(z_1)Y)_{ps}(G(z_2)Y)_{jq}(G(z_2)Y)_{pq}.$$

Then we have as $n \to \infty$

(i) $\mathbf{Var}\,\mathbf{E}_Y H_n = O(n^{-1})$,

(ii) $\mathbf{E}H_n = f(z_1)f^2(z_2) + O(n^{-1})$.

Proof. (i) For the moment we skip the arguments z_1, z_2 in the resolvents and use the same notation $\mathcal{G}$ for $\big(\sum_q G_{jq,kq}\big)_{j,k}$ and $\big(\sum_q G_{qj,qk}\big)_{j,k}$. It follows from (2.1) that

$$\begin{aligned}\mathbf{E}_Y H_n = &\frac{1}{n^4}\sum_{p,q,u,v}\mathcal{G}_{pu}\mathcal{G}_{qv}G_{pq,uv}\\ &+\Bigg[\frac{2}{n^4}\sum_{p,q,u}\mathcal{G}_{pu}(G^2)_{uq,pq}+\frac{1}{n^4}\sum_{p,q,u,v}(G^2)_{pq,uv}G_{pv,uq}\\ &+\frac{2}{n^4}\sum_{p,q,u,v,j}\mathcal{G}_{pu}G_{jq,uv}G_{pq,jv}+\frac{3}{n^4}\sum_{p,q,u,v,j,s}G_{ps,jv}G_{jq,us}G_{pq,uv}\Bigg].\end{aligned} \tag{5.1}$$

It can be shown that the term in the square brackets is of order $O\big(n^{-1/2}\big)$ and that

$$T_n := \frac{1}{n^4}\sum_{p,q,u,v}\mathcal{G}_{pu}\mathcal{G}_{qv}G_{pq,uv} = O(1).$$

Hence, to get (i) it is enough to show that $\mathbf{Var}\,T_n = O\big(n^{-1}\big)$. Similarly to (3.33) we have

$$\mathbf{Var}\,T_n \le \sum_\alpha \mathbf{E}\big\{|T_n-\mathbf{E}_\alpha T_n|^2\big\} = \sum_\alpha \mathbf{E}\big\{|T_n - T_n^\alpha - \mathbf{E}_\alpha(T_n-T_n^\alpha)|^2\big\},$$

where $T_n^\alpha = \frac{1}{n^4}\sum_{p,q,u,v}\mathcal{G}^\alpha_{pu}\mathcal{G}^\alpha_{qv}G^\alpha_{pq,uv}$, $\mathcal{G}^\alpha = \big(\sum_q G^\alpha_{jq,kq}\big)_{j,k}$. We have

$$\begin{aligned}T_n - T_n^\alpha = \frac{1}{n^4}\sum_{p,q,u,v}&\big((\mathcal{G}-\mathcal{G}^\alpha)_{pu}\mathcal{G}_{qv}G_{pq,uv}+\mathcal{G}^\alpha_{pu}(\mathcal{G}-\mathcal{G}^\alpha)_{qv}G_{pq,uv}\\ &+\mathcal{G}^\alpha_{pu}\mathcal{G}^\alpha_{qv}(G-G^\alpha)_{pq,uv}\big) =: S_n^{(1)}+S_n^{(2)}+S_n^{(3)}.\end{aligned}$$

Since $(G-G^\alpha)_{pq,uv} = -(GY_\alpha)_{pq}(G^\alpha Y_\alpha)_{uv}$, applying the Cauchy–Schwartz inequality we get

$$\begin{aligned}\Big|S_n^{(1)}\Big|^2 =&\frac{1}{n^8}\Bigg|\sum_{p,u}\Bigg(\sum_t (GY_\alpha)_{pt}(G^\alpha Y_\alpha)_{ut}\Bigg)\Bigg(\sum_{q,v}\mathcal{G}_{qv}G_{pq,uv}\Bigg)\Bigg|^2\\ &\le \frac{1}{n^8}\sum_{p,t}|(GY_\alpha)_{pt}|^2\sum_{u,t}|(G^\alpha Y_\alpha)_{ut}|^2\sum_{q,v}|\mathcal{G}_{qv}|^2\sum_{p,q,u,v}|G_{pq,uv}|^2 = O\big(n^{-3}\big).\end{aligned}$$

Similarly, $\big|S_n^{(2)}\big|^2 = O\big(n^{-3}\big)$. Also,

$$\Big|S_n^{(3)}\Big|^2 = \frac{1}{n^8}\Bigg|\sum_{p,q,u,v}\mathcal{G}^\alpha_{pu}\mathcal{G}^\alpha_{qv}(GY_\alpha)_{pq}(G^\alpha Y_\alpha)_{uv}\Bigg|^2 \le \frac{1}{n^8}\|\mathcal{B}\|^2_{op}\|GY\|^4 \le \frac{C}{n^8}\|\mathcal{B}\|^2_{op},$$

where $\mathcal{B}$ is a $n^2\times n^2$ matrix such that $\mathcal{B}_{pq,uv} = \mathcal{G}^\alpha_{pu}\mathcal{G}^\alpha_{qv}$. Since for every unit vector $X = (X_{uv})_{u,v=1}^n \in \mathbb{R}^{n^2}$ we have

$$\|\mathcal{B}X\|^2 = \sum_{p,q}\Bigg|\sum_{u,v}\mathcal{G}^\alpha_{pu}\mathcal{G}^\alpha_{qv}X_{uv}\Bigg|^2 = \sum_{p,q}\sum_{u,v,s,t}\mathcal{G}^\alpha_{pu}\mathcal{G}^\alpha_{qv}\overline{\mathcal{G}}^\alpha_{ps}\overline{\mathcal{G}}^\alpha_{qt}X_{uv}X_{st}$$

$$= \sum_{u,v,s,t} (\mathcal{G}^\alpha \overline{\mathcal{G}}^\alpha)_{su} (\mathcal{G}^\alpha \overline{\mathcal{G}}^\alpha)_{tv} X_{uv} X_{st} \leq \operatorname{Tr}(\mathcal{G}^\alpha \overline{\mathcal{G}}^\alpha)^2 \|X\|^2 = O\big(n^5\big)$$

then $\big|S_n^{(3)}\big|^2 = O\big(n^{-3}\big)$. Thus $\mathbf{Var}\, T_n \leq 3 \sum_\alpha \mathbf{E}\Big(\big|S_n^{(1)}\big|^2 + \big|S_n^{(2)}\big|^2 + \big|S_n^{(3)}\big|^2\Big) = O\big(n^{-1}\big)$. This together with (5.1) finishes the proof of the first part.

(ii) It follows from (5.1) that $\mathbf{E} H_n = \mathbf{E} T_n + O\big(n^{-1/2}\big)$. Let

$$f_n^{(3)}(z_1, z_2) := \mathbf{E} T_n(z_1, z_2) = \frac{1}{n^4} \sum_{p,q,u,v} \mathbf{E} G_{pq,uv}(z_2) \mathcal{G}_{qv}(z_1) \mathcal{G}_{pu}(z_2).$$

Repeating the scheme based on (3.13) and (3.15) and omitting the details, we get

$$\begin{aligned} z_2 f_n^{(3)}(z_1, z_2) &= -f_n(z_1) f_n(z_2) + \frac{1}{n^4} \sum_{u,v,p,q} \sum_\alpha \mathbf{E} \frac{Y_{\alpha\, uv} (G^\alpha(z_2) Y_\alpha)_{pv}}{A_\alpha(z_2)} \mathcal{G}^\alpha_{qv}(z_1) \mathcal{G}^\alpha_{pu}(z_2) \\ &\quad + \frac{1}{n^4} \sum_{u,v,p,q,a,b} \sum_\alpha \mathbf{E} \frac{Y_{\alpha\, uv} (G^\alpha(z_1) Y_\alpha)_{pv}}{A_\alpha(z_1)} \big(\mathcal{G}_{qv}(z_1) \mathcal{G}_{pu}(z_2) - \mathcal{G}^\alpha_{qv}(z_2) \mathcal{G}^\alpha_{pu}(z_1)\big) \\ &= -f_n(z_1) f_n(z_2) + \frac{c f_n^{(3)}(z_1, z_2)}{1 + f_n(z_2)} + O\big(n^{-1}\big). \end{aligned}$$

Hence,

$$f_n^{(3)}(z_1, z_2) = f(z_1) f^2(z_2) + O\big(n^{-1}\big).$$

This finishes the proof of the lemma. □

Lemma 5.3. *Let $g_n^{(1)}$ and $g_n^{(2)}$ be defined in (2.4) and (2.5). Then there exists $\eta_0 \in (0, \infty)$ such that we have uniformly in $z_1, z_2 \in \{z \in \mathbb{C} : \Im z \geq \eta_0\}$*

$$\mathbf{Var}\, g_n^{(1)}(z_1, z_2) = O\big(n^{-5/2}\big), \quad \mathbf{Var}\, g_n^{(2)}(z_1, z_2) = O\big(n^{-1/2}\big)$$

as $n \to \infty$.

Proof. Let $V = V(z_1, z_2) := \mathbf{Var}\, g_n^{(1)}(z_1, z_2) = \mathbf{E}\, g_n^{(1)} \overline{g}_n^{(1)\circ}$. By (3.20)–(3.22),

$$z_1 V = \mathbf{E}\Big((-g_n(z_2) + T_1 - T_2) \overline{g}_n^{(1)\circ}\Big) = \mathbf{E}\, T_1 \overline{g}_n^{(1)\circ} - \mathbf{E}\, T_2 \overline{g}_n^{(1)\circ} + O\big(n^{-3}\big), \quad (5.2)$$

where we used that by Lemma 2.1 and Lemma 3.2(ii), $\mathbf{E}\, g_n \overline{g}_n^{(1)\circ} = O\big(n^{-3}\big)$. We have

$$\begin{aligned} \mathbf{E} T_1 \overline{g}_n^{(1)\circ} &= n^{-2} \sum_\alpha \frac{\mathbf{E}\, h_{n\alpha} \overline{g}_n^{(1)\circ}}{\mathbf{E} A_\alpha(z_1)} - \frac{\mathbf{E} h_{n\alpha} \overline{g}_n^{(1)\circ} A^\circ_\alpha(z_1)}{(\mathbf{E} A_\alpha(z_1))^2} + \frac{\mathbf{E} h_{n\alpha} \overline{g}_n^{(1)\circ} A^{\circ\, 2}_\alpha(z_1) A_\alpha^{-1}(z_1)}{(\mathbf{E} A_\alpha(z_1))^2} \\ &=: R_1 + R_2 + R_3. \end{aligned}$$

It follows from the Cauchy-Schwartz inequality, boundedness of $h_{n\alpha} A^{-1}$ and Lemma 3.2 (i) with $p = 4$ that $|R_3| \leq C n^{-2} V^{1/2}$. To treat R_1, we note first that

$$\mathbf{E}\, h_{n\alpha} \overline{g}_n^{(1)\circ} = \mathbf{E}\, (\mathbf{E}_\alpha h_{n\alpha}) \overline{g}_n^{(1)\alpha\circ} + \mathbf{E}\, h^\circ_{n\alpha} (\overline{g}_n^{(1)} - \overline{g}_n^{(1)\alpha}) = V + O\big(n^{-5/2}\big),$$

where we used that similar to (3.11)

$$g_n^{(1)\alpha} = g_n^{(1)} + O\big(n^{-2}\big)$$

and that by Lemma 5.1, $|\mathbf{E}\, h_{n\alpha}^\circ(\overline{g}_n^{(1)} - \overline{g}_n^{(1)\alpha})| \le Cn^{-2}\mathbf{E}\,|h_{n\alpha}^\circ| \le Cn^{-5/2}$. Together with (3.12) this yields

$$|R_1| \le CV + O\big(n^{-5/2}\big).$$

Treating R_2 similarly to R_1 we get

$$\mathbf{E}\, h_{n\alpha}\overline{g}_n^{(1)\circ}A_\alpha^\circ = \mathbf{E}\,(\mathbf{E}_\alpha h_{n\alpha}A_\alpha^\circ)\overline{g}_n^{(1)\alpha\circ} + O\big(n^{-3}\big),$$

where by Lemma 3.2 (i) with $p = 2$ and Lemma 5.1, $\mathbf{E}_\alpha h_{n\alpha}A_\alpha^\circ = O\big(n^{-3/2}\big)$. This leads to

$$|R_2| \le Cn^{-3/2}V^{1/2} + O\big(n^{-3}\big).$$

Hence,

$$\Big|\mathbf{E}T_1\overline{g}_n^{(1)\circ}\Big| \le C\big[V + n^{-3/2}V^{1/2} + n^{-5/2}\big]. \tag{5.3}$$

Consider $\mathbf{E}T_2\overline{g}_n^{(1)\circ}$. It can be written in the form

$$\begin{aligned}\mathbf{E}T_2\overline{g}_n^{(1)\circ} &= \frac{1}{n^3}\sum_\alpha \mathbf{E}\frac{H_{n\alpha}\overline{g}_n^{(1)\alpha\circ}}{A_\alpha(z_1)A_\alpha(z_2)} + \frac{1}{n^3}\sum_\alpha \mathbf{E}\frac{H_{n\alpha}(\overline{g}_n^{(1)} - \overline{g}_n^{(1)\alpha})^\circ}{A_\alpha(z_1)A_\alpha(z_2)}\\ &= \frac{1}{n^3}\sum_\alpha \frac{\mathbf{E}(\mathbf{E}_\alpha H_{n\alpha})^\circ\overline{g}_n^{(1)\alpha\circ}}{\mathbf{E}A_\alpha A_\alpha} - \frac{1}{n^3}\sum_\alpha \mathbf{E}\frac{H_{n\alpha}\overline{g}_n^{(1)\alpha\circ}(A_\alpha A_\alpha)^\circ}{(A_\alpha A_\alpha)\mathbf{E}(A_\alpha A_\alpha)} + O\big(n^{-3}\big).\end{aligned}$$

Since $H_{n\alpha} = O(1)$ and by Lemmas 3.2(iii) and 5.2 we have $\mathbf{Var}(A_\alpha A_\alpha) = O\big(n^{-2}\big)$ and $\mathbf{Var}\,\mathbf{E}_\alpha H_{n\alpha} = O\big(n^{-1}\big)$, then

$$\Big|\mathbf{E}T_2\overline{g}_n^{(1)\circ}\Big| \le C\big[n^{-2}V^{1/2} + n^{-3}\big].$$

This, (5.2) and (5.3) lead to

$$\eta V \le |z_1 V| \le C\big[V + n^{-3/2}V^{1/2} + n^{-5/2}\big].$$

Choosing η big enough we get $V - C_1n^{-3/2}V^{1/2} - C_2n^{-5/2} \le 0$, where $C_1, C_2 > 0$. Hence $V = \mathbf{Var}\, g_n^{(1)} = O\big(n^{-5/2}\big)$.

The proof of the second part of the lemma follows the same scheme, for $g_n^{(2)}$ this scheme is based on (3.30)–(3.31). This finishes the proof of the lemma. □

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References

[1] G.W. Anderson and O. Zeitouni, *CLT for a band matrix model*. Probab. Theory Related Fields **134** (2006), 283–338.

[2] L.V. Arharov, *Limit theorems for the characteristic roots of a sample covariance matrix*. Doklady Akademii Nauk SSSR **12** (1971), 1206–1209.

[3] Z.D. Bai and J.W. Silverstein, *CLT for linear spectral statistics of large dimensional sample covariance matrices*. Ann. Prob. **32** (2004), 553–605.

[4] Z.D. Bai and J.W. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, Springer, New York, 2010.

[5] Z.D. Bai and W. Zhou, *Large sample covariance matrices without independence structures in columns*, Statistica Sinica **18** (2008), No. 2, 425–442.

[6] Z. D. Bai, X. Wang, and W. Zhou, *Functional CLT for sample covariance matrices*, Bernoulli **16** (2010), No. 4, 1086–1113.

[7] S.G. Bobkov, *Remarks on the growth of L^p-norms of polynomials*, Geometric Aspects of Functionnal Analysis, Lecture Notes in Math., **1745**, Springer, Berlin, 2000, 27–35.

[8] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Lecture Notes in Math., **1469**, 1991, 127–137.

[9] T. Cabanal-Duvillard, *Fluctuations de la loi empirique de grandes matrices aléatoires*, Ann. Inst. H. Poincaré Probab. Statist. **37** (2001), No. 3, 73–402 (French).

[10] S.W. Dharmadhikari, V. Fabian, and K. Jogdeo, *Bounds on the moments of martingales*, Ann. Math. Statist. **39** (1968), 1719–1723.

[11] V. Girko, *Theory of Stochastic Canonical Equations*, **I**, **II**, Kluwer, Dordrecht, 2001.

[12] O. Guédon, A. Lytova, A. Pajor, and L. Pastur, *The Central Limit Theorem for linear eigenvalue statistics of the sum of rank one projections on independent vectors*, Spectral Theory and Differential Equations, V.A. Marchenko 90th Anniversary Collection (Eds.: E. Khruslov and L. Pastur, Amer. Math. Soc., Providence, RI, 2014).

[13] A. Guionnet, *Large deviations, upper bounds, and central limit theorems for non-commutative functionals of Gaussian large random matrices*, *Ann. Inst. H. Poincaré Probab. Statist.* **38** (2002), 341–384.

[14] K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices, *Duke Math. J.* **91** (1998), 151–204.

[15] D. Jonsson, *Some limit theorems for the eigenvalues of a sample covariance matrix*, *J. Multivariate Anal.* **12** (1982), 1–38.

[16] B. Khoruzhenko, A. Khorunzhy, and L. Pastur, *Asymptotic properties of large random matrices with independent entries*, J. Physics A: Math. and General **28** (1995), 31–35.

[17] B. Landon, P. Sosoe, *Almost-optimal bulk regularity conditions in the CLT for Wigner matrices*, preprint, https://arxiv.org/abs/2204.03419.

[18] A. Lytova, *Central limit theorem for linear eigenvalue statistics for a tensor product version of sample covariance matrices*, J. Theoret. Probab. **31** (2018), No. 2, 1024–1057.

[19] A. Lytova and L. Pastur, *Central limit theorem for linear eigenvalue statistics of random matrices with independent entries*, Ann. Prob. **37** (2009), No. 5, 1778–1840.

[20] V. Marchenko and L. Pastur, *The eigenvalue distribution in some ensembles of random matrices*, Math. USSR Sbornik, **1** (1967), 457–483.

[21] J. Najim and J. Yao, *Gaussian fluctuations for linear spectral statistics of large random covariance matrices.* Ann. Appl. Probab. **26** (2016), No. 3, 1837–1887.

[22] A. Pajor and L. Pastur, *On the limiting empirical measure of eigenvalues of the sum of rank one matrices with log-concave distribution*, Studia Math. **195** (2009), No. 1, 11–29.

[23] G.M. Pan and W. Zhou, *Central limit theorem for signal-to-interference ratio of reduced rank linear receiver*, Ann. Appl. Probab., **18** (2008), 1232–1270.

[24] L. Pastur, L. and M. Shcherbina, *Eigenvalue Distribution of Large Random Matrices*, Mathematical Surveys and Monographs., Amer. Math. Soc., **171**, 2011.

[25] M. Shcherbina, *Central limit theorem for linear eigenvalue statistics of Wigner and sample covariance random matrices*, J. Math. Phys. Anal. Geom. **7** (2011), No. 2, 176–192.

[26] P. Sosoe and P. Wong, *Regularity conditions in the CLT for linear eigenvalue statistics of Wigner matrices*, Adv. Math. **249** (2013), 37–87.

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Про ЦГТ для для лінійних статистик власних значень тензорної моделі вибіркових коваріаційних матриць

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В [18] було доведено центральну граничну теорему (ЦГТ) для лінійних статистик власних значень $\operatorname{Tr}\varphi(M_n)$ вибіркових коваріаційних матриць $M_n = \sum_{\alpha=1}^{m} \mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)} (\mathbf{y}_\alpha^{(1)} \otimes \mathbf{y}_\alpha^{(2)})^T$, де $(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})_\alpha$ є незалежними копіями вектора $\mathbf{y} \in \mathbb{R}^n$, що задовольняє умови $\mathbf{E}\mathbf{y}\mathbf{y}^T = n^{-1}I_n$, $\mathbf{E}y_i^2 y_j^2 = (1 + \delta_{ij}d)n^{-2} + a(1 + \delta_{ij}d_1)n^{-3} + O(n^{-4})$ для деяких $a, d, d_1 \in \mathbb{R}$. Було показано, що для достатньо гладких тестових функцій φ маємо $\mathbf{Var}\operatorname{Tr}\varphi(M_n) = O(n)$, коли $m, n \to \infty$, $m/n^2 \to c > 0$, крім того

$(\mathrm{Tr}\,\varphi(M_n) - \mathbf{E}\,\mathrm{Tr}\,\varphi(M_n))/\sqrt{n}$ збігається за розподілом до гаусівської випадкової величини з нульовим середнім та дисперсією $V[\varphi]$ пропорційною $a+d$. Зокрема, якщо $\mathbf{y}$ рівномірно розподілено на одиничній сфері, то $a + d = 0$ і $V[\varphi] = 0$. У цій роботі ми показуємо, що в цьому випадку $\mathbf{Var}\,\mathrm{Tr}(M_n - zI_n)^{-1} = O(1)$, так що ЦГТ має бути справедливою для самих лінійних статистик власних значень без нормалізувального коефіцієнта (на відміну від випадку відповідних гаусівських вибіркових коваріаційних матриць).

Ключові слова: вибіркові коваріаційні матриці, центральна гранична теорема, лінійна статистика власних значень