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Long-time asymptotics for Toda shock waves in the modulation region

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To Vladimir Aleksandrovich Marchenko with deep admiration on the occasion of his 100th birthday

We show that a Toda shock wave is asymptotically close to a modulated finite gap solution in the right modulation region. We previously derived formulas for the leading terms of the asymptotic expansion of this shock wave in all five principal regions and conjectured that in two modulation regions the next term is of order $O(t^{-1})$. In the present paper we prove this fact and investigate how resonances and eigenvalues influence the leading asymptotic behaviour. Our main contribution is the solution of the local parametrix Riemann–Hilbert problems and a rigorous justification of the analysis. In particular, this involves the construction of a proper singular matrix model solution.

Key words: Toda equation, Riemann–Hilbert problem, steplike, shock

Mathematical Subject Classification 2020: Primary 37K40, 35Q53; Secondary 37K45, 35Q15

Mathematical Subject Classification 2020: codes

1. Introduction

A Toda shock wave is a solution of the initial value problem for the Toda lattice [34, 35]

$$\begin{aligned} \dot b(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2),\\ \dot a(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \quad (n,t) \in \mathbb{Z}\times\mathbb{R}_+, \end{aligned} \tag{1.1}$$

with a steplike initial profile

$$\begin{aligned} &a(n,0)\to a, && b(n,0)\to b, && \text{as } n\to-\infty,\\ &a(n,0)\to \frac{1}{2}, && b(n,0)\to 0, && \text{as } n\to+\infty, \end{aligned} \tag{1.2}$$

where $a>0$, $b\in\mathbb{R}$ satisfy the condition

$$b + 2a < -1. \tag{1.3}$$

Originally the Toda shock wave was associated with symmetric initial data [36]

$$a(n,0)=a(-n,0)\to\frac{1}{2},\ b(-n,0)=-b(n,0)\to\pm b,\ n\to\pm\infty,\ b>1. \tag{1.4}$$

Such a model is closely related to the motion of driving particles in a container filled with gas ahead of a piston compressing the content of the container. From the viewpoint of spectral theory, this model corresponds to two non-intersecting spectral intervals of equal length, which are associated with left and right background Jacobi operators with constant coefficients; the left background spectrum lies to the left. It is therefore natural to extend the notion of a shock wave to background spectra of different lengths. By scaling and shifting the spectral parameter, one can always assume that the right spectrum coincides with the interval $[-1,1]$.

We are interested in the long-time behavior of the solution of the Cauchy problem (1.1)–(1.3), also referred to as *Toda shock problem*. This problem was first investigated for initial data (1.4) on a physical level of rigor by Bloch and Kodama [3,4] using the Whitham approach. Venakides, Deift and Oba showed in [36] (see also [25]) based on the Lax–Levermore approach that in a middle region of the half plane $(n,t)\in\mathbb{Z}\times\mathbb{R}_+$, the solution to (1.1), (1.4) is asymptotically close to a periodic solution of period two with spectrum $[-1-b,1-b]\cup[-1+b,1+b]$. The classical inverse scattering transform was used to analyze the soliton regions [35] and a transition region behind the leading wave front, where the train of asymptotic solitons was evaluated [5,6]. The nonlinear steepest descent (NSD) method applied to the Toda shock problem yields the most interesting results in the regime $n\to\infty$, $t\to\infty$ with the ratio n/t close to a constant. It was singled out by Deift as an outstanding open problem in [7]. In [18] three of us showed that for the solution of (1.1)–(1.3) there are five principal regions in the (n,t) half plane with different qualitative behavior: the left and right soliton regions, the left and right modulation regions and the elliptic region or middle region first discussed in [36] (see [29] for an overview).

1.1. The main asymptotic regions. The continuous spectrum of the underlying Jacobi operator (the Lax operator for the Toda lattice)

$$\begin{aligned}H(t)y(n)&=a(n-1,t)y(n-1)+b(n,t)y(n)+a(n,t)y(n+1)\\&=\lambda y(n),\quad \lambda\in\mathbb{C},\end{aligned} \tag{1.5}$$

consists of two intervals $[b-2a,b+2a]$ and $[-1,1]$ which are the spectra of the left and right constant background operators,

$$H_\ell y(n)=a\,y(n-1)+a\,y(n+1)+b\,y(n),\qquad n\in\mathbb{Z}, \tag{1.6}$$

$$H_r y(n)=\frac{1}{2}y(n-1)+\frac{1}{2}y(n+1),\qquad n\in\mathbb{Z}. \tag{1.7}$$

Two parameters z and ζ are associated with the right and left background; they are connected with the spectral parameter λ by the Joukowsky transform

$$\lambda=\frac{1}{2}\left(z+z^{-1}\right)=b+a\left(\zeta+\zeta^{-1}\right),\quad |z|\le 1,\quad |\zeta|\le 1. \tag{1.8}$$

In the NSD approach, the behavior of the solution essentially depends on the location of the stationary phase points, that is, the nodal points of the level lines where the real part of the phase function vanishes. In our case both the right phase function

$$\Phi(z,\xi) = \frac{z - z^{-1}}{2} + \xi \log z, \quad \xi := \frac{n}{t}, \tag{1.9}$$

and the left phase function

$$\Phi_\ell(z,\xi) = a(\zeta^{-1} - \zeta) - \xi \log \zeta, \tag{1.10}$$

take part in this characterization. Two soliton regions corresponding to the domains of n and t for which $\frac{n}{t} > \xi_{cr}$ or $\frac{n}{t} < \xi_{cr,1}$ are naturally identified, where the solution to (1.1)–(1.3) is asymptotically close as $t \to \infty$ to the respective constant background solution plus a finite number of solitons generated by the discrete spectrum (if any). Everywhere in this subsection we take positive values of square roots. The right leading wave front

$$\xi_{cr} = \frac{\sqrt{(2a-b)^2 - 1}}{\log(2a - b + \sqrt{(2a-b)^2 - 1})} \tag{1.11}$$

corresponds to the case when the level line $\operatorname{Re}\Phi(z,\xi) = 0$ crosses the real line at the point $z = b - 2a + \sqrt{(2a-b)^2 - 1}$ (the image under the Joukowsky map $z(\lambda)$ of the left endpoint $b - 2a$ of the left spectrum). The left wave front

$$\xi_{cr,1} = \frac{\sqrt{(1-b)^2 - 4a^2}}{\log(2a) - \log\left(1 - b + \sqrt{(1-b)^2 - 4a^2}\right)} \tag{1.12}$$

is the value where the stationary phase point of the left phase Φ_ℓ coincides with the right endpoint of the right spectrum, $z = 1$. The region $\xi_{cr,1} < \frac{n}{t} < \xi_{cr}$ consists of three sectors with different type of quasi-periodic behavior of the solution. These sectors are divided by rays corresponding to the critical values $\xi'_{cr,1}$ and ξ'_{cr} of the parameter ξ such that $\xi_{cr,1} < \xi'_{cr,1} < \xi'_{cr} < \xi_{cr}$. In the modulation regions $\xi_{cr,1} < \frac{n}{t} < \xi'_{cr,1}$ and $\xi'_{cr} < \frac{n}{t} < \xi_{cr}$, the main terms of the expansion of the solution (with respect to large t) are modulated elliptic waves ([18]). In the middle region $\xi'_{cr,1} < \frac{n}{t} < \xi'_{cr}$, the solution is asymptotically close to a finite gap (two band) solution of the Toda equation if the discrete spectrum is absent in the gap $(b + 2a, -1)$.

1.2. Modulated elliptic waves. A finite gap solution of the Toda equation is completely characterized by the geometry of its continuous spectrum and by the initial Dirichlet divisor on the hyperelliptic Riemann surface associated with the spectrum. In the shock problem we deal with spectra consisting of two bands and one initial Dirichlet eigenvalue in the gap between the bands. The sign necessary to lift this eigenvalue to the Riemann surface is the sign of the respective half-axis, where the corresponding eigenvector is supported.

Let us first discuss the region $\xi \in [\xi'_{cr}, \xi_{cr})$. For any such ξ consider a point $\gamma(\xi) \in (b - 2a, b + 2a]$ which moves monotonically and continuously with respect to ξ covering the interval $(b-2a, b+2a]$, with $\gamma(\xi'_{cr}) = b+2a$. Associated with the set

$$\sigma(\xi) := [b-2a, \gamma(\xi)] \cup [-1,1] \tag{1.13}$$

is the two-sheeted Riemann surface $\mathbb{M}(\xi)$. The upper sheet of $\mathbb{M}(\xi)$ is treated as the complex plane of the spectral parameter λ with cuts along $\sigma(\xi)$. Let $\Omega(\lambda,\xi)$ and $\omega(\lambda,\xi)$ be the normalized Abel integrals of the second and the third kind on the upper sheet of $\mathbb{M}(\xi)$, with zero $\mathfrak{a}$-periods along the gap $(\gamma(\xi), -1)$. The linear combination $g(\lambda,\xi) = \Omega(\lambda,\xi) + \xi\omega(\lambda,\xi)$ is then another Abel integral with zero $\mathfrak{a}$-period. The nominator of the function $\frac{\partial g(\lambda,\xi)}{\partial \lambda}$ has two real zeros $\nu(\xi)$ and $\mu(\xi)$ with at least one zero in the gap, say $\mu(\xi) \in (\gamma(\xi), -1)$. Moreover, for $\lambda \to \infty$ the function $g(\lambda,\xi)$ has the same asymptotic behavior as the phase function $\Phi(z(\lambda),\xi)$ up to a constant term. These properties hold for any choice of $\gamma(\xi)$. The peculiarity of our choice for $\gamma(\xi)$ when $\xi \in [\xi'_{cr}, \xi_{cr})$ is that we require the second zero $\nu(\xi)$ of $g(\lambda,\xi)$ to match with $\gamma(\xi)$, that is,

$$\frac{\partial}{\partial\lambda}\left(\Omega(\lambda,\xi) + \xi\,\omega(\lambda,\xi)\right) = \frac{(\lambda-\mu(\xi))\sqrt{\lambda-\gamma(\xi)}}{\sqrt{(\lambda-b+2a)(\lambda^2-1)}}. \tag{1.14}$$

Such a point $\gamma(\xi)$ is unique for every ξ, and $\gamma(\xi)$ satisfies the same continuity and monotonicity properties as described above; it defines the moving edge of the Whitham zone (cf. [4]). From this construction it follows that

$$\xi'_{cr} = -2a - \frac{\int_{b+2a}^{-1}\lambda\, Q(\lambda)d\lambda}{\int_{b+2a}^{-1} Q(\lambda)d\lambda}, \qquad Q(\lambda) = \sqrt{\frac{\lambda-b-2a}{(\lambda-b+2a)(\lambda^2-1)}},$$

and $\sigma(\xi'_{cr}) = [b-2a, b+2a] \cup [-1,1]$. In contrast to the phase function $\Phi(z,\xi)$, the properties of $g(\lambda,\xi)$ allow us to apply the lens construction for the RHP approach to all contours whenever needed. For this reason we replace $\Phi(z,\xi)$ by $g(\lambda,\xi)$, which plays the role of the g-function ([11]) in the NSD method. Given $\gamma(\xi)$, let

$$\left\{\hat{a}\big(n,t,\xi\big), \hat{b}\big(n,t,\xi\big)\right\} \tag{1.15}$$

be the finite gap solution for the Toda lattice associated with the spectrum (1.13) and with an initial Dirichlet eigenvalue defined via the initial scattering data for (1.2), (1.3) by the Jacobi inversion problem. This Dirichlet eigenvalue was computed in [18, Equ. (5.25)]. The functions $\left\{\hat{a}\big(n,t,\frac{n}{t}\big), \hat{b}\big(n,t,\frac{n}{t}\big)\right\}$ are then well defined in the region

$$\left\{(n,t) \in \mathbb{Z}\times\mathbb{R}_+ : \tfrac{n}{t} \in [\xi'_{cr}, \xi_{cr}-\varepsilon]\right\}, \tag{1.16}$$

where ε is arbitrary small. In analogy to the KdV shock case, we call them modulated elliptic waves. They are the main terms of the asymptotic expansion for the Toda shock wave with respect to large t in the region (1.16).

The middle region $\frac{n}{t} \in (\xi'_{cr,1}, \xi'_{cr})$, where

$$\xi'_{cr,1} = b+1 - \frac{\int_{b+2a}^{-1}\lambda\, Q_1(\lambda)d\lambda}{\int_{b+2a}^{-1} Q_1(\lambda)d\lambda}, \qquad Q_1(\lambda) = \sqrt{\frac{\lambda+1}{((\lambda-b)^2-4a^2)(\lambda-1)}},$$

is associated with the gap $(b+2a,-1)$, although we cannot claim that the stationary phase point of $\Phi(z(\lambda),\xi)$ for such ξ is located in this gap. A suitable g-function here is simply $\Omega(\lambda)+\xi\,\omega(\lambda)$, where $\Omega(\lambda)$ and $\omega(\lambda)$ are the Abel integrals as defined above associated with the spectrum

$$\sigma(\xi_{cr}^\prime)=\sigma(\xi_{cr,1}^\prime)=[b-2a,b+2a]\cup[-1,1].$$

The level line $\operatorname{Re} g(\lambda,\xi)=0$ in this case intersects the real axis at a point $\lambda_0(\xi)$ inside the gap, which moves continuously along the gap when ξ moves along $(\xi_{cr,1}^\prime,\,\xi_{cr}^\prime)$. The main asymptotic term for the solution of (1.1)–(1.3) is the classical two band solution of the Toda lattice $\{\hat a(n,t),\hat b(n,t)\}$ with the initial Dirichlet eigenvalue depending on ξ if the discrete spectrum inside the gap is nonempty. The phase summand in the theta function representation for this two band solution (cf. [34, equation (9.48)]) contains information on the initial scattering data for (1.1)–(1.3), and undergoes a shift when $\lambda_0(\xi)$ hits a point of the discrete spectrum. This agrees with the effect of adding a single eigenvalue as can be done using the double commutation method (cf. [21] and [34, Lem. 11.26]). For $\frac{n}{t}=\xi_{cr}^\prime$, the solution (1.15) coincides with the two band solution $\{\hat a(n,t),\hat b(n,t)\}$ above. The same is valid for the second boundary of the middle region $\xi=\xi_{cr,1}^\prime$, because the construction of the g-function in $[\xi_{cr,1}^\prime,\xi_{cr,1})$ is the same as for the modulated elliptic waves above. It is associated with $[b-2a,b+2a]\cup[\gamma(\xi),1]$ where $\gamma(\xi)\in[-1,1)$.

1.3. Main result. In [18] we derived the precise formula for the modulated finite-gap solution (1.15) using the NSD approach for vector RHPs and more restrictive initial data: we assumed that there are no resonances on the edges of the spectrum of $H(t)$ and that the discrete spectrum consists of a single point in the spectral gap. We did not justify the asymptotic expansion for the solution of (1.1)–(1.3) and only conjectured that the next term is of order $O(t^{-1})$. The aim of the present paper is to prove this fact by solving local parametrix problems and finishing the conclusive analysis. We will implement a rigorous asymptotic analysis in the region

$$\mathcal{D}:=\{(n,t)\in\mathbb{Z}\times\mathbb{R}_+:\tfrac{n}{t}\in\mathcal{I}_\varepsilon:=[\xi_{cr}^\prime+\varepsilon,\xi_{cr}-\varepsilon]\}\tag{1.17}$$

to prove

Theorem 1.1. *For $(n,t)\in\mathcal{D}$, $n,t\to\infty$ uniformly with respect to $\frac{n}{t}\in\mathcal{I}_\varepsilon$, the Toda shock wave $\{a(n,t),b(n,t)\}$ given by (1.1)–(1.3) and satisfying (2.1), (2.10) has the following asymptotic behavior:*

$$a(n,t)^2+a(n-1,t)^2=\hat a\big(n,t,\tfrac{n}{t}\big)^2+\hat a\big(n-1,t,\tfrac{n}{t}\big)^2+O(t^{-1}),$$
$$b(n,t)=\hat b\big(n,t,\tfrac{n}{t}\big)+O(t^{-1}),$$

where $\{\hat a(n,t,\xi),\hat b(n,t,\xi)\}$ is the finite gap solution of the Toda lattice associated with the spectrum $[b-2a,\gamma(\xi)]\cup[-1,1]$ and the initial divisor $(\lambda(0,0),\pm)$ which is the only zero of the function $\theta(2A(z)-\frac12-\frac{\Delta}{2\pi}\mid 2\tau)$ (see (2.7), (2.19), (3.23), (3.24), (4.7)) on the Riemann surface $\mathbb{M}(\xi)$ with projection on the gap $[\gamma(\xi),-1]$.

For the remaining two regions the asymptotic analysis can be done similarly, see Section 8. However, we essentially improve the estimate on the error term in the middle region $\frac{n}{t} \in (\xi'_{cr,1}, \xi'_{cr})$ in [14], where we also describe the influence of resonances and the discrete spectrum in the gap on the asymptotic.

1.4. Remarks on the method of proof.

- As in [18] we deal with vector statements of RHPs. They are more natural in the Toda case than matrix statements, because the matrix statements are ill-posed for certain values of n and t in the class of invertible matrices with L^2-integrable singularities. This fact for Toda can be established similarly as for the KdV shock wave in [19]. The vector statement requires additional symmetries to be posed on the contours, jump matrices and on the solution itself to guarantee uniqueness of the solution.
- In [18] the RHP was stated in terms of the spectral variable λ, that is, on the two-sheeted Riemann surface with sheets glued along the cuts $[b-2a, b+2a]\cup[-1,1]$. In the present paper we use the standard approach via the Joukowsky map $z(\lambda)$ in (1.8): the upper sheet of the Riemann surface is identified with the inner part of the circle $|z|<1$ without the cut $[z(b-2a), z(b+2a)]$, and the lower sheet with $|z|>1$ without $[(z(b+2a))^{-1}, (z(b-2a))^{-1}]$. Here $z(\lambda)$ denotes the image of λ under the Joukowsky transform mapping the λ-plane into the unit circle. We formulate the initial RHP and reformulate all transformations leading to the model RHP in terms of z, taking into account the discrete spectrum and resonances, which produce singularities in the jump matrix and require a more sophisticated analysis and additional proofs of the uniqueness results. We also solve the vector model problem independently, and derive the asymptotics using the new, more convenient formula (2.13).
- To prove the asymptotics within the framework of the NSD, the traditional approach first requires to solve the matrix analog of the model RHP, then to find matrix solutions of the local parametrix RHPs, and finally to derive the singular integral equation for the error vector and to estimate its norm. But with this approach one fails to obtain uniform estimates in n and t for both KdV and Toda due to the singular behavior of the matrix model solution ([19]). An alternative approach was proposed in [33] for KdV: instead of constructing a matrix model solution, it evaluates the smallness of the difference between initial and model vector solutions as solutions of the associated singular integral equations with slightly different kernels. But this approach does not seem to work for Toda, because we have less control on the behavior of the vector solutions $m(\lambda)=(m_1(\lambda), m_2(\lambda))$ of the initial and model RHPs at infinity. Indeed, for KdV one knows that $m(\lambda)\to(1,1)$ as $\lambda\to\infty$, but for Toda we only know that $m_1(\lambda)m_2(\lambda)\to 1$, which is not sufficient to apply the technique of [33].

  In [30,31] a singular matrix model solution is proposed for the KdV shock case, which has a pole at $\lambda=0$, but the respective error vector does not have pole-like singularities. We use this idea to construct the matrix model solution for Toda shock. It has simple poles at the edges of the right background spectrum, but the error vector does not (see Theorem 5.5).

- To show that the expansion with respect to z of the product of components of the initial RHP solution is asymptotically close to the expansion for the model RHP solution, we have to prove that the error vector $\nu(z)$ (cf. (6.16), (6.14)) is asymptotically close to the vector $(1,1)$ as $z \to 0$ up to a term $O(t^{-1})$. To achieve this, we carry out all conjugations and deformations related to NSD by strictly respecting the symmetry $m(z) = m(z^{-1})\sigma_1$ and normalization $m_1(0)m_2(0) = 1$ for all subsequent RHPs. Here σ_1 is the first Pauli matrix. To preserve $\nu(z^{-1}) = \nu(z)\sigma_1$ for the error vector, in the respective singular integral equation we have to use a special matrix Cauchy kernel with entries that have zeros at $z = 0$ and $z = \infty$ (cf. [27], Equ. (B.8)).

1.5. Visualisation. In Fig. 1.1 the numerical solution corresponding to the pure step initial data $a(n,0) = \frac{1}{2}$, $b(n,0) = 0$ as $n \geq 0$ and $a(n,0) = 1$, $b(n,0) = -4$ as $n < 0$ is plotted at time $t_0 = 799$ for $n = -5000, \ldots, 5000$.

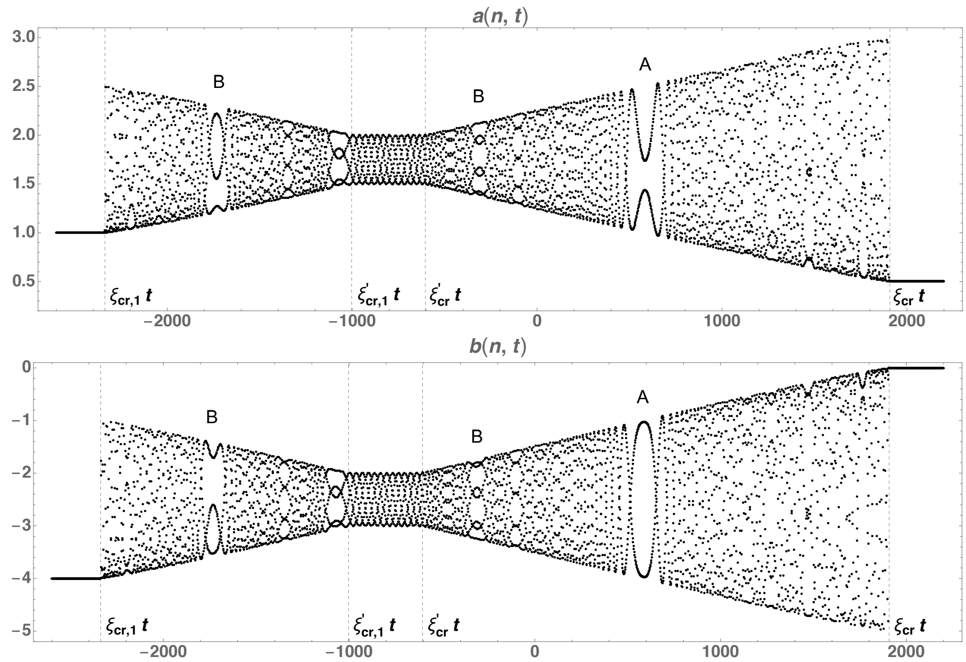

Fig. 1.1: Solutions $a(n,t_0)$, $b(n,t_0)$ at $t_0 = 799$ for $\sigma(H_\ell) = [-6,-2]$ and $\sigma(H_r) = [-1,1]$. The critical values (times t) are plotted as vertical lines, $\xi_{cr} t_0 = 1907.65$, $\xi'_{cr} t_0 = -604.39$, $\xi'_{cr,1} t_0 = -1002.66$, and $\xi_{cr,1} t_0 = -2336.92$.

We can clearly distinguish the left and right regions corresponding to the modulated elliptic waves and the middle region, where the quasi-periodic finite gap solution is associated with the full spectrum $[-6,-2] \cup [-1,1]$. Another phenomenon is nicely visualized in an area as for example A.

Indeed, recall that quasi-periodicity or pure periodicity of the finite gap solution with spectrum $[c,d] \cup [-1,1]$ with $(c < d < -1)$ depends on the ratio r of the frequencies of its quasi-momentum $\omega(\lambda)$, associated with this spectrum,

$$r = \frac{\omega(1) - \omega(-1)}{\omega(d) - \omega(c)}.$$

If $r = \frac{p}{q} \in \mathbb{Q}$, where $\frac{p}{q}$ is an irreducible fraction, then the finite gap solution is periodic with period $p + q$ (for $d - c = 2$ we get a solution of period 2).

Now, if $\omega(\lambda, \xi)$ is the quasi-momentum associated with $[b - 2a, \gamma(\xi)] \cup [-1, 1]$ or $[b - 2a, b + 2a] \cup [\gamma(\xi), 1]$ and

$$r(\xi) = \frac{\omega(1, \xi) - \omega(-1, \xi)}{\omega(\gamma(\xi), \xi) - \omega(b - 2a, \xi)} \quad \text{or} \quad r(\xi) = \frac{\omega(1, \xi) - \omega(\gamma(\xi), \xi)}{\omega(b + 2a, \xi) - \omega(b - 2a, \xi)}$$

is rational, then for such a point ξ the solution is periodic. Recall that if $r(\xi_1)$ is non rational, then the functions $a(n, t_0, \xi_1)$ and $b(n, t_0, \xi_1)$, $t_0 = 799$, take the values in a vicinity of any point between their maximum and minimum. Of course, the set of those ξ which correspond to the periodic solutions is everywhere dense in $\mathbb{R}$. If the period is quite large, there is no visual difference between periodic and almost-periodic (non-periodic) solutions. But small periods are observable. We see that the middle part of area A corresponds to a modulated wave near ξ_0 such that $\gamma(\xi_0) = -4$, where the solution is of period 2. It corresponds to equal lengths of two spectral bands. Note that the solution here appears to consist of two solid lines because due to scaling, the points n and $n + 2$ are visually glued. By perturbation theory arguments, in a small vicinity of $\xi_0 = \frac{n_0}{t_0}$, the modulated wave $a(n, t_0, \xi)$, $b(n, t_0, \xi)$ differs little from the periodic function $a(n, t_0, \xi_0)$, $b(n, t_0, \xi_0)$. This explains the appearance of the solution in area A. In the middle of areas B the solutions are of periods 3 and 6 respectively, and this effect is also observable.

2. Statement of the Riemann–Hilbert problem

In this section we cover some basic facts of the inverse scattering transform and fix notation. For a detailed account of scattering theory for Jacobi operators with steplike backgrounds see [15–17], with zero background see [34, Chapter 10].

Under the assumption that the coefficients of the initial data (1.2) tend to the background constants sufficiently fast, the spectrum of the Jacobi operator $H(t)$ consists of an absolutely continuous part of multiplicity one,

$$\sigma_{ac}(H) = [b - 2a, b + 2a] \cup [-1, 1],$$

plus a finite simple pure point part,

$$\{\lambda_j : j = 1, \dots, N\} \subset \mathbb{R} \setminus \sigma_{ac}(H).$$

To simplify further considerations we assume that the initial data (1.2) decay to their backgrounds exponentially fast

$$\sum_{n=1}^{\infty} \mathrm{e}^{\rho n} \big(|a(-n, 0) - a| + |b(-n, 0) - b| + |a(n, 0) - \tfrac{1}{2}| + |b(n, 0)| \big) < \infty, \quad (2.1)$$

where $\rho > 0$ is a positive number. The operator $H(t)$ is self-adjoint and the diagonal elements of its Green's function $G(\lambda, n, m, t)$ (that is, the kernel of the resolvent operator $(H(t) - \lambda \mathbb{I})^{-1}$) have the following expansion as $\lambda \to \infty$ ([34, Sec. 6.1])

$$G(\lambda, n, n, t) = -\frac{1}{\lambda} \left(1 + \frac{b(n,t)}{\lambda} + \frac{a(n,t)^2 + a(n-1,t)^2 + b(n,t)^2}{\lambda^2} + O(\lambda^{-3}) \right). \tag{2.2}$$

As mentioned in the introduction, instead of the spectral parameter λ we use its Joukowsky transformation,

$$z(\lambda) = \lambda - \sqrt{\lambda^2 - 1},$$

which maps the two sides of the cut along the interval $[-1, 1]$ to the unit circle $\mathbb{T} = \{z : |z| = 1\}$. The map $z \mapsto \lambda$ is one-to-one between the closed domains $\operatorname{clos}(\mathcal{Q})$ and $\operatorname{clos}(\mathbb{C} \setminus \sigma_{ac}(H(t)))$, where

$$\mathcal{Q} := \{z : |z| < 1\} \setminus [q_1, q],$$

We define the closure by adding the upper and lower sides of the cuts as distinct points to the boundary. The points $q_1 = z(b + 2a)$ and $q = z(b - 2a)$ correspond to the edges of $\sigma(H_\ell)$ and $z = -1$ and $z = 1$ correspond to the edges of $\sigma(H_r)$. The eigenvalues λ_j are mapped to $z_j \in ((-1, 0) \cup (0, 1)) \setminus [q_1, q]$, for $j = 1, \dots, N$; we denote them by

$$\sigma_d = \{z_j, j = 1, ..., N\}.$$

In addition to z we also use the Joukowsky transformation $\zeta = \zeta(\lambda)$ associated with the left background and given by (1.8).

Recall that the Jacobi equation (1.5) has two Jost solutions $\psi(z, n, t)$, $\psi_\ell(z, n, t)$ for each $z \in \mathcal{Q}$ with asymptotic behavior

$$\lim_{n \to \infty} z^{-n} \psi(z, n, t) = 1, \quad |z| \le 1; \qquad \lim_{n \to -\infty} \zeta^n \psi_\ell(z, n, t) = 1, \quad |\zeta| \le 1.$$

As functions of z they have slightly different properties on $\mathcal{Q}$. Indeed, since $\zeta^n \psi_\ell(z, n, t)$ is in fact an analytic function of ζ as $|\zeta| < 1$, this function has complex conjugated values on the sides of the cut along $[q_1, q]$, which we denote as $[q_1, q] \pm \mathrm{i}0$. It has equal real values at $z, z^{-1} \in \mathbb{T}$. The function $\psi(z, n, t)$ has complex conjugated values at conjugated points of $\mathbb{T}$, but

$$\psi(z - \mathrm{i}0, n, t) = \psi(z + \mathrm{i}0, n, t) \in \mathbb{R}, \quad \text{for} \quad z \in [q_1, q].$$

Recall that $\psi_\ell(z, n, 0)$ admits a representation via the transformation operator

$$\psi_\ell(z, n, 0) = \sum_{-\infty}^{n} K(n, m) \zeta^{-m}, \quad |\zeta| \le 1.$$

Under condition (2.1) in the domain $1 \le |\zeta| < \mathrm{e}^\rho$ there exists an analytic function which is an extension of $\overline{\psi_\ell}$,

$$\breve{\psi}_\ell(z, n) := \sum_{-\infty}^{n} K(n, m) \zeta^m, \quad \breve{\psi}_\ell(z \pm \mathrm{i}0, n) = \overline{\psi_\ell(z \pm \mathrm{i}0, n, 0)}, \quad z \in [q_1, q]. \tag{2.3}$$

The Jost solutions of (1.5) are connected by the scattering relation

$$T(z,t)\psi_\ell(z,n,t) = \overline{\psi(z,n,t)} + R(z,t)\psi(z,n,t), \quad |z|=1, \tag{2.4}$$

where $R(z,t)$ and $T(z,t)$ are the right reflection and transmission coefficients. Their time evolution is given by

$$R(z,t) = R(z)e^{(z-z^{-1})t}, \quad z\in\mathbb{T}, \qquad |T(z,t)|^2 = |T(z)|^2 e^{(z-z^{-1})t}, \quad z\in[q_1,q],$$

where $R(z)=R(z,0)$, $T(z)=T(z,0)$. The right norming constants

$$\gamma_j(t) = \left(\sum_{n\in\mathbb{Z}} \psi^2(z_j,n,t)\right)^{-2}$$

corresponding to $z_j\in\sigma_d$ evolve as $\gamma_j(t)=\gamma_j e^{(z_j-z_j^{-1})t}$, $\gamma_j=\gamma_j(0)>0$. Let

$$W(z,t) = a(n-1,t)(\psi_\ell(z,n-1,t)\psi(z,n,t) - \psi_\ell(z,n,t)\psi(z,n-1,t))$$

be the Wronskian of the Jost solutions and define $W(z):=W(z,0)$.

Resonant points. *The point $\tilde q\in\{-1,1,q,q_1\}$ is called a resonant point if $W(\tilde q)=0$. If $W(\tilde q)\neq 0$, then $\tilde q$ is non-resonant. Note that $W(\tilde q,t)=0$ iff $W(\tilde q)=0$, that is, the property of being resonant (or not) is preserved with t.*

Under a much weaker decaying condition than (2.1), namely a finite first moment of perturbation

$$\sum_{n=1}^{\infty} n\big(|a(-n,0)-a| + |b(-n,0)-b| + |a(n,0)-\tfrac12| + |b(n,0)|\big) < \infty, \tag{2.5}$$

the set of the associated right initial scattering data

$$\{R(z), z\in\mathbb{T}; \chi(z), z\in[q_1,q]; (z_j,\gamma_j), z_j\in\sigma_d\}, \tag{2.6}$$

where

$$\chi(z) = -2a\frac{\zeta(z-\mathrm{i}0)-\zeta^{-1}(z-\mathrm{i}0)}{z-z^{-1}}|T(z)|^2, \quad z\in[q_1,q], \tag{2.7}$$

defines the solution of the Cauchy problem (1.1)–(1.3) uniquely. For each t this solution also has finite first moments of perturbations ([16]). The scattering data (2.6) satisfy the following properties ([8, 15]):

- The function $R(z)$ is continuous on $\mathbb{T}$ and $R(z^{-1})=\overline{R(z)}=R^{-1}(z)$ for $z\in\mathbb{T}$. If $z=-1$ is non-resonant, then $R(-1)=-1$, and if $z=-1$ is resonant, then $R(-1)=1$.
- The function $T(z)$ can be restored uniquely for $z\in\mathcal{Q}$ from the data (2.6); it is meromorphic with simple poles at z_j.

- The function $\chi(z)$ is continuous for $z \in (q_1, q)$ and vanishes at $\tilde{q} \in \{q, q_1\}$ with
$$\chi(z) = C(z - \tilde{q})^{1/2}, \quad z \to \tilde{q} \in \{q, q_1\}, \tag{2.8}$$
if $\tilde{q}$ is a non-resonant point. If $\tilde{q}$ is a resonant point, then
$$\chi(z) = C(z - \tilde{q})^{-1/2}(1 + o(1)), \quad z \to \tilde{q} \in \{q, q_1\}. \tag{2.9}$$

To apply the nonlinear steepest descent approach in the most general situation which assumes resonances, we choose the number $\rho > 0$ in (2.1) small such that
$$\rho > -\log |q| \tag{2.10}$$
in order to have the inclusion $[q_1, q] \subset \{z : \mathrm{e}^{-\rho} < |z| < 1\}$. Under condition (2.1) the scattering data have additional properties:

- The function $R(z)$ admits an analytic continuation to $\{z : \mathrm{e}^{-\rho} < |z| < 1\} \setminus [q_1, q]$ with simple poles at the points of the discrete spectrum located in this domain.
- The function $\chi(z)$ has an analytic continuation $X(z)$ in a vicinity of $[q_1, q]$ with
$$\chi(z) = \mathrm{i}|\chi(z)| = X(z - \mathrm{i}0), \quad z \in [q_1, q],$$
where
$$X(z) = -\frac{a(\zeta - \zeta^{-1})(z - z^{-1})}{2W(z)\, W(\breve{\psi}_\ell, \psi)(z)}. \tag{2.11}$$
Here $\breve{\psi}_\ell(z, n)$ is defined by (2.3) and $W(\breve{\psi}_\ell, \psi)(z)$ is the Wronskian of $\breve{\psi}_\ell(z, n, 0)$ and $\psi(z, n, 0)$.

Treating the values n and t as parameters, we define a vector-valued function $m(z) = (m_1(z, n, t), m_2(z, n, t))$ on $\mathcal{Q}$ by
$$m(z, n, t) = \big(T(z, t)\psi_\ell(z, n, t)z^n, \quad \psi(z, n, t)z^{-n}\big)\,. \tag{2.12}$$

The first component $m_1(z)$ is a meromorphic function in $\mathcal{Q}$ with poles at z_j. It has continuous limits as z approaches the boundary of $\mathcal{Q}$ except (possibly) at q and q_1, where a square root singularity may appear in the case of resonance. The second component of this vector is a holomorphic function in $\mathcal{Q}$ with continuous limits to the boundary. Both functions have finite positive limits as $z \to 0$ (cf. [18]). For our purpose it will be sufficient to control the product of the components.

Lemma 2.1. *For $z \to 0$,*
$$\begin{aligned} m_1(z, n, t)m_2(z, n, t) &= 1 + 2zb(n, t) \\ &\quad + 4z^2\left(a(n-1, t)^2 + a(n, t)^2 + b(n, t)^2 - \frac{1}{2}\right) + O(z^3). \end{aligned} \tag{2.13}$$

Proof. The Jost solutions ψ and ψ_ℓ can be considered as the Weil solution of $H(t)$, and therefore the Green's function (2.2) considered as a function of z can be represented as

$$G(\lambda(z), n, n, t) = \frac{\psi(z,n,t)\psi_\ell(z,n,t)}{W(z,t)}.$$

Recall that

$$T(z,t) = \frac{z - z^{-1}}{2W(z,t)},$$

that is,

$$m_1(z,n,t)m_2(z,n,t) = \frac{z - z^{-1}}{2}\, G(\lambda(z), n, n, t). \tag{2.14}$$

Taking into account that $\frac{1}{\lambda} = \frac{2z}{1+z^2}$ and $\frac{z-z^{-1}}{2} = -\sqrt{\lambda^2 - 1}$, we obtain (2.13). $\square$

Let $\mathcal{Q}^* := \{z : |z| > 1\} \setminus [q^{-1}, q_1^{-1}]$ be the image of the domain $\mathcal{Q}$ under the map $z \mapsto z^{-1}$. We extend m to $\mathcal{Q}^*$ by $m(z^{-1}) = m(z)\sigma_1$, where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the first Pauli matrix. With this extension, the second component $m_2(z)$ is a meromorphic function on $\mathcal{Q}^*$ with poles at z_j^{-1}, $z_j \in \sigma_d$, and $m_1(z)$ is holomorphic. Being defined now on $\mathbb{C} \setminus \Sigma$, where

$$\Sigma = \mathbb{T} \cup [q, q_1] \cup [q_1^{-1}, q^{-1}], \tag{2.15}$$

the function $m(z)$ can have jumps along Σ. For convenience, from here on we encode the orientation of the contours in $\mathbb{R}$ as follows: assume $-\infty \le c < d \le \infty$, then we write $[d, c]$ for the interval $[c, d]$ with orientation right-to-left. In particular, the contours $[q, q_1]$, and $[q_1^{-1}, q^{-1}]$ in (2.15) are oriented right-to-left, and the unit circle $\mathbb{T}$ is oriented counterclockwise.

Throughout this paper, the plus $(+)$ and minus $(-)$ sides of a contour correspond to the left and right sides according to its orientation, that is, the $+$ side of an oriented contour lies to the left as one traverses the contour in the direction of its orientation. And $m_\pm(z)$ denote the boundary values of $m(z)$ as z tends to the contour from the $\pm$ side. Using this notation implicitly assumes that the limit exists (in the sense that $m(z)$ extends to a continuous function on the boundary except probably at a finite number of points). In this paper, all contours are symmetric with respect to the map $z \mapsto z^{-1}$, i.e., they contain with each point z also z^{-1}. The symmetric part of the contour will be denoted by the same letter, the image of a contour $\mathcal{L} \subset \{z : |z| < 1\}$ is denoted by $\mathcal{L}^*$. Given the orientation on $\mathcal{L}$, the orientation on the starred contour $\mathcal{L}^*$ can be chosen in two ways. For the convenience of tracking this orientation we use the following formal notation. If the points z^{-1} and z simultaneously move in the positive direction of $\mathcal{L}$ and $\mathcal{L}^*$, we encode this as $\mathcal{L}^* \uparrow\uparrow \mathcal{L}$. If z^{-1} moves in the negative direction while z moves in the positive direction, we use the notation $\mathcal{L}^* \downarrow\uparrow \mathcal{L}$. In particular, $[q_1^{-1}, q^{-1}] \downarrow\uparrow [q, q_1]$. The following symmetry should be preserved for the jump matrix of any vector RHP and for its solution.

Symmetry condition. *Let $\hat{\Sigma}$ be a symmetric oriented contour and $\mathcal{L} \cup \mathcal{L}^* \subset (\mathbb{C}\setminus\mathbb{T})$ be any symmetric part. The jump matrix $v(z)$ of the vector problem $m_+(z) = m_-(z)v(z)$, $z \in \Sigma$, satisfies*

$$\begin{aligned} &v(z) = \sigma_1(v(z^{-1}))^{-1}\sigma_1, && z \in \mathcal{L}\cup\mathcal{L}^*, && \text{for } \mathcal{L}^* \downarrow\uparrow \mathcal{L},\\ &v(z) = \sigma_1 v(z^{-1})\sigma_1, && z \in \mathcal{L}\cup\mathcal{L}^*, && \text{for } \mathcal{L}^* \uparrow\uparrow \mathcal{L}. \end{aligned}$$

If $\mathbb{T} \subset \Sigma$ then $v(z) = \sigma_1(v(z^{-1}))^{-1}\sigma_1$, $z \in \mathbb{T}$. Moreover,

$$m(z) = m(z^{-1})\sigma_1, \quad z \in \mathbb{C}\setminus\hat{\Sigma}. \tag{2.16}$$

To preserve the symmetry condition we will always choose symmetric deformations of the contours. Moreover, we will only use conjugations by diagonal matrices which respect the symmetry condition, as outlined in the next lemma.

Lemma 2.2 (Conjugation, [26])**.** *Let m be the solution on $\mathbb{C}$ of the RH problem $m_+(z) = m_-(z)v(z)$, $z \in \hat{\Sigma}$, which satisfies the symmetry condition. Let $d : \mathbb{C}\setminus\tilde{\Sigma} \to \mathbb{C}$ be a sectionally analytic function with jump on a symmetric contour $\tilde{\Sigma} \subset \hat{\Sigma}$. Set*

$$\tilde{m}(z) = m(z)\begin{pmatrix} d(z)^{-1} & 0 \\ 0 & d(z)\end{pmatrix} = m(z)[d(z)]^{-\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}. \tag{2.17}$$

If d satisfies

$$d(z^{-1}) = d(z)^{-1}, \quad z \in \mathbb{C}\setminus\Sigma,$$

then (2.17) respects the symmetry condition. The jump matrix of $\tilde{m}_+ = \tilde{m}_-\tilde{v}$ is given by

$$\tilde{v} = \begin{cases} \begin{pmatrix} v_{11} & v_{12}d^2 \\ v_{21}d^{-2} & v_{22}\end{pmatrix}, & z \in \hat{\Sigma}\setminus\Sigma \\ \begin{pmatrix} \frac{d_-}{d_+}v_{11} & v_{12}d_+d_- \\ v_{21}d_+^{-1}d_-^{-1} & \frac{d_+}{d_-}v_{22}\end{pmatrix}, & z \in \Sigma \end{cases},$$

The symmetry constraints described above will allow us to shorten notations and computations on starred parts of the contours. Indeed, if we know that $d(z)$ has a jump on $\mathcal{L}\cup\mathcal{L}^*$, with $d_+(z) = d_-(z)s(z)$ on $\mathcal{L}$, then the property $d(z^{-1}) = d^{-1}(z)$ used in a vicinity of $\mathcal{L}$ gives a complete information about the jump on $\mathcal{L}^*$. The same is true for the jump matrices.

In $\mathbb{C}\setminus(-\infty,0]$ introduce the *phase function*

$$\Phi(z) := \Phi(z,\xi) = \frac{1}{2}\big(z - z^{-1}\big) + \xi\log z, \quad \xi := \frac{n}{t},$$

which is odd with respect to $z \to z^{-1}$, that is, $\Phi(z^{-1}) = -\Phi(z)$. Note that $\mathrm{e}^{2t\Phi(z)} = z^{2n}\mathrm{e}^{t(z-z^{-1})}$ is well defined in $\mathbb{C}\setminus\{0\}$. The vector function (2.12) extended to $\mathcal{Q}^*$ by symmetry (2.16) solves the following RHP (cf. [8, 19, 27]):

RH problem 1 (Initial meromorphic RHP statement)**.** *Find a vector-valued function $m : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{1\times 2}$ which is meromorphic in $\mathcal{Q} \cup \mathcal{Q}^*$ and continuous up to Σ except at possibly the points q, q_1, q^{-1}, q_1^{-1}. It has simple poles at $z_j^{\pm 1}$, $j = 1, \dots, N$, and satisfies:*

- *the jump condition $m_+(z) = m_-(z)v(z)$, where*

$$v(z) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(z)}\mathrm{e}^{-2t\Phi(z)} \\ R(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in \mathbb{T} \\ \begin{pmatrix} 1 & 0 \\ \chi(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in [q, q_1] \\ \sigma_1(v(z^{-1}))^{-1}\sigma_1, & z \in [q_1^{-1}, q^{-1}] \end{cases};$$

- *the residue conditions*

$$\begin{aligned} \operatorname{Res}_{z=z_j} m(z) &= \lim_{z\to z_j} m(z) \begin{pmatrix} 0 & 0 \\ -z_j\gamma_j \mathrm{e}^{2t\Phi(z_j)} & 0 \end{pmatrix}, \qquad j = 1, \dots, N, \\ \operatorname{Res}_{z=z_j^{-1}} m(z) &= \lim_{z\to z_j^{-1}} m(z) \begin{pmatrix} 0 & z_j^{-1}\gamma_j \mathrm{e}^{2t\Phi(z_j)} \\ 0 & 0 \end{pmatrix}, \qquad j = 1, \dots, N; \end{aligned}$$

- *the symmetry condition $m(z^{-1}) = m(z)\sigma_1$;*
- *the normalization condition $m_1(0)\, m_2(0) = 1$ and $m_1(0) > 0$;*
- *the resonant/non-resonant condition: If $\chi(z)$ satisfies (2.8) at $\tilde q$ then $m(z)$ has finite limits $m(\tilde q^{\pm 1}) \in \mathbb{R}^{1\times 2}$ as $z \to \tilde q^{\pm 1}$, $\tilde q \in \{q, q_1\}$. If (2.9) is fulfilled then*

$$\begin{aligned} m(z) &= \left(\frac{C_1}{(z - \tilde q)^{1/2}}, C_2\right)(1 + o(1)), \quad C_1 C_2 \neq 0, \ \text{or} \\ m(z) &= (C_1, C_2(z - \tilde q))(1 + o(1)), \quad z \to \tilde q, \quad C_1 C_2 \neq 0. \end{aligned} \tag{2.18}$$

At $\tilde q^{-1}$ the analog of (2.18) holds by symmetry (2.16).

Lemma 2.3. *Suppose that the initial data of the Cauchy problem (1.1)–(1.3) satisfy (2.5) and let (2.6) be the associated initial right scattering data. Then the vector function $m(z) = m(z, n, t)$ defined by (2.12), (2.16) is the unique solution of RHP 1.*

The proof of uniqueness is completely analogous to the KdV shock case [19].

The behavior of solutions of such RHPs is determined mostly by the behavior of the real part of the phase function $\Phi(z, \xi)$ which depends on the value of the parameter $\xi = \frac{n}{t}$. The signature table of $\operatorname{Re}\Phi(z, \xi)$ for the region (1.17) is depicted in Fig. 2.1. One part of the eigenvalues in $\mathcal{Q}$ lies in the set $\operatorname{Re}\Phi(z) > 0$ (namely $z_j \in (0, q)$), while the remaining eigenvalues belong to the domain $\operatorname{Re}\Phi(z) < 0$ ($z_k \in (-1, q_1) \cup (0, 1)$). As outlined in [8, 26], one can redefine $m(z)$ on $\mathcal{Q}$ by conjugating it with an invertible bounded matrix-function such that the residue conditions at $z_j \in \sigma_d$ are replaced by jump conditions along

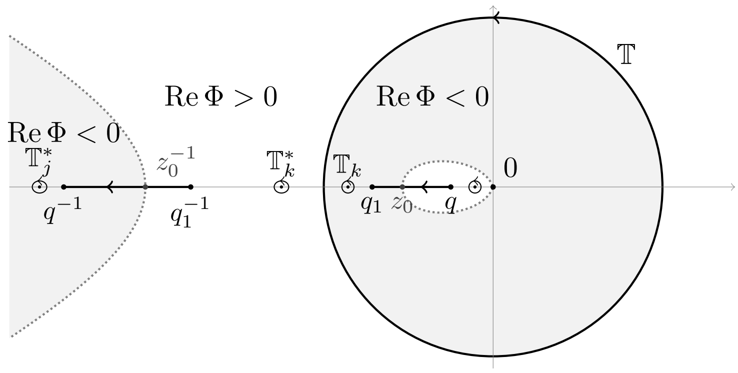

Fig. 2.1: Signature table for $\mathrm{Re}\,\Phi(z,\xi)$ for $\xi \in (\xi_{cr}', \xi_{cr})$.

non-intersecting small circles around points of σ_d. An associated transformation on $\mathcal{Q}^*$ follows immediately from the symmetry condition. The respective jump matrices will be exponentially close as $t \to \infty$ to the unit matrix for all further transformations of RHP 1. By this transformation, the main contour Σ is not changed and the structure of the jump matrices there remains qualitatively the same as in RHP 1 with respect to decay/oscillation in t and symmetry.

Indeed, let $\epsilon > 0$ be sufficiently small such that the circles $\mathbb{T}_j = \{z : |z - z_j| = \epsilon\}$, $z_j \in \sigma_d$, do not intersect, do not contain the origin, and lie away from $\mathbb{T} \cup [q, q_1]$ (the precise value of ϵ will be chosen later). Denote their images under the map $z \mapsto z^{-1}$ by $\mathbb{T}_j^*$. We orient $\mathbb{T}_j$ and $\mathbb{T}_j^*$ counterclockwise, that is, $\mathbb{T}_j^* \uparrow\uparrow \mathbb{T}_j$. Note that the curves $\mathbb{T}_j^*$ are not circles, but they surround z_j^{-1} with minimal distance from the curve to z_j^{-1} given by $\frac{\epsilon}{z_j(z_j-\epsilon)}$. Introduce the Blaschke product

$$\Pi(z) = \prod_{z_j \in (q,0)} |z_j| \frac{z - z_j^{-1}}{z - z_j}, \tag{2.19}$$

and note that $\Pi(z^{-1}) = \Pi^{-1}(z)$, $\Pi(0) > 0$. Set $\mathbb{D}_\epsilon^j = \{z : |z - z_j| < \epsilon, z_j \in \sigma_d\}$ and

$$A(z) = \begin{cases} \begin{pmatrix} 1 & \frac{z-z_j}{z_j \gamma_j \mathrm{e}^{2t\Phi(z_j)}} \\ 0 & 1 \end{pmatrix}, & z \in \mathbb{D}_\epsilon^j, \quad z_j \in (q,0) \\ \begin{pmatrix} 1 & 0 \\ \frac{z_j \gamma_j \mathrm{e}^{2t\Phi(z_j)}}{z-z_j} & 1 \end{pmatrix}, & z \in \mathbb{D}_\epsilon^j, \quad z_j \in (-1,q_1) \cup (0,1) \end{cases}.$$

Let $m(z)$ be the solution of RHP 1 and define $m^{\mathrm{ini}}(z) = m^{\mathrm{ini}}(z,n,t)$ as

$$m^{\mathrm{ini}}(z) = \begin{cases} m(z)A(z)[\Pi(z)]^{-\sigma_3}, & z \in \mathbb{D}_\epsilon^j, \quad z_j \in \sigma_d \\ m(z)[\Pi(z)]^{-\sigma_3}, & z \in \mathcal{Q} \setminus \bigcup_{z_j \in \sigma_d} \overline{\mathbb{D}_\epsilon^j} \\ m^{\mathrm{ini}}(z^{-1})\sigma_1, & z \in \mathcal{Q}^* \end{cases}. \tag{2.20}$$

This vector function is the unique solution of the following RHP (cf. [27]):

RH problem 2 (Holomorphic RHP for $\xi'_{cr} \le \xi \le \xi_{cr}$)**.** *Find a holomorphic vector function away from $\Sigma \cup \bigcup_{j=1}^{N} (\mathbb{T}_j \cup \mathbb{T}_j^*)$ that satisfies*

- *the jump condition $m_+^{\rm ini}(z) = m_-^{\rm ini}(z) v^{\rm ini}(z)$, where*

$$v^{\rm ini}(z) = \begin{cases} \begin{pmatrix} 0 & -\frac{\Pi^2(z)\overline{R(z)}}{\mathrm{e}^{2t\Phi(z)}} \\ \frac{R(z)\mathrm{e}^{2t\Phi(z)}}{\Pi^2(z)} & 1 \end{pmatrix}, & z \in \mathbb{T} \\ \begin{pmatrix} 1 & 0 \\ \Pi^{-2}(z)\chi(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in [q, q_1] \\ \begin{pmatrix} 1 & \frac{(z-z_j)\Pi^2(z)}{z_j\gamma_j \mathrm{e}^{2t\Phi(z_j)}} \\ 0 & 1 \end{pmatrix}, & z \in \mathbb{T}_j, \ z_j \in (q, 0) \\ \begin{pmatrix} 1 & 0 \\ \frac{z_j\gamma_j \mathrm{e}^{2t\Phi(z_j)}}{(z-z_j)\Pi^2(z)} & 1 \end{pmatrix}, & z \in \mathbb{T}_j, \ z_j \in \sigma_d \setminus (q, 0) \\ \sigma_1 v^{\rm ini}(z^{-1})\sigma_1, & z \in \bigcup_{j=1}^{N} \mathbb{T}_j^* \\ \sigma_1 (v^{\rm ini}(z^{-1}))^{-1}\sigma_1 & z \in [q_1^{-1}, q^{-1}] \end{cases};$$

- $m^{\rm ini}(z^{-1}) = m^{\rm ini}(z)\sigma_1$;
- $m_1^{\rm ini}(0) m_2^{\rm ini}(0) = 1$, $m_1^{\rm ini}(0) > 0$;
- *The resonant/non-resonant condition of RHP 1 holds for $m^{\rm ini}(z)$ too.*

To summarize, for all values of $\xi \in [\xi'_{cr}, \xi_{cr}]$ we performed a one-to-one transformation and replaced the meromorphic RHP by the holomorphic RHP,

$$[m(z,n,t); \text{RH problem } 1] \longmapsto [m^{\rm ini}(z,n,t); \text{RH problem } 2].$$

In the next section we list some results established in [18]. We represent them in terms of the variable z and modify them to take the resonances and the additional discrete spectrum into account.

3. Reduction to the model RH problem

Let $\xi \in [\xi'_{cr}, \xi_{cr})$. Before we describe the transformations applicable in (1.17) to obtain the model problem for this region, let us recall the g-function mechanism. For shock waves, the g-function proved its efficiency for several completely integrable equations (cf. [13, 24]). In our case, the z-analog of the g-function constructed in [18] looks as follows. Set

$$Q(z) = \sqrt{\frac{z-y}{z-q}\frac{z-y^{-1}}{z-q^{-1}}}, \qquad z \in \mathbb{C} \setminus \big([q, y] \cup [y^{-1}, q^{-1}]\big), \tag{3.1}$$

where y is a point which can be computed implicitly from the condition

$$\int_{-1}^{y} P(s)\, Q(s) \frac{ds}{s} = 0, \tag{3.2}$$

with

$$P(s) := s + s^{-1} + 2\xi + \frac{1}{2}\big(y + y^{-1} - q - q^{-1}\big). \tag{3.3}$$

The point $\gamma(\xi)$ in the introduction is connected with $y = y(\xi)$ by $\gamma = \frac{y+y^{-1}}{2}$. In the present paper we use the notation λ_y instead of γ (see Remark 3.2). As is shown in [18], equation (3.2) has the unique solution $y = y(\xi) \in (q_1, q)$ for any $\xi \in [\xi'_{cr}, \xi_{cr})$. The function $y(\xi)$ is continuous and monotonous, with $y(\xi'_{cr}) = q_1$ and $y(\xi_{cr}) = q$. Moreover, $y(\xi)$ is differentiable with respect to $\xi \in (\xi'_{cr}, \xi_{cr})$ (cf. [18, App.]). For any y, the function (3.1) satisfies $Q^2(z^{-1}) = Q^2(z)$ which implies evenness, $Q(z^{-1}) = Q(z)$, because $Q(1^{-1}) = Q(1)$. With the chosen orientation on $[q, y] \cup [y^{-1}, q^{-1}]$ we denote $Q_+(z) = Q(z - \mathrm{i}0)$. From the evenness of Q outside of $[q, y] \cup [y^{-1}, q^{-1}]$ we obtain oddness of Q_+, $Q_+(s) = -Q_+(s^{-1})$ for $s \in [q, y] \cup [y^{-1}, q^{-1}]$. Note that we choose the square root in (3.1) such that $Q(z) > 0$ for $z \in (q, +\infty)$. Introduce the g-function by

$$g(z) = g(z, \xi) = \frac{1}{2}\int_1^z P(s)Q(s)\frac{ds}{s}, \quad z \in \mathbb{C} \setminus (-\infty, 1). \tag{3.4}$$

Lemma 3.1. *The function $g(z)$ satisfies the following properties:*

(a) *$g(z)$ is single valued on $\mathbb{C} \setminus [q^{-1}, q]$, moreover,*

$$g(z^{-1}) = -g(z) \quad \textit{for } z \in \mathbb{C} \setminus [q^{-1}, q]; \tag{3.5}$$

(b) $\operatorname{Re} g(z) = 0$ *for* $z \in [q, y] \cup [y^{-1}, q^{-1}] \cup \{z : |z| = 1\}$;

(c) $g(q) = g(q^{-1}) = 0$;

(d) $g_-(z) = -g_+(z)$ *for* $z \in [q, y] \cup [y^{-1}, q^{-1}]$;

(e) $\Phi(z) - g(z) = K(\xi) + O(z)$ *as* $z \to 0$, *where* $K(\xi) \in \mathbb{R}$;

(f) $g_+(z) - g_-(z) = 2\mathrm{i}B$ *for* $z \in [y, y^{-1}]$, *where*

$$B := -\mathrm{i}\int_q^y P(s)Q_+(s)\frac{ds}{s} \in \mathbb{R}_+. \tag{3.6}$$

In particular, $g_\pm(y) = g_\pm(y^{-1}) = \pm\mathrm{i}B$.

Proof. (a)–(c) Since $Q(z^{-1}) = Q(z)$ and $P(z^{-1}) = P(z)$ for $z \in \mathbb{C} \setminus [q^{-1}, q]$, then choosing a contour from 1 to z which does not have common points with the interval $[q^{-1}, q]$, we obtain (3.5) by the simple change of variables $s \to s^{-1}$. Condition (3.2) implies $g_\pm(y) = g_\pm(y^{-1})$. Since the integrand $P(s)Q_\pm(s)s^{-1}$ is purely imaginary for $s \in [q, y] \cup [y^{-1}, q^{-1}]$, we have

$$\operatorname{Re} g(q) = \operatorname{Re} g_\pm(y) = \operatorname{Re} g_\pm(y^{-1}) = \operatorname{Re} g_\pm(q^{-1}). \tag{3.7}$$

Oddness of $Q_+(s)$ also implies $\operatorname{Im} g(q) = \operatorname{Im} g_\pm(q^{-1}) = 0$. Together with (3.7) and the oddness of g this yields $g_+(q^{-1}) = g_-(q^{-1}) = g(q) = 0$. Moreover, for $|s| = 1$ we have $Q^2(\overline{s}) = Q^2(s) \in \mathbb{R}_+$, and therefore $\operatorname{Im} Q(s)P(s) = 0$. Since $\frac{ds}{s} \in \mathrm{i}\mathbb{R}$, this implies $\operatorname{Re} g(z) = 0$ for $|z| = 1$, items (b), (c) are thus proved. They imply that $g(z)$ does not have a jump along $(-\infty, q^{-1})$, and this shows (a). Note that these properties improve [18, Lemma 3.2] (see, e.g., [18, equation (3.25)]). The above considerations imply an additional property,

$$\operatorname{Re} g_{\pm}(-1) = 0.$$

Items (e), (f) are z-analogs of [18, equations (3.21) and (3.24)], and can be obtained by a simple change of variable $(\lambda, +) \mapsto z$. The constant $B = B(\xi)$ in (f) is the same as [18, equation (3.21)]. $\square$

Remark 3.2. The point $y(\xi)$ defines the edge $\lambda_y = \frac{1}{2}(y + y^{-1})$ of the Whitham zone for the Toda shock case. The point $y(\xi)$ coincides with the stationary phase point $z_0(\xi)$ for $\Phi(z, \xi)$ at $\xi = \xi_{cr}$, that is, $z_0(\xi_{cr}) = q$. One can see that $y(\xi'_{cr}) \neq z_0(\xi'_{cr})$. However, as it was shown in [18], there are proper g-functions in the whole diapason $\xi \in (\xi_{cr}, \xi_{cr,1})$, and the respective Whitham point $y_1(\xi)$ for $\xi \to \xi_{cr,1}$ will end at $z_{0,\ell}(\xi_{cr,1}) = 1$, where $z_{0,\ell}(\xi)$ is the stationary phase point for the left phase function $\Phi_\ell(z, \xi)$ in (1.10) connected with the left initial scattering data.

The signature table for the real part of g is depicted in Fig. 3.1. The points y and y^{-1} are nodal points for the curves $\operatorname{Re} g(z) = 0$.

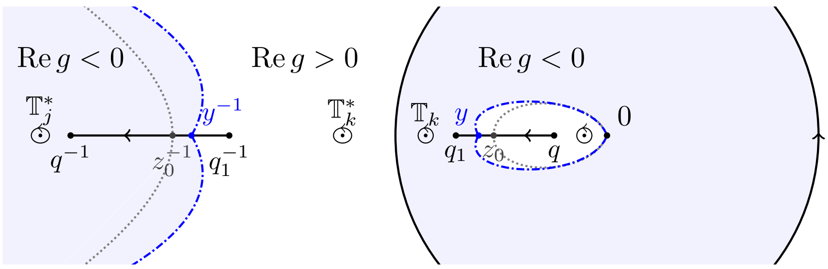

Fig. 3.1: Signature table of $\operatorname{Re} g(z, \xi)$ for $\xi \in (\xi'_{cr}, \xi_{cr})$.

This signature table allows us to choose the radius ϵ of the circles $\mathbb{T}_j$ so small that for $\Phi_j(z) := \Phi(z_j) - \Phi(z) + g(z)$ we will have

$$\operatorname{sign} \operatorname{Re} \Phi_j(z) = \operatorname{sign} \operatorname{Re} \Phi(z_j) \quad \text{for all} \quad z_j \in \sigma_d \quad \text{and} \quad z \in \mathbb{T}_j. \tag{3.8}$$

The radius ϵ should also satisfy

$$8\epsilon < \min \left\{ \min_{j \neq k} |z_j - z_k|; \min_j |z_j + 1|; \min_j |z_j - 1|; \min_j |z_j - q|; \min_j |z_j - q_1| \right\}.$$

Moreover, since we intend to justify the asymptotics uniformly in the regions (1.16) or (1.17) for arbitrary small but fixed positive ε, we also assume that

$$4\epsilon < |y(\xi_{cr} - \varepsilon) - q|, \tag{3.9}$$

$$4\epsilon < |y(\xi'_{cr} + \varepsilon) - q_1|. \tag{3.10}$$

With such a value of ϵ chosen, we next perform three transformations which lead to a model problem. The transformations are analogous to those in [18], modified by additional deformations to wipe out the non-L^2 singularities of the jump matrix in case of resonances at q or q_1.

Step 1: On $\mathbb{T}$ one can factorize v^{ini} using Schur complements

$$v^{\mathrm{ini}} = \begin{pmatrix} 1 & -\Pi^2 \overline{R} \mathrm{e}^{-2t\Phi} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Pi^{-2} R \mathrm{e}^{2t\Phi} & 1 \end{pmatrix}.$$

Let $\xi \in \mathcal{I}_\varepsilon$, where $\mathcal{I}_\varepsilon$ is defined by (1.17). Let $y = y(\xi)$ and let $\mathfrak{r}$, $q_1 < \mathfrak{r} < y$, be a point in a small vicinity of y as depicted in Fig. 3.2 with

$$\frac{\epsilon}{2} \le |\mathfrak{r} - y| \le \epsilon. \tag{3.11}$$

Introduce a closed contour $\mathcal{C}_\mathfrak{r}$ oriented counterclockwise, which starts at $\mathfrak{r}$ and encloses the interval $[q_1, \mathfrak{r}]$ passing through the point $q_1 - \epsilon$. Denote the domain inside this contour by $\Omega_\mathfrak{r}$ (with $[q_1, \mathfrak{r}]$ excluded).

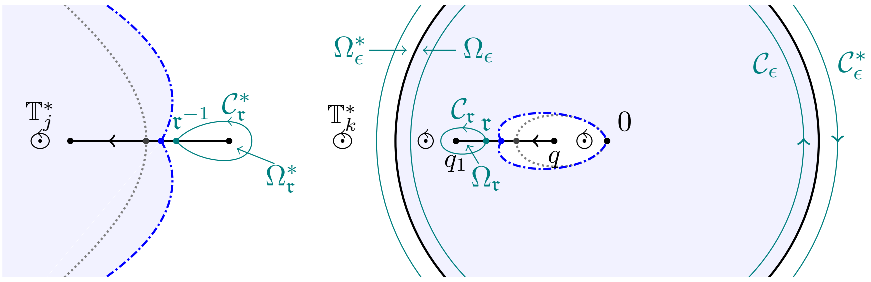

Fig. 3.2: Contour deformation of Step 1.

Let Ω_ϵ be an open annulus between the circles $\mathbb{T}$ and $\mathcal{C}_\epsilon = \{z : |z| = 1 - \epsilon\}$ oriented counterclockwise, with $\Omega_\mathfrak{r}^*$ and Ω_ϵ^* the images of these domains under the map $z \mapsto z^{-1}$. According to (2.1) the reflection coefficient $R(z)$ can be continued as a meromorphic function in the domain $\{z : 1 > |z| > \mathrm{e}^{-\rho}\}$, which covers the interval $[q_1, y(\xi_{cr} - \varepsilon)]$ by (2.10). Thus $R(z)$ is a holomorphic function in $\Omega_\epsilon \cup \Omega_\mathfrak{r}$, because these domains do not contain points of the discrete spectrum by our choice of ϵ. We extend $R(z)$ to $\Omega_\epsilon^* \cup \Omega_\mathfrak{r}^*$ by $R(z) = \overline{R(z^{-1})}$. Redefine m^{ini} by

$$m^{(1)}(z) = m^{\mathrm{ini}}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -\Pi^{-2}(z) R(z) \mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in \Omega_\mathfrak{r} \cup \Omega_\epsilon \\ \begin{pmatrix} 1 & -\Pi^{-2}(z) R(z^{-1}) \mathrm{e}^{-2t\Phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_\mathfrak{r}^* \cup \Omega_\epsilon^* \\ \mathbb{I}, & \text{else} \end{cases} \tag{3.12}$$

and orient $\mathcal{C}_\mathfrak{r}$ and $\mathcal{C}_\mathfrak{r}^*$ counterclockwise. Then the jump along $\mathbb{T}$ disappears as well as the jump along $[\mathfrak{r}, q_1]$, since the Plücker identity implies that $R_-(z) - R_+(z) + \chi(z) = 0$ for $z \in [\mathfrak{r}, q_1]$ (compare [13, Lemma 3.2]). Moreover, since the continuation of $R(z)$ is in agreement with the scattering relation (2.4) and (2.12) is the unique solution of RHP 1, it is straightforward to check that $m^{(1)}(z)$ given by (2.20), (3.12) does not have singularities at q_1 and q_1^{-1} both in the resonant and non-resonant case. In summary, $m^{(1)}(z)$ satisfies

RH problem 3.

- $m_+^{(1)}(z,n,t) = m_-^{(1)}(z,n,t)v^{(1)}(z,n,t)$, *where*

$$v^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \Pi^{-2}(z)\chi(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in [q,\mathfrak{r}] \\ \begin{pmatrix} 1 & 0 \\ \Pi^{-2}(z)R(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in \mathcal{C}_\epsilon \cup \mathcal{C}_\mathfrak{r} \\ \sigma_1 v^{(1)}(z^{-1})\sigma_1, & z \in \mathcal{C}_\mathfrak{r}^* \cup \mathcal{C}_\epsilon^* \\ \sigma_1 (v^{(1)}(z^{-1}))^{-1}\sigma_1, & z \in [\mathfrak{r}^{-1}, q^{-1}] \\ v^{\mathrm{ini}}(z), & z \in \bigcup_j (\mathbb{T}_j \cup \mathbb{T}_j^*) \end{cases};$$

- $m^{(1)}(z^{-1}) = m^{(1)}(z)\sigma_1$;
- $m_1^{(1)}(0) \cdot m_2^{(1)}(0) = 1$, $m_1^{(1)}(0) > 0$;
- *the resonant/non-resonant condition of RHP 1 holds for $m^{(1)}(z)$ only at q, q^{-1}.*

Step 2: The jump matrix on $[q,\mathfrak{r}] \cap \{z : \operatorname{Re}\Phi(z) > 0\}$ contains off-diagonal elements which are exponentially increasing in time. One can get rid of this exponential growth by replacing the phase function with the g-function, which is purely imaginary on $[q,y]$ and has negative real part on $[y,\mathfrak{r}]$. For $z \in \mathbb{C}$ set

$$m^{(2)}(z) = m^{(1)}(z)\mathrm{e}^{-t(\Phi(z)-g(z))\sigma_3} = m^{(1)}(z)\begin{pmatrix} \mathrm{e}^{-t(\Phi(z)-g(z))} & 0 \\ 0 & \mathrm{e}^{t(\Phi(z)-g(z))} \end{pmatrix}.$$

Then Lemma 3.1 and (3.6) imply that $m^{(2)}(z)$ is the unique holomorphic solution of the following problem:

RH problem 4.

- $m_+^{(2)}(z,n,t) = m_-^{(2)}(z,n,t)v^{(2)}(z,n,t)$, *where*

$$v^{(2)}(z) = \begin{cases} \begin{pmatrix} \mathrm{e}^{t(g_+-g_-)} & 0 \\ \Pi^{-2}\chi & \mathrm{e}^{-t(g_+-g_-)} \end{pmatrix}, & z \in [q,y] \\ \begin{pmatrix} \mathrm{e}^{2\mathrm{i}tB} & 0 \\ \Pi^{-2}\chi\mathrm{e}^{2t\operatorname{Re} g} & \mathrm{e}^{-2\mathrm{i}tB} \end{pmatrix}, & z \in [y,\mathfrak{r}] \\ \begin{pmatrix} \mathrm{e}^{2\mathrm{i}tB} & 0 \\ 0 & \mathrm{e}^{-2\mathrm{i}tB} \end{pmatrix}, & z \in [\mathfrak{r},\mathfrak{r}^{-1}] \\ \sigma_1(v^{(2)}(z^{-1}))^{-1}\sigma_1, & z \in [\mathfrak{r}^{-1}, q^{-1}] \\ \mathrm{e}^{-t(\Phi-g)\sigma_3}v^{(1)}\mathrm{e}^{t(\Phi-g)\sigma_3}, & z \in \Gamma, \end{cases}$$

and

$$\Gamma := \mathcal{C}_\mathfrak{r} \cup \mathcal{C}_\mathfrak{r}^* \cup \mathcal{C}_\epsilon \cup \mathcal{C}_\epsilon^* \cup \bigcup_{j=1}^N \left(\mathbb{T}_j \cup \mathbb{T}_j^*\right); \tag{3.13}$$

- $m^{(2)}(z^{-1}) = m^{(2)}(z)\sigma_1$;
- $m_1^{(2)}(0)m_2^{(2)}(0) = 1$, $m_1^{(2)}(0) > 0$;
- *the vector $m^{(2)}(z)$ does not have singularities at q_1 and q_1^{-1}; it has bounded values at $\mathfrak{r}$ and $\mathfrak{r}^{-1}$; its behavior at q and q^{-1} is the same as for $m(z)$ in (2.18).*

Remark 3.3. Our choice of y, $\mathfrak{r}$ and ϵ in (3.8)–(3.9), (3.11) guarantees that

$$v^{(2)}(z) = \mathbb{I} + O(\mathrm{e}^{-c(\varepsilon)t}) \quad \text{for } z \in \Gamma \text{ as } \ t \to \infty, \tag{3.14}$$

uniformly for $\xi \in \mathcal{I}_\varepsilon$.

Step 3: The last step involves the lense mechanism to remove the oscillating terms (with respect to t) in the jump matrix on $[q, y] \cup [y^{-1}, q^{-1}]$. To this end introduce the function

$$\Omega(z, s) = \frac{1}{2s}\frac{s+z}{s-z},$$

which can be considered as the Cauchy kernel for symmetric contours, because $\Omega(z, s) = \frac{1}{z-s}(1 + o(1))$ as $z \to s$, and

$$\Omega(z, s^{-1})\, d(s^{-1}) = \Omega(z^{-1}, s)\, ds.$$

This property implies that for any "good" function $f(s)$ such that $f(s^{-1}) = f(s)$ and

$$\int_{-1}^{q} f(s)\frac{ds}{s} = \int_{-1}^{q} f(s)\Omega(0, s)\, ds = 0, \tag{3.15}$$

the function

$$p(z) = \frac{1}{2\pi\mathrm{i}} \int_q^{q^{-1}} \Omega(z, s) f(s)\, ds, \tag{3.16}$$

solves the jump problem

$$p_+(z) = p_-(z) + f(z), \quad z \in [q, q^{-1}], \tag{3.17}$$

$$p(z^{-1}) = -p(z), \quad z \in \mathbb{C} \setminus [q^{-1}, q] \tag{3.18}$$

$$p(z) = O(z), \quad z \to 0. \tag{3.19}$$

By "good" function we mean a function $f \in C^1((q^{-1}, y^{-1}) \cup (y^{-1}, y) \cup (y, q))$ which has the following behavior in the node points:

$$f(s) = \frac{C(\kappa)}{\sqrt{s-\kappa}}(1 + o(1)), \quad s \to \kappa \in \{q^{-1}, y^{-1}, y, q\}, \quad C(\kappa) \neq 0.$$

Then (cf. [32])

$$p(z) = \frac{C_1(\kappa)}{\sqrt{z-\kappa}}(1 + o(1)), \quad z \to \kappa \in \{q^{-1}, y^{-1}, y, q\}, \quad C_1(\kappa) \neq 0.$$

Set

$$\mathcal{S}(z) = \sqrt{\frac{(z-q)(z-y)(z-y^{-1})(z-q^{-1})}{z^2}}, \quad z \in \mathbb{C} \setminus ([q, y] \cup [y^{-1}, q^{-1}]). \tag{3.20}$$

This function satisfies the following symmetries: $\mathcal{S}(z^{-1}) = \mathcal{S}(z)$ for $z \notin [q, y] \cup [y^{-1}, q^{-1}]$ and $\mathcal{S}_-(z) = \mathcal{S}_+(z^{-1}) = -\mathcal{S}_+(z)$ for $z \in [q, y] \cup [y^{-1}, q^{-1}]$. Define

$$\mathcal{F}(z) := \mathrm{e}^{\mathcal{S}(z)p(z)}, \quad z \in \mathbb{C} \setminus [q, q^{-1}]. \tag{3.21}$$

Lemma 3.4. *The function $\mathcal{F}(z)$ is holomorphic in $\mathbb{C} \setminus [q, q^{-1}]$ and satisfies the property $\mathcal{F}(z^{-1}) = \mathcal{F}^{-1}(z)$. It has bounded limits on the sides of the contour $[q, q^{-1}]$ and solves the jump problem*

$$\begin{aligned} \mathcal{F}_+(z)\mathcal{F}_-(z) &= \mathrm{e}^{f(z)\mathcal{S}_+(z)}, && z \in [q, y] \cup [y^{-1}, q^{-1}]; \\ \mathcal{F}_+(z) &= \mathcal{F}_-(z)\mathrm{e}^{f(z)\mathcal{S}(z)}, && z \in [y, y^{-1}]. \end{aligned}$$

Proof. The proof is immediate from (3.17)–(3.20), the properties of $\mathcal{S}$ and the Sokhotski–Plemelj theorem. □

Let now $f(z)$ be defined as

$$f(s) := \begin{cases} \dfrac{\log(\Pi^{-2}(s)|\chi(s)\mathcal{V}_+^2(s)|)}{\mathcal{S}_+(s)}, & s \in [q, y] \\ \dfrac{\mathrm{i}(\Delta - \frac{\pi \ell_{\mathcal{V}}}{2})}{\mathcal{S}(s)}, & s \in [y, -1] \\ f(s^{-1}), & s \in [-1, q^{-1}] \end{cases}, \tag{3.22}$$

where

$$\Delta = -\mathrm{i} \int_q^y \frac{\log(\Pi^{-2}(s)|\chi(s)\,\mathcal{V}_+^2(z)|)}{\mathcal{S}_+(s)} \frac{ds}{s} \left(\int_y^{-1} \frac{ds}{s\,\mathcal{S}(s)} \right)^{-1} + \frac{\pi \ell_{\mathcal{V}}}{2} \in \mathbb{R}, \tag{3.23}$$

and

$$\begin{cases} \mathcal{V}(z) := \left(\dfrac{z - q^{-1}}{z - q} \right)^{1/4}, & \ell_{\mathcal{V}} = 1, \quad \text{if } \chi(z) \text{ satisfies (2.8) at } q; \\ \mathcal{V}(z) := \left(\dfrac{z - q}{z - q^{-1}} \right)^{1/4}, & \ell_{\mathcal{V}} = -1, \quad \text{if } \chi(z) \text{ satisfies (2.9) at } q; \end{cases} \tag{3.24}$$

with $\mathcal{V}(0) > 0$. We observe that $\mathcal{V}(z^{-1}) = \mathcal{V}^{-1}(z)$ for $z \notin [q, y] \cup [y^{-1}, q^{-1}]$ and

$$\mathcal{V}_+(z) = \mathcal{V}_-(z)\mathrm{e}^{\mathrm{i}\frac{\pi \ell_{\mathcal{V}}}{2}}, \qquad \mathcal{V}_+(z)\mathcal{V}_-(z) = |\mathcal{V}_+(z)|^2, \quad z \in [q, q^{-1}].$$

It is straightforward to see that $f(z)$ satisfies (3.15), and is a "good" function. In addition, $f(s) \in \mathrm{i}\mathbb{R}$, therefore if $p(z)$ is its Cauchy type integral (3.16), then

$$\lim_{z \to 0} p(z)\mathcal{F}(z) > 0.$$

The above considerations imply for $F(z)$ defined by

$$F(z) = \mathcal{F}(z)\mathcal{V}^{-1}(z), \quad z \in \mathbb{C} \setminus [q, q^{-1}], \tag{3.25}$$

the following properties.

Lemma 3.5. *The function $F(z)$ satisfies*

(a) $F_+(z)F_-(z) = \Pi^{-2}(z)|\chi(z)|$ *for* $z \in [q, y]$;
(b) $F_+(z)F_-(z) = \Pi^{-2}(z)|\chi(z)|^{-1}$ *for* $z \in [y^{-1}, q^{-1}]$;
(c) $F_+(z) = F_-(z)e^{\mathrm{i}\Delta}$ *for* $z \in [y, y^{-1}]$;
(d) $F(z^{-1}) = F^{-1}(z)$ *for* $z \in \mathbb{C} \setminus [q, q^{-1}]$;
(e) $F(0) > 0$;
(f) *if* $\chi(z)$ *satisfies* (2.8) *at* $q^{\pm 1}$ *then* $F(z) = C(z - q^{\pm 1})^{\pm 1/4}(1 + o(1))$ *as* $z \to q^{\pm 1}$;
(g) *if* $\chi(z)$ *satisfies* (2.9) *at* $q^{\pm 1}$ *then* $F(z) = C(z - q^{\pm 1})^{\mp 1/4}(1 + o(1))$ *as* $z \to q^{\pm 1}$.

Set

$$G^F(z) = \begin{pmatrix} F^{-1}(z) & -\frac{\Pi^2(z)F(z)}{X(z)}e^{-2tg(z)} \\ 0 & F(z) \end{pmatrix},$$

where the function $X(z)$ is well defined by (2.11) in a vicinity of $[q, y]$ and satisfies the property $X_\pm(z) = \pm\mathrm{i}|\chi(z)|$ for $z \in [q, y]$. Recalling that $g_+(z) = -g_-(z)$ for $z \in [q, y]$, we observe that $v^{(2)}(z)$ can be factorized by

$$v^{(2)}(z) = G_-^F(z)\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} G_+^F(z)^{-1}, \quad z \in [q, y].$$

In the domain of existence of $X(z)$ introduce a subdomain Ω as depicted in Fig. 3.3 with $\Omega^* = \{z : z^{-1} \in \Omega\}$.

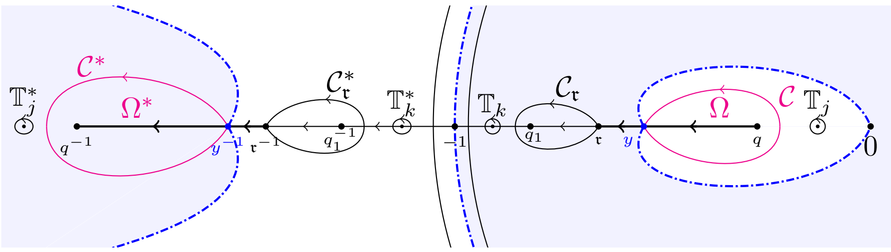

Fig. 3.3: Contour deformation of Step 3.

These domains and their boundaries $\mathcal{C}$ and $\mathcal{C}^*$ should not contain or intersect $\mathbb{T}_j$ and $\mathbb{T}_j^*$ and should be situated inside the regions $\operatorname{Re} g > 0$ and $\operatorname{Re} g < 0$, respectively. We add $\mathcal{C}$ and $\mathcal{C}^*$ (both oriented counterclockwise) to the contour Γ and denote

$$\Xi := \mathcal{C} \cup \mathcal{C}^* \cup \mathcal{C}_\mathfrak{r} \cup \mathcal{C}_\mathfrak{r}^* \cup \mathcal{C}_\epsilon \cup \mathcal{C}_\epsilon^* \cup \bigcup_{j=1}^{N} \left(\mathbb{T}_j \cup \mathbb{T}_j^*\right). \tag{3.26}$$

Define $m^{(3)}(z)$ by

$$m^{(3)}(z) = \begin{cases} m^{(2)}(z)G^F(z), & z \in \Omega \\ m^{(3)}(z^{-1})\sigma_1, & z \in \Omega^* \\ m^{(2)}(z)(F(z))^{-\sigma_3}, & z \in \mathbb{C} \setminus (\overline{\Omega} \cup \overline{\Omega^*}) \end{cases}. \tag{3.27}$$

Theorem 3.6. *For $\xi \in \mathcal{I}_\varepsilon$, RH problem 2 is equivalent to the following RH problem: to find a vector function holomorphic in $\mathbb{C} \setminus (\Xi \cup [q, q^{-1}])$ which has continuous limits on the sides of the contour $\Xi \cup [q, q^{-1}]$ except for the points q, q^{-1} and satisfies*

- *the jump condition $m_+^{(3)}(z,n,t) = m_-^{(3)}(z,n,t)v^{(3)}(z,n,t)$, where*

$$v^{(3)}(z) = \begin{cases} \mathrm{i}\sigma_1, & z \in [q, y], \\ \begin{pmatrix} \mathrm{e}^{2\mathrm{i}tB - \mathrm{i}\Delta} & 0 \\ \frac{\chi(z)\mathrm{e}^{2t(g_+(z)+g_-(z))}}{\Pi^2(z)F_+(z)F_-(z)} & \mathrm{e}^{-2\mathrm{i}tB+\mathrm{i}\Delta} \end{pmatrix}, & z \in [y, \mathfrak{r}], \\ \begin{pmatrix} \mathrm{e}^{2\mathrm{i}tB - \mathrm{i}\Delta} & 0 \\ 0 & \mathrm{e}^{-2\mathrm{i}tB+\mathrm{i}\Delta} \end{pmatrix}, & z \in [\mathfrak{r}, -1], \end{cases} \tag{3.28}$$

$$v^{(3)}(z) = \begin{cases} \begin{pmatrix} 1 & \frac{\Pi^2(z)F^2(z)}{X(z)} e^{-2tg(z)} \\ 0 & 1 \end{pmatrix}, & z \in \mathcal{C}, \\ \sigma_1 v^{(3)}(z^{-1})\sigma_1, & z \in \mathcal{C}^*, \\ \sigma_1 (v^{(3)}(z^{-1}))^{-1}\sigma_1, & z \in [-1, q^{-1}], \\ [F(z)]^{-\sigma_3} v^{(2)}(z)[F(z)]^{\sigma_3}, & z \in -; \end{cases} \tag{3.29}$$

- *the symmetry condition $m^{(3)}(z^{-1}) = m^{(3)}(z)\sigma_1$;*
- *the normalization condition $m_1^{(3)}(0)\, m_2^{(3)}(0) = 1$, $m_1^{(3)}(0) > 0$;*
- *at the points $\{q, q^{-1}, y, y^{-1}, \mathfrak{r}, \mathfrak{r}^{-1}\}$ of discontinuity of the jump matrix, $m^{(3)}(z)$ has the following behavior: it has at most a fourth root singularity*

$$\begin{aligned} m^{(3)}(z) &= O(z-\kappa)^{-1/4} && \text{as } z \to \kappa \in \{q, q^{-1}\} \quad \text{and} \\ m^{(3)}(z) &= O(1) && \text{as } z \to \kappa \in \{\mathfrak{r}, \mathfrak{r}^{-1}, y, y^{-1}\}. \end{aligned} \tag{3.30}$$

Here B and Δ are defined by (3.6) and (3.23), $g(z)$ by (3.4), and $F(z)$ by (3.20), (3.16), (3.25). For large z, $m^{(3)}(z)$ and the solution $m(z)$ of the initial RH problem 1 are connected via

$$m^{(3)}(z) = m(z)\Big[\Pi(z)F(z)\mathrm{e}^{t(\Phi(z)-g(z))}\Big]^{-\sigma_3}. \tag{3.31}$$

Proof. The jump condition is immediate from Lemmas 3.5, 3.1, which imply

$$\frac{F_-(z)}{F_+(z)}\mathrm{e}^{t(g_+(z)-g_-(z))} = \mathrm{e}^{2\mathrm{i}tB-\mathrm{i}\Delta}, \quad z \in [y, y^{-1}]. \tag{3.32}$$

The claim to be discussed in more detail is (3.30). Transformation (3.27) implies that in a vicinity of q,

$$m^{(3)}(z) = \Big(F^{-1}(z)m_1^{(2)}(z), \quad -\tfrac{\Pi^2(z)F(z)}{X(z)}m_1^{(2)}(z)\mathrm{e}^{-2tg(z)} + F(z)m_2^{(2)}(z)\Big).$$

In the non-resonant case (2.8) we have three possibilities for m, and therefore for $m^{(2)}$, which include possible zeros of the Jost solutions,

1. $m^{(2)}(q)=(C_1,C_2)$;
2. $m^{(2)}(z)=(C_1(z-q)^{1/2},C_2)(1+o(1))$;
3. $m^{(2)}(z)=(C_1,C_2(z-q))(1+o(1))$, where $C_1C_2\neq 0$.

The symmetry condition implies the respective behavior at q^{-1}. In the resonant case (2.9) we have (2.18). By use of (f) and (g) of Lemma 3.5 we obtain (3.30). □

In summary, we have transformed the initial RH problem $[m^{\mathrm{ini}}(z,n,t);\text{RHP } 2]$ by Steps 1–3 to an equivalent RH problem $[m^{(3)}(z,n,t);\text{Theorem } 3.6]$ with jump matrix $v^{(3)}$ of the form $v^{(3)}=v^{mod}+v^{err}$, where

$$v^{mod}=\begin{cases}\mathrm{i}\sigma_1, & z\in[q,y]\\ -\mathrm{i}\sigma_1, & z\in[y^{-1},q^{-1}]\\ \begin{pmatrix}\mathrm{e}^{2\mathrm{i}tB-\mathrm{i}\Delta} & 0\\ 0 & \mathrm{e}^{-2\mathrm{i}tB+\mathrm{i}\Delta}\end{pmatrix}, & z\in[y,y^{-1}]\\ \mathbb{I}, & z\in\Xi\end{cases}. \tag{3.33}$$

The matrix v^{mod} on $[q,q^{-1}]$ is the jump matrix of an explicitly solvable RHP and its solution will yield the principal term of the long-time asymptotic expansion of the solution for the initial value problem (1.1)–(1.3), (2.1). We will solve this model RHP in the next section. Note that the jump matrix $v^{(3)}$ on the contour $\Xi\cup[\mathfrak{r},\mathfrak{r}^{-1}]$ is exponentially close to the identity matrix as $t\to\infty$ except for small neighborhoods of the critical (parametrix) points y,y^{-1}. To estimate the error term one has to rescale the equivalent RHP in neighborhoods of the parametrix points and solve the respective local problems, which can be analyzed and controlled individually. This will be done in Section 6.

4. Solution of the vector model RH problem

We have to solve the following jump problem

Model RH problem. *Find a holomorphic vector function in $\mathbb{C}\setminus[q^{-1},q]$ satisfying*

- *the jump condition $m_+^{mod}(z)=m_-^{mod}(z)v^{mod}(z)$ with $v^{mod}(z)$ given by* (3.33);
- $m^{mod}(z^{-1})=m^{mod}(z)\sigma_1$;
- $m_1^{mod}(0)\,m_2^{mod}(0)=1$, $m_1^{mod}(0)>0$;
- *the vector $m^{mod}(z)$ has continuous limits as z approaches the jump contour except for q and q^{-1} and the points of discontinuity of the jump matrix, y, y^{-1}, where the forth-root singularities are admissible.*

Uniqueness of the solution to this problem is proved in [18].

Consider the two-sheeted Riemann surface $\mathbb{X}$ associated with the function

$$\mathcal{R}(z)=\sqrt{(z-q)(z-y)(z-y^{-1})(z-q^{-1})}$$

such that $\mathcal{R}(1)\in\mathbb{R}_+$ and $\mathcal{R}(-1)\in\mathbb{R}_-$. The sheets of $\mathbb{X}$ are glued along the cuts $[q^{-1},y^{-1}]$ and $[y,q]$. Points on $\mathbb{X}$ are denoted by $(z,\pm)$. We first choose a canonical homology basis of cycles $\{\mathfrak{a}, \mathfrak{b}\}$ on $\mathbb{X}$, see Fig. 4.1. The $\mathfrak{b}$ cycle surrounds the interval $[y, q]$ counterclockwise on the upper sheet and the $\mathfrak{a}$ cycle passes from y to y^{-1} on the upper sheet and back from y^{-1} to y on the lower sheet.

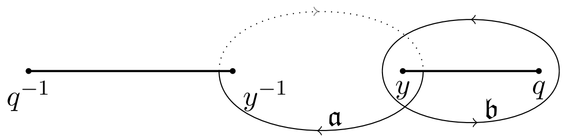

Fig. 4.1: Homology basis on $\mathbb{X}$. Solid curves lie on upper sheet, dotted curve lies on lower sheet.

Consider the normalized holomorphic Abel differential

$$\zeta = \frac{dz}{\Gamma \mathcal{R}(z)}, \qquad \Gamma = \int_{\mathfrak{a}} \frac{dz}{\mathcal{R}(z)} = 2\int_y^{y^{-1}} \frac{dz}{\mathcal{R}(z)} > 0,$$

then $\int_{\mathfrak{a}} \zeta = 1$ and $\tau = \tau(\xi) = \int_{\mathfrak{b}} \zeta \in \mathrm{i}\mathbb{R}_+$. From here on we work on the upper sheet of $\mathbb{X}$ and identify it with the domain $\mathbb{C} \setminus ([q^{-1}, y^{-1}] \cup [y, q])$. On $\mathbb{C} \setminus [q^{-1}, q]$ introduce the Abel map $A(z) = \int_q^z \zeta$. Its properties are determined by those of $\mathcal{R}(z)$, that is, we will take into account that

$$\begin{gathered}\frac{dz}{\mathcal{R}(z)} = -\frac{d(z^{-1})}{\mathcal{R}(z^{-1})}, \quad z \in \mathbb{C} \setminus ([q^{-1}, y^{-1}] \cup [y, q]); \\ \frac{ds}{\mathcal{R}_-(s)} = \frac{d(s^{-1})}{\mathcal{R}_-(s^{-1})}, \quad s \in [y^{-1}, q^{-1}] \cup [q, y], \quad \mathcal{R}_-(z) = \mathcal{R}(z + \mathrm{i}0). \end{gathered} \tag{4.1}$$

Lemma 4.1. *The Abel map $A(z)$ satisfies*

$$A(z^{-1}) = -A(z) + \frac{1}{2}, \qquad A(q^{-1}) = \frac{1}{2}, \qquad A(y) = \frac{\tau}{2} \pmod{\tau}, \tag{4.2}$$

$$A_{\pm}(-1) = \frac{1}{4} \mp \frac{\tau}{2}, \qquad A(1) = \frac{1}{4}, \qquad A(0) = \frac{1}{2} - A(\infty), \tag{4.3}$$

$$A_+(z) = -A_-(z), \qquad z \in [q, y], \tag{4.4}$$

$$A_+(z) = A_-(z) - \tau, \qquad z \in [y, y^{-1}], \tag{4.5}$$

$$A_+(z) = -A_-(z) + 1, \qquad z \in [y^{-1}, q^{-1}]. \tag{4.6}$$

Associated with $\mathbb{X}$ is the Riemann theta function

$$\theta(z) = \theta(z \mid \tau) = \sum_{k \in \mathbb{Z}} \exp\big(\pi \mathrm{i} k^2 \tau + 2\pi \mathrm{i} k z\big). \tag{4.7}$$

It satisfies $\theta(-z) = \theta(z)$ and $\theta(z + l + k\tau) = \exp(-2\pi \mathrm{i} k z - \pi \mathrm{i} k^2 \tau)\theta(z)$ for $l, k \in \mathbb{Z}$.

Lemma 4.2 (Vector solution of the model RHP)**.** *On $\mathbb{C} \setminus [q, q^{-1}]$ define*

$$\delta(z) = \frac{\theta\big(A(z) - \frac{1}{2} + \frac{tB}{2\pi} - \frac{\Delta}{4\pi}\big)\theta\big(A(z) + \frac{tB}{2\pi} - \frac{\Delta}{4\pi}\big)}{\theta\big(A(z) - \frac{1}{2}\big)\theta\big(A(z)\big)} \tag{4.8}$$

and

$$H(z) = \sqrt[4]{\frac{(y-z)(y^{-1}-z)}{(q-z)(q^{-1}-z)}}. \tag{4.9}$$

Then the vector function

$$m^{mod}(z) = \big(\delta(z), \quad \delta(z^{-1})\big)\,\frac{H(z)}{\sqrt{\delta(0)\delta(\infty)}} \tag{4.10}$$

is the unique solution of the model RH problem. Moreover,

$$\delta(z) = \frac{\theta\big(2A(z) - \frac{1}{2} + \frac{tB}{\pi} - \frac{\Delta}{2\pi} \mid 2\tau\big)}{\theta\big(2A(z) - \frac{1}{2} \mid 2\tau\big)}. \tag{4.11}$$

Proof. Using the formula $\theta(v \mid \tau)\theta(v - \frac{1}{2} \mid \tau) = \theta(2v - \frac{1}{2} \mid 2\tau)\theta(\frac{1}{2} \mid 2\tau)$ (cf. [12]) we can rewrite $\delta(z)$ as a quotient of two theta functions with double period 2τ as it is written in (4.11). Applying Lemma 4.1 to (4.8) and using that

$$H(z^{-1}) = H(z),\ z \in \mathbb{C} \setminus [q,y] \cup [y^{-1}, q^{-1}]; \qquad H(0) = 1;$$
$$H_+(z) = \mathrm{i}H_-(z),\ z \in [q,y]; \qquad H_+(z) = -\mathrm{i}H_-(z),\ z \in [y^{-1}, q^{-1}];$$

it is straightforward to check that m^{mod} (4.10) satisfies the jump (3.33) as far as the symmetry and normalization conditions. In fact, (4.9)–(4.10) are the z-analog of [18, equation (5.22)], where the vector model problem solution (4.10) was computed on the Riemann surface $\mathbb{M}(\xi)$ of the function

$$R^{1/2}(\lambda) = -\sqrt{(\lambda^2-1)(\lambda-\lambda_q)(\lambda-\lambda_y)} \tag{4.12}$$

with $\lambda_q = b - 2a = \frac{1}{2}(q + q^{-1})$ and $\lambda_y = \frac{1}{2}(y + y^{-1})$. □

Recall that $B = B(\xi)$ depends on n and t. By [18, Lem. 5.3],

$$2\mathrm{i}tB = -n\Lambda - tU,$$

where Λ and U are the $\mathfrak{b}$-periods of the normalized Abel differentials Ω_0 and $\omega_{\infty_+,\infty_-}$ of the second and third kind on $\mathbb{M}(\xi)$ (cf. [34, Ch. 9]). They do not correspond to the respective Abel differentials on $\mathbb{X}$, but due to (4.1) the constants Λ and U can easily be expressed in terms of the variable z. In fact, $\Omega_0 = \frac{\partial}{\partial\lambda}\Omega(\lambda,\xi)d\lambda$ and $\omega_{\infty_+,\infty_-} = \frac{\partial}{\partial\lambda}\omega(\lambda,\xi)d\lambda$ from (1.14). In particular,

$$\Lambda = 2\int_y^q \frac{(s-h)(s-h^{-1})}{W_-(s)}\frac{ds}{s},$$

where $\lambda_h = \frac{h+h^{-1}}{2}$ is the zero of $\omega_{\infty_+,\infty_-}$. Note that Λ is connected with the Abel map $A(z)$ by (cf. [34])

$$A(\infty) = A(0) - \frac{\Lambda}{4\pi\mathrm{i}}.$$

Remark 4.3. Observe that by (4.3),

$$\delta_\pm(-1) = \frac{\theta\big(-\frac14 \mp \frac{\tau}{2} + \frac{tB}{2\pi} - \frac{\Delta}{4\pi}\big)\theta\big(\frac14 \mp \frac{\tau}{2} + \frac{tB}{2\pi} - \frac{\Delta}{4\pi}\big)}{\theta\big(-\frac14 \mp \frac{\tau}{2}\big)\theta\big(\frac14 \mp \frac{\tau}{2}\big)}.$$

This implies that $m^{mod}_\pm(-1) = (0,0)$ if $2tB - \Delta = \pi + 2\pi k$, $k \in \mathbb{Z}$. As it is shown in [19], for those pairs of n and t which satisfy

$$n\Lambda + tU = \mathrm{i}(2k+1)\pi, \quad k \in \mathbb{Z},$$

a bounded and invertible matrix solution of the model jump problem (3.33) with integrable isolated singularities on the jump contour $[y, y^{-1}]$ does not exist.

In the next section we propose a matrix model solution $M^{mod}(z)$ which is invertible for all n and t, but has poles at the edges of the right background (at $z = 1$ and $z = -1$). We will establish that the determinant of this matrix does not have singularities at these points, and therefore is a nonzero constant. Moreover, $m^{(3)}(z)[M^{mod}(z)]^{-1}$ does not have singularities at these points too, and hence is a suitable vector for the conclusive asymptotic analysis.

5. The matrix model RH problem

Let $\omega(p) = \int_{b-2a}^{p} \omega_{\infty_+\infty_-}$ be the Abel integral of the third kind on the Riemann surface $\mathbb{M}(\xi)$ of (4.12), as introduced in [18]. Let $I(\xi)$ be the closed contour on $\mathbb{M}(\xi)$ with projection on the interval $[\lambda_y, -1]$, which starts at λ_y, passes to -1 on the upper sheet and returns on the lower sheet. Then

$$\omega_+(p) - \omega_-(p) = -\Lambda, \quad p \in I(\xi). \tag{5.1}$$

We associate $z \in \mathcal{Q}(\xi) := \{z : |z| < 1\} \setminus [q, y]$ with $p = (\lambda, +)$ on the upper sheet of $\mathbb{M}(\xi)$, and z^{-1}, $z \in \mathcal{Q}(\xi)$, with $p^* = (\lambda, -)$ on the lower sheet. Calculating the z-analog of (5.1) and taking into account the symmetry property $\omega(p) = -\omega(p^*)$, we obtain that $\mathrm{e}^{\omega(p)} =: G(z)$, defined on $z \in \mathbb{C} \setminus [q^{-1}, q]$ if and only if $p \in \mathbb{M}(\xi) \setminus I(\xi)$, admits the representation

$$G(z) = \exp\left(\int_q^z \frac{(s-h)(s-h^{-1})}{\mathcal{R}(s)}\frac{ds}{s}\right), \quad z \in \mathbb{C} \setminus [q^{-1}, q],$$

and has the following properties.

- The function $G(z)$ is holomorphic on $\mathcal{E} \setminus [y^{-1}, y]$ and satisfies $G(z^{-1}) = G^{-1}(z)$.
- Its jumps are given by

$$\begin{aligned} G_+(z) &= G_-(z)\mathrm{e}^{-\Lambda}, && z \in [y, y^{-1}], \\ G_\pm(z) &= [G_\mp(z^{-1})]^{-1}, && z \in [q, y] \cup [y^{-1}, q^{-1}]. \end{aligned}$$

- The following asymptotic expansion is valid,

$$G(z) = -\frac{\tilde a}{2z}\Big(1 + 2\tilde b z + O(z^2)\Big), \quad G(z^{-1}) = -\frac{2z}{\tilde a}\Big(1 - 2\tilde b z + O(z^2)\Big). \tag{5.2}$$

Here $\tilde{a}$ and $\tilde{b}$ are the coefficients of the asymptotic expansion for $\omega(p)$ as $p \to \infty_\pm$ (cf. [34, equation (9.44)]),

$$\mathrm{e}^{\omega(p)} = -\left(\frac{\tilde{a}}{\lambda}\right)^{\pm 1}\left(1 + \frac{\tilde{b}}{\lambda} + O(\lambda^{-2})\right).$$

Note that in all our considerations the values y, τ, h etc. depend on n via ξ. To emphasize the dependence of (4.11) on n, we abbreviate

$$\begin{aligned} \alpha_n(z) &:= \delta(z)\frac{H(z)}{\sqrt{\delta(0)\delta(\infty)}} = \alpha_n H(z)\frac{\theta\big(2A(z) - \frac{1}{2} - \frac{n\Lambda}{2\pi\mathrm{i}} - \frac{tU}{2\pi\mathrm{i}} - \frac{\Delta}{2\pi} \mid 2\tau\big)}{\theta\big(2A(z) - \frac{1}{2} \mid 2\tau\big)}, \\ \alpha_n &:= \frac{\theta\big(2A(\infty) - \frac{1}{2} \mid 2\tau\big)}{\sqrt{\theta\big(2A(\infty) - \frac{1}{2} - \frac{n\Lambda}{2\pi\mathrm{i}} - \frac{tU}{2\pi\mathrm{i}} - \frac{\Delta}{2\pi} \mid 2\tau\big)\theta\big(2A(0) - \frac{1}{2} - \frac{n\Lambda}{2\pi\mathrm{i}} - \frac{tU}{2\pi\mathrm{i}} - \frac{\Delta}{2\pi} \mid 2\tau\big)}}. \end{aligned} \tag{5.3}$$

We fix y, τ and h in (5.3) and consider this expression shifted to $n+1$,

$$\alpha_{n+1}(z) = \alpha_{n+1}H(z)\frac{\theta\big(2A(z) - \frac{1}{2} - \frac{(n+1)\Lambda}{2\pi\mathrm{i}} - \frac{tU}{2\pi\mathrm{i}} - \frac{\Delta}{2\pi} \mid 2\tau\big)}{\theta\big(2A(z) - \frac{1}{2} \mid 2\tau\big)}.$$

Lemma 5.1. *The vector function*

$$m^{\#}(z) = \big(\beta_n(z), \quad \beta_n(z^{-1})\big), \quad \text{where } \beta_n(z) := \alpha_{n+1}(z)G(z^{-1}), \tag{5.4}$$

solves the jump problem of the model RH problem, that is,

$$m_+^{\#}(z,n,t) = m_-^{\#}(z,n,t)v^{mod}(z), \qquad \text{where}$$

$$v^{mod}(z) = \begin{cases} \mathrm{i}\sigma_1, & z \in [q, y], \\ -\mathrm{i}\sigma_1, & z \in [y^{-1}, q^{-1}], \\ \begin{pmatrix} \mathrm{e}^{-n\Lambda - tU - \mathrm{i}\Delta} & 0 \\ 0 & \mathrm{e}^{n\Lambda + tU + \mathrm{i}\Delta} \end{pmatrix}, & z \in [y, y^{-1}]. \end{cases} \tag{5.5}$$

It satisfies the symmetry condition $m^{\#}(z^{-1}) = m^{\#}(z)\sigma_1$. The normalization condition is not fulfilled, instead $\lim_{z\to 0} m_1^{\#}(z)m_2^{\#}(z) = 1$. More precisely,

$$m_1^{\#}(z) = -\frac{2z}{\tilde{a}}\alpha_{n+1}(0)\big(1+O(z)\big), \quad m_2^{\#}(z) = -\frac{\tilde{a}}{2z}\alpha_{n+1}(\infty)\big(1+O(z)\big), \text{ as } z \to 0.$$

Introduce two functions defined on $\mathbb{C} \setminus ([q^{-1}, q] \cup \{1, -1\})$:

$$\begin{aligned} \Psi_1(z) &= \frac{1}{2}m_1^{mod}(z) + \rho(z)m_1^{\#}(z) = \frac{1}{2}\alpha_n(z) + \rho(z)\beta_n(z), \\ \Psi_2(z) &= \frac{1}{2}m_2^{mod}(z) + \rho(z)m_2^{\#}(z) = \frac{1}{2}\alpha_n(z^{-1}) + \rho(z)\beta_n(z^{-1}), \end{aligned} \tag{5.6}$$

where

$$\rho(z) = -\rho(z^{-1}) = \frac{2K_n}{\tilde{a}\,(z^{-1} - z)}, \qquad K_n^{-1} = \alpha_n(0)\alpha_{n+1}(\infty). \tag{5.7}$$

Lemma 5.2.

(i) *The matrix*

$$M^{mod}(z) = \begin{pmatrix} \Psi_1(z) & \Psi_2(z) \\ \Psi_2(z^{-1}) & \Psi_1(z^{-1}) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \left([q^{-1}, q] \cup \{1, -1\}\right), \tag{5.8}$$

is a meromorphic matrix solution for the model jump problem

$$M_+^{mod}(z) = M_-^{mod}(z) v^{mod}(z), \quad z \in [q, q^{-1}], \tag{5.9}$$

with $v^{mod}(z)$ given by (3.33) or by the equivalent matrix (5.5). It has simple poles at $z = \pm 1$.

(ii) *$M^{mod}(z)$ satisfies the symmetry*

$$M^{mod}(z^{-1}) = \sigma_1 M^{mod}(z) \sigma_1. \tag{5.10}$$

(iii) *The vector function $(1,\, 1) M^{mod}(z)$ has removable singularities at $1, -1$ and integrable singularities at $\{q, q^{-1}, y, y^{-1}\}$ of order $O((z-\kappa)^{-\frac{1}{4}})$ as $z \to \kappa \in \{q, q^{-1}, y, y^{-1}\}$.*

(iv) *The determinant of $M^{mod}(z)$ is constant,*

$$\det M^{mod}(z) = 1, \quad z \in \mathbb{C}. \tag{5.11}$$

Proof. Items (i) and (ii) follow from Lemmas 4.2, 5.1. To prove (iii) observe that

$$\Psi_2(z^{-1}) = m_1^{mod}(z) - \rho(z) m_1^{\#}(z), \quad \Psi_1(z^{-1}) = m_2^{mod}(z) - \rho(z) m_2^{\#}(z). \tag{5.12}$$

Thus,

$$m^{mod}(z) = \left(m_1^{mod}(z), \;\; m_2^{mod}(z)\right) = (1,\, 1) M^{mod}(z). \tag{5.13}$$

The vector function $m^{mod}(z)$ given by (4.8), (4.9) and (4.10), does not have singularities at $z = \pm 1$, and it has fourth root singularities at $\{q, q^{-1}, y, y^{-1}\}$ which proves (iii). We emphasize that (5.13) provides a connection between the unique solution of the vector model RHP and the matrix model problem solution.

(iv) Evaluating $\det M^{mod}(z)$ as $z \to 0$ by use of (5.2) and (5.7) we get

$$\det M^{mod}(z) = \rho(z) \left(m_1^{\#}(z) m_2^{mod}(z) - m_2^{\#}(z) m_1^{mod}(z) \right), \tag{5.14}$$

that is,

$$\begin{aligned} \det M^{mod}(z) &= -2\rho(z) \alpha_{n+1}(z^{-1}) G(z) \alpha_n(z) + O(z^2), \\ &= \frac{2K_n z}{\tilde{a}} \left(-\frac{\tilde{a}}{2z} \right) \alpha_n(0) \alpha_{n+1}(\infty)(1 + O(z)) \\ &= 1 + O(z), \quad z \to 0. \end{aligned}$$

By (5.9), $\det M^{mod}(z)$ does not have jumps in $\mathbb{C}$ and by (5.10), it is an even function,

$$\det M^{mod}(z^{-1}) = \det M^{mod}(z). \tag{5.15}$$

Therefore, $\det M^{mod}(\infty)=1$. This function is even and bounded outside of small vicinities of 1 and -1 and may have simple poles at ±1. The singularities at the points $\{q,q^{-1},y,y^{-1}\}$ are at most of square root order, and therefore removable. Since the Abel map $A(z)$ and $G(z)$ are single valued functions in a vicinity of 1 with $A(1)=1/4$ and $G(1)=1$, we have

$$m_1^{mod}(1)=m_2^{mod}(1),\quad m_1^{\#}(1)=m_2^{\#}(1).$$

This implies together with (5.14) the absence of a singularity at 1. Hence we have a function holomorphic in $\mathbb{C}\setminus\{-1\}$ and bounded at infinity, which has at most a simple pole at the only point $z=-1$ and the additional symmetry (5.15). The only function which satisfies these properties is a constant. □

Corollary 5.3. *The following equality holds,*

$$\lim_{z\to0}\rho(z)m_2^{\#}(z)=-\frac{1}{m_1^{mod}(0)}. \tag{5.16}$$

Proof. By (5.14) and (5.11)

$$\rho(z)\left(m_1^{\#}(z)m_2^{mod}(z)-m_2^{\#}(z)m_1^{mod}(z)\right)\equiv1.$$

But $m_1^{\#}(z)\to0$ as $z\to0$. This proves (5.16). □

Let $\mathcal{B}$ and $\mathcal{B}^*$ be small symmetric vicinities of the points y and y^{-1}. The precise shape of their boundaries will be chosen in the next section. In $\mathbb{C}\setminus(\overline{\mathcal{B}}\cup\overline{\mathcal{B}^*})$ introduce the vector function $\nu(z)=m^{(3)}(z)[M^{mod}(z)]^{-1}$. This function satisfies the symmetry condition $\nu(z^{-1})=\nu(z)\sigma_1$ but the normalization condition is not identified yet. Instead, due to symmetry of ν and the properties of $[M^{mod}(z)]^{-1}$ we have

Lemma 5.4. *We have*

$$\nu_2(0)=\nu_1(\infty)=\frac{1}{2}\left(\frac{m_1^{(3)}(0)}{m_1^{mod}(0)}+\frac{m_1^{mod}(0)}{m_1^{(3)}(0)}\right):=\tau>0. \tag{5.17}$$

Proof. From (5.6), (5.16) and the normalization $m_2^{\mathrm{mod}}(0)=[m_1^{\mathrm{mod}}(0)]^{-1}>0$ we observe that

$$\Psi_2(0)=\frac{1}{2}m_2^{\mathrm{mod}}(0)-\frac{1}{m_1^{\mathrm{mod}}(0)}=-\frac{1}{2\,m_1^{\mathrm{mod}}(0)},\qquad \Psi_1(0)=\frac{1}{2}m_1^{\mathrm{mod}}(0).$$

Then (5.17) follows from the normalization $m_2^{(3)}(0)=[m_1^{(3)}(0)]^{-1}>0$ and

$$\nu_2(0)=-\Psi_2(0)m_1^{(3)}(0)+\Psi_1(0)[m_1^{(3)}(0)]^{-1}.$$ □

Lemma 5.5. *The function $\nu(z)$ does not have singularities in vicinities of the points $q,q^{-1},1,-1$.*

Proof. By (3.33) and (3.27) the vector $\nu(z)$ does not have jumps in small vicinities of q, q^{-1} and 1. By (3.30), (4.9), (5.3), (5.4), (5.6), (5.8) and (5.11) we conclude that $\nu(z) = O(z - q^{\pm 1})^{-1/2}$, and therefore it has no singularities at q and q^{-1}.

At $z = 1$, both $m^{(3)}(z)$ and $M^{mod}(z)$ have no jumps. The same is true for $m^{mod}(z)$, $m^{\#}(z)$ and $G(z)$, which means that the equalities

$$G(z^{-1}) = G^{-1}(z), \qquad m_1^{(3)}(z) = m_2^{(3)}(z^{-1}),$$
$$m_1^{mod}(z) = m_2^{mod}(z^{-1}), \qquad m_1^{\#}(z) = m_2^{\#}(z^{-1}),$$

can be applied in a vicinity of $z = 1$. The differences

$$G(z^{-1}) - G(z),\ m_1^{(3)}(z) - m_1^{(3)}(z^{-1}),\ m_1^{mod}(z) - m_1^{mod}(z^{-1}),\ m_1^{\#}(z) - m_1^{\#}(z^{-1}),$$

are all of order $O(z - 1)$ as $z \to 1$. Thus, from (5.6) and (5.12) it follows that

$$\Psi_2(z) - \Psi_1(z) = O(z-1) + \rho(z)(m_1^{\#}(z) - m_1^{\#}(z^{-1})) \to \Psi_0, \qquad z \to 1,$$
$$\Psi_1(z^{-1}) - \Psi_2(z^{-1}) = O(z-1) - \rho(z)(m_1^{\#}(z) - m_1^{\#}(z^{-1})) \to -\Psi_0, \quad z \to 1,$$

where

$$\Psi_0 = \lim_{z\to 1} \frac{K_n}{\tilde{a}(z^{-1} - z)} \left(\beta_n(z) - \beta_n(z^{-1})\right).$$

On the other hand, $m^{(3)}(z)$ does not have a jump in a vicinity of $z = 1$. Therefore, by the symmetry property, $m_1^{(3)}(1) = m_2^{(3)}(1^{-1}) = m_2^{(3)}(1)$. Hence

$$\begin{aligned}
\nu(z) &= m^{(3)}(z) \begin{pmatrix} \Psi_1(z^{-1}) & -\Psi_2(z) \\ -\Psi_2(z^{-1}) & \Psi_1(z) \end{pmatrix} \\
&= \Big(m_1^{(3)}(z)\Psi_1(z^{-1}) - m_2^{(3)}(z)\Psi_2(z^{-1}),\ \ m_2^{(3)}(z)\Psi_1(z) - m_1^{(3)}(z)\Psi_2(z)\Big) \\
&\to \Big(\Psi_0 m_1^{(3)}(1)\Big)\begin{pmatrix}1, & 1\end{pmatrix}, \quad z \to 1.
\end{aligned}$$

It remains to investigate the behavior of $\nu(z)$ near $z = -1$. Since $m^{(3)}(z)$ and $M^{mod}(z)$ have the same constant jump $v^3(z) = v^{mod}(z) = \mathrm{e}^{(2\mathrm{i}tB - \mathrm{i}\Delta)\sigma_3}$ in a vicinity of this point, we conclude that $\nu(z)$ does not have jumps here, and therefore $z = -1$ is an isolated singularity, which is at most a simple pole. From the symmetry condition it follows that both components $\nu_1(z)$ and $\nu_2(z)$ of $\nu(z)$ have the same behavior, either simple poles or removable singularities. To prove that -1 is in fact a removable singularity, it suffices to check that

$$\begin{aligned}
f(z) &= \nu_1(z)\nu_2(z) \\
&= \Big(m_1^{(3)}(z)\Psi_1(z^{-1}) - m_2^{(3)}(z)\Psi_2(z^{-1})\Big)\Big(m_2^{(3)}(z)\Psi_1(z) - m_1^{(3)}(z)\Psi_2(z)\Big)
\end{aligned}$$

increases not faster than $o((z+1)^{-2})$ from some direction. The behavior of $f(z)$ is determined by the summand which contains $\rho^2(z)$ (cf. (5.6) and (5.12)). Computing this term we get

$$f(z) \sim \rho^2(z)\Big([m_2^{\#}(z)]^2[m_1^{(3)}(z)]^2 + [m_1^{\#}(z)]^2[m_2^{(3)}(z)]^2$$

$$- 2m_1^{\#}(z)m_2^{\#}(z)m_1^{(3)}(z)m_2^{(3)}(z)\Big) = \rho^2(z)\tilde f(z).$$

The function $\tilde f(z)$ has finite limiting values on the sides of the contour $[\mathfrak{r},\mathfrak{r}^{-1}]$, and in particular at $z=-1$. Using the symmetry condition we get $m_{1,\pm}^{\#}(-1)=m_{2,\mp}^{\#}(-1)$, $m_{1,\pm}^{(3)}(-1)=m_{2,\mp}^{(3)}(-1)$, therefore

$$m_{1,\pm}^{\#}(-1)=m_{2,\pm}^{\#}\mathrm{e}^{\mp(2\mathrm{i}tB-\mathrm{i}\Delta)},\qquad m_{1,\pm}^{(3)}(-1)=m_{2,\pm}^{(3)}\mathrm{e}^{\mp(2\mathrm{i}tB-\mathrm{i}\Delta)},$$

that is, $m_{2,\pm}^{\#}(-1)m_{1,\pm}^{(3)}(-1)=m_{1,\pm}^{\#}(-1)m_{2,\pm}^{(3)}(-1)$. Thus

$$\begin{aligned}\tilde f_\pm(-1) &= [m_{2,\pm}^{\#}(-1)]^2[m_{1,\pm}^{(3)}(-1)]^2+[m_{1,\pm}^{\#}(-1)]^2[m_{2,\pm}^{(3)}(-1)]^2\\ &\quad - 2m_{1,\pm}^{\#}(-1)m_{2,\pm}^{\#}(-1)m_{1,\pm}^{(3)}(-1)m_{2,\pm}^{(3)}(-1)=0.\end{aligned}$$ □

Remark 5.6. The jump matrix $v^{(3)}(z)$ given by (3.28), (3.29) satisfies the symmetry

$$v^{(3)}(z)=\sigma_1 v^{(3)}(z^{-1})\sigma_1 \tag{5.18}$$

on the contour Ξ (cf. (3.26)), while on $[q,q^{-1}]$ it satisfies $[v^{(3)}(z)]^{-1}=\sigma_1 v^{(3)}(z^{-1})\sigma_1$. Therefore we reverse the orientation on the part $[-1,q^{-1}]$ such that the property (5.18) is satisfied on the whole jump contour of RHP 3.

6. Solution of the parametrix RH problems

In this section we solve local RHPs in vicinities of the points y,y^{-1}, where the error jump matrix (introduced at the end of Section 3) is not small as $t\to\infty$. Recall that on the contours with y^{-1} as a nodal point we have (with the new orientation on $[q^{-1},-1]$ as introduced in Remark 5.6)

$$v^{(3)}(z)=\begin{cases}\mathrm{i}\sigma_1, & z\in[q^{-1},y^{-1}],\\ \mathrm{e}^{(-2\mathrm{i}tB+\mathrm{i}\Delta)\sigma_3} & z\in[\mathfrak{r}^{-1},-1],\\ \begin{pmatrix}\frac{F_+(z)}{F_-(z)}\mathrm{e}^{t(g_+(z)-g_-(z))} & U(z)\mathrm{e}^{-2t\,\mathrm{Re}\,g(z)}\\ 0 & \frac{F_-(z)}{F_+(z)}\mathrm{e}^{t(g_-(z)-g_+(z))}\end{pmatrix}, & z\in[y^{-1},\mathfrak{r}^{-1}],\\ \begin{pmatrix}1 & 0\\ \frac{e^{2tg(z)}}{U_1(z)} & 1\end{pmatrix}, & z\in\mathcal{C}^*,\end{cases}$$

where we denoted

$$U(z)=\mathrm{i}|\chi(z)|\Pi^2(z)F_+(z)F_-(z),\qquad U_1(z)=\Pi^2(z)F^2(z)X(z), \tag{6.1}$$

and used (3.32). Respectively,

$$v^{err}(z)=v^{(3)}(z)-v^{mod}(z)=\begin{cases}\begin{pmatrix}0 & U(z)\mathrm{e}^{-2t\,\mathrm{Re}\,g(z)}\\ 0 & 0\end{pmatrix}, & z\in[y^{-1},\mathfrak{r}^{-1}],\\ \begin{pmatrix}0 & 0\\ \frac{e^{2tg(z)}}{U_1(z)} & 0\end{pmatrix}, & z\in\mathcal{C}^*,\end{cases}$$

does not vanish as $t \to \infty$ since $\operatorname{Re} g(y^{-1}) = 0$, $U(y^{-1})U_1(y^{-1}) \neq 0$. The local (parametrix) RHPs are similar to those of the KdV shock wave analysis (see e.g., [19, Sec. 7]). Consider first the point y^{-1}. Let $\mathcal{B}^* = \mathcal{B}^*(\varepsilon)$ be a neighborhood of y^{-1} such that its boundary contains $\mathfrak{r}^{-1}$ given by (3.11). To describe the boundary of $\mathcal{B}^*$, introduce a local change of variables

$$w^{3/2}(z) = \frac{3t}{2}\left(g(z) - g_\pm(y^{-1})\right), \quad z \in \mathcal{B}^*, \tag{6.2}$$

with the cut along the interval $J = [q^{-1}, y^{-1}] \cap \overline{\mathcal{B}^*}$. From (3.4) and item (b) of Lemma 3.1 we have

$$\begin{aligned}\frac{3}{2}(g(z) - g_\pm(y^{-1})) &= \frac{3}{2}\int_{y^{-1}\pm \mathrm{i}0}^{z} P(s)\tilde{Q}(s)\,\sqrt{y^{-1} - s}\,\frac{ds}{2s} \\ &= \frac{2}{y}P(y)\tilde{Q}(y)(z - y^{-1})^{3/2}\left(1 + o(1)\right), \quad z \to y^{-1},\end{aligned}$$

with $P(s)$ given by (3.3) and

$$\tilde{Q}(s) := \sqrt{\frac{s - y}{(s - q)(s - q^{-1})}}.$$

Evidently, $\tilde{Q}(y) > 0$. For $\xi \in (\xi_{cr}', \xi_{cr})$ we have (see [18]) $P(s) = s + s^{-1} - \zeta - \zeta^{-1}$, where $\zeta = \zeta(\xi) \in (-1, y)$. Thus, $P(y^{-1}) < 0$ and $\mathcal{T} := 2y^{-1}P(y)\tilde{Q}(y) > 0$. Then

$$w(z) = \mathcal{T}^{2/3} t^{2/3}\,(z - y^{-1})(1 + o(1)), \quad \text{as } z \to y^{-1}, \quad (\mathcal{T}t)^{2/3} > 0. \tag{6.3}$$

Hence $w(z)$ is a holomorphic function in $\mathcal{B}^*$.

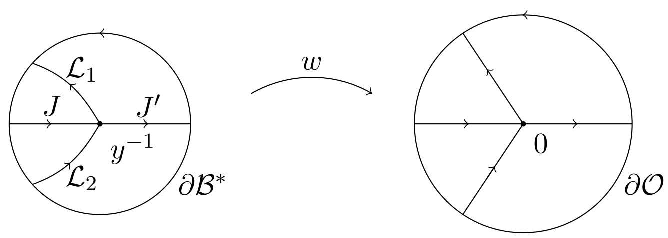

Fig. 6.1: The local change of variables $w(z)$.

Untill now we did not specify the particular shape of the boundary of $\mathcal{B}^*$ and the shape of the contour $\mathcal{C}^*$ inside $\mathcal{B}^*$. Treating $w(z)$ as a conformal map, let us think of $\mathcal{B}^*$ as a pre-image of a disc $\mathcal{O}$ of radius $\mathcal{T}^{2/3}|y^{-1} - \mathfrak{r}^{-1}|t^{2/3}$ centered at the origin. Then $w(z)$ maps the interval $J = [q^{-1}, y^{-1}] \cap \overline{\mathcal{B}^*}$ to the negative half axis and $J' = [y^{-1}, \mathfrak{r}^{-1}]$ to the positive half axis. We also choose the contour $\mathcal{C}^* \cap \mathcal{B}^*$ to be contained in the pre-image of the rays $\arg w = \pm\frac{2\pi \mathrm{i}}{3}$ and divide it in two parts $\mathcal{L}_1$ and $\mathcal{L}_2$, with orientation as depicted in Fig. 6.1. With the new orientation,

$$v^{mod}(z) = \begin{cases} \mathrm{i}\sigma_1 & z \in J \\ \mathrm{e}^{(-2\mathrm{i}tB + \mathrm{i}\Delta)\sigma_3} & z \in J' \end{cases}. \tag{6.4}$$

In $\mathcal{B}^*$ we introduce the function

$$r(z) := \frac{\mathrm{e}^{\mp\frac{\mathrm{i}\pi}{4}}\,\mathrm{e}^{\mp\mathrm{i}tB}}{\sqrt{X(z)}\Pi(z)F(z)}, \quad z \in \mathcal{B}^* \cap \{z : \pm\operatorname{Im} z > 0\}. \tag{6.5}$$

Since

$$X_\pm(z) = \mp\mathrm{i}|\chi(z)|, \quad z \in J \cup J', \tag{6.6}$$

we see that

$$r_\pm(z) = \frac{\mathrm{e}^{\mp\mathrm{i}tB}}{\sqrt{|\chi(z)|}\Pi(z)F_\pm(z)}, \quad z \in J \cup J'.$$

By items (b) and (c) of Lemma 3.5, taking into account the change of direction for the contour in (c), we get

$$r_+(z)r_-(z) = 1, \quad z \in J; \qquad r_+(z) = r_-(z)\mathrm{e}^{\mathrm{i}\Delta - 2\mathrm{i}tB}, \quad z \in J'. \tag{6.7}$$

Due to (6.6), $\sqrt{X_+(z)X_-(z)} = |\chi(z)|$ for $z \in J'$. By use of (6.1) and (6.2), we have for the off-diagonal elements of v^{err}

$$\begin{aligned} U(z)\mathrm{e}^{-2t\operatorname{Re} g(z)} &= \mathrm{i}|\chi(z)|\Pi^2(z)F_+(z)F_-(z)\mathrm{e}^{-t(g_-(z)+g_+(z))} \\ &= \mathrm{i}\,\frac{\mathrm{e}^{-t(g_+(z)-g_+(y^{-1}))}\mathrm{e}^{-t(g_-(z)-g_-(y^{-1}))}}{r_+(z)r_-(z)} = \frac{\mathrm{i}\,\mathrm{e}^{-\frac{4}{3}w^{3/2}(z)}}{r_+(z)r_-(z)}, \end{aligned} \tag{6.8}$$

$$\begin{aligned} \frac{\mathrm{e}^{2tg(z)}}{U_1(z)} &= \pm\mathrm{i}\,r^2(z)\mathrm{e}^{\pm 2\mathrm{i}tB + 2tg(z)} = \pm\mathrm{i}\,r^2(z)\mathrm{e}^{2tg(z) - g_\pm(y^{-1})} \\ &= \pm\mathrm{i}r^2(z)\mathrm{e}^{\frac{4}{3}w^{3/2}(z)}, \quad \pm\operatorname{Im} z > 0. \end{aligned} \tag{6.9}$$

We redefine $m^{(3)}(z)$, $m^{mod}(z)$ and the matrix $M^{mod}(z)$ inside $\mathcal{B}^*$ by

$$\begin{aligned} \widehat{m}^{(3)}(z) = m^{(3)}(z)r(z)^{-\sigma_3}, \quad \widehat{m}^{mod}(z) = m^{mod}(z)r(z)^{-\sigma_3}, \\ \widehat{M}^{mod}(z) = M^{mod}(z)r(z)^{-\sigma_3}, \quad z \in \mathcal{B}^*. \end{aligned} \tag{6.10}$$

Using (6.4), (6.7), (6.8) and (6.9), we obtain that inside $\mathcal{B}^*$

$$\widehat{m}^{(3)}_+(z) = \widehat{m}^{(3)}_-(z)\widehat{v}^{(3)}(z), \quad \widehat{M}^{mod}_+(z) = \widehat{M}^{mod}_-(z)\widehat{v}^{mod}(z),$$

where

$$\widehat{v}^{mod}(z) = \mathrm{i}\sigma_1, \quad z \in J; \qquad \widehat{v}^{mod}(z) = \mathbb{I}, \quad z \in J';$$

$$\widehat{v}^{(3)}(z) = \begin{cases} \mathrm{i}\sigma_1, & z \in J, \\ \begin{pmatrix} 1 & \mathrm{i}\mathrm{e}^{-\frac{4}{3}w^{3/2}(z)} \\ 0 & 1 \end{pmatrix}, & z \in J', \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{4/3w^{3/2}(z)} & 1 \end{pmatrix}, & z \in \mathcal{L}_1, \\ \begin{pmatrix} 1 & 0 \\ \mathrm{i}\mathrm{e}^{4/3w^{3/2}(z)} & 1 \end{pmatrix}, & z \in \mathcal{L}_2. \end{cases}$$

By (6.3) we conclude that $w^{1/4}(z)$ has the following jump along J,

$$w_+^{1/4}(z) = w_-^{1/4}(z)\mathrm{i}, \quad z \in J.$$

Recall that $\mathcal{O} = w(\mathcal{B}^*)$. It is now straightforward to check that the matrix

$$N(w) = \frac{1}{\sqrt{2}} \begin{pmatrix} w^{1/4} & w^{1/4} \\ -w^{-1/4} & w^{-1/4} \end{pmatrix}, \quad w \in \mathcal{O},$$

solves the jump problem

$$N_+(w(z)) = \mathrm{i}N_-(w(z))\sigma_1, \quad z \in J.$$

Therefore, in $\mathcal{B}^*$ we have $\widehat{M}^{mod}(z) = \mathcal{H}(z)N(w(z))$, where $\mathcal{H}(z)$ is a holomorphic matrix function in $\mathcal{B}^*$. Since $\det N(w) = \det[r(z)^{\sigma_3}] = 1$, we have

$$\det \mathcal{H}(z) = \det M^{\mathrm{mod}}(z) = \det \widehat{M}^{mod}(z) = 1. \tag{6.11}$$

According to (6.10) we get

$$M^{mod}(z) = \mathcal{H}(z)N(w(z))r(z)^{\sigma_3}, \quad z \in \partial\mathcal{B}^*.$$

By property (c) of Lemma 3.1, $w_+(z)^{3/2} = -w_-(z)^{3/2}$ for $z \in J$, that is,

$$\widehat{v}^{(3)}(z) = d_-(z)^{-\sigma_3}\mathcal{S}\, d_+(z)^{\sigma_3}, \quad z \in \mathcal{B}^*,$$

where

$$d(z) := \tilde{d}(w(z)), \quad \tilde{d}(w) = \mathrm{e}^{2/3w^{3/2}},$$

and

$$\mathcal{S} = \begin{cases} \mathcal{S}_1, & z \in \mathcal{L}_1, \\ \mathcal{S}_2, & z \in J, \\ \mathcal{S}_3, & z \in \mathcal{L}_2, \\ \mathcal{S}_4, & z \in J'. \end{cases}$$

Here

$$\mathcal{S}_1 = \begin{pmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{pmatrix}; \quad \mathcal{S}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \quad \mathcal{S}_3 = \begin{pmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{pmatrix}; \quad \mathcal{S}_4 = \begin{pmatrix} 1 & \mathrm{i} \\ 0 & 1 \end{pmatrix}.$$

Consider $\mathcal{S}$ as the jump matrix on the contour $w(J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2)$ in $\mathcal{O}$. Let $\mathcal{A}(w)$ be the matrix solution of the jump problem

$$\mathcal{A}_+(w) = \mathcal{A}_-(w)\mathcal{S}, \quad w \in w(J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2),$$

satisfying the boundary condition

$$\mathcal{A}(w) = N(w)\Psi(w)\tilde{d}(w)^{-\sigma_3}, \quad w \in \partial\mathcal{O}, \quad t \to \infty, \tag{6.12}$$

where

$$\Psi(w) = \mathbb{I} + \frac{C}{w^{3/2}}\Big(1 + O\big(w^{-3/2}\big)\Big), \quad w \to \infty, \tag{6.13}$$

is an invertible matrix, and C is a constant matrix with respect to w, t and ξ. The solution $\mathcal{A}(w)$ can be expressed via Airy functions and their derivatives in a standard manner (see, e.g., [9], [2, Ch. 3], [20] or [1]). In particular, in the sector between the contours $w(J')$ and $w(\mathcal{L}_1)$ in $\mathcal{O}$ we have

$$\mathcal{A}(w) = \mathcal{A}_1(w) = \sqrt{2\pi}\begin{pmatrix} -y_1'(w) & \mathrm{i}y_2'(w) \\ -y_1(w) & \mathrm{i}y_2(w)\end{pmatrix},$$

where $y_1(w) = \mathrm{Ai}(w)$ and $y_2(w) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\mathrm{Ai}(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}w)$. In the sector between the lines $w(\mathcal{L}_1)$ and $w(J)$ we get

$$\mathcal{A}(w) = \mathcal{A}_2(w) = \mathcal{A}_1(w)\mathcal{S}_1 = \sqrt{2\pi}\begin{pmatrix} y_3'(w) & \mathrm{i}y_2'(w) \\ y_3(w) & \mathrm{i}y_2(w)\end{pmatrix},$$

where $y_3(w) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\mathrm{Ai}(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}w)$. Here we used the standard equality $y_1(w)+y_2(w)+y_3(w)=0$. Changing orientation on J and $\mathcal{L}_2$ we obtain between $w(J)$ and $w(\mathcal{L}_2)$

$$\mathcal{A}(w) = \mathcal{A}_3(w) = -\mathrm{i}\mathcal{A}_2(w)\sigma_1,$$

and between the lines $w(\mathcal{L}_2)$ and $w(J')$, correspondingly,

$$\mathcal{A}(w) = \mathcal{A}_4(w) = \mathcal{A}_3(w)\mathcal{S}_3^{-1}.$$

The last conjugation with the matrix S_4 will lead to matrix $\mathcal{A}_1(w)$ again, because $\mathcal{S}_1\mathcal{S}_2^{-1}\mathcal{S}_3^{-1}\mathcal{S}_4 = \mathbb{I}$. Note that the constant matrix C in (6.12) is the same for all regions,

$$C = \frac{1}{48}\begin{pmatrix} -1 & 6 \\ -6 & 1\end{pmatrix}.$$

The precise formulas for $\mathcal{A}_j(w)$ are in fact not important for us. The matrix

$$M^{par}(z) := \mathcal{H}(z)\mathcal{A}(w(z))d(z)^{\sigma_3}, \quad z \in \mathcal{B}^*,$$

solves in $\mathcal{B}^*$ the jump problem

$$M_+^{par}(z) = M_-^{par}(z)\widehat{v}^{(3)}(z), \quad z \in J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2,$$

and satisfies for sufficiently large t the boundary condition

$$\begin{aligned} M^{par}(z) &= \mathcal{H}(z)N(w(z))\Psi(w(z)) = \widehat{M}^{mod}(z)\Psi(w(z)) \\ &= M^{mod}(z)r(z)^{-\sigma_3}\Psi(w(z)), \quad z \in \partial\mathcal{B}^*. \end{aligned}$$

Note that (6.13) and (6.2) yield

$$\det\Psi(w(z)) = 1 + O(t^{-1}), \quad z \in \mathcal{B}^*, \quad t \to \infty,$$

uniformly with respect to $\xi \in \mathcal{I}_\varepsilon$. This implies with (6.11) invertibility of $M^{par}(z)$ in $\overline{\mathcal{B}^*}$. In summary, we constructed a matrix with the following properties:

Lemma 6.1. *The vector function*

$$\nu(z) = \widehat{m}^{(3)}(z)M^{par}(z)^{-1} = m^{(3)}(z)r(z)^{-\sigma_3}M^{par}(z)^{-1}, \quad z \in \mathcal{B}^*, \tag{6.14}$$

does not have jumps and isolated singularities in $\mathcal{B}^*$, it is holomorphic there. The function $\nu(z)$ has piecewise continuous limiting values as z approaches $\partial\mathcal{B}^*$ from inside, given by

$$\nu(z) = m^{(3)}(z)r(z)^{-\sigma_3}\Psi(w(z))^{-1}r(z)^{\sigma_3}M^{mod}(z)^{-1}, \quad z \in \partial\mathcal{B}^*. \tag{6.15}$$

Let $\mathcal{B} := \{z : z^{-1} \in \mathcal{B}^*\}$. Define $\nu(z)$ in $\mathcal{B}^*$ by the symmetry $\nu(z) = \nu(z^{-1})\sigma_1$, $z \in \mathcal{B}$. With this extension, $\nu(z)$ is holomorphic in $\mathcal{B}$. Let us extend the definition of $\nu(z)$ to $\mathbb{C} \setminus (\overline{\mathcal{B}^*} \cup \overline{\mathcal{B}})$ by

$$\nu(z) = m^{(3)}(z)M^{mod}(z)^{-1}, \quad z \in \mathbb{C} \setminus (\overline{\mathcal{B}^*} \cup \overline{\mathcal{B}}). \tag{6.16}$$

Theorem 5.5 implies that this function does have jumps on the contour $[q,-1] \cup [q^{-1},-1]$ outside of $\overline{\mathcal{B}^*} \cup \overline{\mathcal{B}}$. We label the parts of $\mathcal{C}$ and $\mathcal{C}^*$ outside $\mathcal{B}$ and $\mathcal{B}^*$ by $\mathcal{C}_{\mathcal{B}}$ and $\mathcal{C}^*_{\mathcal{B}}$, see Fig. 3.3. The jumps of $\nu(z)$ on $\Gamma \cup \mathcal{C}_{\mathcal{B}} \cup \mathcal{C}^*_{\mathcal{B}}$ (cf. (3.13)) are exponentially small with respect to $t \to \infty$. Let us compute the jump of this vector on the boundaries $\partial\mathcal{B}$ and $\partial\mathcal{B}^*$, which we treat as clockwise oriented contours. Since neither $m^{(3)}(z)$, $r(z) = r^{-1}(z^{-1})$ nor $M^{mod}(z)$ have jumps on these contours, we obtain from (6.15) and (6.16)

$$m^{(3)}(z) = \nu_-(z)M^{mod}(z)r(z)^{-\sigma_3}\Psi(w(z))r(z)^{\sigma_3} = \nu_+(z)M^{mod}(z), \quad z \in \partial\mathcal{B}^*.$$

Taking into account (6.13) we find the jump

$$\nu_+(z) = \nu_-(z)(\mathbb{I} + W(z)), \quad z \in \partial\mathcal{B}^* \cup \partial\mathcal{B},$$

where

$$\begin{aligned} W(z) &= M^{mod}(z)r(z)^{-\sigma_3}\big(\Psi(w(z)) - \mathbb{I}\big)r(z)^{\sigma_3}M^{mod}(z)^{-1}, \quad z \in \partial\mathcal{B}^*;\\ W(z) &= \sigma_1 W(z^{-1})\sigma_1, \qquad z \in \partial\mathcal{B}. \end{aligned} \tag{6.17}$$

The jump contour

$$\mathcal{K} := \Gamma \cup \mathcal{C}_{\mathcal{B}} \cup \mathcal{C}^*_{\mathcal{B}} \cup \partial\mathcal{B} \cup \partial\mathcal{B}^*,$$

for the RHP associated with the error vector $\nu(z)$ is depicted in Fig. 6.2.

Merging the results of this section with Lemmas 5.5 and 5.4 we have proven

Theorem 6.2. *The vector $\nu(z)$ is a holomorphic function in the domain $\mathbb{C} \setminus \mathcal{K}$ and bounded on the closure of this domain. On the contour $\mathcal{K}$, $\nu(z)$ has the jump*

$$\nu_+(z) = \nu_-(z)(\mathbb{I} + W(z)), \tag{6.18}$$

where $W(z)$ is given by (6.17) on $\partial\mathcal{B} \cup \partial\mathcal{B}^*$ and

$$W(z) = M^{mod}_-(z)\big(v^{(3)}(z) - \mathbb{I}\big)M^{mod}_+(z)^{-1}, \quad z \in \Gamma \cup \mathcal{C}_{\mathcal{B}} \cup \mathcal{C}^*_{\mathcal{B}} = \mathcal{K} \setminus (\partial\mathcal{B} \cup \partial\mathcal{B}^*). \tag{6.19}$$

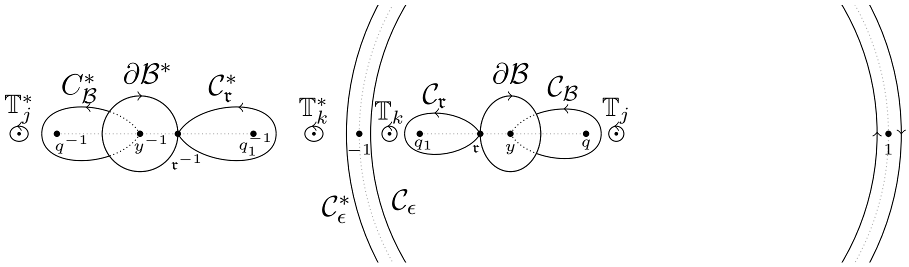

Fig. 6.2: $\mathcal{K} = \bigcup_{j=1}^N (\mathbb{T}_j \cup \mathbb{T}_j^*) \cup \mathcal{C}_\epsilon \cup \mathcal{C}_\epsilon^* \cup \mathcal{C}_{\mathfrak{r}} \cup \mathcal{C}_{\mathfrak{r}}^* \cup \mathcal{C}_{\mathcal{B}} \cup \mathcal{C}_{\mathcal{B}}^* \cup \partial\mathcal{B} \cup \partial\mathcal{B}^*$.

On $\mathcal{K}$, $W(z)$ has the symmetry

$$W(z^{-1}) = \sigma_1 W(z)\sigma_1, \quad z \in \mathcal{K}. \tag{6.20}$$

The vector $\nu(z)$ satisfies

$$\nu(z^{-1}) = \nu(z)\sigma_1. \tag{6.21}$$

Moreover,

$$\nu_-(z^{-1}) = \nu_-(z)\sigma_1, \quad z \in \mathcal{K}. \tag{6.22}$$

In addition,

$$\nu_2(0) = \nu_1(\infty) = \tau > 0.$$

We also observe the following estimate. By definition,

$$\min \operatorname{dist}\left(y^{-1}, \partial\mathcal{B}^*\right) > C(\varepsilon) > 0, \text{ uniformly with respect to } \xi \in \mathcal{I}_\varepsilon.$$

Respectively,

$$\min_{z\in\partial\mathcal{B}*} \frac{1}{|g(z) - g_\pm(y^{-1})|} > C_1(\varepsilon) > 0, \text{ uniformly with respect to } \xi \in \mathcal{I}_\varepsilon. \tag{6.23}$$

Moreover, the matrix functions $M^{mod}(z)$, $[M^{mod}(z)]^{-1}$, $r(z)$, $r^{-1}(z)$ are bounded uniformly with respect to $z \in \mathcal{B} \cup \mathcal{B}^*$ and $\xi \in \mathcal{I}_\varepsilon$. From (6.13), (6.5) and (6.17) we conclude that

$$\sup_{\xi\in\mathcal{I}_\varepsilon} \sup_{z\in\partial\mathcal{B}\cup\partial\mathcal{B}^*} \|W(z)\| \le \frac{C_2(\varepsilon)}{t}. \tag{6.24}$$

On the other hand, (6.23), (6.19), (3.29) and (3.14) imply

$$\sup_{\xi\in\mathcal{I}_\varepsilon} \sup_{z\in\mathcal{K}\setminus(\partial\mathcal{B}\cup\partial\mathcal{B}^*)} \|W(z)\| \le O\Big(\mathrm{e}^{-C_3(\varepsilon)t}\Big). \tag{6.25}$$

7. Completion of the asymptotic analysis

The aim of this section is to establish that the solution $m^{(3)}(z)$ is well approximated by $m^{mod}(z) = \begin{pmatrix}1 & 1\end{pmatrix} M^{\mathrm{mod}}(z)$ as $z \to 0$. We follow the well-known approach via singular integral equations (see, e.g., [10, 22, 28], [23, Ch. 4]). A peculiarity of this approach applied to the Toda equation is generated by the type of normalization condition of the vector RHP and the symmetry condition. In particular, if we want to preserve the symmetry condition (6.21) in the Cauchy-type formula for $\nu(z)$, we should use a matrix Cauchy kernel (cf. [27, equation (B.8)],

$$\hat{\Omega}(s,z)=\begin{pmatrix}\frac{1}{s-z} & 0\\ 0 & \frac{1}{s-z}-\frac{1}{s}\end{pmatrix}ds,\quad s\in\mathcal{K},\quad z\notin\mathcal{K}.$$

Since [27, equation (B.9)]

$$\hat{\Omega}(s,z^{-1})=\sigma_1\hat{\Omega}(s^{-1},z)\sigma_1,$$

this implies with (6.22) and (6.20) the symmetry

$$\int_{\mathcal{K}}\nu_-(s)W(s)\hat{\Omega}(s,z)=\int_{\mathcal{K}}\nu_-(s)W(s)\hat{\Omega}(s,z^{-1})\sigma_1.$$

Note that the $1,1$-entry of the Cauchy kernel $\hat{\Omega}(s,z)$ has zero at $z=\infty$ while the $2,2$-entry has zero at $z=0$. From (6.18) and (5.17) it follows that

$$\begin{aligned}\nu(z)&=\begin{pmatrix}\nu_1(\infty), & \nu_2(0)\end{pmatrix}+\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{K}}\nu_-(s)W(s)\hat{\Omega}(s,z)\\&=\tau\begin{pmatrix}1, & 1\end{pmatrix}+\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{K}}\nu_-(s)W(s)\hat{\Omega}(s,z).\end{aligned}$$

Let $\mathfrak{C}$ denote the Cauchy operator associated with $\mathcal{K}$,

$$(\mathfrak{C}h)(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{K}}h(s)\hat{\Omega}(s,z),\qquad s\in\mathbb{C}\setminus\mathcal{K},$$

where $h=\begin{pmatrix}h_1, & h_2\end{pmatrix}\in L^2(\mathcal{K})$ and satisfies the symmetry $h(s)=h(s^{-1})\sigma_1$. Let $(\mathfrak{C}_+h)(z)$ and $(\mathfrak{C}_-h)(z)$ be the non-tangential limiting values of $(\mathfrak{C}h)(z)$ from the left and right sides of $\mathcal{K}$, respectively. As usual, we introduce the operator $\mathfrak{C}_W: L^2(\mathcal{K})\cap L^\infty(\mathcal{K})\to L^2(\mathcal{K})$ by $\mathfrak{C}_W h=\mathfrak{C}_-(hW)$. By virtue of (6.24) and (6.25) we obtain

$$\|\mathfrak{C}_W\|=\|\mathfrak{C}_W\|_{L^2(\mathcal{K})\to L^2(\mathcal{K})}\le C\|W\|_{L^\infty(\mathcal{K})}=O\left(\frac{1}{t}\right)$$

as well as

$$\|(\mathbb{I}-\mathfrak{C}_W)^{-1}\|=\|(\mathbb{I}-\mathfrak{C}_W)^{-1}\|_{L^2(\mathcal{K})\to L^2(\mathcal{K})}\le\frac{1}{1-O(t^{-1})}$$

for sufficiently large t. Consequently, for $t\gg 1$, on $\mathcal{K}$ we define a vector function

$$\mu(s)=(\tau,\ \tau)+(\mathbb{I}-\mathfrak{C}_W)^{-1}\mathfrak{C}_W\big((\tau,\ \tau)\big)(s),$$

with τ given by (5.17). Then

$$\|\mu(s)-(\tau,\ \tau)\|_{L^2(\mathcal{K})}\le\|(\mathbb{I}-\mathfrak{C}_W)^{-1}\|\|\mathfrak{C}_-\|\|W\|_{L^\infty(\mathcal{K})}=O(t^{-1}).\tag{7.1}$$

With the help of μ, the vector function $\nu(z)$ can be represented as

$$\nu(z)=(\tau,\ \tau)+\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{K}}\mu(s)W(s)\hat{\Omega}(s,z),$$

and by virtue of (7.1), (6.24), and (6.25) we obtain as $z \to 0$

$$\nu(z) = (\tau,\ \tau) + \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{K}} (\tau,\ \tau) W(s) \begin{pmatrix} s^{-1} + zs^{-2} & 0 \\ 0 & zs^{-2} \end{pmatrix} ds + E(z). \tag{7.2}$$

Here $E(z)$ is a holomorphic vector function in a vicinity of $z = 0$ with

$$\|E(z)\| \le \|W\|_{L^2(\mathcal{K})} \|\mu(s) - (\tau,\ \tau)\|_{L^2(\mathcal{K})} (1 + O(z)) = O(t^{-2})(1 + O(z)),$$

and $O(z)$ is uniformly bounded for $\xi \in \mathcal{I}_\varepsilon$. From (7.2) and (5.13) we get

$$m^{(3)}(z) = \nu(z) M^{mod}(z) = \tau m^{\mathrm{mod}}(z) + \tau O(t^{-1}) E_1(z), \tag{7.3}$$

where $E_1(z)$ is a holomorphic vector function in a vicinity of $z = 0$, uniformly bounded with respect to $\xi \in \mathcal{I}_\epsilon$. The normalization conditions for $m^{(3)}$ and m^{mod} imply that $\tau^2(1 + O(t^{-1})) = 1$, that is,

$$\tau = 1 + O(t^{-1}).$$

Together with (7.3), (4.9)–(4.10) and (3.31) this implies

Theorem 7.1. *The following representation holds for $t \to \infty$ and $n \to \infty$*

$$m_1(z) m_2(z) = H^2(z) \frac{\delta(z)\delta(z^{-1})}{\delta(0)\delta(\infty)} + \beta_1(\xi, t) + \beta_2(\xi, t) z + \beta_2(\xi, t) O(z^2), \quad z \to 0, \tag{7.4}$$

where $|\beta_j(\xi, t)| \le \frac{C(\varepsilon)}{t}$ uniformly with respect to $\xi \in \mathcal{I}_\varepsilon$.

Our next aim is to clarify the properties of the function

$$\mathcal{Y}(z) := \frac{\delta(z)\delta(z^{-1})}{\delta(0)\delta(\infty)}. \tag{7.5}$$

To simplify notations denote $\frac{tB}{2\pi} - \frac{\Delta}{4\pi} =: x \in \mathbb{R}$. Then by (4.8),

$$\mathcal{Y}(z) = \frac{\theta\big(A(z) - \frac{1}{2} + x\big)\, \theta\big(A(z) + x\big)\, \theta\big(A(z^{-1}) - \frac{1}{2} + x\big)\, \theta\big(A(z^{-1}) + x\big)}{\delta(0)\delta(\infty)\, \theta\big(A(z) - \frac{1}{2}\big)\, \theta\big(A(z)\big)\, \theta\big(A(z^{-1}) - \frac{1}{2}\big)\, \theta\big(A(z^{-1})\big)}.$$

Since $x \in \mathbb{R}$, the properties of the Abel integral $A(z)$ listed in Lemma 4.1 imply that $\theta\big(A(z) - \frac{1}{2} + x\big)\theta\big(A(z) + x\big)$ has a simple zero at $(\mu(x), \pm)$ on one of the sheets of the Riemann surface $\mathbb{X}$ with projection $\mu(x) \in [y^{-1}, y]$, and a zero $(\mu^{-1}(x), \mp)$ on the other sheet. The function $\theta\big(A(z^{-1}) - \frac{1}{2} + x\big)\theta\big(A(z^{-1}) + x\big)$ has zeros at $(\mu(x), \mp), (\mu^{-1}(x), \pm)$. The denominator of $\mathcal{Y}(z)$ has double zeros (as points on the Riemann surface) at y and y^{-1}. Since y and y^{-1} are the branching points on $\mathbb{X}$, then these zeros are simple in the variable z on the complex plane. We observe that $\mathcal{Y}(z)$, being considered as a function in the domain $\mathbb{C} \setminus [q^{-1}, q]$ identified with the upper sheet of $\mathbb{X}$, does not have jumps on $[q^{-1}, q]$ and tends to 1 as $z \to \infty$.

Hence $\mathcal{Y}(z)$ is a rational function with simple poles at y and y^{-1} and simple zeros at $\mu(x)$ and $\mu^{-1}(x)$. Therefore,

$$\mathcal{Y}(z) = \frac{(z-\mu(x))(z-\mu^{-1}(x))}{(z-y)(z-y^{-1})},$$

and

$$\begin{aligned} m_1^{mod}(z)m_2^{mod}(z) = H^2(z)\mathcal{Y}(z) &= \frac{(z-\mu(x))(z-\mu^{-1}(x))}{\sqrt{(z-q)(z-q^{-1})(z-y)(z-y^{-1})}} \\ &= \frac{z+z^{-1}-\mu(x)-\mu^{-1}(x)}{\sqrt{(z+z^{-1}-q-q^{-1})(z+z^{-1}-y-y^{-1})}} \\ &= \frac{\lambda-\lambda(n,t)}{\sqrt{(\lambda-(b-2a))(\lambda-\lambda_y)}}, \end{aligned} \tag{7.6}$$

where $\lambda(n,t) = \frac{\mu(x)+\mu^{-1}(x)}{2} \in [\lambda_y, -1]$. We emphasize that $\mu(x) = \mu(x(n,t))$ depends on n and t via

$$x = x(n,t) = -\frac{n\Lambda}{4\pi\mathrm{i}} - \frac{tU}{4\pi\mathrm{i}} - \frac{\Delta}{4\pi}.$$

In particular, $\mu(x(0,0)) = \mu(-\frac{\Delta}{4\pi})$.

Let $\Psi(n,t,p,\xi)$, $p \in \mathbb{M}(\xi)$, be the Baker–Akhiezer function of a finite gap Toda lattice solution $\{\hat{a}(n,t,\xi), \hat{b}(n,t,\xi)\}$ associated with the spectrum on the set $[b-2a, \lambda_y] \cup [-1,1]$ and with initial divisor point $(\lambda(0,0), \pm)$, where we choose sign $+$ if $\mu(-\frac{\Delta}{4\pi}) \in [-1,y]$ and sign $-$ if $\mu(-\frac{\Delta}{4\pi}) \in [y^{-1},-1]$. Here we took into account that the set $\{z : |z| < 1\} \setminus [y,q]$ is in one-to-one correspondence with the upper sheet of $\mathbb{M}(\xi)$ (cf. Section 5). Then (cf. [34])

$$\Psi(n,t,p,\xi)\Psi(n,t,p^*,\xi) = \frac{\lambda-\lambda(n,t)}{\lambda-\lambda(0,0)},$$

where $\lambda(n,t)$ is the projection of the zero divisor for Ψ. Equation (4.11) and our considerations above justify this claim. In particular, according to the trace formulas we have

$$\begin{aligned} &\hat{b}(n,t,\xi) = \frac{1}{2}\left(b-2a+\lambda_y(\xi)-2\lambda(n,t)\right), \\ &\hat{a}(n,t,\xi)^2 + \hat{a}(n-1,t,\xi)^2 \\ &\quad = \frac{1}{4}\left(2+(b-2a)^2+\lambda_y(\xi)^2-2\lambda(n,t)^2-\frac{1}{2}\left(b-2a+\lambda_y(\xi)-2\lambda(n,t)\right)^2\right). \end{aligned}$$

The operator $\hat{H}(t) = \hat{H}(t,\xi)$ associated with these coefficients is reflectionless (since it is finite gap) and has the following Green's function (cf. [34])

$$\hat{G}(\lambda,n,n,t) = -\frac{\lambda-\lambda(n,t)}{\sqrt{(\lambda^2-1)(\lambda-(b-2a))(\lambda-\lambda_y)}} = -\frac{m_1^{mod}(z)m_2^{mod}(z)}{\sqrt{\lambda^2-1}}.$$

Combining (2.2), (2.13), (2.14), (7.4), (7.5) and (7.6) we arrive at Theorem 1.1, where we switched back to the notation $\gamma(\xi)$ instead of λ_y.

8. Discussions

In this section we briefly discuss how to derive and justify the asymptotics in the left region $\mathcal{I}_{1,\varepsilon} := [\xi_{cr,1}+\varepsilon, \xi'_{cr,1}-\varepsilon]$. A justification of the asymptotics in the middle region $(\xi'_{cr,1}, \xi'_{cr})$ which takes into account the presence of resonances and the discrete spectrum in the gap $(b+2a,-1)$ is given in [14].

First of all, if left and right background spectra are of equal length, that is, in case $a=1$, there is no need for an independent extensive study. Indeed, for arbitrary $a>0$ let us consider the Toda lattice associated with the functions

$$\breve{a}(n,t) = \frac{1}{2a} a\big(-n-1, \tfrac{t}{2a}\big), \quad \breve{b}(n,t) = \frac{1}{2a}\Big(b - b\big(-n, \tfrac{t}{2a}\big)\Big),$$

where $\{a(n,t), b(n,t)\}$ is the solution of (1.1)–(1.3), (2.1). It is straightforward to check that $\{\breve{a}(n,t), \breve{b}(n,t)\}$ satisfy (1.1) associated with the initial profile

$$\begin{aligned} &\breve{a}(n,0) \to \frac{1}{2a}, && \breve{b}(n,0) \to \frac{b}{2a} && \text{as } n \to -\infty, \\ &\hat{a}(n,0) \to \frac{1}{2}, && \hat{b}(n,0) \to 0 && \text{as } n \to +\infty. \end{aligned}$$

If $a=1$, the region (ξ'_{cr}, ξ_{cr}) for $\{\breve{a}(n,t), \breve{b}(n,t)\}$ coincides with $(\xi'_{cr,1}, \xi_{cr,1})$ for $\{a(n,t), b(n,t)\}$, and therefore we can simply apply the results of Theorem 1.1. This approach is applicable for arbitrary a when $b-2a>1$, that is, for rarefaction waves. Unfortunately, for the shock waves and $a \neq 1$ we are not able to match these regions. Moreover, even if they would match, this approach would still require cumbersome computations. Indeed, applying the asymptotics from Theorem 1.1 to

$$a(m,\breve{t}) = 2a\breve{a}(n,t), \quad b(m,\breve{t}) = b - 2a\breve{b}(n+1,t), \quad \breve{t} = \frac{t}{2a}, \quad m = -n-1,$$

one has to take into account the new time and space variables and recompute $\mathfrak{b}$ and τ periods in theta functions, as far as the initial Dirichlet eigenvalues.

As we see from the above considerations, the form of the g-function in $\mathcal{I}_{1,\varepsilon}$ is dictated by the spectrum $[b-2a, b+2a]\cup[\gamma(\xi), 1]$, where $\gamma(\xi) \in [-1,1]$. Therefore on the z-plane the images of this point will belong to $\mathbb{T}\setminus\{-1,1\}$. Denote them by z_0 and $\overline{z_0}$. It is clear that the piece-wise constant jump matrix for the respective model problem will appear on the union of the real interval and the arc,

$$[q, q^{-1}] \cup \{z \in \mathbb{T} : \operatorname{Re} z < \operatorname{Re} z_0\}.$$

A construction of the vector and matrix model solutions in theta functions for such a contour in terms of z is quite bulky and not transparent for further analysis. For this reason it is more convenient to study the asymptotics for $\xi \in \mathcal{I}_{1,\varepsilon}$ using the other vector RHP stated with respect to the left scattering data $R_\ell(\zeta,t)$, $T_\ell(\zeta,t)$ on the ζ-plane (cf. (1.8)). The left phase function (1.10) is used and replaced by a suitable g-function; the structure of the jump matrices and the further analysis is completely analogous to the one given in this paper. It allows us to conclude that the error term in this region is described in terms of Airy functions and is of order $O(t^{-1})$. The error term in the middle region is of order $O(\mathrm{e}^{-C(\varepsilon)t})$, [14].

Our last remark concerns condition (2.1). The value of ρ given by (2.10) can be significantly reduced up to any $\rho > 0$ if the point q_1 is non-resonant, because in the non-resonant case we do not need to apply the lens mechanism around the domains $\Omega_{\mathfrak{r}}$ and $\Omega_{\mathfrak{r}}^*$. It was used to remove a possible singularity of m at q_1. Moreover, the condition $\rho > -\log|q_1|$ is sufficient to remove the singularity. Condition (2.10) was chosen to achieve less cumbersome formulas for the jump matrices. Note that condition (3.10) is only essential in the resonant case, and one can expect that the asymptotics in Theorem 1.1 hold in the region (1.16) if q_1 is non-resonant.

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Асимптотика за великим часом хвиль стиснення для рівняння Тоди у модуляційному регіоні

Iryna Egorova, Johanna Michor, Anton Pryimak, and Gerald Teschl

Ми показуємо, що хвиля стиснення для ланцюжка Тоди є асимптотично наближеною до модульованого скінченнозонного розв'язку у правому модуляційному регіоні. Раніше нами було виведено формули для головних членів асимптотичного розвинення цієї хвилі стиснення в усіх п'ятьох принципових регіонах, а також було припущено, що у двох модуляційних регіонах наступний член є порядку $O(t^{-1})$. У даній роботі ми доводимо цей факт і досліджуємо, як дискретний спектр та резонанси впливають на провідну асимптотику. Основним внеском є розв'язання локальних задач Рімана–Гільберта (задач параметріксу) і строге обґрунтування заключного асимптотичного аналізу. Зокрема, це включає в себе побудову належного матричного розв'язку модельної задачі Рімана–Гільберта.

Ключові слова: Рівняння Тоди, задача Рімана–Гільберта, тип сходинки, хвиля стиснення