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Inverse Scattering Problem for Linear System of Four-Wave Interaction Problem on the Half-Line with a General Boundary Condition

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The first order hyperbolic system of four equations on the semi-axis in the case of equal numbers of incident and scattered waves are considered when the velocities of the scattering waves are coincident. It is determined the criteria for inverse scattering problem (the problem of finding the potential with respect to scattering operator) in terms of transmission matrices in two different boundary conditions. The uniqueness of the inverse scattering problem is studied by utilizing it to Gelfand–Levitan–Marchenko type linear integral equation.

Key words: Inverse scattering problem, general boundary conditions, first-order hyperbolic system, transformation operator

Mathematical Subject Classification 2020: 35R30, 35L50, 35P25, 37K15, 81U40

1. Introduction

There are many papers dealing with the inverse problems in wave propagation, but only a few of them deal with the solution of the inverse problems for space and time-dependent coefficients, [2, 9, 10]. Inverse scattering problem (ISP) for the first-order hyperbolic system with the space and time depended potentials were studied in [12] and references therein, where the ISP for a one-dimensional hyperbolic system on the whole line was satisfactorily studied (see also [16]). But, there are very few studies on the ISP on the half-line regarding the numbers of incoming and outgoing waves.

Consider the first order hyperbolic system in the following form on the half-line $x \geq 0$ in the case of equal numbers of incoming and outgoing waves:

$$\begin{cases} \sigma_1 \partial_t \psi_1 - \partial_x \psi_1 = Q_{11}\psi_1 + Q_{12}\psi_2 \\ \sigma_2 \partial_t \psi_2 - \partial_x \psi_2 = Q_{21}\psi_1 + Q_{22}\psi_2 \end{cases}, \quad t \in \mathbb{R}, \tag{1.1}$$

where $\psi_1 = \psi_1(x,t)$ and $\psi_2 = \psi_2(x,t)$ are 2−dimensional vector functions,

$$\sigma_1 = \operatorname{diag}[\xi_1, \xi_2], \sigma_2 = \operatorname{diag}[\xi_3, \xi_4]$$

are diagonal matrices with $\xi_1 \geq \xi_2 > 0 > \xi_3 \geq \xi_4$, $Q_{ij} = Q_{ij}(x,t)$, $i,j = 1,2$ are 2×2 matrix functions with measurable complex-valued and square-integrable entries.

The model problem which this type of system occurs can be found in paper [1] which deals with the inverse problem for two-velocity dynamical system:

$$\rho \partial_t^2 u - \partial_x^2 u + V(x) u = 0$$

with constant diagonal matrix $\rho = \mathrm{diag}[\rho_1^2, \rho_2^2]$, $0 < \rho_1 < \rho_2$, and 2×2 matrix potential

$$V = V(x) = \begin{bmatrix} v_{11}(x) & v_{12}(x) \\ v_{21}(x) & v_{22}(x) \end{bmatrix}, \quad x > 0.$$

This system becomes first order system (1.1) where $\sigma_1 = \mathrm{diag}\{\rho_2, \rho_1\}$, $\sigma_2 = \mathrm{diag}\{-\rho_1, -\rho_2\}$ and

$$Q_{11} = \begin{bmatrix} 0 & \frac{v_{21}(x)}{a(t+\rho_2 x)} \\ 0 & 0 \end{bmatrix}, \qquad Q_{12} = \begin{bmatrix} 0 & \frac{v_{22}(x)}{a(t+\rho_2 x)} \\ a(t-\rho_1 x) & 0 \end{bmatrix},$$

$$Q_{21} = \begin{bmatrix} 0 & \frac{v_{11}(x)}{a(t-\rho_1 x)} \\ a(t+\rho_2 x) & 0 \end{bmatrix}, \qquad Q_{22} = \begin{bmatrix} 0 & \frac{v_{12}(x)}{a(t-\rho_1 x)} \\ 0 & 0 \end{bmatrix}$$

with a differentiable nonzero function $a(s)$. From the physical point of view this class of systems is selected by the property of two types of waves (channels), which propagate with different velocities and interact with one another. As examples of two-velocity dynamical systems we could mention the Timoshenko beam in elasticity theory and cable lines in electrical engineering. Various properties of this systems were studied in [11, 14, 15].

The scattering problem for the system (1.1) on the semi-axis is the problem of finding the solution $\psi(x,t) = \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix}$ of the system (1.1) with known incident wave and the boundary condition at $x = 0$

$$\psi_1(0,t) = H\psi_2(0,t), \tag{1.2}$$

where H is the constant transmission matrix of order 2 with $det H \neq 0$.

The following situations are possible for the system (1.1) on the half-line:

1) *Two incident and two scattered waves with different velocities* ($\xi_1 > \xi_2 > 0 > \xi_3 > \xi_4$) *:* The ISP for this situation, under consideration of two problems for the same system but different boundary conditions (1.2) with $H = I$ and $H = E$ were studied in [4], where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The ISP for the first situation and some of its $2n$ generalization for the special forms of potential but more general boundary conditions is studied in [5].
2) *Two incident and two scattered waves with same velocities* ($\xi_1 = \xi_2 > 0 > \xi_3 = \xi_4$): The second situation and its $2n$-generalization are studied in [6] under consideration of single scattering problem.

3) *Two incident waves with same velocities and two scattering waves with different velocities* ($\xi_1 = \xi_2 > 0 > \xi_3 > \xi_4$): The ISP for this situation, under consideration of two problems with the same system but different boundary conditions were studied in [8].
4) *Two incident waves with different velocities and two scattered waves with same velocities* ($\xi_1 > \xi_2 > 0 > \xi_3 = \xi_4$): The conditionally ISP for this situation, under consideration of two problems with the same system but different boundary conditions (1.2) with $H = I$ and $H = E$ were studied in [7].

In this paper, our aim is to study the ISP of finding the potential

$$Q_{11} = \begin{bmatrix} 0 & 0 \\ q_{21} & 0 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} q_{13} & 0 \\ q_{23} & q_{24} \end{bmatrix}, \quad Q_{21} = \begin{bmatrix} q_{31} & 0 \\ q_{41} & q_{42} \end{bmatrix}, \quad Q_{22} = 0 \qquad (1.3)$$

for the system (1.1) in the fourth situation under more general boundary conditions.

The paper is organized as follows. In Section 2, the preliminary results on Volterra integral operators with Hilbert–Schmidt kernel are given. In Section 3 we construct the scattering operator on the half-line corresponding to the scattering problem. In Section 4, we prove that the considered system has a Volterra type of transformation operator as $x \to +\infty$, when matrix coefficients of the system satisfy a certain triangular structure. Using such a transformation operator, in this section it is shown that the scattering operators admit right factorization. In Section 5, we give the formulation of the inverse scattering problem by scattering operators of two scattering problems on the half-line. In this section, it is given a transmission operator which relates the scattering problem on the half-line and the scattering problem on the whole line, when the coefficients are zero for $x < 0$. This relation transforms the uniqueness of inverse scattering problem on the half line to the uniqueness of inverse scattering problem on thewhole line [13]. In this section, two examples are given showing that

(a) one scattering operator is insufficient for unambiguous reconstruction and
(b) that a condition $\det(H_1 - H_2) \neq 0$ on the transmission matrices in boundary conditions of two scattering problems are crucial.

2. Preliminaries

Throughout the paper, we shall write $Ff(t) = \int_{-\infty}^{+\infty} F(t,s)f(s)\,ds$ the Fredholm operator with Hilbert–Schmidt kernel, $A_-f(t) = \int_t^{+\infty} A_-(t,s)f(s)ds$ and $A_+f(t) = \int_{-\infty}^{t} A_+(t,s)f(s)\,ds$ the upper-triangular (upper Volterra) and lower-triangular (lower Volterra) integral operators, respectively. We will say that the operator $I + F$ in the space $\mathbf{L}_2(\mathbb{R}, \mathbb{C}^n)$ admits a right factorization, if it can be represented as $I + F = (I + A_+)(I + A_-)$, where the operators A_- and A_+ are the lower and upper Volterra and Hilbert–Schmidt integral operators, respectively. The left factorization $I + F = (I + A_-)(I + A_+)$ is similarly defined. The left and right factorizations are unique. If an operator F in the space $L_2(\mathbb{R}, \mathbb{C}^n)$ admits the right factorization (or left factorization) then the operators A_+ and A_- are uniquely restored by F since the kernels $A_\pm(t,s)$ of the operators $A_\pm$ are the solution of well-known Gelfand–Levitan–Marchenko type integral equation in the following form:

For the upper-triangular A_- and lower-triangular B_+ integral operators, we obtain from the right factorization $I+F=(I+B_+)^{-1}(I+A_-)$ that

$$A_- = B_+ + F + B_+ F.$$

The kernels of the integral operators A_-, B_+ we denote by $A_-\,(t,s)$ and $B_+\,(t,s)$. Now let us rewrite the operator equations through the kernels

$$\begin{aligned} &B_+\,(t,s) + F\,(t,s) + \int_{-\infty}^{t} B_+\,(t,\tau)\,F\,(\tau,s)\,d\tau = 0, && s \le t,\\ &A_-\,(t,s) - F\,(t,s) - \int_{-\infty}^{t} B_+\,(t,\tau)\,G\,(\tau,s)\,d\tau = 0, && s \ge t, \end{aligned}$$

where $G\,(\tau,s)$ is the kernel of $G=(I+F)^{-1}-I$. These equations are Gelfand–Levitan–Marchenko type and uniquely solvable which follows from factorization of the operator $I+F$ in the following form (see [3, 12]):

$$B_+ = \left[F\,(I+Q_tF)^{-1}\right]_+,\ A_- = \left[G\,(I+E_tG)^{-1}\right]_-,$$

where Q_t is the projection on semi-axis $s<t$:

$$Q_t f(s) = \begin{cases} 0, & s>t \\ f(s), & s<t \end{cases}$$

and E_t is the projection on semi-axis $s>t$:

$$E_t f(s) = \begin{cases} f(s), & s>t \\ 0, & s<t \end{cases},$$

$[K]_+$ and $[K]_-$ are denote the "positive part" and "negative part" of integral operator $Kf(t)=\int_{-\infty}^{+\infty} K\,(t,s)\,f(s)ds$ respectively, i.e., $[K]_+\,f(t)=\int_{-\infty}^{t} K\,(t,s)\,f(s)ds$ and $[K]_-\,f(t)=\int_{t}^{+\infty} K\,(t,s)\,f(s)ds$.

3. Scattering Problem

Consider the problem (1.1), (1.2) with the potential in the form of (1.3). A nonstationary scattering problem for the system (1.1), (1.3) on the semi-axis can be formulated as follows: It is required to find a solution $\psi(x,t)=\begin{bmatrix}\psi_1\,(x,t)\\ \psi_2\,(x,t)\end{bmatrix}$ of (1.1) such that the solution satisfies the asymptotic relation

$$\psi_1\,(x,t) = \Im_{\sigma_1 x} a(t) + o(1), \quad x \to +\infty, \tag{3.1}$$

and the boundary condition (1.2), where H is given $n\times n$ matrix of constants,with $\det H \neq 0$ and $a(t)$ defines the profile of the incident waves,

$$\Im_{\sigma_1 x} = \operatorname{diag}(T_{\xi_1 x}, T_{\xi_2 x}), \Im_{\sigma_2 x} = \operatorname{diag}(T_{\xi_3 x}, T_{\xi_3 x})$$

are shift operators, such that $T_{\xi_i x} h(t) = h(t+\xi_i x)$, $i = 1,2,3$.

We shall consider generalized solutions of system (1.1), which are ordinary functions measurable in x and t. Here, with respect to variable t, these functions belong to the space $\mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^4\right)$ and their $\mathbf{L}_2$-norms are uniformly bounded with respect to x. We refer to such solutions as admissible.

The scattering problem (1.1)–(3.1) is equivalent to following systems of integral equation:

$$\begin{aligned} \psi_1(x,t) &= \Im_{\sigma_1 x} a(t) + \int_x^{+\infty} \Im_{\sigma_1 (x-s)}[q_{11}\psi_1 + q_{12}\psi_2](s,t)\, ds, \\ \psi_2(x,t) &= \Im_{\sigma_2 x} b(t) + \int_x^{+\infty} \Im_{\sigma_2 (x-s)}[q_{21}\psi_1 + q_{22}\psi_2](s,t)\, ds, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} b(t) = Ha(t) + \int_0^{+\infty} &\big\{ H\Im_{-\sigma_1 s}[q_{11}\psi_1 + q_{12}\psi_2](s,t) \\ &- \Im_{-\sigma_2 s}[q_{21}\psi_1 + q_{22}\psi_2](s,t) \big\}\, ds. \end{aligned} \tag{3.3}$$

Theorem 3.1. *If the coefficients of the system* (1.1) *is given by* (1.3)*, then for a given arbitrary incident wave vector* $a(t) \in \mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^2\right)$ *there exists a unique admissible solution of the scattering problem* (1.1)*,* (1.2)*,* (3.1) *and the second component of the solution satisfies the asymptotic relation*

$$\psi_2(x,t) = \Im_{\sigma_2 x} b(t) + o(1), \quad x \to +\infty, \tag{3.4}$$

where $b(t) \in \mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^2\right)$ *defines the profile of the scattered waves.*

The proof of this theorem is omitted since system (3.2)–(3.3) is Volterra integral equation by t with square-integrable kernel and the similar assertion is proved in [5, Theorem 1]. In the view of Theorem 1, for every vector function $a(t) \in \mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^2\right)$, which represents incident waves, when the system (1.1) satisfies the conditions (2.3), (2.4) there exist a unique solution $\psi(x,t) = \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix}$. For this solution there exist scattered waves $b(t) \in \mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^2\right)$ according to (3.2). By comparing the incident and scattered waves,we can define the scattering operator $\mathbf{S}_H$ by

$$b = \mathbf{S}_H H a. \tag{3.5}$$

Operator $\mathbf{S}_H$ is an $n \times n$ matrix operator and defined on $\mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^2\right)$. We call this operator as the scattering operator that corresponds to the scattering problem (1.1), (1.3), (3.1) on the semi-axis.

4. Volterra properties of scattering operator

In solving inverse scattering problems the Volterra type integral representation of the solution plays an important role. Such representation can be taken from transformation operator as $x \to +\infty$. The properties of the scattering operators will be given in detail after studing the transformation operator.

Finding the bounded solution to the system (1.1) with the given asymptotic $\Im_{\sigma_1 x} a(t)$, $\Im_{\sigma_2 x} b(t)$ as $x \to +\infty$ is equivalent to the solvability of the following system of integral equations in $\mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^4\right)$:

$$\begin{aligned}
\psi_1(x,t) &= \Im_{\sigma_1 x} a(t) + \int_x^{+\infty} \Im_{\sigma_1 (x-s)} \left[q_{11}\psi_1 + q_{12}\psi_2\right](s,t)\, ds, \\
\psi_2(x,t) &= \Im_{\sigma_2 x} b(t) + \int_x^{+\infty} \Im_{\sigma_2 (x-s)} \left[q_{21}\psi_1 + q_{22}\psi_2\right](s,t)\, ds.
\end{aligned} \tag{4.1}$$

Theorem 4.1. *Let the coefficients of system (1.1) be given by (1.3). Then for any $a(t), b(t) \in \mathbf{L}_2\left(\mathbb{R}, \mathbb{C}^2\right)$ there exist a unique admissible solution of the system (1.1), and the solution admits the representation*

$$\begin{aligned}
\psi_1(x,t) = \Im_{\sigma_1 x} a(t) &+ \int_t^{+\infty} A_{11}(x,t,s)\, \Im_{\sigma_1 x} a(s)\, ds \\
&+ \int_{-\infty}^{t} A_{12}(x,t,s)\, \Im_{\sigma_2 x} b(s)\, ds,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\psi_2(x,t) = \Im_{\sigma_2 x} b(t) &+ \int_t^{+\infty} A_{21}(x,t,s)\, \Im_{\sigma_1 x} a(s)\, ds \\
&+ \int_{-\infty}^{t} A_{22}(x,t,s)\, \Im_{\sigma_2 x} b(s)\, ds,
\end{aligned} \tag{4.3}$$

where $A_{11} = A_{11}(x,t,s)$, $A_{12} = A_{12}(x,t,s)$, $A_{21} = A_{21}(x,t,s)$, $A_{22} = A_{22}(x,t,s)$ is 2×2 matrix kernels. These kernels are determined uniquely by the coefficients (1.3) of system (1.1) and for the fixed x these kernels are the Hilbert–Schmidt kernels.

Proof. The system (4.1) has unique solution since it is the system of Volterra integral equations by x with square-integrable kernel. If the solution of (4.1) can be represented as in the form of (4.2) for each $a, b \in \mathbf{L}_2$, then substituting (4.2) in (4.1) we obtain the system of equations for the kernels under assumption that $q_{kj} = 0$

$$(k,j) = \{(1,1);(1,2);(1,4);(2,2);(3,2);(3,3);(3,4);(4,3);(4,4)\}$$

we have

$$\begin{aligned}
[A]_{kj}(x,t,\tau) = \frac{\xi_j}{\xi_j - \xi_k} q_{kj}\left(x + \frac{\tau - t}{\xi_j - \xi_k}, t - \frac{\xi_k}{\xi_j - \xi_k}(\tau - t)\right) \\
+ \sum_{p=1}^{4} \int_x^{x + \frac{\tau - t}{\xi_j - \xi_k}} q_{kp}\left(s, t + \xi_k (x - s), \tau + \xi_{n+j}(x - s)\right)
\end{aligned}$$

$$\times [A]_{pj}\left(s, t+\xi_k (x-s), \tau - \frac{\xi_k}{\xi_j}(x-s)\right) ds,$$
$$\tau \le t, \ \ (k,j) \in \left\{(k,j) : \frac{\xi_k}{\xi_j} < 1, \ k,j = 1,2,3,4\right\}, \quad (4.4)$$
$$[A]_{kj}(x,t,\tau) = \sum_{p=1}^{4} \int_x^{+\infty} q_{kp} s, t+\xi_k (x-s), \tau + \xi_{n+j}(x-s))$$
$$\times [A]_{pj}(s, t+\xi_k (x-s), \tau - x + s)\, ds,$$
$$\tau \le t, \ \ (k,j) \in \{(k,j) : \xi_k = \xi_j, \ k,j = 1,2,3,4\},$$

where $[A]_{ij}$ denotes the i,j element of the matrix function $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$. The system of integral equation (4.4) is Volterra by x with square-integrable kernel that is why unique solvable. □

For $i = 1,2$, let us denote that

$$A_{i1-}(x) f(t) = \int_t^{+\infty} A_{i1}(x,t,s) f(s)\, ds, \qquad A_{i1-} = A_{i1-}(0),$$
$$A_{i2+}(x) f(t) = \int_{-\infty}^{t} A_{i2}(x,t,s) f(s)\, ds, \qquad A_{i2+} = A_{i2+}(0).$$

The formulas (4.2) can be written in the form of

$$\psi_1(x,t) = [I + A_{11-}(x)]\, \Im_{\sigma_1 x} a(t) + A_{12+}(x)\, \Im_{\sigma_2 x} b(t),$$
$$\psi_2(x,t) = A_{21-}(x)\, \Im_{\sigma_1 x} a(t) + [I + A_{22+}(x)]\, \Im_{\sigma_2 x} b(t). \quad (4.5)$$

Using the representation (4.5) and the boundary conditions (1.2) we obtain that

$$A_{21-} a(t) + (I + A_{22+})\, b(t) = H\left[(I + A_{11-})\, a(t) + A_{12+} b(t)\right]$$

or

$$(I + A_{22+} - HA_{12+})\, b(t) = \left(I + HA_{11-}H^{-1} - A_{21-}H^{-1}\right) Ha(t) \quad (4.6)$$

By Theorem 4.1, the kernels of the integral operators A_{12+}, A_{22+}, A_{11-} and A_{21-} are Hilbert–Schmidt kernels. Therefore the kernels of the integral Volterra operators $A_+ = A_{22+} - HA_{12+}$ and $A_- = HA_{11-}H^{-1} - A_{21-}H^{-1}$ are Hilbert–Schmidt kernels. Then the operators A_{k+} and A_{k-} are Hilbert–Schmidt operators. Taking into account the definition (3.5) of scattering operator, we obtain from (4.6) the right factorization of scattering operator, as

$$\mathbf{S}_H = (I + A_+)^{-1} (I + A_-). \quad (4.7)$$

If $(I + A_+)^{-1} = I + B_+$, where B_+ is Hilbert–Schmidt integral Volterra operator, then the scattering operators S_H have the form $\mathbf{S}_H = I + F$, where $F = B_+ + A_- + B_+ A_-$. Thus the operator F is Hilbert–Schmidt integral operator.

Remark 4.2. The equalities (4.5) demonstrate that the system (1.1) with the coefficients (1.3) has the transformation operators as $x \to +\infty$ is in the following form

$$P = \begin{bmatrix} I + A_{11-}(x) & A_{12+}(x) \\ A_{21-}(x) & I + A_{22+}(x) \end{bmatrix},$$

where $A_{i1-}(x)$ and $A_{i2+}(x)$ are 2×2 matrix Volterra integral operators such that the kernels satisfy the integral equation (4.4).

5. Inverse scattering problem

Let $\mathbf{S}_H$ be a scattering operator for the system (1.1) with the coefficients giving by (1.3). Inverse scattering problem for the system (1.1) is the problem of finding 4×4 matrix potential

$$\begin{bmatrix} Q_{11}(x,t) & Q_{12}(x,t) \\ Q_{21}(x,t) & Q_{22}(x,t) \end{bmatrix}, \quad x > 0$$

(contains 7 nonzero functions) by the 2×2 matrix integral operator $F = \mathbf{S}_H - I$ (its kernel contains 4 functions on line and 8 function on half-line). The following Example shows that one scattering operator is insufficient for unambiguous reconstruction.

Example 5.1. Let us consider the scattering problem for the system (1.1)–(1.3) in its an explicitly solvable case $q_{31}(x,t) = 0$, $q_{42}(x,t) = 0$. Let us denote

$$\psi_1(x,t) = \begin{bmatrix} \varphi_1(x,t) \\ \varphi_2(x,t) \end{bmatrix}, \psi_2(x,t) = \begin{bmatrix} \varphi_3(x,t) \\ \varphi_4(x,t) \end{bmatrix}, a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix}, b(t) = \begin{bmatrix} b_3(t) \\ b_4(t) \end{bmatrix}.$$

This problem can be explicitly solvable as

$$\begin{aligned}
\varphi_1(x,t) &= a_1(t + \xi_1 x) + \int_x^{+\infty} (q_{14}\varphi_3)(s, t + \xi_1(x - s))\, ds, \\
\varphi_2(x,t) &= a_2(t + \xi_2 x) + \int_x^{+\infty} (q_{21}\varphi_1 + q_{23}\varphi_3)(s, t + \xi_2(x - s))\, ds, \\
\varphi_3(x,t) &= b_3(t + \xi_3 x), \\
\varphi_4(x,t) &= b_4(t + \xi_3 x) + \int_x^{+\infty} (q_{41}\varphi_1)(s, t + \xi_3(x - s))\, ds,
\end{aligned}$$

where

$$\begin{aligned}
\big[I - h_{11}B_{1+} - h_{12}(B_{3+} + B_{4+})\big] b_3(t) \\
= h_{11}a_1(t) + h_{12}a_2(t) + h_{12}A_{1-}a_1(t), \\
b_4(t) + \big[B_{2+} - h_{21}B_{1+} - h_{22}(B_{3+} + B_{4+})\big] b_3(t) \\
= h_{21}a_1(t) + h_{22}a_2(t) + (h_{22}A_{1-} - A_{2-})\, a_1(t).
\end{aligned}$$

Here $B_{k-}, k = 1, 2, 3, 4$ and A_{1-}, A_{2-} are lower and upper Volterra integral operators with the kernels respectively,

$$B_{1-}(s,t) = \frac{1}{\xi_1 - \xi_3} q_{13}\left(\frac{s-t}{\xi_3 - \xi_1}, t - \frac{\xi_1}{\xi_3 - \xi_1}(s-t)\right),$$
$$B_{3-}(s,t) = \frac{1}{\xi_2 - \xi_3} q_{23}\left(\frac{s-t}{\xi_3 - \xi_2}, t - \frac{\xi_2}{\xi_3 - \xi_2}(s-t)\right),$$
$$B_{2-}(s,t) = \frac{1}{\xi_1 - \xi_3}\int_0^{+\infty} q_{41}(\tau, t - \xi_3\tau) q_{13} \times\left(\frac{s - t - (\xi_1 - \xi_3)\tau}{\xi_3 - \xi_1}, t - \xi_3\tau + \frac{\xi_1(t-s)}{\xi_3 - \xi_1}\right) d\tau,$$
$$B_{2-}(s,t) = \frac{1}{\xi_1 - \xi_3}\int_0^{\frac{s-t}{\xi_3-\xi_2}} q_{21}(\tau, t - \xi_2\tau) q_{13} \times\left(\frac{s - t + (\xi_2 - \xi_1)\tau}{\xi_3 - \xi_1}, t - (\xi_1 - \xi_2\tau + \frac{\xi_1(t - s + (\xi_1 - \xi_2)s)}{\xi_3 - \xi_1}\right) d\tau,\ s \le t,$$

and

$$A_{1+}(s,t) = \frac{1}{\xi_1 - \xi_2} q_{21}\left(\frac{s-t}{\xi_1 - \xi_2}, t - \frac{\xi_2}{\xi_1 - \xi_2}(s-t)\right),$$
$$A_{2+}(s,t) = \frac{1}{\xi_1 - \xi_3} q_{41}\left(\frac{s-t}{\xi_1 - \xi_3}, t - \frac{\xi_3}{\xi_1 - \xi_{31}}(s-t)\right),\ s \le t.$$

It is clear from the definition of the scattering operator $S_H : H\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \to \begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$ that it admits right side factorization in the following form

$$\mathbf{S}_H = \begin{bmatrix} I - h_{11}B_{1+} - h_{12}(B_{3+} + B_{4+}) & 0 \\ B_{2+} - h_{21}B_{1+} - h_{22}(B_{3+} + B_{4+}) & I \end{bmatrix}^{-1} \times \begin{bmatrix} h_{11}I + +h_{12}A_{1-} & h_{12}I \\ h_{21}I + h_{22}A_{1-} - A_{2-} & h_{22}I \end{bmatrix} H^{-1}.$$

Then $h_{11}B_{1+} - h_{12}(B_{3+} + B_{4+})$, $B_{2+} - h_{21}B_{1+} - h_{22}(B_{3+} + B_{4+})$, $h_{12}A_{1-}$ and $h_{22}A_{1-} - A_{2-}$ can be uniquely determined by S_H. If $h_{12} = 0$ *with* $\det H = h_{11}h_{22} \neq 0$ then it is clear to see that the coefficients are not uniquely determined by the scattering operator on the semi-axis.

Consider two scattering problems on semi-axis for the system (1.1), (1.3). *First scattering problem:* It is required to find a solution $\psi^1(x,t) = \begin{bmatrix} \psi_1^1(x,t) \\ \psi_2^1(x,t) \end{bmatrix}$ of the system (1.1) such that the asymptotic relation

$$\psi_1^1(x,t) = \Im_{\sigma_1 x} a(t) + o(1), \quad x \to +\infty,$$

and boundary condition

$$\psi_2^1(0,t) = H_1\psi_1^1(0,t), \quad \det H_1 \neq 0$$

are satisfied.

Second scattering problem: It is required to find a solution $\psi^2(x,t) = \begin{bmatrix} \psi_1^2(x,t) \\ \psi_2^2(x,t) \end{bmatrix}$ of the system (1.1) such that the asymptotic relation

$$\psi_1^2(x,t) = \Im_{\sigma_1 x} a(t) + o(1), \quad x \to +\infty,$$

and boundary condition

$$\psi_2^2(0,t) = H_2 \psi_1^2(0,t), \quad \det H_2 \neq 0$$

are satisfied. We are going to investigate the solution of the inverse scattering problem considering the first and the second scattering problems together under the following assumption

$$\det(H_1 - H_2) \neq 0. \tag{5.1}$$

According to the Theorem (3.1) for arbitrary $a(t) \in \mathbf{L}_2(\mathbb{R}, \mathbb{C}^2)$ first and second scattering problems have unique bounded solutions. Moreover, these solutions satisfy the following asymptotic relations

$$\psi_2^k(x,t) = \Im_{\sigma_2 x} b^k(t) + o(1), \quad x \to +\infty, \ k = 1, 2,$$

where $b^k(t) \in \mathbf{L}_2(\mathbb{R}, \mathbb{C}^2)$ defines the profile of the scattered waves. The scattering operators corresponding to the first and the second scattering problems are denoted by $\mathbf{S}_{H_1}$ and $\mathbf{S}_{H_2}$:

$$\mathbf{S}_{H_k} : H_k a(t) \to b^k(t), \quad k = 1, 2. \tag{5.2}$$

The operators $\mathbf{S}_{H_1}$ and $\mathbf{S}_{H_2}$ are evidently matrix operators on $\mathbf{L}_2(\mathbb{R}, \mathbb{C}^2)$. In the subsequent sections, the operators $\mathbf{S}_{H_k}$ $(k = 1, 2)$ will be studied in the space $\mathbf{L}_2(\mathbb{R}, \mathbb{C}^2)$, i.e., under $\mathbf{S}_{H_k}$ we will understand the closure in $\mathbf{L}_2(\mathbb{R}, \mathbb{C}^2)$ by the operator $\mathbf{S}_{H_k}$ contracted on $\mathbf{L}_2(\mathbb{R}, \mathbb{C}^2)$. It is proved in (4.7) that the scattering operators admits right factorizations:

$$\mathbf{S}_{H_k} = (I + A_{k+})^{-1}(I + A_{k-}) \tag{5.3}$$

where $A_{k+} = A_{22+} - H_k A_{12+}$ and $A_{k-} = H_k A_{11-} H_k^{-1} - A_{21-} H_k^{-1}$, $k = 1, 2$. It is known (see [5]) that the transmission matrix operator

$$T \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} = \begin{bmatrix} \psi_1(0,t) \\ \psi_2(0,t) \end{bmatrix}.$$

is a scattering operator on the whole-axis for a system of first order hyperbolic equations, with the coefficients of system (1.1) equal to zero for $x < 0$. From the representation (4.2) we determine that

$$T = \begin{bmatrix} I + A_{11-} & A_{12+} \\ A_{21-} & I + A_{22+} \end{bmatrix}. \tag{5.4}$$

Since the inverse scattering problem for the system of hyperbolic equations on the whole-axis is solved in [13], then using from (5.4) to obtain the following result about scattering problem for the system (1.1) on semi-axis.

Theorem 5.2. *Let* $\mathbf{S}_{H_1}$ *and* $\mathbf{S}_{H_2}$ *be two scattering operators on the semi-axis for the system* (1.1) *with the coefficients giving by* (1.3)*, where the matrices* H_1, H_2 *satisfy condition* (5.1)*. Then the coefficients* (1.3) *of the system* (1.1) *are uniquely determined by the scattering operators* $\mathbf{S}_{H_1}$ *and* $\mathbf{S}_{H_2}$*.*

Proof. Let $\mathbf{S}_{H_k}$ $(k = 1, 2)$ be given scattering operators on the semi-axis. Let us define the operator T by (5.4). For operators $\mathbf{S}_{H_1}$and $\mathbf{S}_{H_2}$ the following formulas are correct with respect to (5.2):

$$I + H_k A_{11-} H_k^{-1} - A_{21-} H_k^{-1} = (I + A_{22+} - H_k A_{12+})\, S_{H_k}, k = 1, 2.$$

From this it follows that

$$I + A_{22+} - H_k A_{12+} = \left(I + H_k A_{11-} H_k^{-1} - A_{21-} H_k^{-1}\right) S_{H_k}^{-1}, k = 1, 2.$$

Since the operators $\mathbf{S}_{H_k} = I + F^k, k = 1, 2$ admit right factorization, equations (5.3) are uniquely solvable with respect to the factorization multiplications $I + H_k A_{11-} H_k^{-1} - A_{21-} H_k^{-1}$ and $I + A_{22+} - H_k A_{12+}, k = 1, 2$. Thus we obtain that

$$\begin{aligned} H_k A_{12+} - A_{22+} &= \Gamma_{k+}, \\ A_{21-} H_k^{-1} - H_k A_{11-} H_k^{-1} &= \Gamma_{k-}, \end{aligned} \tag{5.5}$$

where $\Gamma_{k+} = \left[F_k\,(I + Q_t F_k)^{-1}\right]_+$, $\Gamma_{k-} = \left[G_k\,(I + E_t G_k)^{-1}\right]_-$, $k = 1, 2$. Considering $\det(H_1 - H_2) \neq 0$, then from (5.5) we get

$$\begin{aligned} A_{12+} &= (H_1 - H_2)^{-1}\,(\Gamma_{1+} - \Gamma_{2+})\,, \\ A_{22+} &= H_1\,(H_1 - H_2)^{-1}\,(\Gamma_{1+} - \Gamma_{2+}) - \Gamma_{1+}, \\ A_{11-} &= (H_1 - H_2)^{-1}\,(\Gamma_{2-} H_2 - \Gamma_{1-} H_1)\,, \\ A_{21-} &= \Gamma_{1-} H_1 + H_1\,(H_1 - H_2)^{-1}\,(\Gamma_{2-} H_2 - \Gamma_{1-} H_1)\,. \end{aligned}$$

The theorem is proved. □

The following example shows that the condition (5.1) on the transmission matrices in boundary conditions of two scattering problems is crucial.

Example 5.3. Consider scattering problem for the system in Example (5.1) with boundary condition of the form

$$\psi_2\,(0, t) = \tilde{H} \psi_1\,(0, t)\,, \quad \tilde{H} = \begin{bmatrix} \tilde{h}_{11} & 0 \\ \tilde{h}_{21} & \tilde{h}_{22} \end{bmatrix}$$

with $det \tilde{H} = \tilde{h}_{11} \tilde{h}_{22} \neq 0$. It is clear from Example 5.1 that the scattering operator has the form

$$\mathbf{S}_{\tilde{H}} = \begin{bmatrix} I - \tilde{h}_{11} B_{1+} & 0 \\ B_{2+} - \tilde{h}_{21} B_{1+} - \tilde{h}_{22}(B_{3+} + B_{4+}) & I \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \tilde{h}_{11}I & \tilde{h}_{12}I \\ \tilde{h}_{21}I + \tilde{h}_{22}A_{1-} - A_{2-} & \tilde{h}_{22}I \end{bmatrix} \tilde{H}^{-1}.$$

The operators $h_{11}B_{1+}$, $B_{2+} - h_{21}B_{1+} - h_{22}(B_{3+} + B_{4+})$ and $h_{22}A_{1-} - A_{2-}$ can be uniquely determined by $S_{\tilde{H}}$since it admits left factorization. But the coefficients are not uniquely determined by the scattering operator $S_{\tilde{H}}$.

If $\det\left[H - \tilde{H}\right] \neq 0$ then the coefficients are unique determined by the scattering operators S_H and $S_{\tilde{H}}$. If $\det\left[H - \tilde{H}\right] = 0$ by $h_{11} \neq \tilde{h}_{11}$, $h_{22} = \tilde{h}_{22}$ then the unique restoration of coefficients are also violated.

6. Conclusion

This paper considers the ISP for the first order hyperbolic system of four equations on the semi-axis in the case of two incident and two scattered waves. The transmission matrix in boundary condition is general nonsingular matrix but the matrix coefficients of the system satisfy some triangular structures. Such type of systems occur in elasticity theory and cable lines in electrical engineering. The ISP for the first order hyperbolic system of $2n$ ($n > 2$) equations on the semi-axis in the case of equal number of incident and scattered waves partially studied in [5], but the problem were not generally studied. The same sort of uniqueness results should be true in general case, which suggests a line for further investigation.

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Обернена задача розсіювання для лінійної системи чотирихвильової проблеми взаємодії на півпрямій із загальною крайовою умовою

Mansur I. Ismailov

Розглянуто гіперболічну систему чотирьох рівнянь першого порядку на півосі у випадку рівної кількості падної і розсіяних хвиль за умови, що швидкості розсіяних хвиль збігаються. Установлено критерії для оберненої задачі розсіювання (задачі знаходження потенціалу за оператором розсіювання) в термінах матриць передачі у двох різних крайових умовах. Вивчено єдиність оберненої задачі розсіювання за допомогою зведення задачі до лінійного інтегрального рівняння типу Ґельфанда–Левітана–Марченка.

Ключові слова: обернена задача розсіювання, загальні крайові умови, гіперболічна система першого порядку, оператор перетворення