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Dressing for Fokker–Planck Equations: the Cases of $1+1$ and $1+\ell$ Dimensions

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Dedicated with admiration
to the 100th birthday of V.A. Marchenko

We consider dressing and explicit solutions for the scalar reduced Fokker–Planck equation in the $(1+1)$-dimensional case and for the matrix Fokker–Planck system in the $(1+\ell)$-dimensional case. For this purpose, we use our generalised Bäcklund–Darboux transformation (GBDT). There are only several works on the dressing for the important Fokker–Planck equation and those works deal with $1+1$ and $1+2$ cases.

Key words: reduced Fokker–Planck equation, matrix Fokker–Planck system, dressing, Darboux transformation, matrix identity, explicit solution

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1. Introduction

Fokker–Planck (FP) equation is one of the most well-known and important equations in statistical and quantum mechanics. Correspondingly, its explicit solutions are of great interest in theory and applications. Following seminal works of Bäcklund, Darboux and Jacobi, different kinds of Darboux transformations and related dressing, commutation and factorization methods are fruitfully used to obtain explicit solutions of linear and nonlinear equations (see, e.g. [3–5, 8–10, 12, 16, 19, 20, 23, 36] and numerous references therein). In particular, matrix and operator identities were successfully used in these constructions. See, for instance, the well-known book [19] by V.A. Marchenko and later works [1, 2, 14, 15, 18, 24, 25, 30] (by different authors) for various results, discussions and references.

In this paper, we apply a *generalised Bäcklund–Darboux transformation* (GBDT), which is based on the matrix identities and *generalised matrix eigenvalues*. GBDT was first introduced in our work [24] (for this approach see also [3, 14, 16, 25–27, 30] and references therein).

In the $1+1$ case (of one time and one space variables) Bäcklund–Darboux transformations have been applied to the FP equation in [13, 22, 35] (see also some references therein). The $1+2$ case of FP, where the term with second derivatives has the form $d\Delta u$ and d is constant, was studied in [34] using corresponding results for Schrödinger equation. A related interesting work on the symmetries of an important case of FP equation (in $1+2$ dimensions) [17] should be mentioned as well.

The present work may be considered as an important development of the classical work on the Darboux transformation for the 1+1 case of FP by C. Rogers [22]. Namely, in his work C. Rogers introduced Darboux transformation in the $1+1$ matrix FP case, where coefficients are matrix valued functions (matrix functions), and studied in greater detail the applications to FP in the scalar case. In both scalar and matrix cases, we apply a more developed iterated binary GBDT. Even more essential is the fact that we apply GBDT to the FP with one time and ℓ space variables (in the matrix case) for the arbitrary $\ell \in \mathbb{N}$. Here, $\mathbb{N}$ stands for the set of positive integer numbers. Note that FP systems with coefficients given by matrix functions are studied, for instance, in [6, 21]. Since real-valued entries of the matrix coefficients and solutions are considered, some natural requirements on the constructions are added.

We note that various forms of FP equations and systems appear in the literature, including quite general ones (see, e.g. [22, (2)] and [17, (1)]). Here, we deal with the dynamical scalar reduced FP equation [22, (29)]:

$$\frac{\partial}{\partial t}u(t,x)+\frac{\partial^2}{\partial x^2}u(t,x)+q(x)u(t,x)=0 \quad (q(x)\in\mathbb{R}) \tag{1.1}$$

in the case of one space variable, and with the matrix FP system (or FP-type system):

$$\frac{\partial}{\partial t}\Upsilon(t,x_1,\dots,x_\ell)+\sum_{k=1}^{\ell}\sum_{p=1}^{2}q_{kp}(t)\frac{\partial^p}{\partial x_k^p}\Upsilon(t,x_1,\dots,x_\ell)=0, \tag{1.2}$$

$\Upsilon(t,x_1,\dots,x_\ell)\in\mathbb{R}^m$, $q_{kp}(t)\in\mathbb{R}^{m\times m}$, in the case of ℓ space variables. The notation $\mathbb{R}$ in (1.1) stands, as usually, for the real axis, and by $\mathbb{R}^{m\times n}$ we denote $m\times n$ matrices with real-valued entries ($\mathbb{R}^{m\times 1}=\mathbb{R}^m$).

Remark 1.1. GBDT is applied to dynamical systems as follows (see, e.g. [26–29]). First, GBDT is applied to some auxiliary system in one variable. Next, we multiply certain construction from this GBDT by a matrix exponent depending on other variables (see formulas (2.9) and (4.6)). In this way, one obtains wide families of solutions of much greater variety than in the case of the standard separation of variables (i.e. than in the case of a scalar exponent instead of the matrix $n\times n$ exponent, where we may increase the number n).

While constructing solutions of (1.1) we apply GBDT first to Schrödinger equation depending on x, and for the construction of solutions of (1.2) we apply GBDT to a system (with second order poles of the spectral parameter) depending on t.

In the next section, we apply GBDT to the scalar reduced FP system (in the terminology of C. Rogers). The necessary information on GBDT in the case of one variable (and with the second order poles of the spectral parameter) is presented in Section 3. The following Section 4 is dedicated to GBDT for the FP systems (1.2). Some non-stationary examples are presented in the appendix.

Notations. Some notations were explained above. As usually, $\mathbb{C}$ stands for the complex plane. The real part of $a \in \mathbb{C}$ is denoted by $\Re(a)$, the imaginary part of a is denoted by $\Im(a)$, the complex conjugate of a is denoted by $\overline{a}$, and the absolute value of a is denoted by $|a|$. By $\mathbb{C}^{n\times m}$ we denote $n \times m$ matrices with complex-valued entries and $\mathbb{C}^{n\times 1} = \mathbb{C}^n$. The notation i stands for the imaginary unit ($\mathrm{i}^2 = -1$) and I_m denotes the $m \times m$ identity matrix. The notation q^* stands for conjugate transpose of the matrix q, $\sigma(A)$ stands for the spectrum of the matrix A and the matrix inequality $S > 0$ ($S \geq 0$) means that the eigenvalues of the matrix $S = S^*$ are positive (nonnegative). In a similar way, one interprets the inequalities $S < 0$ and $S \leq 0$. The notation $\mathrm{diag}\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_s\}$ means diagonal matrix with the entries (or blocks) $\mathcal{A}_1, \mathcal{A}_2, \dots$ on the main diagonal.

2. GBDT for the scalar reduced FP equation

1. Dynamical scalar reduced FP equation (1.1) coincides with the equation [27, (7.12)] (for the case $\omega(x) \equiv 1$ and $p(x) \equiv -1$ in [27, (7.12)]). Based on [27], we present GBDT for the equation (1.1). Without loss of generality, similar to [30] we assume that $x \in \mathcal{I}$, where $\mathcal{I}$ is an interval which contains 0 ($0 \in \mathcal{I}$).

Each GBDT for the scalar reduced FP equations is determined by some initial ("seed") equation (1.1) (or, equivalently, the initial potential $q(x)$) and a triple of matrices $\{A, S(0), \Pi(0)\}$ such that

$$AS(0) - S(0)A^* = \Pi(0)J\Pi(0)^*, \quad S(0) = S(0)^*, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.1}$$

$$A,\, S(0) \in \mathbb{C}^{n\times n}; \quad \Pi(0) \in \mathbb{C}^{n\times 2}, \quad n \in \mathbb{N}. \tag{2.2}$$

Note that $q(x)$ is the "seed" potential in the terminology of the Bäcklund–Darboux transformations. The transformed equation and its solutions are expressed in terms of the matrix functions $\Pi(x)$ and $S(x)$, uniquely determined by the values $\Pi(0)$ and $S(0)$ and first order systems (2.4) below. In order to write down these systems, we partition Π into column vector function blocks Λ_1 and Λ_2:

$$\Pi(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}, \quad \Pi(0) = \begin{bmatrix} \Lambda_1(0) & \Lambda_2(0) \end{bmatrix}, \quad \Lambda_i(0) = \vartheta_i \quad (i = 1, 2). \tag{2.3}$$

The following special case of [27, Proposition 7.3] presents the potential and solutions of the transformed FP equation.

Proposition 2.1. *Let an initial real-valued potential $q(x)$ and a triple of matrices $\{A, S(0), \Pi(0)\}$ satisfying (2.1), (2.2) be given. Introduce matrix functions $\Pi(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}$ and $S(x)$ using differential systems*

$$\Lambda_1'(x) = A\Lambda_2(x) - \Lambda_2(x)q(x), \quad \Lambda_2'(x) = \Lambda_1(x); \quad S'(x) = \Lambda_2(x)\Lambda_2(x)^*, \tag{2.4}$$

where $\Lambda_1' = \frac{d}{dx}\Lambda_1$. Then, in the points of invertibility of $S(x)$, the transformed scalar reduced FP equation has the form

$$\frac{\partial}{\partial t}\widetilde{u}(t,x) + \frac{\partial^2}{\partial x^2}\widetilde{u}(t,x) + \widetilde{q}(x)\widetilde{u}(t,x) = 0, \tag{2.5}$$

$$\widetilde{q}(x) = q(x) + 2\big(\mathcal{X}_{11}(x) - \mathcal{X}_{22}(x)\big) - 2\mathcal{X}_{12}(x)^2 = \overline{\widetilde{q}(x)}, \tag{2.6}$$

where $\mathcal{X}_{ik}$ are the entries of X:

$$X(x) = \{\mathcal{X}_{ik}(x)\}_{i,k=1}^{2} := J\Pi(x)^* S(x)^{-1}\Pi(x), \tag{2.7}$$

J is given in (2.1) and the equalities

$$\mathcal{X}_{12}(x) = \overline{\mathcal{X}_{12}(x)}, \quad \mathcal{X}_{22}(x) = -\overline{\mathcal{X}_{11}(x)}. \tag{2.8}$$

hold. Moreover, the functions

$$\widetilde{u}(t,x) = \begin{bmatrix} 1 & 0 \end{bmatrix} J\Pi(x)^* S(x)^{-1} \mathrm{e}^{-tA} h = \Lambda_2(x)^* S(x)^{-1} \mathrm{e}^{-tA} h \quad (h \in \mathbb{C}^{n\times 1}) \tag{2.9}$$

satisfy the transformed FP equation (2.5).

In view of (2.4), A and $\Pi(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}$ are so called *generalised matrix eigenvalue* and *generalised eigenfunction*, respectively, for GBDT. A scheme of the proof of Proposition 2.1 is given after remarks below.

Remark 2.2. In view of the last equality in (2.4), the inequality $S(0) > 0$ yields $S(x) > 0$ for $x \geq 0$. In particular, $S(x)$ is invertible for $x \geq 0$.

Remark 2.3. We note that the vectors h in (2.9) determine various linear combinations of the entries of the row vector $\Lambda_2(x)^* S(x)^{-1} \mathrm{e}^{-tA}$, which itself is a vector solution of (2.5).

Remark 2.4. Clearly, both real and imaginary parts of $\widetilde{u}$ are real-valued solutions of (2.5). That is, we set

$$\text{either} \quad \psi(t,x) = \Re\big(\widetilde{u}(t,x)\big) \quad \text{or} \quad \psi(t,x) = \Im\big(\widetilde{u}(t,x)\big) \tag{2.10}$$

in order to obtain real-valued solutions.

The matrix identity in (2.1) and relations (2.4) yield important identities

$$AS(x) - S(x)A^* = \Pi(x)J\Pi(x)^* \quad \text{for} \quad x \in \mathcal{I}. \tag{2.11}$$

Scheme of the proof of Proposition 2.1. Relations (2.4) correspond to the GBDT for Schrödinger equation $y'' + q(x)y = \lambda y$, where the sign before y'' differs from the standard one (for the convenience of the further application). There is a general GBDT formula for the derivative $\big(J\Pi^* S^{-1}\big)'$ (see, e.g. [30, Chapter 7]) and a formula for $\big(J\Pi^* S^{-1}\big)'$ in the case of general Sturm–Liouville equation in [27]. One may also obtain the corresponding relations in case of our $\Pi = \begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix}$ and S from (2.4) and (2.11). These relations have the form:

$$\big(\Lambda_1^* S^{-1}\big)' = \Lambda_2^* S^{-1} A - q\Lambda_2^* S^{-1} + \mathcal{X}_{12}\Lambda_1^* S^{-1} - \mathcal{X}_{11}\Lambda_2^* S^{-1} + \mathcal{X}_{22}\Lambda_2^* S^{-1},$$
$$\big(\Lambda_2^* S^{-1}\big)' = \Lambda_1^* S^{-1} - \mathcal{X}_{12}\Lambda_2^* S^{-1}.$$

Thus, we easily obtain:

$$\big(\Lambda_2^* S^{-1}\big)'' = -\big(q + \mathcal{X}_{12}' + \mathcal{X}_{11} - \mathcal{X}_{22} - \mathcal{X}_{12}^2\big)\Lambda_2^* S^{-1} + \Lambda_2^* S^{-1} A. \tag{2.12}$$

According to [27, (6.9)] (the formula follows from (2.4) and (2.11) as well), we have

$$\mathcal{X}_{12}' = \mathcal{X}_{11} - \mathcal{X}_{22} - \mathcal{X}_{12}^2. \tag{2.13}$$

In view of (2.6) and (2.13), the equality (2.12) may be rewritten in the form

$$\big(\Lambda_2(x)^* S(x)^{-1}\big)'' = -\widetilde{q}(x)\Lambda_2(x)^* S(x)^{-1} + \Lambda_2(x)^* S(x)^{-1} A. \tag{2.14}$$

Clearly, (2.14) implies that $\widetilde{u}$ given by (2.9) satisfies FP equation (2.5). □

2. Next, we consider the case of the trivial initial system (i.e. of the trivial "seed" $q(x) \equiv 0$) and construct explicit potentials and solutions of the transformed FP equations (2.5) determined by the triples $\{A, S(0), \Pi(0)\}$. We note that the case $q(x) \equiv c$ ($c \in \mathbb{R}$) is equivalent to the case $q(x) \equiv 0$ with the matrix $A - cI_n$ in the triple (taken instead of A).

The main part of the construction consists in solving the first two equations in (2.4) in order to recover $\Lambda_k(x)$ ($k = 1, 2$) and so to recover $\Pi(x)$. It is convenient to look for Λ_2 after presenting it in the form $\Lambda_2 := \mathrm{e}^{xQ} f_1 + \mathrm{e}^{-xQ} f_2$, where Q is a square root of A, that is,

$$A = Q^2. \tag{2.15}$$

(The situation is similar to the presentation of the solution of Schrödinger equation as a linear combination of $\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}$ and $-\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}$.) It is well known that a square root of Q always exists if $\det A \neq 0$ and in many cases where $\det A = 0$ (see [7] and some results and discussion in [29]). After $\Lambda_2(x)$ is recovered, one easily constructs $S(x)$.

If $q(x) \equiv 0$ and (2.15) holds, relations (2.4) may be rewritten in the form

$$\Lambda_1'(x) = Q^2\Lambda_2(x), \quad \Lambda_2'(x) = \Lambda_1(x); \tag{2.16}$$

$$S(x) = S(0) + \int_0^x \Lambda_2(\xi)\Lambda_2(\xi)^* d\xi. \tag{2.17}$$

Similar to the slightly different [28, Lemma 2.1], it is easy to see that the following lemma is valid.

Lemma 2.5. *Let an $n \times n$ matrix Q be a square root of A: $Q^2 = A$. Then, the vector functions*

$$\Lambda_1(x) := Q\big(\mathrm{e}^{xQ} f_1 - \mathrm{e}^{-xQ} f_2\big), \quad \Lambda_2(x) := \mathrm{e}^{xQ} f_1 + \mathrm{e}^{-xQ} f_2, \tag{2.18}$$

where $f_1, f_2 \in \mathbb{C}^n$, satisfy the equalities (2.16).

The equalities $\Lambda_i(0) = \vartheta_i$ in (2.3) are equivalent to

$$Q(f_1 - f_2) = \vartheta_1, \quad f_1 + f_2 = \vartheta_2. \tag{2.19}$$

Thus, in the case (2.19) *the matrix function* $\begin{bmatrix}\Lambda_1(x) & \Lambda_2(x)\end{bmatrix}$ *takes the required value* $\begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}$ *(determined by the triple* $\{A, S(0), \Pi(0) = \begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}\}$*) at* $x = 0$*. Moreover, the integral part in* (2.17) *may be explicitly calculated (for each* Q *and* $\Pi(0)$*).*

Example 2.6. In the simplest case, we assume that

$$A = Q = 0, \quad \vartheta_1 = 0, \tag{2.20}$$

and derive stationary solutions (see also [11,28] on the stationary case).

Proposition 2.7. *Let* (2.20) *hold. Then, we have*

$$\Lambda_1(x) \equiv 0, \quad \Lambda_2(x) \equiv \vartheta_2, \quad \Pi(x) \equiv \begin{bmatrix}0 & \vartheta_2\end{bmatrix}, \quad S(x) = S(0) + x\vartheta_2\vartheta_2^*. \tag{2.21}$$

If $S(0)$ *is invertible and* $\vartheta_2 \neq 0$*, the corresponding stationary solutions of FP* (2.5) *have the form*

$$\widetilde{u}(t,x) = \widetilde{u}(x) = \frac{b}{1+ax} \quad (a := \vartheta_2^* S(0)^{-1}\vartheta_2, \quad b \in \mathbb{C}). \tag{2.22}$$

The solutions are real-valued iff $b \in \mathbb{R}$*.*

Proof. In view of (2.20), identity (2.1) holds for any $S(0) = S(0)^*$ and any $\vartheta_2 \in \mathbb{C}^n$. The equalities (2.19) are valid in the case $f_1 + f_2 = \vartheta_2$. Thus, fixing ϑ_2 and $S(0)$, by virtue of (2.17)–(2.19) we have (2.21). Using (2.6), (2.7) and (2.21), we obtain

$$\widetilde{q}(x) = -2\big(\vartheta_2^*\big(S(0) + x\vartheta_2\vartheta_2^*\big)^{-1}\vartheta_2\big)^2. \tag{2.23}$$

If $S(0)$ is invertible, expression (2.23) is easily simplified by using the equality [28, (3.3)]:

$$\big(I_n + xS(0)^{-1}\vartheta_2\vartheta_2^*\big)^{-1} = I_n - \frac{x}{1+ax}S(0)^{-1}\vartheta_2\vartheta_2^*, \quad a = \vartheta_2^* S(0)^{-1}\vartheta_2. \tag{2.24}$$

Namely, we derive

$$\widetilde{q}(x) = -\frac{2a^2}{(1+ax)^2}. \tag{2.25}$$

In view of (2.9), (2.21), (2.24) and Proposition 2.1, the functions

$$\begin{aligned}\widetilde{u}(t,x) &= \vartheta_2^*(I_n + xS(0)^{-1}\vartheta_2\vartheta_2^*)^{-1}S(0)^{-1}h \\ &= \left(1 - \frac{ax}{1+ax}\right)\vartheta_2^* S(0)^{-1}h = \frac{b}{1+ax}, \quad b := \vartheta_2^* S(0)^{-1}h\end{aligned} \tag{2.26}$$

are stationary solutions of the corresponding FP equation (2.5). Since $\vartheta_2 \neq 0$, it is immediate that b may take all complex values (by different values of h.) □

Several non-stationary examples are presented in the appendix.

3. GBDT—general case of second order poles (preliminaries to GBDT for FP systems)

A general case of the first order system depending rationally on the spectral parameter was studied in [30, pp. 219–221] (see also the references therein). In this paper, we will need a particular case of the second order poles

$$y' = Gy, \quad G(t,z) = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}(z-c_k)^{-p}q_{kp}(t) \quad (\ell \in \mathbb{N},\ c_k \in \mathbb{C}). \tag{3.1}$$

Similar to [30], we assume that $c_i \neq c_s$ for $i \neq s$, that $t \in \mathcal{I}$, where $\mathcal{I}$ is an interval which contains 0 ($0 \in \mathcal{I}$), and that the coefficients $q_{kp}(t)$ are $m \times m$ locally summable matrix functions ($m \in \mathbb{N}$). Note that the interesting case of GBDT for systems with first order poles of the spectral parameter and for the corresponding dynamical systems was studied in the recent paper [29]. The GBDT of the initial ("seed") system (3.1) is determined by a number $n \in \mathbb{N}$, by $n \times n$ matrices A_i ($i = 1, 2$) and $S(0)$, and by $n \times m$ matrices $\Pi_i(0)$ ($i = 1, 2$). It is required that these matrices form an S-node, that is, the matrix identity

$$A_1S(0) - S(0)A_2 = \Pi_1(0)\Pi_2(0)^* \tag{3.2}$$

holds. The matrix functions $\Pi_i(t)$ are introduced via their values at $t = 0$ and equations

$$(\Pi_1)' = \sum_{k=1}^{\ell}\sum_{p=1}^{2}(A_1 - c_kI_n)^{-p}\Pi_1 q_{kp}, \quad (\Pi_2^*)' = -\sum_{k=1}^{\ell}\sum_{p=1}^{2} q_{kp}\Pi_2^*(A_2 - c_kI_n)^{-p}. \tag{3.3}$$

Clearly, we assume above that $c_k \notin \sigma(A_i)$ for $i = 1, 2$ and $1 \leq k \leq \ell$. Compare (3.1) with (3.3) to see that Π_2^* can be viewed as a generalised eigenfunction of the system $u' = Gu$.

Matrix function $S(t)$ is introduced via $S(0)$ and via $S'(t)$ given by the equality

$$S' = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}\sum_{r=1}^{p}(A_1 - c_kI_n)^{r-p-1}\Pi_1 q_{kp}\Pi_2^*(A_2 - c_kI_n)^{-r}. \tag{3.4}$$

Equations (3.2)–(3.4)) yield the identity

$$A_1S(t) - S(t)A_2 = \Pi_1(t)\Pi_2(t)^*, \quad t \in \mathcal{I}, \tag{3.5}$$

of which (3.2) is a particular case of $t = 0$. For initial system (3.1), the GBDT-transformed system is

$$\widetilde{y}' = \widetilde{G}\widetilde{y}, \quad \widetilde{G}(t,z) = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}(z-c_k)^{-p}\widetilde{q}_{kp}(t), \tag{3.6}$$

where the transformed coefficients $\widetilde{q}_{kp}$ are given by the formulas

$$\widetilde{q}_{kp} = q_{kp} + \sum_{r=p}^{2}\Big(q_{kr}Y_{k,p-r-1} - X_{k,p-r-1}q_{kr} - \sum_{i=p}^{r}X_{k,i-r-1}q_{kr}Y_{k,p-i-1}\Big), \tag{3.7}$$

and the matrix functions $X_{kp}(t)$, $Y_{kp}(t)$ have the form

$$X_{kp} = \Pi_2^* S^{-1}(A_1 - c_k I_n)^p \Pi_1, \quad Y_{kp} = \Pi_2^*(A_2 - c_k I_n)^p S^{-1}\Pi_1. \tag{3.8}$$

Our theorem below is a special case of a more general [30, Theorem 7.4]. It shows that the transfer function $w_A(z) = I_m - \Pi_2^* S^{-1}(A_1 - zI_n)^{-1}\Pi_1$ (that is, the transfer matrix function in Lev Sakhnovich form [31–33]) is the so called Darboux matrix for the systems (3.1) and (3.6).

Theorem 3.1. *Let the first order system* (3.1) *and five matrices* A_1, A_2, $S(0)$ *and* $\Pi_1(0)$, $\Pi_2(0)$ *be given. Assume that the identity* (3.2) *holds and that* $\{c_k\} \cap \sigma(A_i) = \varnothing$ $(i = 1, 2)$. *Then* (*in the points of invertibility of* S), *the transfer matrix function*

$$w_A(t, z) = I_m - \Pi_2(t)^* S(t)^{-1}(A_1 - zI_n)^{-1}\Pi_1(t), \tag{3.9}$$

where $\Pi_i(t)$ *and* $S(t)$ *are determined by* (3.3) *and* (3.4) *respectively, satisfies the equation*

$$w_A'(t, z) = \widetilde{G}(t, z) w_A(t, z) - w_A(t, z) G(t, z), \tag{3.10}$$

where $\widetilde{G}$ *is determined by the formulas* (3.6)–(3.8).

Theorem 3.1 means that for y satisfying the initial system (3.1) the expression $\widetilde{y} = w_A(t, z)y$ satisfies the transformed system (3.6).

4. GBDT for FP system with ℓ space variables

1. As already mentioned in the introduction, we start with GBDT for auxiliary systems with second order poles of the spectral parameter. General type GBDT for systems with second order poles, that is, for the systems

$$y' = Gy, \quad G(t, z) = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}(z - c_k)^{-p} q_{kp}(t) \quad \left(y' = \frac{d}{dt}y, \ \ell \in \mathbb{N}\right), \tag{4.1}$$

where $y(t, z) \in \mathbb{C}^m$ and $c_i \neq c_s$ for $i \neq s$, is described in the previous Section 3 and is determined by five matrices A_1, A_2, $S(0)$, $\Pi_1(0)$ and $\Pi_2(0)$ satisfying (3.2). The coefficients q_{kp} of (4.1) and of the corresponding initial system (1.2) are so called “seed” coefficients in the terminology of Bäcklund–Darboux transformations theory.

Recall that we deal here with the real-valued coefficients and entries of matrices. Thus, we add the corresponding requirements on the initial system (4.1) and the matrices A_i $(i = 1, 2)$, $\Pi_i(0)$ $(i = 1, 2)$ and $S(0)$, which determine GBDT (or, in other words, on the initial system and matrices which determine the transformed system (3.6)). More precisely, we require that

$$c_k \in \mathbb{R}, \quad q_{kp}(t) \in \mathbb{R}^{m\times m} \quad (1 \leq k \leq \ell, \ p = 1, 2); \tag{4.2}$$

$$A_i, S(0) \in \mathbb{R}^{n\times n} \quad (i = 1, 2); \quad \Pi_i(0) \in \mathbb{R}^{n\times m} \quad (i = 1, 2). \tag{4.3}$$

Remark 4.1. Recall that the generalised matrix eigenfunctions $\Pi_i(t)$ are introduced by the systems (3.3) and $S(t)$ is introduced via (3.4). In view of (3.3) and (3.4), the requirement that the scalars and matrix entries in (4.2) and (4.3) are real-valued implies that the entries of $\Pi_i(t)$ and $S(t)$ are also real-valued. It follows that the entries of the transformed coefficients $\widetilde{q}_{kp}(t)$ given by (3.7) and (3.8) are real-valued as well.

In further constructions (similar to Proposition 2.1), formula [30, (7.61)] on the derivative of $\Pi_2(t)^*S(t)^{-1}$ plays a fundamental role. We require that

$$c_k \notin \sigma(A_i) \quad (i = 1, 2). \tag{4.4}$$

Then, for the case of the initial system (4.1), formula (7.61) from [30] takes (in the points of invertibility of $S(t)$) the form

$$\big(\Pi_2(t)^*S(t)^{-1}\big)' = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}\widetilde{q}_{kp}(t)\Pi_2(t)^*S(t)^{-1}(A_1 - c_kI_n)^{-p}. \tag{4.5}$$

Using (4.5), we derive the main assertion of this section (where some of the formulas from Section 3 are repeated for convenience). *Note that we omit sometimes the variable t in $\Pi_p(t)$, $q_{kp}(t)$, $X_{kp}(t)$, $Y_{kp}(t)$ and some other matrix functions in order to make formulas shorter.*

Theorem 4.2. *Let some locally summable matrix coefficients $q_{kp}(t)$ and a set of numbers $\{c_k\}$ $(1 \le k \le \ell,\ p = 1, 2)$ as well as five matrices A_1, A_2, $S(0)$ and $\Pi_1(0)$, $\Pi_2(0)$ be given. Assume that the relations (4.2)–(4.4) and the matrix identity $A_1S(0) - S(0)A_2 = \Pi_1(0)\Pi_2(0)^*$ hold. Introduce $\Pi_1(t)$, $\Pi_2(t)$ and $S(t)$ by the values $\Pi_1(0)$, $\Pi_2(0)$ and $S(0)$ and by the equations*

$$(\Pi_1)' = \sum_{k=1}^{\ell}\sum_{p=1}^{2}(A_1 - c_kI_n)^{-p}\Pi_1q_{kp}, \qquad (\Pi_2^*)' = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}q_{kp}\Pi_2^*(A_2 - c_kI_n)^{-p},$$

$$S' = -\sum_{k=1}^{\ell}\sum_{p=1}^{2}\sum_{r=1}^{p}(A_1 - c_kI_n)^{r-p-1}\Pi_1q_{kp}\Pi_2^*(A_2 - c_kI_n)^{-r}.$$

Then (in the points of invertibility of S), the $m \times n$ matrix function

$$\Psi(t, x_1, \dots, x_\ell) = \Pi_2(t)^*S(t)^{-1}\exp\Big\{\sum_{k=1}^{\ell}x_k(A_1 - c_kI_n)^{-1}\Big\} \tag{4.6}$$

satisfies the transformed FP system below (or, equivalently, each column of Ψ satisfies that system):

$$\frac{\partial}{\partial t}\Psi(t, x_1, \dots, x_\ell) + \sum_{k=1}^{\ell}\sum_{p=1}^{2}\widetilde{q}_{kp}(t)\frac{\partial^p}{\partial x_k^p}\Psi(t, x_1, \dots, x_\ell) = 0, \tag{4.7}$$

where $\Psi(t, x_1, \ldots, x_\ell) \in \mathbb{R}^{m\times n}$,

$$\widetilde{q}_{k2}(t) = w_A(t, c_k)q_{k2}(t)w_A(t, c_k)^{-1}, \tag{4.8}$$

$$\widetilde{q}_{k1}(t) = w_A(t, c_k)q_{k1}(t)w_A(t, c_k)^{-1} + w_A(t, c_k)q_{k2}(t)Y_{k,-2}(t) \tag{4.9}$$

$$- X_{k,-2}(t)q_{k2}(t)w_A(t, c_k)^{-1}, \tag{4.10}$$

$$X_{kp} = \Pi_2^* S^{-1}(A_1 - c_k I_n)^p \Pi_1, \quad Y_{kp} = \Pi_2^*(A_2 - c_k I_n)^p S^{-1}\Pi, \tag{4.11}$$

and $w_A(t, c_k) = I_m - \Pi_2(t)^* S(t)^{-1}(A_1 - c_k I_n)^{-1}\Pi_1(t)$.

Proof. Formula (3.7) for the transformed coefficients $\widetilde{q}_{kp}$ may be easily rewritten in a more convenient way:

$$\widetilde{q}_{k2} = (I_m - X_{k,-1})q_{k2}(I_m + Y_{k,-1}), \tag{4.12}$$

$$\widetilde{q}_{k1} = (I_m - X_{k,-1})q_{k1}(I_m + Y_{k,-1}) + (I_m - X_{k,-1})q_{k2}Y_{k,-2} - X_{k,-2}q_{k2}(I_m + Y_{k,-1}), \tag{4.13}$$

where X_{kp} and Y_{kp} are introduced in (4.11) and commas between the indices of $X_{k,-1}$, $Y_{k,-1}$, ... are used in order to avoid ambiguousness. According to (3.9) and (4.11), we have

$$I_m - X_{k,-1} = I_m - \Pi_2(t)^* S(t)^{-1}(A_1 - c_k I_n)^{-1}\Pi_1(t) = w_A(t, c_k). \tag{4.14}$$

The inverse to the transfer matrix function $w_A(t, c_k)$ has the form

$$w_A(t, c_k)^{-1} = I_m + \Pi_2(t)^*(A_2 - c_k I_n)^{-1} S(t)^{-1}\Pi_1(t) \tag{4.15}$$

(see [31] or [30, Section 1.2.1]). Taking into account (4.15) and (4.11), we obtain

$$I_m + Y_{k,-1} = w_A(t, c_k)^{-1}. \tag{4.16}$$

Substituting (4.14) and (4.16) into (4.12) and (4.13), we derive (4.8) and (4.10).

After the substitution of (4.6) into (4.7), the left-hand side of (4.7) transforms into the expression

$$\Big(\big(\Pi_2(t)^* S(t)^{-1}\big)' + \sum_{k=1}^{\ell}\sum_{p=1}^{2} \widetilde{q}_{kp}(t)\Pi_2(t)^* S(t)^{-1}(A_1 - c_k I_n)^{-p} \Big) \times \exp\Big\{ \sum_{k=1}^{\ell} x_k (A_1 - c_k I_n)^{-1} \Big\},$$

which (according to (4.5)) equals zero (zero matrix). Thus, Ψ of the form (4.6) satisfies (4.7). □

2. It is interesting that for the case of the scalar coefficients q_{kp} (i.e. for the case $m = 1$) formula (4.8) yields $q_{k2} = \widetilde{q}_{k2}$. Moreover, (4.10) implies that $q_{k1} = \widetilde{q}_{k1}$ in this case. Indeed, in case $m = 1$ we rewrite (4.10) as

$$\widetilde{q}_{k1} = q_{k1} + q_{k2}\big(w_A(t, c_k)Y_{k,-2} - X_{k,-2}w_A(t, c_k)^{-1}\big). \tag{4.17}$$

In view of (4.14), (4.15), and (4.11), we have

$$\begin{aligned} & w_A(t,c_k)Y_{k,-2} - X_{k,-2}w_A(t,c_k)^{-1} \\ & = \Pi_2^*(A_2-c_kI_n)^{-2}S^{-1}\Pi_1 - \Pi_2^*S^{-1}(A_1-c_kI_n)^{-1}\Pi_1\Pi_2^*(A_2-c_kI_n)^{-2}S^{-1}\Pi_1 \\ & - \Pi_2^*S^{-1}(A_1-c_kI_n)^{-2}\Pi_1 - \Pi_2^*S^{-1}(A_1-c_kI_n)^{-2}\Pi_1\Pi_2^*(A_2-c_kI_n)^{-1}S^{-1}\Pi_1. \end{aligned} \tag{4.18}$$

Using (3.5), we substitute the identity $\Pi_1\Pi_2^* = (A_1 - c_kI_n)S - S(A_2 - c_kI_n)$ into (4.18). Thus, after simple calculations, we derive

$$w_A(t,c_k)Y_{k,-2} - X_{k,-2}w_A(t,c_k)^{-1} = 0, \tag{4.19}$$

and the equality $q_{k1} = \widetilde{q}_{k1}$ follows from (4.17) and (4.19). Therefore, we used another modification of GBDT for the scalar case (in Section 2).

3. Next, we modify the results of Section 3 for an interesting special case of FP systems (4.7) and of GBDT determined by a triple of matrices $\{A, S(0), \Pi(0)\}$ instead of the five matrices in the general case. For this purpose, we consider a special class of systems (4.1), namely, the systems

$$y' = Gy, \quad G(t,z) = -\sum_{k=1}^{2\ell}\sum_{p=1}^{2}(z-c_k)^{-p}q_{kp}(t) \quad (\ell \in \mathbb{N}), \tag{4.20}$$

where 2ℓ poles appear (instead of ℓ in (4.1)), $c_i \neq c_s$ for $i \neq s$ and the following requirements are added for $1 \leq k \leq \ell$:

$$c_k > 0, \quad c_{k+\ell} = -c_k, \quad (-1)^{p+1}jq_{kp}(t)^*j \equiv q_{k+\ell,p}(t) \quad (1 \leq k \leq \ell). \tag{4.21}$$

(Recall that we separate sometimes the indices by commas, see above.) Switching from five to three parameter matrices in GBDT, we introduce (and partly change) notations:

$$A_1 = A, \quad \Pi_1(t) = \Pi(t), \quad j := \mathrm{diag}\{I_{m_1}, -I_{m_2}\}, \tag{4.22}$$

and assume that

$$A_2 = -A^*, \ c_k \notin \sigma(A) \text{ for } 1 \leq k \leq 2\ell, \ S(0) = S(0)^*, \ \Pi_2(0)^* = j\Pi(0)^*. \tag{4.23}$$

According to (4.22) and (4.23), our GBDT transformed system is determined by the triple of matrices $\{A, S(0), \Pi(0)\}$ satisfying (3.2), which in this case takes the form

$$AS(0) + S(0)A^* = \Pi(0)j\Pi(0)^*. \tag{4.24}$$

Conditions (4.2), (4.3) are substituted by the conditions

$$q_{kp}(t) \in \mathbb{R}^{m\times m} \ (1 \leq k \leq \ell, \ p = 1,2); \ A, S(0) \in \mathbb{R}^{n\times n}, \ \Pi(0) \in \mathbb{R}^{n\times m}, \tag{4.25}$$

which again imply that the entries of $\Pi(t)$, $S(t)$ and $\widetilde{q}_{kp}(t)$ are real-valued.

Proposition 4.3. *Let relations (4.21)–(4.25) hold. Then, the transformed coefficients $\widetilde{q}_{kp}(t)$ given by (4.8) and (4.10) satisfy the same equalities as q_{kp}, namely,*

$$(-1)^{p+1} j \widetilde{q}_{kp}(t)^* j \equiv \widetilde{q}_{k+\ell,p}(t) \quad (1 \leq k \leq \ell). \tag{4.26}$$

Proof. In view of (4.21)–(4.23), we rewrite the first equation in (3.3) (for the case of 2ℓ poles c_k) in the form

$$\begin{aligned}(j\Pi^*)' &= \sum_{k=1}^{2\ell}\sum_{p=1}^{2}(-1)^p j q_{kp}^* j \big(j\Pi^*\big)(A_2 + c_k I_n)^{-p} \\ &= -\sum_{k=1}^{2\ell}\sum_{p=1}^{2} q_{kp}\big(j\Pi^*\big)(A_2 - c_k I_n)^{-p}.\end{aligned} \tag{4.27}$$

Comparing (4.27) with the second equation in (3.3), we see that Π_2^* and $j\Pi^*$ satisfy the same first order differential equation. Moreover, we have $\Pi_2(0)^* = j\Pi(0)^*$. Hence,

$$\Pi_2(t)^* = j\Pi(t)^*. \tag{4.28}$$

The first equation in (3.3) takes the form

$$\Pi'(t) = \sum_{k=1}^{2\ell}\sum_{p=1}^{2}(A - c_k I_n)^{-p}\Pi(t) q_{kp}(t). \tag{4.29}$$

The matrices A and $\Pi(0)$ together with (4.29) determine $\Pi(t)$. Then, the matrix $S(0)$ and equation (3.4) (for the case of 2ℓ poles c_k), which we rewrite as

$$S'(t) = \sum_{k=1}^{2\ell}\sum_{p=1}^{2}\sum_{r=1}^{p}(-1)^{r+1}(A - c_k I_n)^{r-p-1}\Pi(t) q_{kp}(t) j \Pi(t)^*(A^* + c_k I_n)^{-r}, \tag{4.30}$$

determine $S(t)$. The matrix identity (3.5) for $S(t)$ takes the form

$$AS(t) + S(t)A^* = \Pi(t) j \Pi(t)^*, \quad t \in \mathcal{I}. \tag{4.31}$$

In order to show that $S'(t) = S'(t)^*$ we split the first sum on the right-hand side of (4.30) into the sum from 1 to ℓ and another sum from $\ell + 1$ to 2ℓ and set

$$\widetilde{r} = p - r + 1, \quad \widetilde{k} = k - \ell.$$

Clearly, $(-1)^{r+1} = (-1)^{\widetilde{r}+1}(-1)^{r-\widetilde{r}} = (-1)^{\widetilde{r}+1}(-1)^{p+1}$. Now, we rewrite (4.30) as

$$S' = \sum_{k=1}^{\ell}\sum_{p=1}^{2}\sum_{r=1}^{p}(-1)^{r+1}(A - c_k I_n)^{r-p-1}\Pi q_{kp} j \Pi^*(A^* + c_k I_n)^{-r}$$

$$+\sum_{\widetilde{k}=1}^{\ell}\sum_{p=1}^{2}\sum_{\widetilde{r}=1}^{p}(-1)^{\widetilde{r}+1}(-1)^{p+1}(A+c_{\widetilde{k}}I_n)^{-\widetilde{r}}\Pi q_{\widetilde{k}+\ell,p}j\Pi^*(A^*-c_{\widetilde{k}}I_n)^{\widetilde{r}-p-1}. \quad (4.32)$$

According to (4.21), we have $(-1)^{p+1}q_{\widetilde{k}+\ell,p}j = jq^*_{\widetilde{k}p}$. Therefore, (4.32) yields $S'(t) = S'(t)^*$. Taking into account that $S(0) = S(0)^*$ and $S'(t) = S'(t)^*$, we derive

$$S(t) = S(t)^*. \quad (4.33)$$

Using (4.22), (4.23) and (4.28), we rewrite formulas for $w_A(t, c_k)$ and $w_A(t, c_k)^{-1}$ in (4.14), (4.15) as:

$$w_A(t, c_k) = I_m - j\Pi(t)^*S(t)^{-1}(A - c_kI_n)^{-1}\Pi(t), \quad (4.34)$$

$$w_A(t, c_k)^{-1} = I_m - j\Pi(t)^*(A^* + c_kI_n)^{-1}S(t)^{-1}\Pi(t). \quad (4.35)$$

Since $c_{k+\ell} = -c_k$, equalities (4.33)–(4.35) easily yield

$$jw_A(t, c_k)^*j = w_A(t, c_{k+\ell})^{-1} \quad (1 \le k \le \ell). \quad (4.36)$$

Relations (4.8), (4.21) and (4.36) imply (4.26) for $p = 2$.

In a similar to (4.36) way one shows that

$$jY^*_{k,-2}j = X_{k+\ell,-2}, \quad jX^*_{k,-2}j = Y_{k+\ell,-2} \quad (1 \le k \le \ell). \quad (4.37)$$

where X_{kp} and Y_{kp} are given (in the general case) by (3.8). Finally, (4.26) for $p = 1$ follows from the relations (4.10), (4.21) and (4.36), (4.37). □

In view of (4.28), formula (4.5) may be rewritten as:

$$\left(\Pi^*S^{-1}\right)' = -\sum_{k=1}^{2\ell}\sum_{p=1}^{2} j\widetilde{q}_{kp}j\Pi^*S^{-1}(A - c_kI_n)^{-p}. \quad (4.38)$$

Correspondingly, Theorem 4.2 takes the following form.

Theorem 4.4. *Let some locally summable matrix coefficients $q_{kp}(t)$ and a set of numbers $\{c_k\}$ $(1 \le k \le \ell,\ p = 1, 2)$ as well as a triple of matrices $\{A, S(0), \Pi(0)\}$ be given. Assume that the relations (4.21) and (4.25) as well as the second and third relations in (4.23) and the matrix identity (4.24) hold. Then (in the points of invertibility of S), the $m \times n$ matrix function*

$$\Psi(t, x_1, \ldots, x_{2\ell}) = \Pi(t)^*S(t)^{-1}\exp\left\{\sum_{k=1}^{2\ell} x_k(A - c_kI_n)^{-1}\right\} \quad (4.39)$$

satisfies the transformed FP system

$$\frac{\partial}{\partial t}\Psi(t, x_1, \ldots, x_{2\ell}) + \sum_{k=1}^{2\ell}\sum_{p=1}^{2} j\widetilde{q}_{kp}(t)j\frac{\partial^p}{\partial x_k^p}\Psi(t, x_1, \ldots, x_{2\ell}) = 0, \quad (4.40)$$

where the equalities $(-1)^{p+1}j\widetilde{q}_{kp}(t)^*j \equiv \widetilde{q}_{k+\ell,p}(t)$ $(1 \le k \le \ell)$ are valid.

A. Non-stationary solutions: examples

In the following examples, we consider rational non-stationary solutions. In our first example, the parameter matrix $S(0)$ is degenerate (i.e. $\det S(0) = 0$) and the non-stationary part of the solution contains strong singularity at $x = 0$ (although the solution is nonsingular on $(0, \infty)$).

Example A.1. Let

$$n = 3, \quad \vartheta_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vartheta_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{A.1}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and so let } A = Q^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.2}$$

Clearly, the matrix

$$S(0) = S(0)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.3}$$

satisfies (2.1) and we fix $S(0)$ given above. In formula (2.9), we fix

$$h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{A.4}$$

Proposition A.2. *Let relations* (A.1)–(A.4) *hold. Then, the function*

$$\psi(t, x) = -\frac{3x^2 + 15x + 15}{2(x^3 + 6x^2 + 15x + 15)} - t\frac{15x^2 + 45x + 45}{x^2(x^3 + 6x^2 + 15x + 15)}. \tag{A.5}$$

is a real-valued non-stationary solution of the corresponding transformed FP. We note that $x^3 + 6x^2 + 15x + 15 > 0$ $(x > 0)$ *in the denominators above.*

Proof. According to (2.18), (2.19) and (A.1), we have

$$\Lambda_1(x) = \begin{bmatrix} x + 1 \\ 0 \\ 0 \end{bmatrix}, \quad \Lambda_2(x) \begin{bmatrix} (x/2)(x + 2) \\ 0 \\ 1 \end{bmatrix}. \tag{A.6}$$

It follows from (2.17), (A.3) and (A.6) that

$$S(x) = \begin{bmatrix} (x^3/60)(3x^2 + 15x + 20) & 0 & (x^2/6)(x + 3) \\ 0 & 1 & 0 \\ (x^2/6)(x + 3) & 0 & x + 1 \end{bmatrix}, \tag{A.7}$$

which yields

$$S(x)^{-1} = \frac{1}{\det S(x)} \begin{bmatrix} x+1 & 0 & -(x^2/6)(x+3) \\ 0 & 1 & 0 \\ -(x^2/6)(x+3) & 0 & (x^3/60)(3x^2+15x+20) \end{bmatrix}, \qquad (A.8)$$

$$\det S(x) = (x^3/45)(x^3+6x^2+15x+15). \qquad (A.9)$$

Taking into account (2.9), (2.10) and (A.2), we see that

$$\widetilde{u}(t,x) = \psi(t,x) = \Lambda_2(x)^* S(x)^{-1}(I_3 - tA)h. \qquad (A.10)$$

Finally, relations (A.2), (A.4), (A.6), (A.8)–(A.10) imply that the solution ψ of the corresponding FP is given by the formula (A.5). □

4. In the same way, wide families of solutions are constructed.

Example A.3. Let

$$n = 3, \quad \vartheta_1 = \begin{bmatrix} c^2 \\ 0 \\ 0 \end{bmatrix}, \quad \vartheta_2 = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \quad (a, b, c \in \mathbb{C}); \qquad (A.11)$$

$$Q = c \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and so let } A = Q^2 = c^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (A.12)$$

It is easy to see that

$$S(0) = S(0)^* = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ \overline{a} & \overline{b} & 1 \end{bmatrix} \qquad (A.13)$$

satisfies (2.1). In view of (A.11) and (A.12), the vectors f_1 and f_2 given by

$$f_1 = \begin{bmatrix} a \\ (b+c)/2 \\ 1/2 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ (b-c)/2 \\ 1/2 \end{bmatrix} \qquad (A.14)$$

satisfy (2.19). Next, we calculate explicitly $\Lambda_i(x)$ and $S(x)$ whereupon the expressions for $\widetilde{q}(x)$ and $\widetilde{u}(t,x)$ follow from (2.7), (2.6) and (2.9). One does not need (A.14) in those calculations. Indeed, according to (2.18), (2.19), (A.11) and (A.12), we have

$$\Lambda_1(x) = Q(f_1 - f_2) + xQ^2(f_1 + f_2) = \begin{bmatrix} c^2(x+1) \\ 0 \\ 0 \end{bmatrix}; \qquad (A.15)$$

$$\Lambda_2(x) = f_1 + f_2 + xQ(f_1 - f_2) + x^2Q^2(f_1+f_2)/2 = \begin{bmatrix} g_1(x) \\ b \\ 1 \end{bmatrix}, \qquad (A.16)$$

$$g_1(x) := (c^2/2)x^2 + c^2x + a. \tag{A.17}$$

Relations (2.17), (A.13) and (A.16) yield

$$S(x) = S(x)^* = \begin{bmatrix} g_3(x)+1 & \overline{b}\, g_2(x) & g_2(x)+a \\ b\,\overline{g_2(x)} & |b|^2x+1 & b(x+1) \\ \overline{g_2(x)}+\overline{a} & \overline{b}(x+1) & x+1 \end{bmatrix}, \tag{A.18}$$

$$g_2(x) := (c^2/6)x^3 + (c^2/2)x^2 + ax, \tag{A.19}$$

$$g_3(x) := (|c|^4/20)x^5 + (|c|^4/4)x^4 + ((\overline{a}c^2 + a\overline{c}^2 + 2|c|^4)x^3/6) + ((\overline{a}c^2 + a\overline{c}^2)x^2/2) + |a|^2x. \tag{A.20}$$

Finally, for A given in (A.12), the expression e^{-tA} in (2.9) takes the form

$$\mathrm{e}^{-tA} = I_3 - tA. \tag{A.21}$$

Clearly, the expressions for $\widetilde{q}(x)$ and $\widetilde{u}(t,x)$ get more transparent if one sets $b=0$, which we do in the next proposition.

Proposition A.4. *Let relations (A.11)–(A.13) hold and assume that $b=0$. Then, we have*

$$\widetilde{q}(x) = \frac{4|c|^4}{g_0(x)}(x+1)\big((x^3/3)+x^2+x\big) - \frac{2}{g_0(x)^2}\Big((x+1)|g_1(x)|^2 + g_3(x) + 1 - \overline{g_1(x)}(g_2(x)+a) - g_1(x)(\overline{g_2(x)}+\overline{a})\Big)^2; \tag{A.22}$$

$$\widetilde{u}(t,x) = \frac{1}{g_0(x)}\Big(-|c|^4t\big((x^3/3)+x^2+x\big) + g_3(x) - \overline{g_1(x)}\big(g_2(x)+a\big) + 1\Big), \tag{A.23}$$

where

$$g_0(x) = |c|^4\Big(\frac{x^6}{45} + \frac{2x^5}{15} + \frac{x^4}{3} + \frac{x^3}{3}\Big) + (1-|a|^2)x + 1, \tag{A.24}$$

and other g_k are given by (A.17), (A.19) and (A.20).

Proof. It easily follows from (A.18)–(A.20) that $g_0(x) := \det S(x)$ has the form (A.24) and

$$S(x)^{-1} = \frac{1}{g_0(x)}\begin{bmatrix} x+1 & 0 & -g_2(x)-a \\ 0 & 1 & 0 \\ -\overline{g_2(x)}-\overline{a} & 0 & g_3(x)+1 \end{bmatrix} \quad (\text{for } b=0). \tag{A.25}$$

Relation (2.9) together with (2.3), (A.16) and (A.25) yields formula (A.23) for a time-dependent solution $\widetilde{u}$ of (2.5). Here, we take into account that $(x+1)\overline{g_1(x)} - \overline{g_2(x)} - \overline{a} = \overline{c}^2\big((x^3/3)+x^2+x\big)$. Similar computations for the formula (2.6) imply that the transformed potential $\widetilde{q}$ in (2.5) is given by (A.22). □

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Процедура одягнення для рівняння Фоккера–Планка: випадки розмірностей $1+1$ та $1+\ell$

Alexander Sakhnovich

Ми розглядаємо процедуру одягнення та явні розв'язки скороченого скалярного рівняння Фоккера–Планка у випадку розмірності $1+1$ і матричної системи Фоккера–Планка у випадку розмірності $1+\ell$. Для цього ми використовуємо наше узагальнене перетворення Беклунда–Дарбу (УПБД). Є лише декілька праць щодо процедури одягнення для важливого рівняння Фоккера–Планка і ці праці стосуються випадків розмірностей $1+1$ та $1+2$.

Ключові слова: скорочене рівняння Фоккера–Планка, матрична система Фоккера–Планка, процедура одягнення, перетворення Дарбу, матрична тотожність, явний розв'язок