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The Maximal Operator on the Amalgam Space

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We prove the boundedness of the Hardy–Littlewood maximal operator on the amalgam spaces $(L^p, \ell^q)(\mathbb{R}^n)$. As a consequence, we obtain the boundedness of the commutators on these spaces.

Key words: amalgam spaces, maximal operator, commutator

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1. Introduction

Amalgam spaces were first introduced by N. Wiener, who employed some special cases in his study of representation of functions by trigonometrical integrals [13] and as part of his Tauberian theorems [14]. The first systematic study of amalgam spaces on the real line is due to F. Holland [9]. He also obtained several results for the more general amalgam (E, ω_ρ), where E is the Cartesian product of a family $\{E_n\}_{n\in\mathbb{Z}}$ of normed spaces and ω_ρ is a partially ordered vector space of real sequences endowed with a Riesz norm ρ. Another relevant generalization was studied by J. Stewart [12], who defined the amalgam $(L^p, \ell^q)(G)$ for a locally compact Abelian group G. We will focus on the amalgam spaces $(L^p, \ell^q)(\mathbb{R}^n)$.

Let us denote by Q_0 the unit cube $[0,1)^n$ in $\mathbb{R}^n$ and, for each $z \in \mathbb{R}^n$, we denote $Q_z = z + Q_0$. The *amalgam space* $(L^p, \ell^q)(\mathbb{R}^n)$ is defined, for $1 \le p, q \le \infty$, as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{p,q} = \|\{\|f\|_{L^p(Q_w)}\}_{w\in\mathbb{Z}^n}\|_{\ell^q(\mathbb{Z}^n)} < \infty.$$

It is well known that $(L^p, \ell^q)(\mathbb{R}^n)$ is a Banach space [9, Theorem 1] when endowed with the norm $\|\cdot\|_{p,q}$. Notice that for $p, q < \infty$,

$$\|f\|_{p,q} = \left\{ \sum_{w\in\mathbb{Z}^n} \left(\int_{Q_w} |f(z)|^p \, dz \right)^{q/p} \right\}^{1/q}.$$

So, if $p = q < \infty$, then $(L^p, \ell^q)(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. The same conclusion follows for $p = q = \infty$.

If $1 \leq q < p \leq \infty$, then the inclusion $(L^p, \ell^q)(\mathbb{R}^n) \subsetneq L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ holds. In this case, the amalgam space does not contain any new functions besides the ones already contained in $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$. A more interesting case is $1 \leq p < q \leq \infty$ for $L^p(\mathbb{R}^n) \cup L^q(\mathbb{R}^n) \subsetneq (L^p, \ell^q)(\mathbb{R}^n)$. For example, consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^{-1/q} & \text{if } 0 < x \leq 1, \\ x^{-1/p} & \text{if } 1 < x. \end{cases}$$

It is not hard to prove that $f \in (L^p, \ell^q)(\mathbb{R})$ but $f \notin L^p(\mathbb{R}) \cup L^q(\mathbb{R})$ for $q > 1$.

Other important inclusions are $(L^p, \ell^{q_1})(\mathbb{R}^n) \subsetneq (L^p, \ell^{q_2})(\mathbb{R}^n)$ when $q_1 < q_2$ and $(L^{p_2}, \ell^q)(\mathbb{R}^n) \subsetneq (L^{p_1}, \ell^q)(\mathbb{R}^n)$ when $p_1 < p_2$. These and other basic properties of amalgam spaces can be found in [6] and [12]. Regarding the dual space of an amalgam space, it is characterized in the next theorem [9]:

Theorem 1.1. *Let $1 < p, q < \infty$. Then the dual space of $(L^p, \ell^q)(\mathbb{R}^n)$ is $(L^{p'}, \ell^{q'})(\mathbb{R}^n)$, where p' and q' are the conjugate exponents of p and q respectively.*

We finish this section with the amalgam version of Hölder's inequality [6]:

Theorem 1.2. *If $f \in (L^p, \ell^q)(\mathbb{R}^n)$ and $g \in (L^{p'}, \ell^{q'})(\mathbb{R}^n)$, then $gf \in L^1(\mathbb{R}^n)$ and*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{p,q} \|g\|_{p',q'}.$$

2. Boundedness of the Hardy–Littlewood maximal

Recall that the Hardy–Littlewood maximal operator is defined as

$$(Mf)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes Q, with sides parallel to he axis, containing x, and $|Q|$ denotes the Lebesgue measure of Q. All cubes considered in this paper are supposed to have sides satisfying this property. An equivalent operator is obtained by using balls instead of cubes. It is known that M defines a bounded operator on L^p. Even more, it is bounded on L^p spaces with certain weights as stated in the next result, which is essential for proving the boundedness of the Hardy–Littlewood maximal operator on the amalgam spaces [7, Corollary 1.13, p. 393]:

Theorem 2.1. *Let $(u, v) \in A_p$. Then, for every q with $1 < p < q < \infty$, the maximal operator M is bounded from $L^q(v)$ to $L^q(u)$, that is, there exists a constant C such that for every $f \in L^1_{loc}(\mathbb{R}^n)$:*

$$\int_{\mathbb{R}^n} |Mf(x)|^q u(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^q v(x) \, dx. \tag{2.1}$$

We should remember that a pair of weights (u,v) belongs to the class A_p, for $1<p<\infty$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1}<\infty.$$

We will prove the following result:

Theorem 2.2. *The maximal operator $M:(L^p,\ell^q)(\mathbb{R}^n)\longrightarrow(L^p,\ell^q)(\mathbb{R}^n)$ is bounded for $1<p,q<\infty$.*

To prove it, we require first the following weaker property.

Lemma 2.3. *Let $f\in(L^p,\ell^q)(\mathbb{R}^n)$ be bounded with compact support. Then $Mf\in(L^p,\ell^q)(\mathbb{R}^n)$.*

Notice that unlike many other proofs for boundedness, where it is simultaneously proved that an operator is well defined and bounded, we do require the Hardy–Littlewood maximal operator to be well defined on certain functions in order to prove it is bounded on the amalgam spaces.

Proof. Let $K=\operatorname{supp}(f)$. If Q is any cube in $\mathbb{R}^n$, then

$$\frac{1}{|Q|}\int_Q|f|\le\frac{1}{|Q|}\int_Q\|f\|_\infty=\|f\|_\infty.$$

This shows Mf is bounded on $\mathbb{R}^n$, although we will require a better bound to show that $Mf\in(L^p,\ell^q)(\mathbb{R}^n)$.

Since K is a compact, there exists $R>0$ such that $B_R(0)$, the ball with center at 0 and radius R, contains K. In consequence, the restriction of f to any ball contained in the complement of $B_R(0)$ is identically zero. Let $x\in\mathbb{R}^n$ with the euclidean norm $\|x\|>R+1$, and let $B_r(x_0)$ be a ball containing x such that $B_r(x_0)\cap B_R(0)\neq\varnothing$. Then $r\ge(\|x\|-R)/2$. Since the volume of an n-dimensional sphere is Cr^n for some constant C that depends only on n, we obtain

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|f|\lesssim\frac{\|f\|_\infty}{r^n}|B_r(x_0)\cap B_R(0)|,$$

where the symbol $\lesssim$ means that the inequality holds up to some constant that depends only on n, p or q. One could obtain an explicit expression for the quantity $|B_r(x_0)\cap B_R(0)|$ as the sum of the volume of two $n-$dimensional spherical caps. However, we do not require such a sharp bound in order to prove this lemma. If we take $C(f)=\|f\|_\infty|B_R(0)|$, since $r\ge(\|x\|-R)/2$, we obtain

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|f|\lesssim\frac{C(f)}{r^n}\le\frac{2^n}{(\|x\|-R)^n}C(f)\lesssim\frac{1}{\|x\|^n}C(f),$$

where the last inequality holds since

$$\sup_{\|x\|>R+1}\frac{2\|x\|}{\|x\|-R}<\infty.$$

For any cube Q in $\mathbb{R}^n$, there exist two balls B_1^Q and B_2^Q such that $B_1^Q \subset Q \subset B_2^Q$ and $|B_2^Q|/|B_1^Q| \lesssim 1$. Then, if Q is a cube that contains x, with the radius of B_2^Q greater than $(\|x\| - R)/2$, we get

$$\frac{1}{|Q|}\int_Q |f| \le \frac{1}{|B_1^Q|}\int_{B_2^Q} |f| = \frac{|B_2^Q|}{|B_1^Q|}\frac{1}{|B_2^Q|}\int_{B_2^Q} |f| \lesssim \frac{C(f)}{||x||^n}.$$

Therefore,

$$(Mf)(x) \lesssim \begin{cases} \|f\|_\infty & \text{if } \|x\| \le R+1, \\ \dfrac{C(f)}{\|x\|^n} & \text{if } \|x\|\ R+1. \end{cases}$$

Now we focus on the series

$$\sum_{m\in\mathbb{Z}^m} \left(\int_{Q_m} |Mf(x)|^p\,dx\right)^{q/p}.$$

Since we are only interested in its convergence, by the bounds we found for $(Mf)(x)$ it is enough to prove that

$$\sum_{\substack{m\in\mathbb{Z}^n\\ \|m\|>R_1}} \left(\int_{Q_m} (1/\|x\|^n)^p d\,x\right)^{q/p} < \infty$$

for some R_1 large enough. We can also take R_1 such that $\|x\| > \|m\|/2$ whenever $x \in Q_m$. Then

$$\begin{aligned}\sum_{\substack{m\in\mathbb{Z}^n\\ \|m\|>R_1}} \left(\int_{Q_m} (1/\|x\|^n)^p\,dx\right)^{q/p} &\le \sum_{\substack{m\in\mathbb{Z}^n\\ \|m\|>R_1}} \left(\int_{Q_m} (2^n/\|m\|^n)^p\,dx\right)^{q/p} \\ &= 2^{nq} \sum_{\substack{m\in\mathbb{Z}^n\\ \|m\|>R_1}} \|m\|^{-nq}.\end{aligned}$$

Observe now that

$$\begin{aligned}\sum_{\substack{m\in\mathbb{Z}^n\\ m\neq 0}} \|m\|^{-nq} &= \sum_{\substack{m\in\mathbb{Z}^n\\ m\neq 0}} \int_{\widetilde{Q}_m} \|m\|^{-nq}\chi_{\widetilde{Q}_m}(x)dx \\ &\lesssim \sum_{\substack{m\in\mathbb{Z}^n\\ m\neq 0}} \int_{\widetilde{Q}_m} h(x)^{nq}\,dx = \int_{\mathbb{R}^n} h(x)^{nq}\,dx,\end{aligned}$$

where $\widetilde{Q}_m$ is the cube with side 1 centered in m, and the positive function h is defined as

$$h(x) = \begin{cases} 1, & \text{if } \|x\| \le 1, \\ \|x\|^{-1}, & \text{if } \|x\| > 1. \end{cases}$$

It remains to verify the integrability of h^{nq} over $\mathbb{R}^n$. By using hyperspherical coordinates, it is enough to verify that

$$\int_1^\infty r^{-nq+n-1}\, dr < \infty,$$

which is true for $q > 1$. $\square$

Before proceeding with the proof of Theorem 2.2, we are to introduce a discrete version of the Hardy–Littlewood maximal. For every $x \in \mathbb{Z}^n$ and for every function defined on $\mathbb{Z}^n$, the discrete maximal operator $\mathcal{M}^d$ is defined as

$$\mathcal{M}^d f(x) = \sup_{t>0} \frac{1}{\#(G_t(x) \cap \mathbb{Z}^n)} \sum_{y \in G_t(x) \cap \mathbb{Z}^n} f(y),$$

where $G_t(x) = \{y \in \mathbb{R}^n : \|x - y\|_\infty \le t\}$ and $\#(A)$ denotes the cardinality of the set A. This operator is known to be bounded on $\ell^r(\mathbb{Z}^n)$ for $1 < r < \infty$. Some recent researches regarding bounds for the norm of $\mathcal{M}^d$ can be found in [2] and [3]. Of particular interest to us is the "uncentered" version of the discrete Hardy–Littlewood maximal operator defined as

$$M^d f(x) = \sup_{S \ni x} \frac{1}{\#(S)} \sum_{y \in S} |f(y)|,$$

where the supremum is taken over all sets of the form $S = G_t(z)$ for some $z \in \mathbb{Z}^n$ and $t > 0$ that contain x. It is known that $\mathcal{M}^d$ and M^d are equivalent [8], that is,

$$M^d f(x) \lesssim \mathcal{M}^d f(x) \lesssim M^d f(x)$$

holds for all $f : \mathbb{Z}^n \to \mathbb{C}$ and every $x \in \mathbb{Z}^n$.

Proof of Theorem 2.2. We follow the argument of the proof contained in [4] adjusting it for cubes in $\mathbb{R}^n$ instead of intervals. But before that, let us establish some notation that will simplify the following calculations. For $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$:

- $x \le y$ means that $x_j \le y_j$ for $j = 1, 2, \dots, n$.
- For $x \le y$ and $a, b \ge 0$, we define the set $[[x, a; y, b]]$ as $[x_1 - a, y_1 + b] \times [x_2 - a, y_2 + b] \times \cdots \times [x_n - a, y_n + b]$, and $[[x, a; y, b]]_{\mathbb{Z}} = [[x, a; y, b]] \cap \mathbb{Z}^n$.

Notice that every cube in $\mathbb{R}^n$ can be written as $[[x, h; x, h]]$ for some $x \in \mathbb{R}^n$ and $h > 0$. Consider $f \in (L^p, \ell^q)(\mathbb{R}^n)$ bounded with compact support, so we have $Mf \in (L^p, \ell^q)(\mathbb{R}^n)$. This analysis is divided into two cases:

i. Suppose that $1 < p < q < \infty$. Take $r = (q/p)'$. For any $b = \{b_m\}_{m \in \mathbb{Z}^n} \in \ell^r(\mathbb{Z}^n)$, Hölder's inequality gives us

$$\sum_{m \in \mathbb{Z}^n} \left(\int_{Q_m} |Mf|^p \right) |b_m| \le \left\{ \sum_{m \in \mathbb{Z}^n} \left(\int_{Q_m} |Mf|^p \right)^{q/p} \right\}^{p/q} \left\{ \sum_{m \in \mathbb{Z}^n} |b_m|^r \right\}^{1/r}$$

$$= \|Mf\|_{p,q}^p \|b\|_{\ell^r}. \tag{2.2}$$

By taking $\lambda > 0$ and

$$b_m = \lambda \Big(\int_{Q_m} |Mf|^p \Big)^{(q/p)/r} = \lambda \Big(\int_{Q_m} |Mf|^p \Big)^{(q-p)/p},$$

the equality is attained in (2.2). For this particular choice, $b \in \ell^r$ if and only if $Mf \in (L^p, \ell^q)(\mathbb{R}^n)$. Thus, the assumptions of f being bounded with compact support are necessary in order to apply Lemma 2.3. Then we can take λ such that $\|b\|_{\ell^r} = 1$ to obtain

$$\|Mf\|_{p,q}^p = \sum_{m \in \mathbb{Z}^n} \Big(\int_{Q_m} |Mf|^p \Big) b_m = \int_{\mathbb{R}^n} |Mf(x)|^p u(x)\, dx, \tag{2.3}$$

where $u(x) = \sum_{m\in\mathbb{Z}^n} b_m \chi_{Q_m}(x)$.

Let us define, for $m \in \mathbb{Z}^n$,

$$\alpha_m := M^d(\{b_k\}_{k\in\mathbb{Z}^n})(m) \text{ and } \beta_m = \sum_{k\in[[m,2;m,2]]_{\mathbb{Z}}} b_k.$$

Take now $\Lambda_m = \max\{\alpha_m, \beta_m\}$ and consider the weight

$$v(x) = \sum_{m\in\mathbb{Z}^n} \Lambda_m \chi_{Q_m}(x).$$

We will see now that $(u, v) \in A_{p_0}$ for $1 < p_0 < p$. Let $Q = [[x_0, h; x_0, h]]$ be some cube, with $x_0 \in \mathbb{R}^n$ and $h > 0$. Then there exists a unique $m_0 \in \mathbb{Z}^n$ such that $x_0 \in Q_{m_0}$. We consider two cases:

A) Suppose $0 < h < 1$. Then $Q \subset \bigcup_{k\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} Q_k$ and in consequence,

$$\begin{aligned}\frac{1}{|Q|}\int_Q u &= \frac{1}{|Q|}\int_Q \sum_{k\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k \chi_{Q_k}(x)dx = \frac{1}{|Q|}\sum_{k\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k \int_Q \chi_{Q_k}(x)dx \\ &\le \frac{1}{|Q|}\sum_{k\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k \int_Q dx = \sum_{k\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k.\end{aligned}$$

On the other hand, for $x \in Q$,

$$v(x)^{1/(1-p_0)} = \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} \Lambda_m^{1/(1-p_0)} \chi_{Q_m}(x).$$

It is easy to see that for $m \in [[m_0, 1; m_0, 1]]_{\mathbb{Z}}$ we have

$$\Lambda_m \ge \beta_m \ge \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k.$$

Then

$$\left(\frac{1}{|Q|}\int_Q v^{1/(1-p_0)}\right)^{p_0-1} = \left(\frac{1}{|Q|}\int_Q \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} \Lambda_m^{1/(1-p_0)}\chi_{Q_m}(x)dx\right)^{p_0-1}$$
$$\le \left(\left\{\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k\right\}^{1/(1-p_0)} \frac{1}{|Q|}\int_Q \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} \chi_{Q_m}(x)dx\right)^{p_0-1}$$
$$= \left(\left\{\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k\right\}^{1/(1-p_0)} \frac{1}{|Q|}\int_Q dx\right)^{p_0-1} = \left\{\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} b_k\right\}^{-1}.$$

Therefore, for $0 < h < 1$, we obtain

$$\left(\frac{1}{|Q|}\int_Q u\right)\left(\frac{1}{|Q|}\int_Q v^{1/(1-p_0)}\right)^{p_0-1} \le 1. \tag{2.4}$$

B) Now we consider $1 \le h < \infty$ and denote by $\lfloor h\rfloor$ its integer part. In this case,

$$Q \subset Q^h := [[n_0, \lfloor h\rfloor + 2; n_0, \lfloor h\rfloor + 2]]$$

for some $n_0 \in \mathbb{Z}^n$, and thus we obtain

$$\frac{1}{|Q|}\int_Q u = \frac{1}{(2h)^n}\int_Q \sum_{m\in Q^h_{\mathbb{Z}}} b_m\chi_{Q_m}(x)\,dx$$
$$\le \frac{1}{(2h)^n}\frac{\#(Q^h_{\mathbb{Z}})}{\#(Q^h_{\mathbb{Z}})}\int_{Q^h} \sum_{m\in Q^h_{\mathbb{Z}}} b_m\chi_{Q_m}(x)\,dx \le \frac{(2\lfloor h\rfloor + 5)^n}{(2h)^n}\frac{1}{\#(Q^h_{\mathbb{Z}})}\sum_{m\in Q^h_{\mathbb{Z}}} b_m,$$

where $Q^h_{\mathbb{Z}} = Q^h \cap \mathbb{Z}^n$. Since $1 \le h$, it follows that

$$\frac{2\lfloor h\rfloor + 5}{2h} \le \frac{2h+5}{2h} = 1 + \frac{5}{2h} \le \frac{7}{2}.$$

On the other hand, by the definition of the discrete maximal operator M^d,

$$\frac{1}{\#(Q^h_{\mathbb{Z}})}\sum_{m\in Q^h_{\mathbb{Z}}} b_m \le M^d(\{b_j\}_j)(k)$$

for each $k \in Q^h$. Then, by taking

$$\gamma(m_0, \lfloor h\rfloor) := \min\{M^d(\{b_k\}_k)(m) : m \in Q^h_{\mathbb{Z}}\},$$

we obtain

$$\frac{1}{|Q|}\int_Q u \le \left(\frac{7}{2}\right)^n \gamma(n_0, \lfloor h\rfloor).$$

Also, for $x \in Q$, we have

$$v(x)^{1/(1-p_0)} = \sum_{m\in Q^h_{\mathbb{Z}}} \Lambda_m^{1/(1-p_0)}\chi_{Q_m}(x).$$

By the construction, for $m \in Q^h \cap \mathbb{Z}^n$, we get

$$\Lambda_m \geq \alpha_m = M^d(\{b_k\}_k)(m) \geq \gamma(n_0, \lfloor h \rfloor),$$

and in consequence,

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q v^{1/(1-p_0)}\right)^{p_0-1} &= \left(\frac{1}{|Q|}\int_Q \sum_{m\in Q^h_{\mathbb{Z}}} \Lambda_m^{1/(1-p_0)}\chi_{Q_m}(x)\,dx\right)^{p_0-1}\\
&\leq \left(\frac{1}{|Q|}\int_Q \sum_{m\in Q^h_{\mathbb{Z}}} \gamma(n_0,\lfloor h\rfloor)^{1/(1-p_0)}\chi_{Q_m}(x)\,dx\right)^{p_0-1}\\
&= \left(\gamma(n_0,\lfloor h\rfloor)^{1/(1-p_0)}\frac{1}{|Q|}\int_Q \sum_{m\in Q^h_{\mathbb{Z}}} \chi_{Q_m}(x)\,dx\right)^{p_0-1}\\
&= \left(\gamma(n_0,\lfloor h\rfloor)^{1/(1-p_0)}\frac{1}{|Q|}\int_Q dx\right)^{p_0-1} = \gamma(n_0,\lfloor h\rfloor)^{-1}.
\end{aligned}$$

Therefore,

$$\left(\frac{1}{|Q|}\int_Q u\right)\left(\frac{1}{|Q|}\int_Q v^{1/(1-p_0)}\right)^{p_0-1} \leq \left(\frac{7}{2}\right)^n. \tag{2.5}$$

By (2.4) and (2.5), we conclude that $(u, v) \in A_{p_0}$. Since $p_0 < p$, we can use Theorem 2.1 to guarantee the existence of a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x)\,dx \leq C\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx.$$

Notice that this constant C may depend on u and v since Theorem 2.1 deals with the boundedness of the maximal from $L^p(u)$ into $L^p(v)$. However, analysing the proofs of [7, Theorem 1.12, p. 393] and Theorem 2.1, one concludes that C can be taken as a constant of the same order of some upper bound for

$$\left(\frac{1}{|Q|}\int_Q u\right)\left(\frac{1}{|Q|}\int_Q v^{1/(1-p_0)}\right)^{p_0-1},$$

which we have bounded by $(7/2)^n$, a constant depending only on n.

On the other hand, $1 < p < q < \infty$. So, we can also use (2.3) to obtain

$$\|Mf\|^p_{p,q} = \int_{\mathbb{R}^n} |Mf(x)|^p u(x)\,dx \leq C\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx.$$

To finish the case $1 < p < q < \infty$, we only have to prove that

$$\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \lesssim \|f\|^p_{p,q}.$$

We have taken r such that $r' = q/p$. Then, by Hölder's inequality, we obtain

$$\int_{\mathbb{R}^n} |f|^p v = \int_{\mathbb{R}^n} |f(x)|^p \left(\sum_{m\in\mathbb{Z}^n} \Lambda_m\chi_{Q_m}(x)\right) dx$$

$$= \sum_{m\in\mathbb{Z}^n} \Lambda_m \int_{\mathbb{R}^n} |f(x)|^p \chi_{Q_m}(x)dx = \sum_{m\in\mathbb{Z}^n} \Lambda_m \int_{Q_m} |f(x)|^p dx$$
$$\leq \left(\sum_{m\in\mathbb{Z}^n} \Lambda_m^r\right)^{1/r} \left(\sum_{m\in\mathbb{Z}^n} \left\{\int_{Q_m} |f|^p\right\}^{q/p}\right)^{p/q} = \left(\sum_{m\in\mathbb{Z}^n} \Lambda_m^r\right)^{1/r} \|f\|_{p,q}^p.$$

Since $\Lambda_m = \max\{\alpha_m, \beta_m\} \leq \alpha_m + \beta_m$, Minkowski's inequality implies that

$$\left(\sum_{m\in\mathbb{Z}^n} \Lambda_m^r\right)^{1/r} \leq \left(\sum_{m\in\mathbb{Z}^n} \alpha_m^r\right)^{1/r} + \left(\sum_{m\in\mathbb{Z}^n} \beta_m^r\right)^{1/r}.$$

Recall that

$$\alpha_m := M^d(\{b_k\}_{k\in\mathbb{Z}^n})(m) \text{ and } \beta_m = \sum_{\substack{k\in\mathbb{Z}^n \\ k\in[[m,2;m,2]]_{\mathbb{Z}}}} b_k.$$

Since the discrete maximal is bounded on $\ell^r(\mathbb{Z}^n)$, by the construction of the sequence $\{b_k\}_k$, we conclude that

$$\left(\sum_{m\in\mathbb{Z}^n} \alpha_m^r\right)^{1/r} \leq \|M^d\|\|\{b_k\}_k\|_{\ell^r} = \|M^d\|.$$

Regarding the remaining term, consider the shift operators

$$\begin{aligned} \varphi_k : \quad & \ell^r \longrightarrow \ell^r \\ & \{a_j\}_j \longmapsto \{a_{j+k}\}_j \end{aligned}$$

for $k \in \mathbb{Z}^n$. It is clear that these operators are isometries on ℓ^r. Hence, they are all bounded, their norms are equal to 1, and

$$\{\beta_m\}_m = \sum_{k\in[[0,2;0,2]]_{\mathbb{Z}}} \varphi_k(\{b_j\}_j).$$

Then

$$\|\{\beta_m\}_m\|_{\ell^r} = \left\|\left(\sum_{k\in[[0,2;0,2]]_{\mathbb{Z}}} \varphi_k\right)(\{b_j\}_j)\right\|_{\ell^r} \leq \left\|\sum_{k\in[[0,2;0,2]]_{\mathbb{Z}}} \varphi_k\right\| \|\{b_j\}_j\|_{\ell^r}$$
$$\leq \sum_{k\in[[0,2;0,2]]_{\mathbb{Z}}} \|\varphi_k\| = \#([[0,2;0,2]]_{\mathbb{Z}}) = 5^n,$$

which ends this half of the proof.

ii. Suppose now that $1 < q < p < \infty$. If β is such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{p\beta}$, then $\beta = \frac{q}{p-q} > 1$. Given $\{u_k\}_{k\in\mathbb{Z}^n} \in \ell^\beta(\mathbb{Z}^n)$, with $u_k > 0$. By Hölder's inequality, for any $h \in (L^p, \ell^q)(\mathbb{R}^n)$, we obtain

$$\|h\|_{p,q} = \left\|\left\{\left(\int_{Q_m} |h|^p\right)^{1/p}\right\}_{m\in\mathbb{Z}^n}\right\|_{\ell^q}$$

$$= \left\| \left\{ \left(\int_{Q_m} |h|^p u_m^{-1} \right)^{1/p} u_m^{1/p} \right\}_{m\in\mathbb{Z}^n} \right\|_{\ell^q}$$

$$\leq \left\| \left\{ \left(\int_{Q_m} |h|^p u_m^{-1} \right)^{1/p} \right\}_{m\in\mathbb{Z}^n} \right\|_{\ell^p} \|\{u_m^{1/p}\}_{m\in\mathbb{Z}^n}\|_{\ell^{p\beta}}$$

$$= \left\{ \sum_{m\in\mathbb{Z}^n} u_m^{-1} \int_{Q_m} |h|^p \right\}^{1/p} \left\{ \sum_{m\in\mathbb{Z}^n} |u_m^{1/p}|^{p\beta} \right\}^{1/p\beta}$$

$$= \left\{ \int_{\mathbb{R}^n} |h(x)|^p \left(\sum_{m\in\mathbb{Z}^n} u_m^{-1} \chi_{Q_m}(x) \right) dx \right\}^{1/p} \left\{ \left(\sum_{m\in\mathbb{Z}^n} |u_m|^\beta \right)^{1/\beta} \right\}^{1/p}. \tag{2.6}$$

Notice that $\{u_k\}_{k\in\mathbb{Z}^n} \in \ell^\beta(\mathbb{Z}^n)$ if and only if $\{u_k^{1/p}\}_{k\in\mathbb{Z}^n} \in \ell^{p\beta}$. Then we can take, just like in the previous case, u with $\|u\|_{\ell^\beta} = 1$ such that equality is attained in Hölder's inequality. For this particular sequence u, inequality (2.6) gives us the identity

$$\|f\|_{p,q} = \left\{ \int_{\mathbb{R}^n} |f(x)|^p \left(\sum_{m\in\mathbb{Z}^n} u_m^{-1} \chi_{Q_m}(x) \right) dx \right\}^{1/p}. \tag{2.7}$$

For a sequence $\{v_k\}_{k\in\mathbb{Z}^n}$, with $v_k > 0$, we get the inequality

$$\|Mf\|_{p,q} \leq \left\{ \sum_{m\in\mathbb{Z}^n} v_m^{-1} \int_{Q_m} |Mf|^p \right\}^{1/p} \left\{ \sum_{m\in\mathbb{Z}^n} |v_m|^\beta \right\}^{1/p\beta}. \tag{2.8}$$

Now it should be noticed that since $1 < q < p < \infty$, then there exists $p_0 < p$ such that $p_0 - 1 > p - q$. Then $(p_0 - 1)\beta = q(p_0 - 1)/(p - q) > q > 1$. By the continuity of the discrete maximal operator on $\ell^{\beta(p_0-1)}(\mathbb{Z}^n)$, we can take $A = \|M^d\|^{p_0-1}_{\ell^{\beta(p_0-1)}\to\ell^{\beta(p_0-1)}}$. Then we obtain

$$\left\| \left\{ \left[M^d(\{u_k^{1/(p_0-1)}\}_{k\in\mathbb{Z}})(m) \right]^{p_0-1} \right\}_{m\in\mathbb{Z}^n} \right\|_{\ell^\beta(\mathbb{Z}^n)} =$$

$$= \left\{ \sum_{m\in\mathbb{Z}^n} \left| M^d(\{u_k^{1/(p_0-1)}\}_{k\in\mathbb{Z}^n})(m) \right|^{(p_0-1)\beta} \right\}^{1/\beta}$$

$$= \left\| \left\{ M^d(\{u_k^{1/(p_0-1)}\}_{k\in\mathbb{Z}^n})(m) \right\}_{m\in\mathbb{Z}^n} \right\|^{p_0-1}_{\ell^{\beta(p_0-1)}} \leq A \left\| \left\{ u_k^{1/(p_0-1)} \right\}_{k\in\mathbb{Z}^n} \right\|^{p_0-1}_{\ell^{\beta(p_0-1)}}$$

$$= A \left\{ \sum_{m\in\mathbb{Z}^n} |u_k^{1/(p_0-1)}|^{\beta(p_0-1)} \right\}^{(p_0-1)/(\beta(p_0-1))} = A \left\{ \sum_{m\in\mathbb{Z}^n} |u_k|^\beta \right\}^{1/\beta} = A.$$

By choosing $v_m = \left[M^d(\{u_k^{1/(p_0-1)}\}_{k\in\mathbb{Z}^n})(m) \right]^{p_0-1}$, we obtain

$$\left(\sum_{m\in\mathbb{Z}^n} v_n^\beta \right)^{1/\beta} \leq \|M^d\|^{p_0-1}_{\ell^{\beta(p_0-1)}\to\ell^{\beta(p_0-1)}}. \tag{2.9}$$

Now we will consider the weights

$$v(x) = \sum_{m\in\mathbb{Z}^n} v_m^{-1}\chi_{Q_m}(x) \quad \text{and} \quad u(x) = \sum_{m\in\mathbb{Z}^n} u_m^{-1}\chi_{Q_m}(x).$$

Let us prove that the pair (v, u) belongs to the class A_{p_0}. Let Q be a cube, with $x_0 \in \mathbb{R}^n$ and $h > 0$ such that $Q = [[x_0, h; x_0, h]]$. Then there exists a unique $m_0 \in \mathbb{Z}^n$ that satisfies $x_0 \in Q_{m_0}$. Further we will separate different possibilities for h.

A) Suppose $h > 1$. Then

$$Q \subset \bigcup_{k\in Q^h_{\mathbb{Z}}} Q_k,$$

where $Q^h = [[m_0, \lfloor h\rfloor + 1; m_0, \lfloor h\rfloor + 1]]$. Then we can write the weights u and v as

$$v(x) = \sum_{m\in Q^h_{\mathbb{Z}}} v_m^{-1}\chi_{Q_m}(x)$$

and

$$u(x)^{1/(1-p_0)} = \sum_{m\in Q^h_{\mathbb{Z}}} u_m^{1/(p_0-1)}\chi_{Q_m}(x)$$

whenever $x \in Q$. It follows that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q v &= \frac{1}{|Q|}\int_Q \sum_{m\in Q^h_{\mathbb{Z}}} v_m^{-1}\chi_{Q_m}(x)dx = \frac{1}{|Q|}\sum_{m\in Q^h_{\mathbb{Z}}}\int_Q v_m^{-1}\chi_{Q_m}(x)dx \\
&\le \frac{1}{|Q|}\sum_{m\in Q^h_{\mathbb{Z}}}\int_{Q_m} v_m^{-1}\chi_{Q_m}(x)dx = \frac{1}{(2h)^n}\sum_{m\in Q^h_{\mathbb{Z}}} v_m^{-1} \\
&= \frac{1}{(2h)^n}\sum_{m\in Q^h_{\mathbb{Z}}} [M^d(\{u_k^{1/(p_0-1)}\}_k)(m)]^{1-p_0} \\
&\le \frac{1}{(2h)^n}\#(Q^h\cap\mathbb{Z}^n)\max_{m\in Q^h_{\mathbb{Z}}}[M^d(\{u_k^{1/(p_0-1)}\}_k)(m)]^{1-p_0}.
\end{aligned}$$

Just like before, since $h > 1$, we have

$$\frac{\#(Q^h\cap\mathbb{Z}^n)}{(2h)^n} \le \frac{(2h+5)^n}{(2h)^n} \le \left(\frac{7}{2}\right)^n.$$

Notice that $1 - p_0 < 0$, so we also have

$$\max_{m\in Q^h_{\mathbb{Z}}}[M^d(\{u_j^{1/(p_0-1)}\}_j)(m)]^{1-p_0} = \left(\min_{m\in Q^h_{\mathbb{Z}}} M^d(\{u_k^{1/(p_0-1)}\}_k)(m)\right)^{1-p_0}.$$

On the other hand,

$$\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} = \left(\frac{1}{(2h)^n}\int_Q \sum_{m\in Q^h_{\mathbb{Z}}} u_m^{1/(p_0-1)}\chi_{Q_m}(x)\right)^{p_0-1}$$

$$
\begin{aligned}
&\le \left(\frac{1}{(2h)^n} \sum_{m\in Q^h_{\mathbb{Z}}} \int_{Q_m} u_m^{1/(p_0-1)} \chi_{Q_m}(x) \right)^{p_0-1} \\
&\le \left(\frac{1}{(2h)^n} \frac{\#(Q^h\cap\mathbb{Z}^n)}{\#(Q^h\cap\mathbb{Z}^n)} \sum_{m\in Q^h_{\mathbb{Z}}} u_m^{1/(p_0-1)} \right)^{p_0-1} \\
&\le \left(\left\{\frac{7}{2}\right\}^n \frac{1}{\#(Q^h\cap\mathbb{Z}^n)} \sum_{m\in Q^h_{\mathbb{Z}}} u_m^{1/(p_0-1)} \right)^{p_0-1}.
\end{aligned}
$$

Since $p_0-1>0$, this implies that

$$
\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} \le \left(\frac{7}{2}\right)^{n(p_0-1)} \left(M^d(\{u_m^{1/(p_0-1)}\}_m)(k)\right)^{p_0-1}
$$

for every $k\in Q^h\cap\mathbb{Z}^n$. In particular,

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} &\le \left(\frac{7}{2}\right)^{n(p_0-1)} \min_{m\in Q^h_{\mathbb{Z}}} \left(M^d(\{u_k^{1/(p_0-1)}\}_k)(m)\right)^{p_0-1} \\
&= \left(\frac{7}{2}\right)^{n(p_0-1)} \left(\min_{m\in Q^h_{\mathbb{Z}}} M^d(\{u_k^{1/(p_0-1)}\}_k)(m)\right)^{p_0-1},
\end{aligned}
$$

where the last equality holds since $p_0-1>0$. Therefore, we obtain

$$
\left(\frac{1}{|Q|}\int_Q v\right)\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} \le \left(\frac{7}{2}\right)^{np_0}
$$

for $h>1$.

B) Suppose that $0<h<1$. Then $Q\subset[[m_0,1;m_0,1]]$, so

$$
v(x) = \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_m^{-1}\chi_{Q_m}(x)
$$

and

$$
u(x)^{1/(1-p_0)} = \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} u_m^{1/(p_0-1)}\chi_{Q_m}(x)
$$

whenever $x\in Q$. Then

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q v &= \frac{1}{|Q|}\int_Q \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_m^{-1}\chi_{Q_m}(x)\,dx \\
&= \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_m^{-1}\frac{1}{|Q|}\int_Q \chi_{Q_m}(x)\,dx \\
&\le \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_m^{-1}\frac{1}{|Q|}\int_Q 1\,dx = \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_m^{-1}.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} &= \left(\frac{1}{|Q|}\int_Q \sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} u_m^{1/(p_0-1)}\chi_{Q_m}(x)\,dx\right)^{p_0-1}\\
&= \left(\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} u_m^{1/(p_0-1)}\frac{1}{|Q|}\int_Q \chi_{Q_m}(x)\,dx\right)^{p_0-1}\\
&\le \left(\frac{\#([[m_0,1;m_0,1]]_{\mathbb{Z}})}{\#([[m_0,1;m_0,1]]_{\mathbb{Z}})}\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} u_m^{1/(p_0-1)}\frac{1}{|Q|}\int_Q 1\,dx\right)^{p_0-1}\\
&= \left(3^n\frac{1}{\#([[m_0,1;m_0,1]]_{\mathbb{Z}})}\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} u_m^{1/(p_0-1)}\right)^{p_0-1}.\end{aligned}$$

Since $p_0-1>0$, we obtain

$$\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} \le 3^{n(p_0-1)}\left[M^d(\{u_k^{1/(p_0-1)}\}_k)(m)\right]^{p_0-1} = 3^{n(p_0-1)}v_m$$

for every $m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}$. In consequence,

$$\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1} \le 3^{n(p_0-1)}\min_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_k.$$

This implies that

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q v\right)\left(\frac{1}{|Q|}\int_Q u^{1/(1-p_0)}\right)^{p_0-1}\\
&\qquad\le 3^{n(p_0-1)}\left(\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_m^{-1}\right)\left(\min_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_k\right)\\
&\qquad= 3^{n(p_0-1)}\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}}\frac{1}{v_m}\min_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} v_k\\
&\qquad\le 3^{n(p_0-1)}\sum_{m\in[[m_0,1;m_0,1]]_{\mathbb{Z}}} 1 = 3^{n(p_0-1)}3^n = 3^{np_0}.\end{aligned}$$

This proves that the pair (v,u) belongs to the class A_{p_0}. Then there exists a constant $C>0$ such that

$$\int_{\mathbb{R}^n}|Mf(x)|^p v(x)\,dx \le C\int_{\mathbb{R}^n}|f(x)|^p u(x)\,dx.$$

By the same argument as in the case $p<q$, the constant C will depend only on n and p_0. By taking into account (2.7), (2.8) and (2.9), we obtain

$$\|Mf\|_{p,q}^p \le \|M^d\|_{\ell^{\beta(p_0-1)}\to\ell^{\beta(p_0-1)}}^{p(p_0-1)}\int_{\mathbb{R}^n}|Mf(x)|^p v(x)\,dx$$

$$\leq C\|M^d\|^{p(p_0-1)}_{\ell^{\beta(p_0-1)}\to\ell^{\beta(p_0-1)}} \int_{\mathbb{R}^n} |f(x)|^p u(x)\,dx$$
$$= C\|M^d\|^{p(p_0-1)}_{\ell^{\beta(p_0-1)}\to\ell^{\beta(p_0-1)}} \|f\|^p_{p,q}.$$

So far we have proved, for $1 < p, q < \infty$, the existence of some positive constant K such that if f is a bounded function with compact support on $\mathbb{R}^n$, then

$$\|Mf\|_{p,q} \leq K\|f\|_{p,q}.$$

To finish the demonstration, we let $f \in (L^p, \ell^q)(\mathbb{R}^n)$ and take a sequence $\{f_k\}_{k=1}^\infty$ of bounded functions with compact support that converges to f in the norm of $(L^p, \ell^q)(\mathbb{R}^n)$. □

3. On the sharp and the commutator

After verifying the boundedness of the maximal operator on the amalgam spaces $(L^p, \ell^q)(\mathbb{R}^n)$, we focus our attention on the sharp function. Recall that it is defined as

$$f^\sharp(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy.$$

It is known that the operator $f \mapsto f^\sharp$ is bounded on L^p. We adapt the argument used in [11, Theorem 2, p.148] in order to obtain that boundedness on the amalgam spaces.

Theorem 3.1. *Suppose $1 < p, q < \infty$. Then there exist positive constants C_1 and C_2 such that for every $f \in (L^p, \ell^q)(\mathbb{R}^n)$ the following inequalities hold:*

$$C_1\|f^\sharp\|_{p,q} \leq \|f\|_{p,q} \leq C_2\|f^\sharp\|_{p,q}.$$

Proof. Let $f \in (L^p, \ell^q)(\mathbb{R}^n)$. It is easy to verify the pointwise inequality

$$f^\sharp(x) \leq 2Mf(x).$$

Thus, $f^\sharp \in (L^p, \ell^q)(\mathbb{R}^n)$ and

$$\|f^\sharp\|_{p,q} \leq 2\|Mf\|_{p,q} \leq C\|f\|_{p,q}.$$

Let $g \in (L^{p'}, \ell^{q'})(\mathbb{R}^n)$. Consider a pair of sequences $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ satisfying the following properties:

(i) $f_k \to f$ a.e.;
(ii) f_k is bounded;
(iii) $|g_k| \to |g|$ a.e.;
(iv) g_k has compact support;
(v) The g_k are dominated by some function in $(L^{p'}, \ell^{q'})(\mathbb{R}^n)$;
(vi) $\int_{\mathbb{R}^n} g_k = 0$.

Properties (i)–(v) can be satisfied if we obtain the sequences $\{f_k\}_k$ and $\{g_k\}_k$ by applying the construction used at the end of the previous demonstration. To satisfy (vi), consider the function

$$\varphi_k(r) = \left| \int_{|x|\leq r} g_k(y)\, dy \right| - \left| \int_{|x|\geq r} g_k(y)\, dy \right|.$$

This function is continuous on $[0, \infty)$ and clearly, $\varphi_k(0) \leq 0$, while $\varphi_k(t) \geq 0$ for t large enough. Then there exists $R > 0$ such that $\varphi_k(R) = 0$, i.e.,

$$\left| \int_{|x|\leq R} g_k(x)\, dx \right| = \left| \int_{|x|\geq R} g_k(x)\, dx \right|.$$

Now we can find a constant λ, with $|\lambda| = 1$, such that the function $\widetilde{g}_k$, defined as

$$\widetilde{g}_k(x) = \begin{cases} \lambda g_k(x) & \text{if } |x| \leq R \\ g_k(x) & \text{if } |x| > R. \end{cases}$$

satisfies the required property $\displaystyle\int_{\mathbb{R}^n} \widetilde{g}_k(x)dx = 0$ and the previous ones.

The conditions imposed in [11, Theorem 2, p.148] are used to prove that g_k belongs to the Hardy space H^1 in order to apply the following result:

If u is bounded and $v \in H^1$, then

$$\left| \int_{\mathbb{R}^n} u(x)v(x)\, dx \right| \lesssim \int_{\mathbb{R}^n} u^\sharp(x)\mathcal{M}v(x)\, dx. \tag{3.1}$$

The $\mathcal{M}$ in the previous inequality is a maximal operator whose definition can be found in [11, §1.2, p. 90]. It is shown in that same section that the following pointwise inequality holds:

$$\mathcal{M}f(x) \lesssim Mf(x). \tag{3.2}$$

By applying (3.1) to f_k and g_k and using (3.2) together with Hölder's inequality for the amalgam spaces, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f_k(x)g_k(x)|dx &\leq \int_{\mathbb{R}^n} f_k^\sharp(x)\mathcal{M}g_k(x)dx \lesssim \int_{\mathbb{R}^n} f_k^\sharp(x)Mg_k(x)dx \\ &\leq \|f_k^\sharp\|_{p,q}\|Mg_k\|_{p',q'} \lesssim \|f_k^\sharp\|_{p,q}\|g_k\|_{p',q'}. \end{aligned}$$

By letting $k \to \infty$, it follows that

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \int_{\mathbb{R}^n} |f(x)g(x)|dx \lesssim \|f^\sharp\|_{p,q}\|g\|_{p',q'}.$$

The previous inequality holds for any $f \in (L^p, \ell^q)(\mathbb{R}^n)$ and $g \in (L^{p'}, \ell^{q'})(\mathbb{R}^n)$. If we take the supremum on the left side over all the g's with $\|g\|_{p',q'} \leq 1$, we get

$$\|f\|_{p,q} \leq C_2\|f^\sharp\|_{p,q}. \qquad \square$$

Our last result is obtained as a Corollary of the previous one. For any $b \in BMO(\mathbb{R}^n)$, and T a Calderón-Zygmund operator on $\mathbb{R}^n$, let $[b, T]$ denote the commutator of T and the multiplication operator $\phi \mapsto b \cdot \phi$. The commutator acts on $C_0^\infty(\mathbb{R}^n)$ by the rule

$$[b, T]\phi = b \cdot (T\phi) - T(b \cdot \phi).$$

The boundedness of $[b, T]$ on L^p was first proved by Coifman, Rochberg and Weiss in 1976 [5]. Other sources, where one can find this result include [1] and [10]. The standard proof employs the pointwise inequality

$$([b, T]\phi)^\sharp(z) \leq C_r \|b\|_{BMO}\big[(M(\phi^r)(z))^{1/r} + (M((T\phi)^r)(z))^{1/r}\big], \tag{3.3}$$

which holds whenever $\phi \in L^p$, $1 < p < \infty$ and $1 < r < p$. In particular, it will hold for every ϕ bounded with compact support. If we approximate any $f \in (L^p, \ell^q)(\mathbb{R}^n)$ using bounded functions with compact support, by applying (3.3) and the boundedness of both the Hardy–Littlewood maximal operator and the operator $\phi \mapsto \phi^\sharp$ on the amalgam spaces, we obtain:

Theorem 3.2. *Let T be a Calderón–Zygmund operator on $\mathbb{R}^n$ and $b \in BMO(\mathbb{R}^n)$. Then the commutator operator $[b, T]$ is bounded on $(L^p, \ell^q)(\mathbb{R}^n)$ for $1 < p, q < \infty$.*

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Максимальний оператор на амальгамному просторі

Antonio L. Baisón, Jorge Bueno-Contreras, and Victor A. Cruz

Ми доводимо обмеженість максимального оператора Харді–Літтлвуда на амальгамних просторах $(L^p, \ell^q)(\mathbb{R}^n)$. Як наслідок, одержуємо обмеженість комутаторів на цих просторах.

Ключові слова: амальгамні простори, максимальний оператор, комутатор