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On Some Weighted Classes of m-Subharmonic Functions

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In this paper, we study the class $\mathcal{E}_m(\Omega)$ of m-subharmonic functions introduced by Lu in [18]. We prove that the convergence of the Hessian measures is deduced from the convergence in m-capacity for the functions that belong to $\mathcal{E}_m(\Omega)$ satisfying certain additional properties. Then we extend those results to the class $\mathcal{E}_{m,\chi}(\Omega)$ that depends on a given increasing real function χ. A complete characterization of those classes using the Hessian measure is given as well as a subextension theorem relative to $\mathcal{E}_{m,\chi}(\Omega)$.

Key words: m-subharmonic function, capacity, Hessian operator, convergence in m-capacity

Mathematical Subject Classification 2020: 32W20, 32U05, 32U15, 32U40

1. Introduction

In complex analysis, the Monge–Ampère operator represents the objective of several studies since Bedford and Taylor [1, 2] demonstrated that the operator $(dd^c\cdot)^n$ is well defined in the set of locally bounded plurisubharmonic (psh) functions defined on a hyperconvex domain Ω of $\mathbb{C}^n$. This domain was extended by Cegrell [7, 8] by introducing and investigating the classes $\mathcal{E}_0(\Omega)$, $\mathcal{F}(\Omega)$ and $\mathcal{E}(\Omega)$ that contain unbounded psh functions. He proved that $\mathcal{E}(\Omega)$ is the largest domain of definition of the complex Monge–Ampère operator if we want the operator to be continuous for decreasing sequences. These works were taken up by Lu [18, 19] to define the complex Hessian operator H_m on the set of m-subharmonic functions which coincides with the set of psh functions in the case $m = n$. By giving an analogy to Cegrell's classes, Lu [18] studied some analogous classes denoted by $\mathcal{E}^0_m(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$. One of the most well-known problems in this direction is the link between the convergence in capacity Cap_m and the convergence of the complex Hessian operator. This problem was studied firstly by Xing [22] in the case of plurisubharmonic function and then it was generalized in different directions in several works [11, 13, 23] which aim essentially to give a connection between the convergence in capacity and the convergence of the associated Hessian measure. In this work, we continue the study of the stated problem. The paper is organized as follows. In Section 2, we recall some preliminaries on the pluripotential theory for an m-subharmonic function as well as the different energy classes which will be studied throughout the paper. In section 3, we give a connection between the convergence in capacity Cap_m of a sequence of m-subharmonic functions f_j towards f, $\liminf_{j\to+\infty} H_m(f_j)$ and $H_m(f)$ when the function $f \in \mathcal{E}_m(\Omega)$. We generalize the result established by Hiep [13] in the case of plurisubharmonic functions to the general case of m-subharmonic functions. More precisely, we prove the following theorem.

Theorem A. *If $(f_j)_j$ is a sequence of m-subharmonic functions that belong to $\mathcal{E}_m(\Omega)$ and satisfies $f_j \to f \in \mathcal{E}_m(\Omega)$ in Cap_m-capacity, then*

$$1_{\{f>-\infty\}} H_m(f) \le \liminf_{j\to+\infty} H_m(f_j).$$

As a consequence on Theorem A, we obtain several results of the convergence and especially we prove that if we modify the sufficient condition in the previous theorem, we may obtain the weak convergence of $H_m(f_j)$ to $H_m(f)$.

In Section 4, we study the classes $\mathcal{E}_{m,\chi}(\Omega)$ introduced by Hung [15] for a given increasing function χ. These classes generalize the weighted pluricomplex energy classes investigated by Benelkourchi, Guedj and Zeriahi [5] and studied in [3,4,11]. First, we prove that the class $\mathcal{E}_{m,\chi}(\Omega)$ is fully included in the Cegrell class $\mathcal{E}_m(\Omega)$ and hence the Hessian operator $H_m(f)$ is well defined for every $f \in \mathcal{E}_{m,\chi}(\Omega)$. Then we give several results on the class $\mathcal{E}_{m,\chi}(\Omega)$ depending on some condition on the function χ. These results generalize the well-known works in [4] and [5] it suffices to take $m = n$ to recover them. The most important result that we prove in this context is a complete characterization of functions that belong to $\mathcal{E}_{m,\chi}(\Omega)$ using the class $\mathcal{N}_m(\Omega)$. In other words, we show that

$$\mathcal{E}_{m,\chi}(\Omega) = \left\{ f \in \mathcal{N}_m(\Omega) \,/\, \chi(f) \in L^1(H_m(f)) \right\}.$$

In the end, we extend Theorem A to the class $\mathcal{E}_{m,\chi}(\Omega)$ by proving the following result.

Theorem B. *Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a continuous increasing function such that $\chi(-\infty) > -\infty$ and $f, f_j \in \mathcal{E}_m(\Omega)$ for all $j \in \mathbb{N}$. Suppose that there is a function $g \in \mathcal{E}_m(\Omega)$ satisfying $f_j \ge g$. Then:*

1. *If f_j converges to f in Cap_{m-1}-capacity, then $\liminf_{j\to+\infty} -\chi(f_j)H_m(f_j) \ge -\chi(f)H_m(f)$.*
2. *If f_j converges to f in Cap_m-capacity, then $-\chi(f_j)H_m(f_j)$ converges weakly to $-\chi(f)H_m(f)$.*

2. Preliminaries

2.1. m-subharmonic functions. This section is aimed to recall some basic properties of m-subharmonic functions introduced by Blocki [6]. These functions are admissible for the complex Hessian equation. Throughout the paper, we denote by $d := \partial + \overline{\partial}$, $d^c := i(\overline{\partial} - \partial)$ and by $\Lambda_p(\Omega)$, the set of $(p,p)-$forms in Ω. The standard Kähler form defined on $\mathbb{C}^n$ is denoted by $\beta := dd^c|z|^2$.

Definition 2.1 ([6])**.** Let $\zeta \in \Lambda_1(\Omega)$ and $m \in \mathbb{N} \cap [1, n]$. The form ζ is called m-positive if it satisfies

$$\zeta^j \wedge \beta^{n-j} \geq 0, \quad j = 1, \ldots, m,$$

at every point of Ω.

Definition 2.2 ([6])**.** Let $\zeta \in \Lambda_p(\Omega)$ and $m \in \mathbb{N} \cap [p, n]$. The form ζ is said to be m-positive on Ω if the measure

$$\zeta \wedge \beta^{n-m} \wedge \psi_1 \wedge \cdots \wedge \psi_{m-p}$$

is positive at every point of Ω, where $\psi_1, \ldots, \psi_{m-p} \in \Lambda_1(\Omega)$.

We denote by $\Lambda_p^m(\Omega)$ the set of all (p,p)-forms on Ω that are m-positive. In 2005, Blocki [6] introduced the notion of m-subharmonic functions and developed an analogous pluripotential theory. This notion is given in the following definition.

Definition 2.3. Let $f : \Omega \to \mathbb{R} \cup \{-\infty\}$. The function f is called m-subharmonic (m-sh for short) if it satisfies the following conditions:

1. The function f is subharmonic.
2. For all $\zeta_1, \cdots, \zeta_{m-1} \in \Lambda_1^m(\Omega)$, one has

$$dd^c f \wedge \beta^{n-m} \wedge \zeta_1 \wedge \cdots \wedge \zeta_{m-1} \geq 0.$$

We denote by $\mathcal{SH}_m(\Omega)$ the cone of m-subharmonic functions defined on Ω.

Remark 2.4. In the case $m = n$, we have the following:

1. The definition of m-positivity coincides with the classic definition of positivity given by Lelong for forms.
2. The set $\mathcal{SH}_n(\Omega)$ coincides with the set of psh functions on Ω.

One can refer to [6, 16, 18, 21] for more details about the properties of m-subharmonicity.

Example 2.5.

1. If $\zeta := i(4.dz_1 \wedge d\bar{z}_1 + 4.dz_2 \wedge d\bar{z}_2 - dz_3 \wedge d\bar{z}_3)$, then $\zeta \in \Lambda_1^2(\mathbb{C}^3) \setminus \Lambda_1^3(\mathbb{C}^3)$.
2. If $f(z) := -|z_1|^2 + 2|z_2|^2 + 2|z_3|$, then $f \in \mathcal{SH}_2(\mathbb{C}^3) \setminus \mathcal{SH}_3(\mathbb{C}^3)$. It is easy to see that $f \in \mathcal{SH}_2$. However, the restriction of f on the line $(z_1, 0, 0)$ is not subharmonic, and thus f is not plurisubharmonic.

Following Bedford and Taylor [2], by induction, one can define a closed nonnegative current when the m-sh function f is locally bounded,

$$dd^c f_1 \wedge \ldots \wedge dd^c f_k \wedge \beta^{n-m} := dd^c(f_1 dd^c f_2 \wedge \ldots \wedge dd^c f_k \wedge \beta^{n-m}),$$

where $f_1, \ldots, f_k \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$. In particular, for a given m-sh function $f \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$, we define the nonnegative Hessian measure of f as follows:

$$H_m(f) = (dd^c f)^m \wedge \beta^{n-m}.$$

2.2. Cegrell's classes of m-sh functions and m-capacity.

Definition 2.6.

1. A bounded domain Ω in $\mathbb{C}^n$ is said to be m-hyperconvex if the following property holds for some continuous m-sh function $\rho : \Omega \to \mathbb{R}^-$:

$$\{\rho < c\} \Subset \Omega$$

for every $c < 0$.

2. A set $M \subset \Omega$ is called m-polar if there exist $u \in \mathcal{SH}_m(\Omega)$ such that

$$M \subset \{u = -\infty\}.$$

3. A positive measure μ defined on Ω is said to be absolutely continuous with respect to the capacity Cap_m ($\mu << \mathrm{Cap}_m$ for short) on a Borel subset E in Ω if

$$\forall t > 0 \ \exists s > 0 \ \forall E_1 \subset E \quad \mathrm{Cap}_m(E_1) < s \Rightarrow \mu(E_1) < t.$$

Throughout the rest of the paper, we denote by Ω an m-hyperconvex domain of $\mathbb{C}^n$. In [18,19], Lu introduced the following classes of m-sh functions to generalize Cegrell's classes. Below, we recall the definitions of these classes.

Definition 2.7. We denote:

$$\mathcal{E}_m^0(\Omega) = \{f \in \mathcal{SH}_m^-(\Omega) \cap L^\infty(\Omega) : \forall \xi \in \partial\Omega \ \lim_{z \to \xi} f(z) = 0 \text{ and } \int_\Omega H_m(f) < +\infty\},$$

$$\mathcal{F}_m(\Omega) = \{f \in \mathcal{SH}_m^-(\Omega) : \exists (f_j) \subset \mathcal{E}_m^0 \ \ f_j \searrow f \text{ in } \Omega \text{ and } \sup_j \int_\Omega H_m(f_j) < +\infty\}.$$

$$\mathcal{E}_m(\Omega) = \{f \in \mathcal{SH}_m^-(\Omega) : \forall U \Subset \Omega \ \exists \ f_U \in \mathcal{F}_m(\Omega) \ \ f_U = f \text{ on } U\}.$$

Definition 2.8. A function $f \in \mathcal{SH}_m(\Omega)$ is said to be m-maximal if for every $g \in \mathcal{SH}_m(\Omega)$ such that $g \le f$ outside a compact subset of Ω, then $g \le f$ in Ω.

The previous notion represents an essential tool in the study of the Hessian operator since Blocki [6] showed that every m-maximal function $f \in \mathcal{E}_m(\Omega)$ satisfies $H_m(f) = 0$. Take $(\Omega_j)_j$, a sequence of strictly m-pseudoconvex subsets of Ω such that $\Omega_j \Subset \Omega_{j+1}$, $\bigcup_{j=1}^\infty \Omega_j = \Omega$, and for every j there exists a smooth strictly m-subharmonic function φ in a neighborhood V of Ω_j such that $\Omega_j := \{z \in V / \varphi(z) < 0\}$.

Definition 2.9. Let $f \in \mathcal{SH}_m^-(\Omega)$ and $(\Omega_j)_j$ be the sequence defined above. Take f^j, the function defined by

$$f^j = \sup\left\{\psi \in \mathcal{SH}_m(\Omega) : \psi_{|\Omega \setminus \Omega_j} \le f\right\} \in \mathcal{SH}_m(\Omega).$$

The function $\widetilde{f} := (\lim_{j \to +\infty} f^j)^*$ is called the smallest maximal m-subharmonic majorant function of f.

It is clear that $f \leq f^j \leq f^{j+1}$, so $\lim_{j\to+\infty} f^j$ exists on Ω except at an m-polar set, we deduce that $\widetilde{f} \in \mathcal{SH}_m(\Omega)$. Moreover, if $f \in \mathcal{E}_m(\Omega)$, then by [19] and [6], $\widetilde{f} \in \mathcal{E}_m(\Omega)$ and it is m-maximal on Ω. We denote by $\mathcal{MSH}_m(\Omega)$ the family of m-maximal functions in $\mathcal{SH}_m(\Omega)$.

We cite below some useful properties of $\mathcal{MSH}_m(\Omega)$.

Proposition 2.10 ([6])**.** *Let $f, g \in \mathcal{E}_m(\Omega)$ and $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then we have:*

1. $\widetilde{f+g} \geq \widetilde{f} + \widetilde{g}$.
2. $\widetilde{\alpha f} = \alpha \widetilde{f}$.
3. *If $f \leq g$, then $\widetilde{f} \leq \widetilde{g}$.*
4. $\mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega) = \{f \in \mathcal{E}_m : \ \widetilde{f} = f\}$.

In [20], the author introduced a new Cegrell class $\mathcal{N}_m(\Omega) := \{f \in \mathcal{E}_m : \widetilde{f} = 0\}$. It is easy to check that $\mathcal{N}_m(\Omega)$ is a convex cone satisfying

$$\mathcal{E}_m^0(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega).$$

Definition 2.11. Let $\mathcal{L}_m \in \{\mathcal{F}_m, \mathcal{N}_m, \mathcal{E}_m\}$. We define

$$\mathcal{L}_m^a(\Omega) := \{f \in \mathcal{L}_m : \forall P \ m\text{-polar set} \ \ H_m(f)(P) = 0\}.$$

Definition 2.12.

1. Let E be a Borel subset of Ω. The Cap_s-capacity of E with respect to Ω is given as follows:
$$\mathrm{Cap}_s(E) = \mathrm{Cap}_s(E,\Omega) = \sup\left\{\int_E H_s(f) : f \in \mathcal{SH}_m(\Omega),\ -1 \leq f \leq 0\right\},$$
where $1 \leq s \leq m$.
2. We say that a sequence $(f_j)_j$ of real-valued Borel measurable functions, defined on Ω, converges to f in Cap_s-capacity, when $j \to +\infty$, if for every compact subset K of Ω and $\varepsilon > 0$ the following limit holds:
$$\lim_{j\to+\infty} \mathrm{Cap}_s(\{z \in K : |f_j(z) - f(z)| > \varepsilon\}) = 0.$$
3. For a given Borel subset $E \subset \Omega$, the outer s-capacity $\mathrm{Cap}_s^\star$ of E is defined as
$$\mathrm{Cap}_s^\star(E,\Omega) := \inf\{\mathrm{Cap}_s(F,\Omega) : E \subset F \text{ and } F \text{ is an open subset of } \Omega\}.$$

Remark 2.13. For a given subset E of Ω, one can define $h_{E,\Omega}$ as follows:

$$h_{E,\Omega} := \sup\{f(z) : f \in \mathcal{SH}^-(\Omega),\ f \leq -1 \text{ on } E\}.$$

Using the definitions above and Theorem 2.20 from [18], we have the following:

$$\mathrm{Cap}_m^\star(E,\Omega) = \int_\Omega H_m(h_{E,\Omega}^*),$$

where $h_{E,\Omega}^*$ is the smallest upper semicontinuous majorant function of $h_{E,\Omega}$.

The following results, due to V.V. Hung and N.V. Phu [16], will be used frequently in our paper so we are to give some statements.

Theorem 2.14 ([16]). *Let $f, f_1, \dots, f_{m-1} \in \mathcal{E}_m(\Omega)$, $g \in \mathcal{SH}_m^-(\Omega)$. Then*

$$dd^c \max(f,g) \wedge dd^c f_1 \wedge \cdots \wedge dd^c f_{m-1} \wedge \beta^{n-m}|_{\{f>g\}} = dd^c f \wedge dd^c f_1 \wedge \cdots \wedge dd^c f_{m-1} \wedge \beta^{n-m}|_{\{f>g\}}$$

Theorem 2.15 ([16]). *Let $f_j, g_j, h \in \mathcal{E}_m(\Omega)$ be such that $f_j, g_j \geq h$ for all $j \geq 1$. Assume that $|f_j - g_j| \to 0$ in Cap_m-capacity. Then*

$$\lim_{j\to+\infty} u[H_m(f_j) - H_m(g_j)] = 0$$

for all $u \in SH_m \cap L^\infty_{loc}(\Omega)$.

3. Convergence in Cap_m-capacity

Proposition 3.1 ([16, 17]).

1. *For every $f, g \in \mathcal{E}_m(\Omega)$ such that $g \leq f$, one has*

$$1_{\{f=-\infty\}} H_m(f) \leq 1_{\{g=-\infty\}} H_m(g).$$

2. *If $f \in \mathcal{E}_m(\Omega)$ and $g \in \mathcal{E}_m^a(\Omega)$, then*

$$1_{\{f+g=-\infty\}} H_m(f+g) \leq 1_{\{f=-\infty\}} H_m(f).$$

Proposition 3.2. *For each of non-negative measures μ, ν on Ω, satisfying $(\mu+\nu)(\Omega) < \infty$ and $\int_\Omega -f d\mu \geq \int_\Omega -f d\nu$ for all $f \in \mathcal{E}_m^0(\Omega)$, one has $\mu(K) \geq \nu(K)$ for all complete m-polar subsets K in Ω.*

Proof. Using Theorem 1.7.1 in [19], we get

$$\int_\Omega -f d\mu \geq \int_\Omega -f\, d\nu, \quad f \in \mathcal{SH}_m^-(\Omega) \cap L^\infty(\Omega).$$

Take $g \in \mathcal{SH}_m^-(\Omega)$ such that $K = \{g = -\infty\}$. Then, for all $\varepsilon > 0$, we have

$$\int_\Omega -\max(\varepsilon g, -1)\, d\mu \geq \int_\Omega -\max(\varepsilon g, -1)\, d\nu.$$

The result follows by letting $\varepsilon \to 0$. □

We consider the sets $\mathcal{P}_m(\Omega)$ and $\mathcal{Q}_m(\Omega)$ defined as follows:

$$\mathcal{P}_m(\Omega) = \{f \in \mathcal{E}_m(\Omega) : \exists P_1, \dots, P_n \text{ polar in } \mathbb{C}/1_{\{f=-\infty\}}\ \ H_m(f)(\Omega \backslash P_1 \times ... \times P_n) = 0\}.$$

$$\mathcal{Q}_m(\Omega) = \{(f,g) \in (\mathcal{E}_m(\Omega))^2 : \forall z \in \Omega\ \exists V \text{ a neighborhood of } z\ \ u_V \in \mathcal{E}_m^a(V)/f + u_V \leq g \text{ on } V\}.$$

We cite below some properties of the class $\mathcal{P}_m(\Omega)$ that will be useful further.

Proposition 3.3.

1. *If $f \in \mathcal{SH}_m^-(\Omega)$, $g \in \mathcal{P}_m(\Omega)$ and $f \geq g$, then $f \in \mathcal{P}_m(\Omega)$.*
2. *If $f, g \in \mathcal{P}_m(\Omega)$, then $f + g \in \mathcal{P}_m(\Omega)$.*

Proof. 1. Since $g \in \mathcal{E}_m(\Omega)$, so is f. Now assume that there exists $P_1, \ldots, P_n$ polar in $\mathbb{C}$ such that $1_{\{g=-\infty\}} H_m(g)(\Omega \backslash P_1 \times \cdots \times P_n) = 0$. Then, by Proposition 3.1, we deduce that

$$1_{\{f=-\infty\}} H_m(f)(\Omega \backslash P_1 \times \cdots \times P_n) = 0.$$

It follows that $f \in \mathcal{P}_m(\Omega)$. The proof of the first assertion is completed.

2. By [19], the set $\mathcal{E}_m(\Omega)$ is a convex cone. Hence, if $f, g \in \mathcal{E}_m(\Omega)$, so is $f + g$. We have

$$H_m(f+g) = \sum_{k=0}^{m} \binom{m}{k} (dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}.$$

If we fix $k \in \{1, \ldots, m-1\}$, then

$$(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m} = \mu + \mathbf{1}_{\{f=g=-\infty\}} (dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m},$$

where $\mu := (dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}|_{\{f>-\infty\} \cup \{g>-\infty\}}$. The measure μ has no mass on m-polar sets. Indeed, by Theorem 2.14, we have

$$\begin{aligned}(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}|_{\{f>-j\}} \\ = dd^c \max(f, -j) \wedge (dd^c f)^{k-1} \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}|_{\{f>-j\}}.\end{aligned}$$

Using Proposition 3.4 from [16], we get that $(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}|_{\{f>-j\}} << \mathrm{Cap}_m$. We deduce that $\mu << \mathrm{Cap}_m$ in every $B \Subset \Omega$. So, μ has no mass on all m-polar sets.

The same reason remains true for the cases $k = 0$ and $k = m$ to obtain that

$$H_m(f) = \mu_1 + 1_{\{f=-\infty\}} H_m(f) \text{ and } H_m(g) = \mu_2 + 1_{\{g=-\infty\}} H_m(g),$$

where μ_1 and μ_2 are two measures that have no mass on all m-polar sets. We deduce that

$$\begin{aligned}\mathbf{1}_{\{f+g=-\infty\}} H_m(f+g) = \sum_{k=1}^{m-1} \binom{m}{k} \mathbf{1}_{\{f=g=-\infty\}} (dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m} \\ + 1_{\{f=-\infty\}} H_m(f) + 1_{\{g=-\infty\}} H_m(g).\end{aligned}$$

Take $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$, polar sets in $\mathbb{C}$ such that $1_{\{f=-\infty\}} H_m(f)(\Omega \backslash P_1 \times \cdots \times P_n) = 0$ and $1_{\{g=-\infty\}} H_m(g)(\Omega \backslash Q_1 \times \cdots \times Q_n) = 0$. Now, by Lemma 5.6 from [16], we get

$$\int_{\Omega \backslash ((P_1 \cup Q_1) \times \cdots \times (P_n \cup Q_n))} \mathbf{1}_{\{f+g=-\infty\}} H_m(f+g)$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1} \binom{m}{k} \int_{\Omega\setminus((P_1\cup Q_1)\times\cdots\times(P_n\cup Q_n))} \mathbf{1}_{\{f=g=-\infty\}} (dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m} \\
&\quad + \int_{\Omega\setminus((P_1\cup Q_1)\times\cdots\times(P_n\cup Q_n))} \mathbf{1}_{\{f=-\infty\}} H_m(f) + \mathbf{1}_{\{g=-\infty\}} H_m(g) \\
&\le \sum_{k=1}^{m-1} \binom{m}{k} \left(\int_{\Omega\setminus(P_1\times\cdots\times P_n)\cap\{f=g=-\infty\}} H_m(f) \right)^{\frac{k}{m}} \\
&\quad \times \left(\int_{\Omega\setminus(Q_1\times\cdots\times Q_n)\cap\{f=g=-\infty\}} H_m(g) \right)^{\frac{m-k}{m}} \\
&\quad + \left(\int_{\Omega\setminus(P_1\times\cdots\times P_n)\cap\{f=-\infty\}} H_m(f) \right) + \left(\int_{\Omega\setminus(P_1\times\cdots\times P_n)\cap\{g=-\infty\}} H_m(g) \right) \\
&\le \Big(\left[1_{\{f=-\infty\}} H_m(f)(\Omega\setminus P_1 \times \cdots \times P_n) \right]^{\frac{1}{m}} \\
&\quad + \left[1_{\{g=-\infty\}} H_m(g)(\Omega\setminus Q_1 \times \cdots \times Q_n) \right]^{\frac{1}{m}} \Big)^m = 0.
\end{aligned}
$$

We conclude that $f+g \in \mathcal{P}_m(\Omega)$. □

The following theorem represents the first main result of this paper.

Theorem 3.4. *If f_j is a sequence of m-subharmonic functions belonging to $\mathcal{E}_m(\Omega)$ and it satisfies $f_j \to f \in \mathcal{E}_m(\Omega)$ in Cap_m-capacity, then*

$$1_{\{f>-\infty\}} H_m(f) \le \liminf_{j\to+\infty} H_m(f_j).$$

Proof. Take $0 \le \varphi \in C_0^\infty(\Omega)$ and $\Omega_1 \Subset \Omega$ such that $\operatorname{supp}\varphi \Subset \Omega_1$. It suffices to show that

$$\liminf_{j\to+\infty} \int_\Omega \varphi H_m(f_j) \ge \int_\Omega 1_{\{f>-\infty\}} \varphi H_m(f).$$

For each $a > 0$, one has

$$\int_\Omega \varphi H_m(f_j) - \int_\Omega 1_{\{f>-\infty\}} \varphi H_m(f) = A_1 + A_2 + A_3,$$

where

$$
\begin{aligned}
A_1 &= \int_\Omega \varphi \left(H_m(f_j) - H_m(\max(f_j, -a)) \right) + \int_\Omega 1_{\{f=-\infty\}} \varphi H_m(f), \\
A_2 &= \int_\Omega \varphi \left(H_m(\max(f_j, -a)) - H_m(\max(f, -a)) \right), \\
A_3 &= \int_\Omega \varphi \left(H_m(\max(f, -a)) - H_m(f) \right).
\end{aligned}
$$

Using Theorem 2.14, we obtain

$$A_1 = \int_{\{f_j \le -a\}} \varphi (H_m(f_j) - H_m(\max(f_j, -a))) + \int_\Omega 1_{\{f=-\infty\}} \varphi H_m(f)$$

$$
\begin{aligned}
&\geq -\int_{\{f_j\leq -a\}} \varphi H_m(\max(f_j,-a)) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f)\\
&\geq -\int_{\{f_j\leq -a\}\cap\{|f_j-f|\leq 1\}} \varphi H_m(\max(f_j,-a))\\
&\quad -\int_{\{|f_j-f|>1\}} \varphi H_m(\max(f_j,-a)) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f)\\
&\geq -\int_{\{f<-a+2\}} \varphi H_m(\max(f_j,-a))\\
&\quad - a^n \operatorname{Cap}_m(\{|f_j-f|>1\}\cap\Omega_1) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f)\\
&\geq \int_\Omega h_{\{f<-a+2\}\cap\Omega_1,\Omega}\varphi H_m(\max(f_j,-a))\\
&\quad - a^n \operatorname{Cap}_m(\{|f_j-f|>1\}\cap\Omega_1) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f).
\end{aligned}
$$

Letting $j\to+\infty$, by Theorem 2.15 we obtain

$$
\liminf_{j\to+\infty} A_1 \geq \int_\Omega h_{\{f<-a+2\}\cap\Omega_1,\Omega}\varphi H_m(\max(f,-a)) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f).
$$

By Theorem 2.15, for all $s>0$, one has

$$
\begin{aligned}
\liminf_{a\to+\infty}(\liminf_{j\to+\infty} A_1) &\geq \liminf_{a\to+\infty}\int_\Omega h_{\{f<-a+2\}\cap\Omega_1,\Omega}\varphi H_m(\max(f,-a))\\
&\qquad\qquad + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f)\\
&\geq \liminf_{a\to+\infty}\int_\Omega h_{\{f<-s\}\cap\Omega_1,\Omega}\varphi H_m(\max(f,-a))\\
&\qquad\qquad + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f))\\
&= \int_\Omega h_{\{f<-s\}\cap\Omega_1,\Omega}\varphi H_m(f) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f).
\end{aligned}
$$

Since $\lim_{s\to+\infty}\operatorname{Cap}_m(\{f<-s\}\cap\Omega_1)=0$, then there exists a subset A of Ω with $\operatorname{Cap}_m(A)=0$ such that the function $h_{\{f<-s\}\cap\Omega_1,\Omega}$ increases to 0 as $s\to+\infty$ on $\Omega\backslash A$. Now, by a decomposition theorem from [19], we get that if $s\to+\infty$, then

$$
\liminf_{a\to+\infty}(\liminf_{j\to+\infty} A_1) \geq \int_\Omega -1_A\varphi H_m(f) + \int_\Omega 1_{\{f=-\infty\}}\varphi H_m(f) \geq 0.
$$

By Theorem 2.15, it follows that

$$
\begin{aligned}
&\liminf_{j\to+\infty}\left(\int_\Omega \varphi H_m(f_j) - \int_\Omega 1_{\{f>-\infty\}}\varphi H_m(f)\right)\\
&\qquad\qquad\qquad \geq \liminf_{a\to+\infty}\liminf_{j\to+\infty} A_1 + \liminf_{a\to+\infty} A_3 \geq 0.
\end{aligned}
$$

The theorem is proved. □

Corollary 3.5. *Let $(f_j)_j \subset \mathcal{E}_m(\Omega)$ such that $f_j \to f \in \mathcal{E}_m(\Omega)$ in* Cap_m*-capacity. If $f_j, f \in \mathcal{Q}_m(\Omega)$ for all $j \geq 1$, then*

$$H_m(f) \leq \liminf_{j\to+\infty} H_m(f_j).$$

Proof. By combining the definition of $\mathcal{Q}_m(\Omega)$ with Proposition 3.1, we get

$$1_{\{f=-\infty\}} H_m(f) \leq 1_{\{f_j=-\infty\}} H_m(f_j) \leq H_m(f_j).$$

Using Theorem 3.4, we get the result. □

Corollary 3.6. *Let $(f_j)_j \subset \mathcal{F}_m(\Omega)$ such that $f_j \to f \in \mathcal{F}_m(\Omega)$ in* Cap_m*-capacity. If $f_j, f \in \mathcal{Q}_m(\Omega)$ for all $j \geq 1$ and*

$$\lim_{j\to+\infty} \int_\Omega H_m(f_j) = \int_\Omega H_m(f),$$

then $H_m(f_j) \to H_m(f)$ weakly as $j \to +\infty$.

Proof. Without loss of generality, one can assume that $H_m(f_j) \to \mu$ weakly as $j \to +\infty$. Using Corollary 3.5, we obtain that $\mu \geq H_m(f)$. On the other hand,

$$\mu(\Omega) \leq \liminf_{j\to+\infty} \int_\Omega H_m(f_j) = \int_\Omega H_m(f).$$

Hence $\mu = H_m(f)$. □

Theorem 3.7. *Let $f_j, g \in \mathcal{E}_m(\Omega)$, $f \in \mathcal{P}_m(\Omega)$, and $D \Subset \Omega$. Assume that*

1. *$f_j \to f$ in* Cap_m*-capacity.*
2. *For all $j \geq 1$, $f_j \geq g$ on $\Omega \backslash D$.*

Then $H_m(f_j) \to H_m(f)$ weakly as $j \to \infty$.

Proof. As $f \in \mathcal{P}_m(\Omega)$, then there exist $P_1, \ldots, P_n$, m-polar subsets in $\mathbb{C}$, such that

$$1_{\{f=-\infty\}} H_m(f)(\Omega \backslash P_1 \times \cdots \times P_n) = 0.$$

Take

$$\tilde{f}_j = \max(f_j, g), \quad \tilde{f} = \max(f, g).$$

It is easy to check that $\tilde{f}_j, f \in \mathcal{E}_m(\Omega)$ and $\tilde{f}_j \to \tilde{f}$ in Cap_m-capacity. Moreover, $\tilde{f}_j|_{\Omega\backslash D} = f_j|_{\Omega\backslash D}$ and $\tilde{f}|_{\Omega\backslash D} = f|_{\Omega\backslash D}$. Using Theorem 2.15, we get that $H_m(\tilde{f}_j) \to H_m(\tilde{f})$ weakly as $j \to \infty$. Let Ω_1 be an m-hyperconvex domain such that $D \Subset \Omega_1 \Subset \Omega$. By Stokes' theorem, we have

$$\limsup_{j\to+\infty} \int_{\Omega_1} H_m(f_j) = \limsup_{j\to+\infty} \int_{\Omega_1} H_m(\tilde{f}_j) \leq \int_{\bar{\Omega}_1} H_m(\tilde{f}) < \infty.$$

Hence, without loss of generality, one may assume that there exists a positive measure μ such that $H_m(f_j) \to \mu$ weakly as $j \to \infty$. The proof will be completed if we show that $\mu = H_m(f)$ on Ω_1. For this, we take $u \in \mathcal{E}_m^0(\Omega_1)$. Further, by Stokes' theorem, we obtain

$$\int_{\Omega_1} -ud\mu = \lim_{j\to+\infty}\int_{\Omega_1} -uH_m(f_j) \geq \lim_{j\to+\infty}\int_{\Omega_1} -uH_m(\tilde{f}_j) \geq \lim_{j\to+\infty}\int_{\Omega_1} -uH_m(\tilde{f}).$$

Moreover, by Proposition 3.2 and [13], we get

$$H_m(f)(K) \leq \mu(K) \tag{3.1}$$

for all compact subsets K of $P_1 \times ... \times P_n$. Using (3.1) and the fact that $f \in \mathcal{P}_m(\Omega)$, we deduce that $\mu \geq 1_{\{f=-\infty\}}H_m(f)$. So, by Theorem 3.4, we obtain

$$H_m(f) \leq \mu \text{ on } \Omega_1.$$

Now, let Ω_2 be a domain satisfying $D \Subset \Omega_2 \Subset \Omega_1$. By Stokes' theorem, we obtain

$$\begin{aligned}\mu(\Omega_2) &\leq \liminf_{j\to+\infty}\int_{\Omega_2} H_m(f_j) = \liminf_{j\to+\infty}\int_{\Omega_2} H_m(\tilde{f}_j)\\ &\leq \int_{\bar{\Omega}_2} H_m(\tilde{f}) \leq \int_{\Omega_1} H_m(\tilde{f}) = \int_{\Omega_1} H_m(f).\end{aligned}$$

It follows that

$$\mu(\Omega_1) \leq H_m(f)(\Omega_1). \tag{3.2}$$

Using (3.1) and (3.2), we deduce that $\mu = H_m(f)$ on Ω_1. □

Remark 3.8. The previous theorem is a different version of Theorems 3.8 and 3.10 in [16]. On the one hand, the assumption $f_j \geq g$ is sufficient outside a relatively compact set D, but in Theorem 3.8 [16] this assumption is required to be true in the whole Ω. On the other hand, the function f taken in our result belongs to $\mathcal{P}_m(\Omega)$ which is not the case for Theorem 3.8 in [16] since $f \in \mathcal{E}_m(\Omega)$.

The following lemma will be useful in the proof of several results of this paper.

Lemma 3.9. *Fix $f \in \mathcal{F}_m(\Omega)$. Then, for all $s > 0$ and $t > 0$, one has*

$$t^m \operatorname{Cap}_m(f < -s-t) \leq \int_{\{f<-s\}} H_m(f) \leq s^m \operatorname{Cap}_m(f < -s). \tag{3.3}$$

Proof. Let $t, s > 0$, and let K be a compact subset satisfying $K \subset \{f < -s - t\}$. We have

$$\begin{aligned}\operatorname{Cap}_m(K) = \int_\Omega H_m(h_K^*) &= \int_{\{f<-s-t\}} H_m(h_K^*)\\ &= \int_{\{f<-s+th_K^*\}} H_m(h_K^*) = \frac{1}{t^m}\int_{\{f<g\}} H_m(g),\end{aligned}$$

where $g := -s + th_K^*$. Using Theorem 2.14, we obtain

$$\frac{1}{t^m}\int_{\{f<g\}} H_m(g) = \frac{1}{t^m}\int_{\{f<\max(f,g)\}} H_m(\max(f,g))$$

$$\leq \frac{1}{t^m} \int_{\{f<\max(f,g)\}} H_m(f) = \frac{1}{t^m} \int_{\{f<-s+th_K\}} H_m(f) \leq \frac{1}{t^m} \int_{\{f<-s\}} H_m(f).$$

The left-hand side of inequality (3.1) follows by taking the supremum over all compact sets $K \subset \Omega$.
For the right-hand side of inequality, we have

$$\begin{aligned}\int_{\{f\leq -s\}} H_m(f) &= \int_\Omega H_m(f) - \int_{f>-s} H_m(f) \\ &= \int_\Omega H_m(\max(f,-s)) - \int_{f>-s} H_m(\max(f,-s)) \\ &= \int_{f\leq -s} H_m(\max(f,-s)) \leq s^m \operatorname{Cap}_m\{f \leq -s\}.\end{aligned}$$

The result follows. □

Remark 3.10. Using the previous lemma, we deduce the following results:

1. $f \in \mathcal{F}_m(\Omega)$ if and only if $\limsup_{s\to 0} s^m \operatorname{Cap}_m(\{f < -s\}) < +\infty$.
2. If $f \in \mathcal{F}_m(\Omega)$, then

$$\int_\Omega H_m(f) = \lim_{s\to 0} s^m \operatorname{Cap}_m(\{f < -s\})$$

and

$$\int_{\{f=-\infty\}} H_m(f) = \lim_{s\to+\infty} s^m \operatorname{Cap}_m(\{f < -s\}).$$

3. The function $f \in \mathcal{F}_m^a(\Omega)$ if and only if $\lim_{s\to+\infty} s^n \operatorname{Cap}_m(\{f < -s\}) = 0$. Indeed, it is known that if f is an m−sh function on Ω, then $H_m(f)(P) = 0$ for every m−polar set $P \subset \Omega$ if and only if $H_m(f)(\{f = -\infty\}) = 0$ which follows directly from the previous assertion of this remark.

4. The class $\mathcal{E}_{m,\chi}(\Omega)$

Throughout this section, $\chi : \mathbb{R}^- \to \mathbb{R}^-$ will be an increasing function. In [15], Hung introduced the class $\mathcal{E}_{m,\chi}(\Omega)$ to generalize the fundamental weighted energy classes introduced firstly by Benelkourchi, Guedj, and Zeriahi [5]. Such a class is defined as follows.

Definition 4.1. We say that $f \in \mathcal{E}_{m,\chi}(\Omega)$ if there exits $(f_j)_j \subset \mathcal{E}_m^0(\Omega)$ such that $f_j \searrow f$ in Ω and

$$\sup_{j\in\mathbb{N}} \int_\Omega (-\chi(f_j))H_m(f_j) < +\infty.$$

Remark 4.2. It is clear that the class $\mathcal{E}_{m,\chi}(\Omega)$ generalizes all analogous Cegrell classes defined by Lu in [18] and [19]. Indeed,

1. $\mathcal{E}_{m,\chi}(\Omega) = \mathcal{F}_m(\Omega)$ when $\chi(0) \neq 0$ and χ is bounded.

2. $\mathcal{E}_{m,\chi}(\Omega) = \mathcal{E}^p_m(\Omega)$ in the case when $\chi(t) = -(-t)^p$.
3. $\mathcal{E}_{m,\chi}(\Omega) = \mathcal{F}^p_m(\Omega)$ in the case when $\chi(t) = -1 - (-t)^p$.

Notice that if we take $m = n$ for all previous cases, we recover the classic Cegrell classes defined in [7] and [8].

Notice also that in the case $\chi(0) \neq 0$ one has $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{F}_m(\Omega)$, and thus the Hessian operator is well defined in $\mathcal{E}_{m,\chi}(\Omega)$ and is with finite total mass on Ω. So, in the rest of this paper, we will consider the case $\chi(0) = 0$.

In the following theorem, we will prove that the Hessian operator is well defined on $\mathcal{E}_{m,\chi}(\Omega)$. Notice that this result was proved in [15] but with an extra condition ($\chi(2t) \leq a.\chi(t)$). Here we omit this condition and the proof of the result is completely different.

Theorem 4.3. *Assume that* $\chi \not\equiv 0$. *Then*

$$\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega).$$

So, for every $f \in \mathcal{E}_{m,\chi}(\Omega)$, $H_m(f)$ *is well defined and* $-\chi(f) \in L^1(H_m(f))$.

Proof. Since $\chi \not\equiv 0$, then there exists $t_0 > 0$ such that $\chi(-t_0) < 0$. Take an increasing function χ_1 satisfying $\chi_1' = \chi_1'' = 0$ on $[-t_0, 0]$, χ_1 is convex on $]-\infty, -t_0]$ and $\chi_1 \geq \chi$. Let $g \in \mathcal{SH}^-_m(\Omega)$, then

$$dd^c\chi_1(g) \wedge \beta^{n-m} = \chi_1''(g)dg \wedge d^cg \wedge \beta^{n-m} + \chi_1'(g)dd^c\chi_1(g) \wedge \beta^{n-m} \geq 0.$$

Hence the function $\chi_1(g) \in \mathcal{SH}^-_m(\Omega)$. Now consider $f \in \mathcal{E}_{m,\chi}(\Omega)$. By the definition, there exists a sequence $f_j \in \mathcal{E}^0_m(\Omega)$ that decreases to f and satisfies

$$\sup_{j\in\mathbb{N}} \int_\Omega -\chi(f_j)H_m(f_j) < \infty.$$

By the definition of the class $\mathcal{E}_m(\Omega)$, it remains to prove that f coincides locally with a function in $\mathcal{F}_m(\Omega)$. For this, take $G \Subset \Omega$ be a domain and consider the function

$$f_j^G := \sup\{g \in \mathcal{SH}^-_m(\Omega) : g \leq f_j \text{ on } G\}.$$

We have $f_j^G \in \mathcal{E}^0_m(\Omega)$ and $f_j^G \searrow f$ on G. Take $\varphi \in \mathcal{E}^0_m(\Omega)$ such that $\chi_1(f_1) \leq \varphi$. Using integration by parts, we obtain

$$\begin{aligned}\sup_{j\in\mathbb{N}} \int_\Omega -\varphi H_m(f_j^G) &\leq \sup_{j\in\mathbb{N}} \int_\Omega -\varphi H_m(f_j) \leq \sup_{j\in\mathbb{N}} \int_\Omega -\chi_1(f_1) H_m(f_j) \\ &\leq \sup_{j\in\mathbb{N}} \int_\Omega -\chi_1(f_j) H_m(f_j) \leq \sup_{j\in\mathbb{N}} \int_\Omega -\chi(f_j) H_m(f_j) < \infty.\end{aligned}$$

We deduce that

$$\sup_{j\in\mathbb{N}} \int_\Omega H_m(f_j^G) \leq (-\sup_G \varphi)^{-1} \sup_{j\in\mathbb{N}} \int_\Omega -\varphi H_m(f_j^G) < \infty.$$

It follows that the limit $\lim_{j\to+\infty} f_j^G \in \mathcal{F}_m(\Omega)$ and therefore $f \in \mathcal{E}_m(\Omega)$.

For the second assertion, we have that every $f \in \mathcal{E}_{m,\chi}(\Omega)$ is upper semicontinuous, so the sequence of measures $\mu_j := -\chi(f_j)H_m(f_j)$ is bounded. Take μ, a cluster point of μ_j, then $-\chi(f)H_m(f) \le \mu$. Hence, $\int_\Omega -\chi(f)H_m(f) < \infty$ and the desired result follows. □

Proposition 4.4. *Then the following statements are equivalent:*

(1) $\chi(-\infty) = -\infty$,

(2) $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m^a(\Omega)$.

Proof. We will prove that (1) ⇒ (2). For this, assume that $\chi(-\infty) = -\infty$ and take $f \in \mathcal{E}_{m,\chi}(\Omega)$. By the definition of the class $\mathcal{E}_{m,\chi}(\Omega)$, there exists a sequence $\{f_j\} \subset \mathcal{E}_m^0$ such that $f_j \searrow f$ and

$$\sup_j \int_\Omega -\chi(f_j)H_m(f_j) < +\infty.$$

Since χ is increasing, then for all $t > 0$,

$$\int_{\{f_j<-t\}} H_m(f_j) \le \int_{\{f_j<-t\}} \frac{\chi(f_j)}{\chi(-t)} H_m(f_j) \le (\chi(-t))^{-1} \sup_j \int_\Omega \chi(f_j)H_m(f_j).$$

Since the sequence $\{f_j < -t\}$ is increasing to $\{f < -t\}$, then letting $j \to \infty$, we get

$$\int_{\{f<-t\}} H_m(f) \le (\chi(-t))^{-1} \sup_j \int_\Omega \chi(f_j)H_m(f_j).$$

Now, if we let $t \to +\infty$, we can deduce that

$$\int_{\{f=-\infty\}} H_m(f) = 0.$$

Hence $f \in \mathcal{E}_m^a(\Omega)$.

(2) ⇒ (1). Assume that $\chi(-\infty) > -\infty$, then $\mathcal{F}_m(\Omega) \subset \mathcal{E}_{m,\chi}(\Omega)$. But it is known that $\mathcal{F}_m(\Omega)$ is not a subset of $\mathcal{E}_m^a(\Omega)$. Then we deduce that $\mathcal{E}_{m,\chi}(\Omega) \not\subset \mathcal{E}_m^a(\Omega)$. □

The rest of this section is devoted to giving a connection between the class $\mathcal{E}_{m,\chi}(\Omega)$ and the Cap_m-capacity of sublevels $\mathrm{Cap}_m(\{f < -t\})$. As a consequence, we deduce a complete characterization of the class $\mathcal{E}_m^p(\Omega)$ introduced by Lu [18] in term of the Cap_m-capacity of sublevel. For this, we introduce the class $\hat{\mathcal{E}}_{m,\chi}(\Omega)$ as follows.

Definition 4.5. Denote

$$\hat{\mathcal{E}}_{m,\chi}(\Omega) := \left\{ \varphi \in \mathcal{SH}_m^-(\Omega) : \int_0^{+\infty} t^m \chi'(-t)\,\mathrm{Cap}_m(\{\varphi < -t\})dt < +\infty \right\}.$$

The previous class coincides with the class $\hat{\mathcal{E}}_{\chi}(\Omega)$ given by Benelkourchi, Guedj, and Zeriahi [5], it suffices to take $m = n$ to recover it. In the following proposition, we cite some properties of $\hat{\mathcal{E}}_{m,\chi}(\Omega)$ and give a relationship between $\mathcal{E}_{m,\chi}(\Omega)$ and $\hat{\mathcal{E}}_{m,\chi}(\Omega)$:

Proposition 4.6.

1. *The class* $\hat{\mathcal{E}}_{m,\chi}(\Omega)$ *is convex.*
2. *For every* $f \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$ *and* $g \in \mathcal{SH}_m^-(\Omega)$*, one has that* $\max(f, g) \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$*.*
3. $\hat{\mathcal{E}}_{m,\chi}(\Omega) \subset \mathcal{E}_{m,\chi}(\Omega)$*.*
4. *If we denote by* $\hat{\chi}(t)$ *the function defined by* $\hat{\chi}(t) := \chi(t/2)$*, then*

$$\mathcal{E}_{m,\chi}(\Omega) \subset \hat{\mathcal{E}}_{m,\hat{\chi}}(\Omega).$$

Proof. 1. Let $f, g \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$ and $0 \le \alpha \le 1$. Since we have

$$\{\alpha f + (1-\alpha) g < -t\} \subset \{f < -t\} \cup \{g < -t\},$$

then $f + \alpha g \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$. The result follows.

2. The proof of this assertion is obvious.

3. Take $f \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$. It remains to construct a sequence $f_j \in \mathcal{E}_m^0(\Omega)$ satisfying

$$\int_{\Omega} -\chi(f_j)\, H_m(f_j) < \infty.$$

Without loss of generality, we may assume that $f \le 0$. If we set $f_j := \max(f, -j)$, then $f_j \in \mathcal{E}_m^0(\Omega)$. Using Lemma 3.9, we get

$$\begin{aligned}\int_{\Omega} -\chi(f_j)\, H_m(f_j) &= \int_0^{+\infty} \chi'(-t) H_m(f_j)(f_j < -t)\, dt \\ &\le \int_0^{+\infty} \chi'(-t) t^m \operatorname{Cap}_m(f < -t)\, dt < +\infty.\end{aligned}$$

It follows that $f \in \mathcal{E}_{m,\chi}(\Omega)$.

4. The proof of this assertion follows directly by using the same argument as in 3 and the second inequality in Lemma 3.9 for $t = s$. □

Proposition 4.7. *Assume that for all* $t < 0$ *one has* $\chi(t) < 0$*. Then, for all* $f \in \mathcal{E}_{m,\chi}(\Omega)$*, one has*

$$\limsup_{z \to w} f(z) = 0, \quad w \in \partial\Omega.$$

Proof. Since by hypothesis we have for all $t < 0$; $\chi(t) < 0$ so, without loss of generality, we can assume that the length of the set $\{t > 0 : t < t_0 \text{ and } \chi'(-t) \neq 0\}$ is positive for all $t_0 > 0$. By contradiction, we suppose that there is $w_0 \in \partial\Omega$ such that $\limsup_{z \to w_0} f(z) = c < 0$. Then there is a ball B_0 centered at w_0 satisfying $B_0 \cap \Omega \subset \{f < \frac{c}{2}\}$. If we consider $(K_j)_j$ to a sequence of regular compact subsets such that for all j one has $K_j \subset K_{j+1}$ and $B_0 \cap \Omega = \cup K_j$, then the extremal function $h_{K_j,\Omega}$ belongs to $\mathcal{E}_m^0(\Omega)$ and decreases to $h_{E,\Omega}$. It is easy to check that $h_{E,\Omega} \notin \mathcal{F}_m(\Omega)$. By the definition of the class $\mathcal{F}_m(\Omega)$, we obtain

$$\sup_j \operatorname{Cap}_m(K_j) = \sup_j \int_\Omega H_m(f_{K_j,\Omega}) = +\infty.$$

So,

$$\operatorname{Cap}_m(B_0 \cap \Omega) = +\infty.$$

We deduce that

$$\operatorname{Cap}_m(\{f < -s\}) = +\infty, \quad s \leq -c/2,$$

and hence

$$\int_0^{+\infty} t^m \chi'(-t) \operatorname{Cap}_m(\{f < -t\}) dt = +\infty.$$

We get a contradiction with the fact that $\mathcal{E}_{m,\chi}(\Omega) \subset \hat{\mathcal{E}}_{m,\hat{\chi}}(\Omega)$. □

Proposition 4.8. *Assume that* $\chi \not\equiv 0$. *If there exists a sequence* $(f_k) \subset \mathcal{E}_m^0(\Omega)$ *such that*

$$\sup_{k\in\mathbb{N}} \int_\Omega -\chi(f_k) H_m(f_k) < \infty,$$

then the function $f := \lim_{k\to+\infty} f_k \not\equiv -\infty$ *and therefore* $f \in \mathcal{E}_{m,\chi}(\Omega)$.

Proof. Using the hypothesis, we observe that the length of the set $\{t > 0 : t < t_0 \text{ and } \chi'(-t) \neq 0\}$ is positive. By Lemma 3.9, we get

$$s^m \operatorname{Cap}_m(\{f_k < -2s\}) \leq \int_{\{f_k<-s\}} H_m(f_k).$$

Then

$$\int_0^{+\infty} t^m \chi'(-t) \operatorname{Cap}_m(\{f < -t\}) dt = \lim_{k\to\infty} \int_0^{+\infty} t^m \chi'(-t) \operatorname{Cap}_m(\{f_k < -t\}) dt$$
$$\leq \lim_{k\to\infty} 2^m \int_0^{+\infty} \chi'(-t) \int_{\{f_k<-t\}} H_m(f_k) dt \leq 2^m \sup_{k\in\mathbb{N}} \int_\Omega -\chi(f_k) H_m(f_k) < \infty.$$

Notice that in the previous inequality the convergence monotone theorem was used. We conclude that $f \not\equiv -\infty$ and therefore $f \in \mathcal{E}_{m,\chi}(\Omega)$. □

Theorem 4.9. *Assume that one has* $\chi(t) < 0$ *for all* $t < 0$. *Then*

$$\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{N}_m(\Omega).$$

Proof. By Proposition 4.6, it suffices to prove that every maximal function $f \in \mathcal{E}_{m,\chi}(\Omega)$ is identically equal to 0. Take a sequence $f_j \in \mathcal{E}_m^0(\Omega)$ as in the definition of the class $\mathcal{E}_{m,\chi}(\Omega)$. By using Lemma 3.9, we obtain

$$\int_0^{+\infty} \chi'\left(\frac{-s}{2}\right) f^m \operatorname{Cap}_m(\{f < -s\}) ds$$

$$= \lim_{j\to\infty}\int_0^{+\infty}\chi'\left(\frac{-s}{2}\right)s^m\,\mathrm{Cap}_m(\{f_j<-s\})ds$$
$$\le 2^m\lim_{j\to\infty}\int_0^{+\infty}\chi'(-s)\int_{(f_j<-s)}H_m(f_j)ds = 2^m\lim_{j\to\infty}\int_\Omega -\chi(f_j)H_m(f_j).$$

Since the maximality of $f\in\mathcal{E}_m(\Omega)$ is equivalent to $H_m(f)=0$, we deduce that

$$\lim_{j\to\infty}\int_\Omega -\chi(f_j)H_m(f_j)=0.$$

Thus, $\mathrm{Cap}_m(\{f<-s\})=0$ for all $s>0$. It follows that $f\equiv 0$. The proof of the theorem is completed. □

Now we give a complete characterization of $\mathcal{E}_{m,\chi}(\Omega)$ in term of $\mathcal{N}_m(\Omega)$. We will prove essentially the following result.

Corollary 4.10. *If for all $t<0$, $\chi(t)<0$, then*

$$\mathcal{E}_{m,\chi}(\Omega)=\left\{f\in\mathcal{N}_m(\Omega):\chi(f)\in L^1(H_m(f))\right\}.$$

Proof. The first inclusion is a direct deduction from Theorem 4.3 and Theorem 4.9. It suffices to prove the reverse inclusion

$$\left\{f\in\mathcal{N}_m(\Omega):\chi(f)\in L^1(H_m(f))\right\}\subset\mathcal{E}_{m,\chi}(\Omega).$$

Take $f\in\mathcal{N}_m(\Omega)$ satisfying $\int_\Omega -\chi(f)H_m(f)<\infty$. It suffices to construct the sequence $f_j\in\mathcal{E}_m^0(\Omega)$ that decreases to f and satisfies

$$\sup_j\int_\Omega -\chi(f_j)H_m(f_j)<\infty.$$

Let ρ be an exhaustion function for Ω ($\Omega=\{\rho<0\}$). Theorem 5.9 in [16] guarantees that for all $j\in\mathbb{N}$ there is a function $f_j\in\mathcal{E}_m^0(\Omega)$ satisfying $H_m(f_j)=1_{\{f>j\rho\}}H_m(f)$. We have $H_m(f_j)\le H_m(f_{j+1})\le H_m(f)$, so we get that $f_j\ge f_{j+1}$ using the comparison principle and the fact that $(f_j)_j$ converges to a function $\widetilde{f}$. It is easy to check that $\widetilde{f}\ge f$. Now, following the proof of Theorem 4.3, we deduce the existence of a negative m$-$sh function g satisfying $\int_\Omega -gH_m(f)<\infty$. If follows by Theorem 2.10 of [17] that $\widetilde{f}=f$. Thus, the monotone convergence theorem gives

$$\int_\Omega -\chi(f_j)H_m(f_j)=\int_\Omega -\chi(f_j)1_{\{f>j\rho\}}H_m(f)\to\int_\Omega -\chi(f)H_m(f)<\infty.\quad\square$$

Now we extend Theorem A to the class $\mathcal{E}_{m,\chi}(\Omega)$.

Theorem 4.11. *Assume that χ is continuous, $\chi(-\infty)>-\infty$, and $f,f_j\in\mathcal{E}_m(\Omega)$ for all $j\in\mathbb{N}$. If there exists $g\in\mathcal{E}_m(\Omega)$ satisfying $f_j\ge g$ on Ω, then:*

1. *If f_j converges to f in Cap_{m-1}-capacity, then* $\liminf_{j\to+\infty}-\chi(f_j)H_m(f_j)\ge -\chi(f)H_m(f)$.

2. *If f_j converges to f in* Cap_m*-capacity, then $-\chi(f_j)H_m(f_j)$ converges weakly to $-\chi(f)H_m(f)$.*

Proof. 1. Take a test function $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi \le 1$. Using [19], there exist $\psi_k \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ with $\psi_k \ge f$ and $\psi_k \searrow f$ in Ω. Following [14], for a fixed integer $k \ge 1$ there exists $j_0 \in \mathbb{N}$ such that $f_j \ge \psi_k$ on $\operatorname{supp}\varphi$ for all $j \ge j_0$. So, by Theorem 3.10 from [16], we obtain that for all $k \ge 1$ one has

$$\liminf_{j\to+\infty} \int_\Omega -\varphi\chi(f_j)H_m(f_j) \ge \liminf_{j\to+\infty} \int_\Omega -\varphi\chi(\psi_k)H_m(f_j) = \int_\Omega -\varphi\chi(\psi_k)H_m(f).$$

Now, if we let k tend to $+\infty$, then, by the Lebesgue monotone convergence theorem, we get

$$\liminf_{j\to+\infty} \int_\Omega -\varphi\chi(f_j)H_m(f_j) \ge \int_\Omega -\varphi\chi(f)H_m(f).$$

The result follows.

2. Without loss of generality, one can assume that $\chi(-\infty) = -1$. Let $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi \le 1$. We claim that

$$\limsup_{j\to+\infty} \int_\Omega -\varphi\chi(f_j)H_m(f_j) \le \int_\Omega -\varphi\chi(f)H_m(f). \tag{4.1}$$

Indeed, by the quasicontinuity of f and g with respect to the capacity Cap_m, we obtain that for every $k \in \mathbb{N}$ there exists an open subset O_k of Ω and a function $\widetilde{f_k} \in \mathcal{C}(\Omega)$ such that $\mathrm{Cap}_m(O_k) \le \frac{1}{2^k}$ and $\widetilde{f_k} = f$ on $\Omega \setminus O_k$ and $g \ge -\alpha_k$ on $\operatorname{supp}\varphi \setminus O_k$ for some $\alpha_k > 0$. Let $\varepsilon > 0$. Then, by Theorem 3.6 in [13], one has

$$\begin{aligned}
\int_\Omega -\varphi\chi(f_j)H_m(f_j) &= \int_{\Omega\setminus O_k} -\varphi\chi(f_j)H_m(f_j) + \int_{O_k} -\varphi\chi(f_j)H_m(f_j)\\
&\le \int_{\Omega\setminus O_k} -\varphi\chi(f_j)H_m(f_j) + \int_{O_k} -\varphi H_m(f_j)\\
&\le \int_{\{f_j\le f-\varepsilon\}\setminus O_k} -\varphi\chi(f_j)H_m(f_j)\\
&\quad + \int_{\{f_j> f-\varepsilon\}\setminus O_k} -\varphi\chi(f_j)H_m(f_j) + \int_{O_k} -\varphi H_m(f_j)\\
&\le \int_{\{f_j\le f-\varepsilon\}\setminus O_k} -\varphi H_m(f_j)\\
&\quad + \int_{\Omega\setminus O_k} -\varphi\chi(f-\varepsilon)H_m(f_j) + \int_\Omega -\varphi h_{O_k,\Omega} H_m(f_j)\\
&\le \int_{\{f_j< f-\varepsilon\}\setminus O_k} H_m(\max(f_j,-\alpha_k))\\
&\quad + \int_{\Omega\setminus O_k} -\varphi\chi(\widetilde{f_k}-\varepsilon)H_m(f_j) + \int_\Omega -\varphi h_{O_k,\Omega} H_m(f_j)\\
&\le \alpha_k^m \,\mathrm{Cap}_m(\{f_j < f-\varepsilon\} \cap \operatorname{supp}\varphi)
\end{aligned}$$

$$+\int_{\Omega\setminus O_k}-\varphi\chi(\widetilde{f}_k-\varepsilon)H_m(f_j)+\int_{\Omega}-\varphi h_{O_k,\Omega}H_m(f_j).$$

If we let j go to $+\infty$, then, by using Theorem 2.15, we get

$$\limsup_{j\to+\infty}\int_{\Omega}-\varphi\chi(f_j)H_m(f_j)\leq\int_{\Omega\setminus O_k}-\varphi\chi(\widetilde{f}_k-\varepsilon)H_m(f)+\int_{\Omega}-\varphi h_{O_k,\Omega}H_m(f).$$

If we let $\varepsilon\to 0$, we obtain

$$\begin{aligned}\limsup_{j\to+\infty}\int_{\Omega}-\varphi\chi(f_j)H_m(f_j)&\leq\int_{\Omega\setminus O_k}-\varphi\chi(\widetilde{f}_k)H_m(f)+\int_{\Omega}-\varphi h_{O_k,\Omega}H_m(f)\\&\leq\int_{\Omega\setminus\{f=-\infty\}}-\varphi\chi(f)H_m(f)+\int_{\Omega}-\varphi h_{\bigcup_{l=k}^{\infty}O_l,\Omega}H_m(f).\end{aligned}\tag{4.2}$$

Now, as $\bigcup_{l=k}^{\infty}O_l\searrow O$ when $k\to+\infty$, then

$$\mathrm{Cap}_m(O)\leq\lim_{k\to\infty}\mathrm{Cap}_m\left(\bigcup_{l=k}^{\infty}O_l\right)\leq\lim_{k\to\infty}\sum_{l=k}^{\infty}\mathrm{Cap}_m(O_l)\leq\lim_{k\to\infty}\frac{1}{2^{k-1}}.$$

So, there exists an m-polar set M such that $h_{\bigcup_{l=k}^{\infty}O_l,\Omega}\nearrow 0$ when $k\to+\infty$ on $\Omega\setminus M$. Thus, if we take $k\to+\infty$ in 4.2, we obtain

$$\begin{aligned}\limsup_{j\to+\infty}\int_{\Omega}-\varphi\chi(f_j)H_m(f_j)&\leq\int_{\Omega\setminus\{f=-\infty\}}-\varphi\chi(f)H_m(f)+\int_{M}\varphi H_m(f)\\&\leq\int_{\Omega\setminus\{f=-\infty\}}-\varphi\chi(f)H_m(f)+\int_{\{f=-\infty\}}-\varphi\chi(f)H_m(f)\\&=\int_{\Omega}-\varphi\chi(f)H_m(f).\end{aligned}$$

This proves the claim 4.1. Moreover, since f_j converges in Cap_m-capacity, then it converges in Cap_{m-1}-capacity. Using the assertion (a), we obtain

$$\liminf_{j\to+\infty}\int_{\Omega}-\varphi\chi(f_j)H_m(f_j)\geq\int_{\Omega}-\varphi\chi(f)H_m(f).$$

If we combine the last inequality with 4.2, we get

$$\lim_{j\to+\infty}\int_{\Omega}-\varphi\chi(f_j)H_m(f_j)=\int_{\Omega}-\varphi\chi(f)H_m(f)$$

for every $\varphi\in\mathcal{C}_0^{\infty}(\Omega)$ with $0\leq\varphi\leq 1$. Hence we get the desired result. □

Notice that if we take $m=n$ in the above theorem, we obtain the result from [11] established for the particular case of plurisubharmonic functions.

Now we are interested in the problem of subextention in the class $\mathcal{E}_{m,\chi}(\Omega)$. For $\Omega\Subset\tilde{\Omega}\Subset\mathbb{C}^n$ and $f\in\mathcal{E}_{m,\chi}(\Omega)$, we say that $\tilde{f}\in\mathcal{E}_{m,\chi}(\tilde{\Omega})$ is a subextention of f if $\tilde{f}\leq f$ on Ω.

The problem of subextention in the case $m = n$ and $\chi \equiv -1$ was studied by Cegrell and Zeriahi [9] in 2003 and it was investigated by Cegrell, Kolodziej, and Zeriahi [5] for the case of plurisubharmonic functions with weak singularities. Then this problem was studied for the class $\mathcal{E}_\chi(\Omega)$ $(m = n)$ by Benelkourchi in [3]. In this paper, we study the problem in the class $\mathcal{E}_{m,\chi}(\Omega)$. In the following theorem, we prove that every function $f \in \mathcal{E}_{m,\chi}(\Omega)$ has a subextention.

Theorem 4.12. *Let $\tilde{\Omega}$ be an $m-$hyperconvex domain such that $\Omega \Subset \tilde{\Omega} \Subset \mathbb{C}^n$. If $\chi(t) < 0$ for all $t < 0$ and $f \in \mathcal{E}_{m,\chi}(\Omega)$, then there is $\tilde{f} \in \mathcal{E}_{m,\chi}(\tilde{\Omega})$ satisfying*

$$\int_{\tilde{\Omega}} -\chi(\tilde{f})H_m(\tilde{f}) \leq \int_{\Omega} -\chi(f)H_m(f)$$

and $\tilde{f} \leq f$ on Ω.

Proof. Let $f \in \mathcal{E}_{m,\chi}(\Omega)$ and $f_k \in \mathcal{E}^0_m(\Omega)$ be the sequence as in the definition of the class $\mathcal{E}_{m,\chi}(\Omega)$. By using Lemma 3.2 from [12], we obtain that for every $k \in \mathbb{N}$, there exists a subextension $\tilde{f}_k$ of f_k. It follows that

$$\begin{aligned} \int_{\tilde{\Omega}} -\chi(\tilde{f}_k)H_m(\tilde{f}_k) &= \int_{\{\tilde{f}_k = f_k\}\cap\Omega} -\chi(\tilde{f}_k)H_m(\tilde{f}_k) \\ &\leq \int_{\{\tilde{f}_k = f_k\}\cap\Omega} -\chi(f_k)H_m(f_k) \leq \int_{\Omega} -\chi(f_k)H_m(f_k). \end{aligned}$$

So, we obtain

$$\sup_k \int_{\tilde{\Omega}} -\chi(\tilde{f}_k)H_m(\tilde{f}_k) \leq \int_{\Omega} -\chi(f)H_m(f) < \infty. \tag{4.3}$$

Using Proposition 4.8, we get that the function $\tilde{f} = \lim_{k\to\infty} \tilde{f}_k \not\equiv -\infty$ and $\tilde{f} \in \mathcal{E}_{m,\chi}(\tilde{\Omega})$. Then, by 4.3,

$$\int_{\tilde{\Omega}} -\chi(\tilde{f})H_m(\tilde{f}) \leq \int_{\Omega} -\chi(f)H_m(f) < \infty.$$

By the Comparison Principle, it follows that for all $k \in \mathbb{N}$ one has $\tilde{f}_k \leq f_k$ on Ω. If we let k go to ∞, we deduce that $\tilde{f} \leq f$ on Ω. □

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Про деякі вагові класи m-субгармонічних функцій

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У цій роботі ми вивчаємо клас $\mathcal{E}_m(\Omega)$ m-субгармонічних функцій, введений Лю в [18]. Ми доводимо, що збіжність мір Гессе виводиться зі збіжності відносно m-ємності для функцій, що належать $\mathcal{E}_m(\Omega)$ та задовольняють певні додаткові умови. Далі ми розповсюджуємо ці результати на клас $\mathcal{E}_{m,\chi}(\Omega)$, який залежить від заданої дійсної функції χ. Дано повну характеризацію цих класів за допомогою міри Гессе, а також теорему підпродовження відносно $\mathcal{E}_{m,\chi}(\Omega)$.

Ключові слова: m-субгармонічна функція, ємність, оператор Гессе, збіжність відносно m-ємності