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Controllability Problems for the Heat Equation on a Half-Plane Controlled by the Neumann Boundary Condition with a Point-Wise Control

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In the paper, the problems of controllability and approximate controllability are studied for the control system $w_t = \Delta w$, $w_{x_1}(0, x_2, t) = u(t)\delta(x_2)$, $x_1 > 0$, $x_2 \in \mathbb{R}$, $t \in (0, T)$, where $u \in L^\infty(0, T)$ is a control. To this aid, it is investigated the set $\mathcal{R}_T(0) \subset L^2((0, +\infty) \times \mathbb{R})$ of its end states which are reachable from 0. It is established that a function $f \in \mathcal{R}_T(0)$ can be represented in the form $f(x) = g\big(|x|^2\big)$ a.e. in $(0, +\infty) \times \mathbb{R}$ where $g \in L^2(0, +\infty)$. In fact, we reduce the problem dealing with functions from $L^2((0, +\infty) \times \mathbb{R})$ to a problem dealing with functions from $L^2(0, +\infty)$. Both a necessary and sufficient condition for controllability and a sufficient condition for approximate controllability in a given time T under a control u bounded by a given constant are obtained in terms of solvability of a Markov power moment problem. Using the Laguerre functions (forming an orthonormal basis of $L^2(0, +\infty)$), necessary and sufficient conditions for approximate controllability and numerical solutions to the approximate controllability problem are obtained. It is also shown that there is no initial state that is null-controllable in a given time T. The results are illustrated by an example.

Key words: heat equation, controllability, approximate controllability, half-plane

Mathematical Subject Classification 2020: 93B05, 35K05, 35B30

1. Introduction

In the paper, the controllability problems for the heat equation are studied on a half-plane. Note that these problems for the heat equation were studied both in bounded and unbounded domains. However, most of the papers studying these problems deal with domains bounded with respect to the spatial variables (see some recent papers: [3, 4, 6, 21, 23, 36, 37, 43], and the references therein). At the same time, there are quite a few papers considering domains unbounded with respect to the spatial variables [2, 5, 7–9, 15–20, 24, 30–33, 35, 39, 40, 42].

A point-wise control is a mathematical model of a source supported in a domain of very small size with respect to the while domain. That is why studying control problems under a point-wise control is an important issue in control theory (see, e.g. [10, 11, 27, 29, 34] and others).

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In [32], the boundary controllability of the 2-d heat equation was studied in a half-space. Using similarity variables and weighted Sobolev spaces and developing solutions in Fourier series reduce the control problem to a sequence of one-dimensional controlled systems. The null-controllability properties of these systems had been studied in [31]. It had been proved that no initial datum belonging to any Sobolev space of negative order may be driven to zero in finite time. In [32], it was established that if all the corresponding 1-d problems are null-controllable, then the multidimensional problem is null-controllable. However, it was also proved that if there exists at least one 1-d problem which is not null-controllable, then the multi-dimensional problem is not null-controllable. The results of the one-dimensional case was applied to obtain the corresponding results for the multi-dimensional case.

The controllability problems for the heat equation on a half-plane controlled by the Dirichlet boundary condition with a point-wise control were studied in [17]. Both necessary and sufficient conditions for controllability and sufficient conditions for approximate controllability in a given time under a control bounded by a given constant were obtained in terms of solvability of a Markov power moment problem. Orthogonal bases in special spaces of Sobolev type (consisting of functions of two variables) were constructed by using the generalized Laguerre polynomials. Applying these bases, necessary and sufficient conditions for approximate controllability and numerical solutions to the approximate controllability problem were obtained.

The boundary controllability of the wave equation on a half-plane $x_1 > 0$, $x_2 \in \mathbb{R}$ with a pointwise control on the boundary was studied in [12–14].

Consider the following control system on a half-plane

$$w_t = \Delta w, \qquad x_1 > 0,\ x_2 \in \mathbb{R},\ t \in (0,T), \tag{1.1}$$

$$w_{x_1}\big(0, (\cdot)_{[2]}, t\big) = \delta_{[2]} u(t), \qquad x_2 \in \mathbb{R},\ t \in (0,T), \tag{1.2}$$

$$w\big((\cdot)_{[1]}, (\cdot)_{[2]}, 0\big) = w^0, \qquad x_1 > 0,\ x_2 \in \mathbb{R}, \tag{1.3}$$

where $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$, $T > 0$, $u \in L^\infty(0,T)$ is a control, $\delta_{[m]}$ is the Dirac distribution with respect to x_m, $m = 1, 2$. The subscripts [1] and [2] associate with the variable numbers, e.g., $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. This control system is considered in spaces of Sobolev type (see details in Section 2). We treat equality (1.2) as the value of the distribution w_{x_1} on the line $x_1 = 0$ (see Definition 2.3 below).

In Section 2, some notation and definitions are given.

In Section 3, control problem (1.1)–(1.3) is reduced to control problem (3.1), (3.2) (see below) by using the even extension with respect to x_1 for $w(\cdot,t)$ and w^0, $t \in [0,T]$. It is proved that systems (1.1)–(1.3) and (3.1), (3.2) are equivalent so, basing on this reason, we consider control system (3.1), (3.2) (dealing with functions defined on $\mathbb{R}^2$) instead of control system (1.1)–(1.3) (dealing with functions defined on $[0,+\infty) \times \mathbb{R}$). The set $\mathcal{R}_T(0) \subset L^2\big(\mathbb{R}^2\big)$ of its states reachable from 0 (i.e. the set which is formed by the end states $w(\cdot,T)$ of control system (3.1), (3.2) when controls $u \in L^\infty(0,T)$) and the set $\mathcal{R}_T^L(0) \subset \mathcal{R}_T(0) \subset L^2\big(\mathbb{R}^2\big)$ of its states reachable from 0 by using the controls $u \in L^\infty(0,T)$ satisfying the restriction $\|u\|_{L^\infty(0,T)} \le L$ (where $L > 0$ is a given constant) are studied. In particular, properties of the solutions (Theorem 3.4) and properties of the reachability sets $\mathcal{R}_T(0)$ and $\mathcal{R}_T^L(0)$ (Theorem 3.5) are proved for this system. It is also established that a function $f \in \mathcal{R}_T(0)$ can be represented in the form $f(x) = g\big(|x|^2\big)$ a.e. in $\mathbb{R}^2$ where $g \in L^2(0,+\infty)$. Therefore, the functions g form the dual sets $\mathbf{R}_T$ and $\mathbf{R}_T^L$ for the sets $\mathcal{R}_T(0)$ and $\mathcal{R}_T^L(0)$, respectively. In fact, the problem dealing with functions from $L^2\big(\mathbb{R}^2\big)$ is reduced to a problem dealing with functions from $L^2(0,+\infty)$. To this aid, operators Ψ and Φ are introduced and studied. The results mentioned above are applied in Sections 3–6. In Section 3 the following assertions are formulated for system (3.1), (3.2):

1) some additional properties of the set $\mathcal{R}_T^L(0)$ (Theorems 3.6–3.9, 3.11, and 3.13);
2) necessary and sufficient conditions for controllability in a given time under the control bounded by a given constant (Corollary 3.10);
3) sufficient conditions for approximate controllability in a given time under the control bounded by a given constant (Corollary 3.12);
4) necessary and sufficient conditions for approximate controllability in a given time (Theorem 3.14);
5) the lack of controllability to the origin (Theorem 3.15).

In Section 4, properties of the sets $\mathbf{R}_T$ and $\mathbf{R}_T^L$ are established (Theorems 4.1, 4.2, 4.4–4.6, and 4.8). In the proof of Theorem 4.8, an algorithm for construction of controls solving the approximate controllability problem for system (3.1), (3.2) is given.

In Section 5, Theorem 3.14 is illustrated by an example.

The results of Section 4 are applied in the proofs of Theorems 3.6–3.9, 3.11, and 3.13 in Section 6. In this section Theorems 3.4 and 3.15 are also proved.

The main results of the present paper are rather similar to those of [17]. However, the methods of obtaining them are essentially different in these two papers. Roughly speaking, we deal with the two-dimensional case studying reachability sets and constructing the solutions to controllability and approximate controllability problems in [17], but reducing the two-dimensional reachability sets to the one-dimensional ones, we deal with the one-dimensional case studying these problems and constructing their solutions in the present paper. In addition, the methods used to study the one-dimensional reachability sets in this paper principally differ from those used for two-dimensional sets in [17]. That is why Theorems 3.9, 3.11 and Corollaries 3.10, 3.12 in the present paper also differ from their analogues from [17]. Moreover, Theorems 3.6–3.8 have not analogues in [17].

2. Notation and preliminary results

Let $n \in \mathbb{N}$. By $|\cdot|$, we denote the Euclidean norm in $\mathbb{R}^n$.

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions [38]. Put $\mathscr{S} = \mathscr{S}(\mathbb{R})$. Let $\mathscr{S}'(\mathbb{R}^n)$ ($\mathscr{S}'$) be the dual space for $\mathscr{S}(\mathbb{R}^n)$ ($\mathscr{S}$, respectively).

Denote $\mathbb{R}_+ = (0, +\infty)$. Let $\mathscr{D}(\mathbb{R}_+)$ be the space of infinitely differentiable functions on $\mathbb{R}$ whose supports are compact and they are contained in $\mathbb{R}_+$.

Let $D = \big(-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n\big)$, $D^\alpha = \big(-i(\partial/\partial x_1)^{\alpha_1}, \ldots, -i(\partial/\partial x_n)^{\alpha_n}\big)$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Consider the following Sobolev spaces [22, Chap. 1]

$$H^s(\mathbb{R}^n) = \left\{\varphi \in L^2(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n \quad \big(|\alpha| \le s \Rightarrow D^\alpha \varphi \in L^2(\mathbb{R}^n)\big)\right\}, \quad s = \overline{0,2},$$

with the norm

$$\|\varphi\|^s = \left(\sum_{|\alpha| \le s} \frac{s!}{(s - |\alpha|)!\alpha!} \Big(\|D^\alpha \varphi\|_{L^2(\mathbb{R}^n)}\Big)^2\right)^{1/2}, \quad \varphi \in H^s(\mathbb{R}^n).$$

We have $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^*$ with the norm $\|\cdot\|^{-s}$ associated with the strong topology of the adjoint space. Evidently, $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n) = \big(H^0(\mathbb{R}^n)\big)^*$.

For $s = \overline{0,2}$, denote

$$H^s_{⓪} = \Big\{\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \;\Big|\; \Big(\forall \alpha \in \mathbb{N}_0^2 \quad \big(\alpha_1 + \alpha_2 \le s \Rightarrow D^\alpha \varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})\big)\Big) \\ \wedge \Big(\forall k = \overline{0, s-1} \quad D^{(k,0)} \varphi(0^+, (\cdot)_{[2]}) = 0\Big)\Big\}$$

with the norm

$$\|\varphi\|^s_{⓪} = \left(\sum_{\alpha_1 + \alpha_2 \le s} \frac{s!}{(s - (\alpha_1 + \alpha_2))!\alpha_1!\alpha_2!} \Big(\|D^\alpha \varphi\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}\Big)^2\right)^{1/2}, \quad \varphi \in H^s_{⓪},$$

and $H^{-s}_{⓪} = \Big(H^s_{⓪}\Big)^*$ with the norm $\|\cdot\|^{-s}_{⓪}$ associated with the strong topology of the adjoint space. Obviously, $H^0_{⓪} = L^2(\mathbb{R}_+ \times \mathbb{R}) = \Big(H^0_{⓪}\Big)^*$.

Consider also the spaces [22, Chap. 1]

$$H_m(\mathbb{R}^n) = \left\{\psi \in L^2_{\text{loc}}(\mathbb{R}^n) \;\Big|\; \big(1 + |\sigma|^2\big)^{m/2} \psi \in L^2(\mathbb{R}^n)\right\}, \quad m = \overline{-2, 2},$$

with the norm

$$\|\psi\|_m = \Big\|\big(1 + |\sigma|^2\big)^{m/2} \psi\Big\|_{L^2(\mathbb{R}^n)}, \quad \psi \in H_m(\mathbb{R}^n).$$

Evidently, $H_{-m}(\mathbb{R}^n) = (H_m(\mathbb{R}^n))^*$. It is easy to see that $H_0(\mathbb{R}^n) = H^0(\mathbb{R}^n)$.

By $\langle f, \varphi\rangle$, denote the value of a distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ on a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

Let $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ be the Fourier transform operator with the domain $\mathscr{S}'(\mathbb{R}^n)$. This operator is an extension of the classical Fourier transform operator and is given by the formula

$$\langle \mathscr{F} f, \varphi\rangle = \langle f, \mathscr{F}^{-1} \varphi\rangle, \quad f \in \mathscr{S}'(\mathbb{R}^n),\; \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Due to [22, Chap. 1], the operator $\mathscr{F}$ is an isometric isomorphism of $H^m(\mathbb{R}^n)$ and $H_m(\mathbb{R}^n)$, $m = \overline{-2, 2}$.

In the spaces $H_m(\mathbb{R}^2)$ and $H^m(\mathbb{R}^2)$, $m = \overline{-2, 2}$, we consider the following inner products

$$\langle f, g\rangle_m = \left\langle \left(1+|\sigma|^2\right)^{m/2} f, \left(1+|\sigma|^2\right)^{m/2} g \right\rangle_0, \quad f \in H_m(\mathbb{R}^2),\ g \in H_m(\mathbb{R}^2),$$
$$\langle h, p\rangle^m = \langle \mathscr{F}h, \mathscr{F}p\rangle_m, \qquad h \in H^m(\mathbb{R}^2),\ p \in H^m(\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle_0$ is the inner product in $L^2(\mathbb{R}^2)$. Note that $\langle\cdot,\cdot\rangle^0 = \langle\cdot,\cdot\rangle_0$.

A distribution $f \in \mathscr{S}'(\mathbb{R}^2)$ is said to be *odd with respect to* x_1, if $\langle f, \varphi\big((\cdot)_{[1]}, (\cdot)_{[2]}\big)\rangle = -\langle f, \varphi\big(-(\cdot)_{[1]}, (\cdot)_{[2]}\big)\rangle$, where $\varphi \in \mathscr{S}(\mathbb{R}^2)$. A distribution $f \in \mathscr{S}'(\mathbb{R}^2)$ is said to be *even with respect to* x_1, if $\langle f, \varphi\big((\cdot)_{[1]}, (\cdot)_{[2]}\big)\rangle = \langle f, \varphi\big(-(\cdot)_{[1]}, (\cdot)_{[2]}\big)\rangle$, where $\varphi \in \mathscr{S}(\mathbb{R}^2)$.

Let $m = \overline{-2, 2}$. By $\widehat{H}^m\left(\mathbb{R}^2\right)$ (or $\widehat{H}_m\left(\mathbb{R}^2\right)$), denote the subspace of all distributions in $H^m\left(\mathbb{R}^2\right)$ (or $H_m\left(\mathbb{R}^2\right)$, respectively) that are even with respect to x_1. Evidently, $\widehat{H}^m\left(\mathbb{R}^2\right)$ (or $\widehat{H}_m\left(\mathbb{R}^2\right)$) is a closed subspace of $H^m\left(\mathbb{R}^2\right)$ (or $H_m\left(\mathbb{R}^2\right)$, respectively).

Let $f \in L^2\big(\mathbb{R}^2\big)$. Let also $f(x) = g(|x|^2)$, $x \in \mathbb{R}^2$, where g is a function defined on $\mathbb{R}_+$. Setting $x_1 = \sqrt{r}\cos\phi$, $x_2 = \sqrt{r}\sin\phi$, $r \in \mathbb{R}_+$, $\phi \in [0, 2\pi)$, we get

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}^2)} &= \left(\iint_{\mathbb{R}^2} |f(x)|^2\,dx\right)^{1/2} = \left(\iint_{\mathbb{R}^2} |g(|x|^2)|^2\,dx\right)^{1/2} \\ &= \left(\pi\int_0^\infty |g(r)|^2\,dr\right)^{1/2} = \sqrt{\pi}\|g\|_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{2.1}$$

Thus, if $f \in L^2\big(\mathbb{R}^2\big)$ and $f(x) = g(|x|^2)$, $x \in \mathbb{R}^2$, for some g defined on $\mathbb{R}_+$, then $g \in L^2(\mathbb{R}_+)$ and (2.1) holds; and vice versa: if $g \in L^2(\mathbb{R}_+)$, then for $f(x) = g(|x|^2)$, $x \in \mathbb{R}^2$, we have $f \in L^2\big(\mathbb{R}^2\big)$.

Taking this into account, we can introduce the space

$$\mathcal{H} = \left\{ f \in L^2\big(\mathbb{R}^2\big) \;\middle|\; \exists g \in L^2(\mathbb{R}_+) \quad f(x) = g(|x|^2) \text{ a.e. on } \mathbb{R}^2 \right\} \tag{2.2}$$

and the operator $\Psi : \mathcal{H} \to L^2(\mathbb{R}_+)$ with the domain $D\big(\Psi\big) = \mathcal{H}$ for which

$$\Psi f = g \Leftrightarrow \big(f(x) = g(|x|^2) \text{ a.e. on } \mathbb{R}^2\big), \quad f \in D\big(\Psi\big) = \mathcal{H}.$$

One can see that Ψ is invertible, $\Psi^{-1} : L^2(\mathbb{R}_+) \to \mathcal{H}$, and $\big(\Psi^{-1}g\big)(x) = g(|x|^2)$, $x \in \mathbb{R}^2$ for $g \in D\big(\Psi^{-1}\big) = L^2(\mathbb{R}_+)$.

Summarising, we obtain the following proposition.

Proposition 2.1. *The following assertions hold:*

(i) Ψ *is an isomorphism of* $\mathcal{H}$ *and* $L^2(\mathbb{R}_+)$;
(ii) $\mathcal{H}$ *is a subspace of* $\widehat{H}^0\left(\mathbb{R}^2\right) = \widehat{H}_0\left(\mathbb{R}^2\right) \subset L^2\big(\mathbb{R}^2\big)$;
(iii) $\mathcal{H}$ *is a Hilbert space with respect to the inner product* $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^2)}$;
(iv) $\langle f, h\rangle_{L^2(\mathbb{R}^2)} = \pi\langle \Psi f, \Psi h\rangle_{L^2(\mathbb{R}_+)}$, $f \in \mathcal{H}$, $h \in \mathcal{H}$;

(v) $\|\Psi\| = 1/\sqrt{\pi}$;

(vi) $\mathscr{F}\mathcal{H} = \mathcal{H}$.

Let $f \in \mathcal{H}$, $F = \mathscr{F}f$, $g = \Psi f$, and $G = \Psi F$. Then $G = \Psi\mathscr{F}\Psi^{-1}g$.

Let us introduce the operator $\Phi : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with the domain $D(\Phi) = L^2(\mathbb{R}_+)$ by the rule

$$\Phi g = \Psi\mathscr{F}\Psi^{-1}g, \quad g \in D(\Phi) = L^2(\mathbb{R}_+).$$

Since the operator Ψ is an isomorphism of $\mathcal{H}$ and $L^2(\mathbb{R}_+)$ (see Proposition 2.1) and $\mathscr{F}f = \mathscr{F}^{-1}f$ for $f \in \mathcal{H}$, we conclude that Φ is invertible and $\Phi^{-1} = \Phi$, in particular, Φ is an isometric isomorphism of $L^2(\mathbb{R}_+)$.

Let us find a formula for calculating Φg if $g \in L^2(\mathbb{R}_+)$. Put $G = \Phi g$. Setting $x_1 = \sqrt{r}\cos\phi$, $x_2 = \sqrt{r}\sin\phi$, $r \in \mathbb{R}_+$, $\phi \in [0, 2\pi)$, and $\sigma_1 = \sqrt{\rho}\cos\theta$, $\sigma_2 = \sqrt{\rho}\sin\theta$, $\rho \in \mathbb{R}_+$, $\theta \in [0, 2\pi)$, we get

$$\begin{aligned} G(\rho) = \big(\mathscr{F}\Psi^{-1}g\big)(\sigma) &= \frac{1}{2\pi}\lim_{N\to\infty}\iint_{|x|\le N^2}\big(\Psi^{-1}g\big)(x)e^{-i\langle x,\sigma\rangle}\,dx \\ &= \frac{1}{4\pi}\lim_{N\to\infty}\int_0^N g(r)\int_0^{2\pi} e^{-i\sqrt{r\rho}\cos\phi}\,d\phi\,dr \\ &= \frac{1}{2}\lim_{N\to\infty}\int_0^N g(r)J_0\big(\sqrt{r\rho}\big)\,dr, \quad \rho \in \mathbb{R}_+, \end{aligned} \tag{2.3}$$

where J_0 is the Bessel function of order 0. Here the relation

$$J_0(\xi) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\xi\cos\phi}\,d\phi, \quad \xi \in \mathbb{R},$$

has been used.

Summarising, we obtain the following proposition.

Proposition 2.2. *The following assertions hold:*

(i) Φ *is invertible and* $\Phi^{-1} = \Phi$;

(ii) $\big(\Phi g\big)(\rho) = \dfrac{1}{2}\lim_{N\to\infty}\displaystyle\int_0^N g(r)J_0\big(\sqrt{r\rho}\big)\,dr$, $\rho \in \mathbb{R}_+$, $g \in L^2(\mathbb{R}_+)$.

With regard to Proposition 2.2 (ii), one can see that the transform providing by the operator Φ is a modification of the well-known Hankel transform of order 0.

Let $g \in H_{-s}(\mathbb{R}^2)$, $s = \overline{0,2}$. By a similar reasoning to that of [17], we get $g\big(\sigma_1, (\cdot)_{[2]}\big) \in H_{-s}(\mathbb{R})$ for almost all $\sigma_1 \in \mathbb{R}$, and $g\big((\cdot)_{[1]}, \sigma_2\big) \in H_{-s}(\mathbb{R})$ for almost all $\sigma_2 \in \mathbb{R}$.

Let $\psi \in H_s(\mathbb{R})$, $s = \overline{0,2}$. Denote

$$\langle g, \psi\rangle_{[1]}(\sigma_2) = \int_{-\infty}^{\infty}\big(1+\sigma_1^2\big)^{-s} g(\sigma_1,\sigma_2)\big(1+\sigma_1^2\big)^{s}\,\overline{\psi(\sigma_1)}\,d\sigma_1, \qquad \sigma_2 \in \mathbb{R},$$

$$\langle g, \psi\rangle_{[2]}(\sigma_1) = \int_{-\infty}^{\infty}\big(1+\sigma_2^2\big)^{-s} g(\sigma_1,\sigma_2)\big(1+\sigma_2^2\big)^{s}\,\overline{\psi(\sigma_2)}\,d\sigma_2, \qquad \sigma_1 \in \mathbb{R}.$$

Then,

$$\langle g, \psi\rangle_{[1]} \in H_{-s}(\mathbb{R}) \quad \text{and} \quad \langle g, \psi\rangle_{[2]} \in H_{-s}(\mathbb{R}).$$

Let $f \in H^{-s}(\mathbb{R}^2)$, $\varphi \in H^s(\mathbb{R})$, $s = \overline{0,2}$. Denote

$$\langle f, \varphi\rangle_{[1]} = \mathscr{F}^{-1}_{\sigma_2 \to x_2} \left(\langle \mathscr{F}_{x\to\sigma} f, \mathscr{F}\varphi\rangle_{[1]}\right) \text{ and } \langle f, \varphi\rangle_{[2]} = \mathscr{F}^{-1}_{\sigma_1 \to x_1} \left(\langle \mathscr{F}_{x\to\sigma} f, \mathscr{F}\varphi\rangle_{[2]}\right).$$

Since the operator $\mathscr{F}$ is an isometric isomorphism of $H^m(\mathbb{R}^n)$ and $H_m(\mathbb{R}^n)$, $m = \overline{-2,2}$, we get

$$\langle f, \varphi\rangle_{[1]} \in H^{-s}(\mathbb{R}) \quad \text{and} \quad \langle f, \varphi\rangle_{[2]} \in H^{-s}(\mathbb{R}).$$

The following definition is given with regard to the definition of a distribution's value at a point [1, Chap. 1] and to the definition of a distribution's value on a line [14].

Definition 2.3. Let $s = 1, 2$. We say that a distribution $f \in H^{-s}_{⓪}$ has the value $f_0 \in H^{-s}(\mathbb{R})$ on the line $x_1 = 0$, i.e. $f\big(0^+, (\cdot)_{[2]}\big) = f_0\big((\cdot)_{[2]}\big)$, if for each $\varphi \in H^s(\mathbb{R})$ and $\psi \in \mathscr{D}(\mathbb{R}_+)$ we have

$$\left\langle \left\langle f(\alpha(\cdot)_{[1]}, (\cdot)_{[2]}), \varphi((\cdot)_{[2]}) \right\rangle_{[2]}, \psi((\cdot)_{[1]}) \right\rangle_{[1]} \to \left\langle \langle f_0, \varphi\rangle_{[2]}, \psi \right\rangle_{[1]} \quad \text{as } \alpha \to 0^+, \tag{2.4}$$

where $\langle h(\alpha(\cdot)), \psi\rangle = \left\langle h((\cdot)), \frac{1}{\alpha}\psi\left(\frac{(\cdot)}{\alpha}\right)\right\rangle$ for $h \in H^{-s}(\mathbb{R})$.

Remark 2.4. Let $\varphi \in H^s_{⓪}$, $s = \overline{0,2}$. Let $\widehat{\varphi}$ be its even extension with respect to x_1, i.e., $\widehat{\varphi}(x_1, x_2) = \varphi(x_1, x_2)$ if $x_1 \geq 0$ and $\widehat{\varphi}(x_1, x_2) = \varphi(-x_1, x_2)$ if $x_1 < 0$, $x_2 \in \mathbb{R}$. Then $\widehat{\varphi} \in \widehat{H}^s\left(\mathbb{R}^2\right)$, $s = \overline{0,2}$. The converse assertion is true for $s = 0, 1$, and it is not true for $s = 2$. That is why the even extension with respect to x_1 of a distribution $f \in H^{-2}_{⓪}$ may not belong to $\widehat{H}^{-2}\left(\mathbb{R}^2\right)$. However, the following theorem holds.

Theorem 2.5. *Let $f \in H^0_{⓪}$ and there exist $f_{x_1}\big(0^+, (\cdot)_{[2]}\big) \in H^{-1}(\mathbb{R})$. Then $f_{x_1x_1} \in H^{-2}_{⓪}$ can be extended to a distribution $F \in \widehat{H}^{-2}\left(\mathbb{R}^2\right)$ such that F is even with respect to x_1. This distribution is given by the formula*

$$F = \widehat{f}_{x_1x_1} - 2f_{x_1}(0^+, (\cdot)_{[2]})\delta_{[1]}, \tag{2.5}$$

where $\widehat{f}$ is the even extension of f with respect to x_1.

In the case $f \in H^{1/2}_{⓪}$, corresponding theorem has been proved in [14]. The proof of Theorem 2.5 is analogous to the proof of the mentioned theorem.

3. Problem formulation and main results

We consider control system (1.1)–(1.3) in $H^{-l}_{⓪}$, $l = \overline{0,2}$, i.e. $\left(\frac{d}{dt}\right)^s w : [0,T] \to H^{-2s}_{⓪}$, $s = 0, 1$, $w^0 \in H^0_{⓪}$. We treat equality (1.2) as the value of the distribution w at $x_1 = 0$ with regard to Definition 2.3.

Let $w^0, w(\cdot,t) \in H^0_{\textcircled{0}}$, $t \in [0,T]$. Let W^0 and $W(\cdot,t)$ be the even extensions of w^0 and $w(\cdot,t)$ with respect to x_1, respectively, $t \in [0,T]$. Consider the control system

$$W_t = \triangle W - 2\delta u(t), \qquad t \in (0,T), \tag{3.1}$$

$$W\big((\cdot)_{[1]}, (\cdot)_{[2]}, 0\big) = W^0, \tag{3.2}$$

where $\left(\frac{d}{dt}\right)^s W : [0,T] \to \widehat{H}^{-2s}\left(\mathbb{R}^2\right)$, $s = 0, 1$, $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$, δ is the Dirac distribution in $\mathscr{S}'\left(\mathbb{R}^2\right)$.

Theorem 3.1.

1. *Let $w^0 \in H^0_{\textcircled{0}}$. If w is a solution to control system (1.1)–(1.3), then W, its even extension with respect to x_1, is a solution to control system (3.1), (3.2).*
2. *Let $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$. If W is a solution to control system (3.1), (3.2), then w, its restriction to $\mathbb{R}_+ \times \mathbb{R} \times [0,T]$, is a solution to control system (1.1)–(1.3).*

Proof. 1. Let w be a solution to control system (1.1)–(1.3). According to Theorem 2.5, W is a solution to control system (3.1), (3.2).

2. Let W be a solution to (3.1), (3.2). Let w^0 and $w(\cdot,t)$ be the restrictions of W^0 and $W(\cdot,t)$ to $\mathbb{R}_+ \times \mathbb{R}$, respectively, $t \in [0,T]$. According to Lemma 6.1 (see below),

$$W_{x_1}\big(0^+, (\cdot)_{[2]}, t\big) = \delta_{[2]} u(t) \quad \text{for almost all } t \in (0,T]. \tag{3.3}$$

Therefore, w is a solution to (1.1)–(1.3). □

Due to Theorem 3.1, control systems (1.1)–(1.3) and (3.1), (3.2) are equivalent. Therefore, basing on this reason, we will further consider control system (3.1), (3.2) instead of original system (1.1)–(1.3).

Let $T > 0$, $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$. By $\mathcal{R}_T\left(W^0\right)$, denote the set of all states $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ for which there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution W to system (3.1), (3.2) such that $W\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) = W^T$.

Definition 3.2. A state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ is said to be controllable to a target state $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if $W^T \in \mathcal{R}_T\left(W^0\right)$.

In other words, a state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ is said to be controllable to a target state $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution W to system (3.1), (3.2) and $W\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) = W^T$.

Definition 3.3. A state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ is said to be approximately controllable to a target state $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if $W^T \in \overline{\mathcal{R}_T\left(W^0\right)}$, where the closure is considered in the space $\widehat{H}^0\left(\mathbb{R}^2\right)$.

In other words, a state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ is approximately controllable to a target state $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if for each $\varepsilon > 0$, there exists a control $u_\varepsilon \in L^\infty(0,T)$ such that there exists a unique solution W_ε to system (3.1), (3.2) with $u = u_\varepsilon$ and $\left\|W_\varepsilon\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) - W^T\right\|^0 < \varepsilon$.

Using the Poisson integral (see, e.g., [41]), we obtain the unique solution to system (3.1), (3.2)

$$W(x,t) = \mathcal{W}_0(x,t) + \mathcal{W}_u(x,t), \quad x \in \mathbb{R}^2,\ t \in [0,T], \tag{3.4}$$

where

$$\mathcal{W}_0(x,t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}} * W^0(x), \qquad x \in \mathbb{R}^2,\ t \in [0,T], \tag{3.5}$$

$$\mathcal{W}_u(x,t) = -\frac{1}{\pi}\int_0^t \frac{1}{2\xi} e^{-\frac{|x|^2}{4\xi}} u(t-\xi)\, d\xi, \qquad x \in \mathbb{R}^2,\ t \in [0,T]. \tag{3.6}$$

Set $U_T^L = \{v \in L^\infty(0,T) \mid \|v\|_{L^\infty(0,T)} \le L\}$ for $L > 0$ and $T > 0$.

According to (3.4), we have

$$\mathcal{R}_T\big(W^0\big) = \left\{W^T \in \widehat{H}^0\left(\mathbb{R}^2\right) \,\middle|\, \exists u \in L^\infty(0,T) \;\; W^T = \mathcal{W}_0(\cdot,T) + \mathcal{W}_u(\cdot,T)\right\}, \tag{3.7}$$

in particular,

$$\mathcal{R}_T(0) = \left\{W^T \in \widehat{H}^0\left(\mathbb{R}^2\right) \,\middle|\, \exists u \in L^\infty(0,T) \;\; W^T = \mathcal{W}_u(\cdot,T)\right\}. \tag{3.8}$$

Denote also

$$\mathcal{R}_T^L\big(W^0\big) = \left\{W^T \in \widehat{H}^0\left(\mathbb{R}^2\right) \,\middle|\, \exists u \in U_T^L \;\; W^T = \mathcal{W}_0(\cdot,T) + \mathcal{W}_u(\cdot,T)\right\}, \tag{3.9}$$

$$\mathcal{R}_T^L(0) = \left\{W^T \in \widehat{H}^0\left(\mathbb{R}^2\right) \,\middle|\, \exists u \in U_T^L \;\; W^T = \mathcal{W}_u(\cdot,T)\right\}. \tag{3.10}$$

We obtain the following properties of a solution to system (3.1), (3.2)

Theorem 3.4. *Let $u \in L^\infty(0,T)$, $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$. Then,*

(i) $\mathcal{W}_0(\cdot,t) \in \widehat{H}^0(\mathbb{R}^2)$, $t \in [0,T]$;

(ii) $\mathcal{W}_0(\cdot,t) \in C^\infty(\mathbb{R}^2)$, $t \in (0,T]$;

(iii) *if* $W^0 \in \mathcal{H}$, *then* $\mathcal{W}_0(\cdot,t) \in \mathcal{H}$, $t \in [0,T]$;

(iv) $\mathcal{W}_u(\cdot,t) \in \mathcal{H}$ *and* $\|\mathcal{W}_u(\cdot,t)\|^0 \le \frac{2}{\sqrt{\pi}}(t+1)\|u\|_{L^\infty(0,T)}$, $t \in [0,T]$.

The proof of the theorem is given in Section 6.

With regard to Theorem 3.4, we get the following properties of the sets $\mathcal{R}_T(g)$ and $\mathcal{R}_T^L(g)$.

Theorem 3.5. *Let $T > 0$, $g \in \widehat{H}^0\left(\mathbb{R}^2\right)$. We have*

(i) $\mathcal{R}_T(0) = \bigcup_{L>0} \mathcal{R}_T^L(0) \subset \mathcal{H}$;

(ii) $\mathcal{R}_T^L(0) \subset \mathcal{R}_T^{L'}(0)$, $0 < L < L'$;

(iii) $f \in \mathcal{R}_T^1(0) \Leftrightarrow Lf \in \mathcal{R}_T^L(0)$, $L > 0$;

(iv) $f \in \mathcal{R}_T^L(g) \Leftrightarrow \left(f - \frac{1}{4\pi T} e^{-\frac{|\cdot|^2}{4T}} * g\right) \in \mathcal{R}_T^L(0)$, $L > 0$;

(v) $f \in \mathcal{R}_T(g) \Leftrightarrow \left(f - \frac{1}{4\pi T} e^{-\frac{|\cdot|^2}{4T}} * g\right) \in \mathcal{R}_T(0)$.

Consider also the sets

$$\mathbf{R}_T = \Psi \mathcal{R}_T(0) \quad \text{and} \quad \mathbf{R}_T^L = \Psi \mathcal{R}_T^L(0), \tag{3.11}$$

where Theorem 3.4 (iv) is taken into account. Put $\mathcal{Y}_u(\cdot, t) = \Psi \mathcal{W}_u(\cdot, t)$, $t \in [0, T]$. Then, we have

$$\mathcal{Y}_u(r, t) = -\frac{1}{\pi} \int_0^t \frac{1}{2\xi} e^{-\frac{r}{4\xi}} u(t - \xi)\, d\xi, \quad r \in \mathbb{R}_+. \tag{3.12}$$

With regard to (3.8) and (3.10), we obtain

$$\mathbf{R}_T = \left\{ Y \in L^2(\mathbb{R}_+) \mid \exists u \in L^\infty(0, T) \quad Y = \mathcal{Y}_u(\cdot, T) \right\}, \tag{3.13}$$

$$\mathbf{R}_T^L = \left\{ Y \in L^2(\mathbb{R}_+) \mid \exists u \in U_T^L \quad Y = \mathcal{Y}_u(\cdot, T) \right\}. \tag{3.14}$$

Since Ψ is an isomorphism of $\mathcal{H}$ and $L^2(\mathbb{R}_+)$ (see Proposition 2.1 (i)), we have

$$\overline{\mathbf{R}_T} = \Psi \overline{\mathcal{R}_T(0)} \quad \text{and} \quad \overline{\mathbf{R}_T^L} = \Psi \overline{\mathcal{R}_T^L(0)}. \tag{3.15}$$

Properties of the sets $\mathbf{R}_T$, $\overline{\mathbf{R}_T}$, $\mathbf{R}_T^L$, and $\overline{\mathbf{R}_T^L}$ are studied in Section 4. By using these properties, we obtain the main results of the paper.

3.1. Controllability under controls bounded by a hard constant. Let us find conditions under which an initial state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ is controllable to a target state $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$.

First, consider necessary conditions for $f \in \mathcal{R}_T^L(0)$.

Theorem 3.6. *Let $L > 0$ and $T > 0$. If $f \in \mathcal{R}_T^L(0)$, then $f \in \mathcal{H}$ and we have*

$$|f(x)| \le \frac{L}{2\pi} e^{-\frac{|x|^2}{4T}} \ln\left(1 + \frac{4T}{|x|^2}\right), \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{3.16}$$

The proof of the theorem is given in Section 6.

Theorem 3.7. *Let $L > 0$ and $T > 0$. Let also $f \in \mathcal{R}_T^L(0)$, $F = \mathscr{F} f$ and $G = \Psi F$. Then G can be extended to an entire function G_e of order ≤ 1 and type $\le T$. Moreover, F can be also extended to an entire function F_e and*

$$F_e(s) = G_e\left(s_1^2 + s_2^2\right), \quad s = (s_1, s_1) \in \mathbb{C}^2, \tag{3.17}$$

and F_e is of order ≤ 2 and type $\le T$. In addition,

$$\left|F_e(s)\right| \le \left|G_e\left(s_1^2 + s_2^2\right)\right| \le \frac{L}{\pi} \frac{e^{T|s|^2} - 1}{|s|^2}, \quad s \in \mathbb{C}^2. \tag{3.18}$$

The proof of the theorem is given in Section 6.

According to Example 4.3 below, condition (3.16) is only necessary for $f \in \mathcal{R}_T^L(0)$, but it is not sufficient. However, if f satisfies (3.16), its Fourier transform can be extended to an entire function of order ≤ 2 and type $\leq T$ (cf. Theorem 3.7).

Theorem 3.8. *Let $L > 0$, $T > 0$, $f \in \mathcal{H}$, and condition (3.16) hold for f. Let also $F = \mathscr{F} f$ and $G = \Psi F$. Then G can be extended to an entire function G_e of order ≤ 1 and type $\leq T$. Moreover, F can be also extended to an entire function F_e, the extension F_e is given by (3.17), and F_e is of order ≤ 2 and type $\leq T$.*

The proof of the theorem is given in Section 6.

Thus, condition (3.16) is not sufficient for $f \in \mathcal{R}_T^L(0)$, but it guarantees the necessary condition from Theorem 3.7 holds for $\mathscr{F} f$.

Now, we consider a necessary and sufficient condition for controllability in a given time $T > 0$ under controls bounded by a hard constant. Denote

$$W_0^T = W^T - \mathcal{W}_0(\cdot, T). \qquad (3.19)$$

Theorem 3.9. *Let $L > 0$, $T > 0$, $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$, $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$. Let also $W_0^T \in \mathcal{H}$, condition (3.16) hold for W_0^T, and*

$$\omega_n = -2\frac{n!}{(2n)!}\int_0^\infty \int_0^\infty x_1^{2n} W_0^T(x_1, x_2)\, dx_1\, dx_2, \qquad n \in \mathbb{N}_0. \qquad (3.20)$$

Then $W^T \in \mathcal{R}_T^L\left(W^0\right)$ iff

$$\exists u \in U_T^L \ \forall n \in \mathbb{N}_0 \quad \int_0^T \xi^n u(T - \xi)\, d\xi = \omega_n. \qquad (3.21)$$

The proof of the theorem is given in Section 6.

Taking into account Definition 3.2, we get

Corollary 3.10. *Let $L > 0$, $T > 0$, $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$, $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$. Let also $W_0^T \in \mathcal{H}$, condition (3.16) hold for W_0^T, and $\{\omega_n\}_{n=1}^\infty$ be determined by (3.20). Then the state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ is controllable to the target state $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ iff (3.21) holds.*

Now, we consider a sufficient condition for approximate controllability in a given time $T > 0$ under controls bounded by a hard constant.

Theorem 3.11. *Let $L > 0$, $T > 0$, $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$, $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$, $W_0^T \in \mathcal{H}$, and condition (3.16) hold for W_0^T. Let $\{\omega_n\}_{n=0}^\infty$ be defined by (3.20). If for each $N \in \mathbb{N}$ there exists $u_N \in U_T^L$ such that*

$$\int_0^T \xi^n u_N(T - \xi) d\xi = \omega_n, \quad n = \overline{0, N}, \qquad (3.22)$$

then $W^T \in \overline{\mathcal{R}_T^L\left(W^0\right)}$, where the closure is considered in $\widehat{H}^0\left(\mathbb{R}^2\right)$.

The proof of the theorem is given in Section 6.

Taking into account Definition 3.3, we get

Corollary 3.12. *Let* $L > 0$, $T > 0$, $W^0 \in \widehat{H}^0(\mathbb{R}^2)$, $W^T \in \widehat{H}^0(\mathbb{R}^2)$, $W_0^T \in \mathcal{H}$, *and condition* (3.16) *hold for* W_0^T. *Let* $\{\omega_n\}_{n=0}^{\infty}$ *be defined by* (3.20). *If for each* $N \in \mathbb{N}$ *there exists* $u_N \in U_T^L$ *such that* (3.22) *holds, then the state* $W^0 \in \widehat{H}^0(\mathbb{R}^2)$ *is approximately controllable to the target state* $W^T \in \widehat{H}^0(\mathbb{R}^2)$ *in a given time* $T > 0$.

We can see that the controllability problems were reduced to the Markov power moment problems in Theorems 3.9 and 3.11. These Markov power moment problems may be solved by using the algorithms given in [25, 28]. Similar results were obtained for controllability problems for the heat equation on a half-axis [15, 16] and on a half-plane [17]. However, the description of the set $\mathcal{R}_T^L(W^0)$ is given in principally different way in the present paper (see Theorems 3.6, 3.7, 3.8). As a result, the necessary condition (3.16) essentially differs from the necessary conditions obtained in the mentioned papers: it is given in the form of an estimate for a function belonging to $\mathcal{R}_T^L(W^0)$ in contrast to the conditions in the form of estimates for integrals with special weights of a such function in [15–17]. In addition, as a consequence of the different necessary condition (3.16), the proofs of Theorems 3.9 and 3.11 also differ from their analogues in the mentioned papers.

3.2. Approximate controllability. Consider the problem of approximate controllability for system (3.1), (3.2) under controls from $L^\infty(0,T)$ unlike Subsection 3.1, where we consider this system under controls bounded by a hard constant. We have the following main theorem.

Theorem 3.13. *Let* $T > 0$. *We have* $\overline{\mathcal{R}_T(0)} = \mathcal{H}$.

The proof of the theorem is given in Section 6. This theorem yields

Theorem 3.14. *Let* $T > 0$. *A state* $W^0 \in \widehat{H}^0(\mathbb{R}^2)$ *is approximately controllable to a state* $W^T \in \widehat{H}^0(\mathbb{R}^2)$ *in a given time* T *iff* $W_0^T \in \mathcal{H}$.

3.3. Lack of controllability to the origin. For $W^0 \in \widehat{H}^0(\mathbb{R}^2)$ and $W^T \in \widehat{H}^0(\mathbb{R}^2)$ we have $W^T \in \overline{\mathcal{R}_T(W^0)}$ iff $W_0^T \in \mathcal{H}$ according to Theorem 3.14. However, $0 \notin \mathcal{R}_T(W^0)$ for all nonzero $W^0 \in \widehat{H}^0(\mathbb{R}^2)$, i.e. the following theorem holds.

Theorem 3.15. *If a state* $W^0 \in \widehat{H}^0(\mathbb{R}^2)$ *is controllable to the state* $W^T = 0$ *in a given time* $T > 0$, *then* $W^0 = 0$.

The proof of the theorem is given in Section 6.

4. Properties of the sets $\mathbf{R}_T$ and $\mathbf{R}_T^L$

First, consider necessary conditions for $g \in \mathbf{R}_T^L$.

Theorem 4.1. *Let $T > 0$, $L > 0$ and $g \in \mathbf{R}_T^L$. Then*

$$\frac{e^{\frac{(\cdot)}{4T}} g}{\ln\left(1 + \frac{4T}{(\cdot)}\right)} \in L^\infty(\mathbb{R}_+). \tag{4.1}$$

In addition, we have

$$\left\| \frac{e^{\frac{(\cdot)}{4T}} g}{\ln\left(1 + \frac{4T}{(\cdot)}\right)} \right\|_{L^\infty(\mathbb{R}_+)} \leq \frac{L}{2\pi}.$$

Proof. According to (3.14), there exists $u \in U_T^L$ such that

$$g(r) = -\frac{1}{\pi} \int_0^T \frac{e^{-\frac{r}{4\xi}}}{2\xi} u(T-\xi)\, d\xi, \quad r \in \mathbb{R}_+.$$

Setting $y = r/(4\xi)$ and taking into account [26, 5.1.1, 5.1.20], we get

$$|g(r)| \leq \frac{L}{2\pi} \int_{r/(4T)}^\infty \frac{e^{-y}}{y}\, dy = \frac{L}{2\pi} E_1\left(\frac{r}{4T}\right) \leq \frac{L}{2\pi} e^{-\frac{r}{4T}} \ln\left(1 + \frac{4T}{r}\right), \quad r > 0,$$

where $E_1(\xi) = \int_\xi^\infty \left(e^{-t}/t\right) dt$, $\xi \in \mathbb{R}$. Therefore, (4.1) holds and the estimate for the norm is true. □

We need the following formula

$$\Phi\left(e^{-\alpha(\cdot)}\right) = \frac{1}{2\alpha} e^{-\frac{(\cdot)}{4\alpha}}, \quad \alpha \in \mathbb{R}_+. \tag{4.2}$$

To prove it, set $\alpha \in \mathbb{R}_+$ and $q(r) = e^{-\alpha r}$, $r \in \mathbb{R}_+$. Expanding the Bessel function into the power series, we get

$$(\Phi q)(\rho) = \frac{1}{2} \int_0^\infty e^{-\alpha r} J_0\big(\sqrt{r\rho}\big)\, dr = \frac{1}{2} \sum_{m=0}^\infty \frac{(-1)^m \rho^m}{(m!)^2 2^{2m}} \int_0^\infty r^m e^{-\alpha r}\, dr, \quad \rho \in \mathbb{R}_+.$$

Since

$$\begin{aligned} \int_0^\infty r^m e^{-\alpha r}\, dr &= (-1)^m \left(\frac{d}{d\alpha}\right)^m \int_0^\infty e^{-\alpha r}\, dr \\ &= (-1)^m \left(\frac{d}{d\alpha}\right)^m \frac{1}{\alpha} = \frac{m!}{\alpha^{m+1}}, \end{aligned} \tag{4.3}$$

we get

$$(\Phi q)(\rho) = \frac{1}{2\alpha} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \left(\frac{\rho}{4\alpha}\right)^m = \frac{1}{2\alpha} e^{-\frac{\rho}{4\alpha}}, \quad \rho \in \mathbb{R}_+,$$

i.e. (4.2) holds.

Theorem 4.2. *Let $T > 0$, $L > 0$, $g \in \mathbf{R}_T^L$, and $G = \Phi g$. Then G can be extended to an entire function G_e of order ≤ 1 and type $\le T$ and*

$$|G_e(z)| \le \frac{L}{\pi}\frac{e^{T|z|}-1}{|z|}, \quad z \in \mathbb{C}. \tag{4.4}$$

Proof. Since $g \in \mathbf{R}_T^L$, there is $u \in U_T^L$ such that $g = \mathcal{Y}_u(\cdot, T)$ according to (3.14). With regard to (4.2), we get

$$G(\rho) = -\frac{1}{\pi}\int_0^T e^{-\xi\rho} u(T-\xi)\, d\xi, \quad \rho \in \mathbb{R}_+.$$

Hence G can be extended to the entire function G_e by the formula

$$G_e(z) = -\frac{1}{\pi}\int_0^T e^{-\xi z} u(T-\xi)\, d\xi, \quad z \in \mathbb{C}. \tag{4.5}$$

Evidently,

$$|G_e(z)| \le \frac{L}{\pi}\int_0^T e^{\xi|z|}\, d\xi = \frac{L}{\pi}\frac{e^{T|z|}-1}{|z|}, \quad z \in \mathbb{C}.$$

The theorem is proved. □

Example 4.3. Let $T > 0$ and

$$g(r) = -\frac{2}{\pi T} e^{-\frac{r}{2T}}, \quad r \ge 0.$$

Let us verify condition (4.1) for g. Put $v(\xi) = 2\ln(1+1/\xi) - e^{-\xi}$, $\xi > 0$. Then $v'(\xi) = -2/(\xi+\xi^2) + e^{-\xi}$, $\xi > 0$. Since $\xi + \xi^2 < 2 + 2\xi + \xi^2 < 2e^{\xi}$, $\xi > 0$, we have $v'(\xi) < 0$, $\xi > 0$, i.e. v decreases on $(0, +\infty)$. We have $v(\xi) \to +\infty$ as $\xi \to 0^+$ and $v(\xi) \to 0$ as $\xi \to +\infty$. Therefore, $v(\xi) > 0$, $\xi > 0$, i.e.

$$e^{-\xi} \le 2\ln\left(1 + \frac{1}{\xi}\right), \quad \xi > 0.$$

Setting $\xi = r/(4T)$ and applying this estimate to g, we get

$$|g(r)| \le \frac{4}{\pi T} e^{-\frac{r}{4T}} \ln\left(1 + \frac{4T}{r}\right), \quad r > 0.$$

Therefore, condition (4.1) holds for g.

Let us try to find $u \in U_T^L$ such that

$$g(r) = \mathcal{Y}_u(x, T) = -\frac{1}{\pi}\int_0^T \frac{1}{2\xi} e^{-\frac{r}{4\xi}} u(T-\xi)\, d\xi, \quad r \in \mathbb{R}_+.$$

Applying the operator Φ, we get

$$-\frac{2}{\pi} e^{-T\rho/2} = \big(\Phi g\big)(\rho) = -\frac{1}{\pi}\int_0^T e^{-\xi\rho} u(T-\xi)\, d\xi$$

$$= -\sqrt{\frac{2}{\pi}}\left(\mathscr{F}\mathcal{U}_T\right)(-i\rho), \quad \rho \in \mathbb{R}_+, \tag{4.6}$$

where

$$\mathcal{U}_T(\xi) = \begin{cases} u(T-\xi), & \xi \in [0,T], \\ 0, & \xi \in \mathbb{R} \setminus [0,T]. \end{cases}$$

Due to the Paley–Wiener theorem, $\mathscr{F}\mathcal{U}_T$ can be extended to an entire function. Replacing ρ by $i\mu$, we obtain

$$\sqrt{\frac{2}{\pi}} e^{-iT\mu/2} = \left(\mathscr{F}\mathcal{U}_T\right)(\mu), \quad \mu \in \mathbb{C}. \tag{4.7}$$

Therefore, $\mathcal{U}_T(\xi) = 2\delta(\xi - T/2)$ is the unique solution to equation (4.7). Hence $u(\xi) = 2\delta(\xi - T/2)$ is the unique solution to equation (4.6). But this function u does not belong to $L^\infty(0,T)$. Therefore, $g \notin \mathbf{R}_T^L$ for any $T > 0$ and $L > 0$ although condition (4.1) holds for it.

Thus, condition (4.1) is only necessary for $g \in \mathbf{R}_T^L$, but it is not sufficient. However, if g satisfies (4.1), Φg can be extended to an entire function of order ≤ 1 and type $\le T$ (cf. Theorem 4.2).

Theorem 4.4. *Let $T > 0$, $g \in L^2(\mathbb{R}_+)$, $G = \Phi g$, and condition (4.1) hold for g. Then G can be extended to an entire function G_e of order ≤ 1 and type $\le T$.*

Proof. According to Proposition 2.2 (ii), we have

$$G(\rho) = (\Phi g)(\rho) = \frac{1}{2} \lim_{N\to\infty} \int_0^N g(r) J_0\left(\sqrt{r\rho}\right) dr, \quad \rho \in \mathbb{R}_+. \tag{4.8}$$

Setting $M = \left\| e^{\frac{(\cdot)}{4T}} g \Big/ \ln\left(1 + \frac{4T}{(\cdot)}\right) \right\|_{L^\infty(\mathbb{R}_+)}$, we obtain from (4.1) that

$$|g(r)| \le M e^{-\frac{r}{4T}} \ln\left(1 + \frac{4T}{r}\right) \le 2\sqrt{2T} M \frac{e^{-\frac{r}{4T}}}{\sqrt{r}}, \quad r \in \mathbb{R}_+, \tag{4.9}$$

where the estimate

$$\ln\left(1 + y^2\right) \le \sqrt{2} y, \quad y \in \mathbb{R}_+, \tag{4.10}$$

obtained from the obvious estimate

$$1 + y^2 \le 1 + \sqrt{2}y + y^2 \le e^{\sqrt{2}y}, \quad y \in \mathbb{R}_+,$$

has been also used. It follows from (4.9) that

$$G_e(z) = \frac{1}{2} \int_0^\infty g(r) J_0\left(\sqrt{rz}\right) dr, \quad z \in \mathbb{C}, \tag{4.11}$$

is an entire function because J_0 is an entire function. Due to (4.8), we have

$$G_e(r) = G(r), \quad r \in \mathbb{R}_+. \tag{4.12}$$

Taking into account (4.9), we get

$$|G_e(z)| \le \sqrt{2T}M\int_0^\infty \left|J_0\big(\sqrt{rz}\big)\right|\frac{e^{-\frac{r}{4T}}}{\sqrt{r}}\,dr = 2\sqrt{2T}M\int_0^\infty \left|J_0\big(y\sqrt{z}\big)\right|e^{-\frac{y^2}{4T}}\,dy$$
$$\le 2\sqrt{2T}M\sum_{m=0}^\infty \frac{|z|^m}{(m!)^2 2^{2m}}\int_0^\infty y^{2m}e^{-\frac{y^2}{4T}}\,dy, \quad z\in\mathbb{C}. \tag{4.13}$$

Let us calculate the last integral. We have

$$\int_0^\infty y^{2m}e^{-\alpha y^2}\,dy = (-1)^m\left(\frac{d}{d\alpha}\right)^m\int_0^\infty e^{-\alpha y^2}\,dy = \frac{(-1)^m}{2}\left(\frac{d}{d\alpha}\right)^m\sqrt{\frac{\pi}{\alpha}}$$
$$= \frac{\sqrt{\pi}}{2}\frac{(2m-1)!!}{2^m\alpha^{m+1/2}}, \quad \alpha\in\mathbb{R}_+,\ m\in\mathbb{N}_0, \tag{4.14}$$

where we set $(-1)!!=1$. Setting $\alpha = 1/(4T)$ and continuing (4.13), we obtain

$$|G_e(z)| \le 2\sqrt{2\pi}TM\sum_{m=0}^\infty \frac{(4T)^m|z|^m}{m!2^{2m}}\frac{(2m-1)!!}{(2m)!!}$$
$$\le 2\sqrt{2\pi}TM\sum_{m=0}^\infty \frac{T^m|z|^m}{m!} \le 4\sqrt{2}TMe^{T|z|}, \quad z\in\mathbb{C}.$$

Thus, G_e is an entire function of order ≤ 1 and type $\le T$, and condition (4.12) holds for it. □

Thus, condition (4.1) is not sufficient for $g\in\mathbf{R}_T^L$, but it guarantees the necessary condition from Theorem 4.2 holds for Φg.

Theorem 4.5. *Let $L>0$, $T>0$, $g\in L^2(\mathbb{R}_+)$, and condition (4.1) hold for g. Let also*

$$\gamma_n = -\frac{\pi}{2^{2n+1}n!}\int_0^\infty r^n g(r)\,dr, \quad n\in\mathbb{N}_0. \tag{4.15}$$

Then $g\in\mathbf{R}_T^L$ iff

$$\exists u\in U_T^L\ \forall n\in\mathbb{N}_0 \quad \int_0^T \xi^n u(T-\xi)\,d\xi = \gamma_n. \tag{4.16}$$

Proof. Due to (3.14), we have

$$g\in\mathbf{R}_T^L \Leftrightarrow \big(\exists u\in U_T^L \quad g = \mathcal{Y}_u(\cdot,T)\big). \tag{4.17}$$

Put $G=\Phi g$, $\mathcal{G}_u(\cdot,T) = \Phi\mathcal{Y}_u(\cdot,T)$. According to Theorem 4.2, G can be extended to an entire function G_e. Taking into account (4.11), we obtain

$$G_e(z) = \frac{1}{2}\int_0^\infty g(r)J_0\big(\sqrt{rz}\big)\,dr$$
$$= \frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n z^n}{(n!)^2 2^{2n}}\int_0^\infty r^n g(r)\,dr = -\frac{1}{\pi}\sum_{n=0}^\infty \frac{(-1)^n\gamma_n}{n!}z^n, \quad z\in\mathbb{C}. \tag{4.18}$$

Taking into account (3.12) and (4.2), we get

$$\begin{aligned}\big(\Phi\mathcal{Y}_u(\cdot,T)\big)(z) &= -\frac{1}{\pi}\int_0^T e^{-\xi z}u(T-\xi)\,d\xi\\ &= -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\int_0^T \xi^n u(T-\xi)\,d\xi, \quad z\in\mathbb{C}. \qquad (4.19)\end{aligned}$$

It follows from (4.17)–(4.19) that $g \in \mathbf{R}_T^L$ iff (4.16) holds. □

Theorem 4.6. *Let $L>0$, $T>0$, $g\in L^2(\mathbb{R}_+)$, condition (4.1) hold for g, and $\{\gamma_n\}_{n=0}^{\infty}$ be defined by (4.15). If for each $N\in\mathbb{N}$ there exists $u_N\in U_T^L$ such that*

$$\int_0^T \xi^n u_N(T-\xi)d\xi = \gamma_n, \quad n=\overline{0,N}, \qquad (4.20)$$

then $g\in\overline{\mathbf{R}_T^L}$, where the closure is considered in $L^2(\mathbb{R}_+)$.

Proof. Let $N\in\mathbb{N}$. Set $g_N=\mathcal{Y}_{u_N}(\cdot,T)$, $G=\Phi g$, and $G_N=\Phi g_N$. According to Theorem 4.4, we have G can be extended to an entire function G_e of order ≤ 1 and type $\le T$. In addition, (4.18) holds for it. Setting

$$\gamma_n^N = \int_0^T \xi^n u_N(T-\xi)d\xi, \quad n\in\mathbb{N}_0, \qquad (4.21)$$

and taking into account (4.19), we get

$$\begin{aligned}G_N(z) &= -\frac{1}{\pi}\int_0^T e^{-\xi z}u_N(T-\xi)\,d\xi\\ &= -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n^N}{n!}z^n, \quad z\in\mathbb{C}. \qquad (4.22)\end{aligned}$$

It follows from (4.18), (4.20), and (4.22) that

$$G_e(z)-G_N(z) = -\frac{1}{\pi}\sum_{n=N+1}^{\infty}\frac{(-1)^n}{n!}\big(\gamma_n-\gamma_n^N\big)z^n, \quad z\in\mathbb{C}. \qquad (4.23)$$

Let $\varepsilon>0$ be fixed. Then there exists $A_\varepsilon>0$ such that

$$\int_{A_\varepsilon}^{\infty}|G(\rho)-G_N(\rho)|^2\,d\rho<\varepsilon^2. \qquad (4.24)$$

With regard to (4.23), we get

$$|G(\rho)-G_N(\rho)| \le \frac{1}{\pi}\sum_{n=N+1}^{\infty}\frac{1}{n!}\big|\gamma_n-\gamma_n^N\big|A_\varepsilon^n, \quad \rho\in(0,A_\varepsilon]. \qquad (4.25)$$

Now, let us estimate γ_n and γ_n^N. It follows from (4.1) that

$$|g(r)|\le Me^{-\frac{r}{4T}}\ln\left(1+\frac{4T}{r}\right), \quad r\in\mathbb{R}_+,$$

for some $M>0$. Taking into account (4.10) and (4.15), we get

$$|\gamma_n| \le \frac{M\pi}{2^{2n+1}n!}\int_0^\infty r^n e^{-\frac{r}{4T}} \ln\left(1+\frac{4T}{r}\right) dr$$
$$\le \frac{\sqrt{2T}M\pi}{2^{2n-1}n!}\int_0^\infty r^n e^{-\frac{r}{4T}} \frac{dr}{2\sqrt{r}} = \frac{\sqrt{2T}M\pi}{2^{2n-1}n!}\int_0^\infty y^{2n} e^{-\frac{y^2}{4T}}\,dr, \quad n\in\mathbb{N}_0.$$

Taking into account (4.14), we obtain

$$|\gamma_n| \le \frac{M\pi^{3/2}}{2^{2n+1/2}}\frac{(2n-1)!!}{(2n)!!}(4T)^{n+1} \le M(2\pi)^{3/2}T^{n+1}, \quad n\in\mathbb{N}_0. \tag{4.26}$$

It follows from (4.21) that

$$\left|\gamma_n^N\right| \le L\int_0^T \xi^n\,d\xi = L\frac{T^{n+1}}{n+1}, \quad n\in\mathbb{N}_0. \tag{4.27}$$

According to (4.26) and (4.27), we get

$$\left|\gamma_n-\gamma_n^N\right| \le \left(M(2\pi)^{3/2}+\frac{L}{n+1}\right)T^{n+1} \le \pi C T^n, \quad n\in\mathbb{N}_0,$$

where $C=T(M(2\pi)^{3/2}+L/(n+1))$. With regard to (4.25), we have

$$|G(\rho)-G_N(\rho)| \le C\sum_{n=N+1}^\infty \frac{(TA_\varepsilon)^n}{n!} = C\left(e^{TA_\varepsilon}-\sum_{n=0}^N\frac{(TA_\varepsilon)^n}{n!}\right)$$
$$\le Ce^{TA_\varepsilon}\frac{(TA_\varepsilon)^{N+1}}{(N+1)!}, \quad \rho\in(0,A_\varepsilon],$$

because

$$\left|e^y-\sum_{n=0}^N\frac{y^n}{n!}\right| \le e^{|y|}\frac{|y|^{N+1}}{(N+1)!}, \quad N\in\mathbb{N},$$

according to the Taylor formula. Therefore,

$$\left(\int_0^{A_\varepsilon}|G(\rho)-G_N(\rho)|^2\,d\rho\right)^{1/2} \le C\sqrt{A_\varepsilon}e^{TA_\varepsilon}\frac{(TA_\varepsilon)^{N+1}}{(N+1)!} \underset{N\to\infty}{\to} 0.$$

Taking into account (4.24), we conclude that

$$\|g-g_N\|_{L^2(\mathbb{R}_+)} = \|G-G_N\|_{L^2(\mathbb{R}_+)} \underset{N\to\infty}{\to} 0.$$

Therefore, $g\in\overline{\mathbf{R}_T^L}$. □

Put

$$\varphi_n(\rho)=\rho^n e^{-T\rho}, \qquad \rho\in\mathbb{R}_+, \qquad n\in\mathbb{N}_0, \tag{4.28}$$

$$\varphi_n^l(\rho) = \rho^n e^{-T\rho}\left(\frac{e^{\rho/l}-1}{\rho/l}\right)^{n+1}, \qquad \rho\in\mathbb{R}_+, \qquad n\in\mathbb{N}_0, \qquad l\in\mathbb{N}. \tag{4.29}$$

First we consider the system $\{\varphi_n\}_{n=0}^\infty$. It is well-known that it is complete in $L^2(\mathbb{R}_+)$.

The following lemma describes the relation between the systems $\{\varphi_n\}_{n=0}^\infty$ and $\{\varphi_n^l\}_{n=0}^\infty$, $l\in\mathbb{N}$.

Lemma 4.7. *Let $n\in\mathbb{N}_0$ and $l>(n+1)/T$. Then*

$$\left\|\varphi_n-\varphi_n^l\right\|_{L^2(\mathbb{R}_+)} \le \frac{n+1}{2^{n+5/2}l}\frac{\sqrt{(2n+2)!}}{(T-(n+1)/l)^{n+3/2}} \underset{l\to\infty}{\to} 0. \tag{4.30}$$

Proof. Let $l>\frac{n+1}{T}$, $n\in\mathbb{N}_0$. Since

$$\begin{aligned}(1+y)^{n+1}-1 &\le (n+1)y(1+y)^n, && y\in\mathbb{R}_+,\\ e^\xi-1-\xi &\le \frac12\xi^2e^\xi \quad\text{and}\quad e^\xi-1\le \xi e^\xi, && \xi\in\mathbb{R}_+,\end{aligned}$$

we have

$$\begin{aligned}\left|\varphi_n(\rho)-\varphi_n^l(\rho)\right| &= \left(\left(\frac{e^{\rho/l}-1}{\rho/l}\right)^{n+1}-1\right)\rho^ne^{-T\rho}\\ &\le (n+1)\left(\frac{e^{\rho/l}-1}{\rho/l}\right)^n\left(\frac{e^{\rho/l}-1}{\rho/l}-1\right)\rho^ne^{-T\rho}\\ &\le \frac{n+1}{2l}\rho^{n+1}e^{-(T-(n+1)/l)\rho}, \quad \rho\in\mathbb{R}_+.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\varphi_n-\varphi_n^l\right\|_{L^2(\mathbb{R}_+)} &\le \frac{n+1}{2l}\left(\int_0^\infty \rho^{2(n+1)}e^{-2(T-(n+1)/l)\rho}\,d\rho\right)^{1/2}\\ &\le \frac{n+1}{2^{n+5/2}l}\frac{\sqrt{(2n+2)!}}{(T-(n+1)/l)^{n+3/2}} \underset{l\to\infty}{\to} 0, \quad n\in\mathbb{N}_0,\end{aligned}$$

that was to be proved. □

Theorem 4.8. *Let $T>0$. We have $\overline{\mathbf{R}_T}=L^2(\mathbb{R}_+)$.*

To prove the theorem, we need to construct controls $\{u_n\}_{n=0}^\infty$ from $L^\infty[0,T]$ such that

$$\mathcal{Y}_{u_n}(\cdot,T) \underset{n\to\infty}{\to} g \quad \text{in } L^2(\mathbb{R}_+) \tag{4.31}$$

for a given $g\in L^2(\mathbb{R}_+)$.

To this aid, we need an appropriate basis in $L^2(\mathbb{R}_+)$. Consider the Laguerre polynomials [26, pp. 773–775, 22.1.1, 22.1.2, 22.2.13]:

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n\left(e^{-x}x^n\right) = \sum_{k=0}^n\binom{n}{k}\frac{(-1)^k}{k!}x^k, \quad n\in\mathbb{N}_0. \tag{4.32}$$

It is well-known that the system $\{\mathrm{e}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\mathbb{R}_+)$ where $\mathrm{e}_n(x) = L_n(x)e^{-x/2}$, $x \in \mathbb{R}_+$. Put

$$\psi_n(r) = \frac{1}{\sqrt{2T}} L_n\left(\frac{r}{2T}\right) e^{-\frac{r}{4T}}, \qquad r \in \mathbb{R}_+, \qquad n \in \mathbb{N}_0, \tag{4.33}$$

$$\widehat{\psi}_n(\rho) = (-1)^n \sqrt{2T} L_n(2T\rho)e^{-T\rho}, \qquad \rho \in \mathbb{R}_+, \qquad n \in \mathbb{N}_0. \tag{4.34}$$

Evidently, each of the systems $\{\psi_n\}_{n=0}^{\infty}$ and $\{\widehat{\psi}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\mathbb{R}_+)$. In addition, we have

$$\widehat{\psi}_n = \Phi\psi_n, \quad n \in \mathbb{N}_0. \tag{4.35}$$

Let us prove this formula. Let $n \in \mathbb{N}_0$. With regard to Theorem 2.2 (ii) and (4.3), we obtain

$$\begin{aligned}
\big(\Phi\psi_n\big)(\rho) &= \frac{1}{2\sqrt{2T}} \int_0^{\infty} J_0\big(\sqrt{r\rho}\big) L_n\left(\frac{r}{2T}\right) e^{-\frac{r}{4T}}\, dr \\
&= \frac{1}{2\sqrt{2T}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!(2T)^k} \sum_{m=0}^{\infty} \frac{(-1)^m \rho^m}{(m!)^2 2^{2m}} \int_0^{\infty} r^{m+k} e^{-\frac{r}{4T}}\, dr \\
&= \sqrt{2T} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 2^k}{k!} \sum_{m=0}^{\infty} \frac{(-1)^m (T\rho)^m}{m!} \frac{(m+k)!}{m!}, \quad \rho \in \mathbb{R}_+.
\end{aligned} \tag{4.36}$$

Due to (4.32), we have for the series with $\xi = T\rho$ that

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(m+k)!}{m!} \xi^m &= \left(\frac{d}{d\xi}\right)^k \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \xi^{m+k} \\
&= \left(\frac{d}{d\xi}\right)^k \big(\xi^k e^{-\xi}\big) = k! L_k(\xi) e^{-\xi}, \quad \xi \in \mathbb{R}.
\end{aligned}$$

Taking this into account and continuing (4.36), we obtain

$$\big(\Phi\psi_n\big)(\rho) = \sqrt{2T} \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^k L_k(T\rho) e^{-T\rho}, \quad \rho \in \mathbb{R}_+.$$

By using multiple argument formula (see [26, p. 785, 22.12.7]):

$$L_n(\mu\xi) = \sum_{k=0}^{n} \binom{n}{k} \mu^k (1-\mu)^{n-k} L_k(\xi), \quad \mu \in \mathbb{R},\ \xi \in \mathbb{R},$$

with $\mu = 2$ and $\xi = T\rho$, we get (4.34).

Proof of Theorem 4.8. Let $g \in L^2(\mathbb{R}_+)$. Put $G = \Phi g$. Hence, $G \in L^2(\mathbb{R}_+)$. Set $g_n = \langle g, \psi_n \rangle_{L^2(\mathbb{R}_+)}$, $n \in \mathbb{N}_0$. We have

$$g = \sum_{n=0}^{\infty} g_n \psi_n$$

and

$$G = \sum_{n=0}^{\infty} g_n \widehat{\psi}_n = G^N + \sum_{n=N+1}^{\infty} g_n \widehat{\psi}_n,$$

where $N \in \mathbb{N}$ and

$$G^N = \sum_{n=0}^{N} g_n \widehat{\psi}_n.$$

Then

$$\left\| G - G^N \right\|_{L^2(\mathbb{R}_+)} = \left(\sum_{n=N+1}^{\infty} |g_n|^2 \right)^{1/2} \underset{N\to\infty}{\to} 0. \tag{4.37}$$

With regard to (4.34), we get

$$G^N = \sqrt{2T} \sum_{n=0}^{N} g_n (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} (2T)^k \varphi_k = \sqrt{2T} \sum_{k=0}^{N} \frac{(-1)^k}{k!} (2T)^k d_k^N \varphi_k,$$

where

$$d_k^N = \sum_{n=k}^{N} \binom{n}{k} (-1)^n g_n, \quad k = \overline{0, N}. \tag{4.38}$$

Put

$$G_l^N = \sqrt{2T} \sum_{k=0}^{N} \frac{(-1)^k}{k!} (2T)^k d_k^N \varphi_k^l, \quad l \in \mathbb{N}. \tag{4.39}$$

Taking into account Lemma 4.7, we conclude that for $l > (N+1)/T$, we have

$$\begin{aligned} \left\| G^N - G_l^N \right\|_{L^2(\mathbb{R}_+)} &\le \sqrt{2T} \sum_{k=0}^{N} \frac{(2T)^k}{k!} |d_k^N| \left\| \varphi_k^l - \varphi_k \right\|_{L^2(\mathbb{R}_+)} \\ &\le \frac{\sqrt{T}}{4l} \sum_{k=0}^{N} \frac{T^k \sqrt{(2k+2)!}}{(T-(k+1)/l)^{k+3/2}} \frac{k+1}{k!} |d_k^N| \underset{l\to\infty}{\to} 0. \end{aligned} \tag{4.40}$$

Let $g_l^N = \Phi^{-1} G_l^N$. We have

$$\begin{aligned} \left\| g - g_l^N \right\|_{L^2(\mathbb{R}_+)} &= \left\| G - G_l^N \right\|_{L^2(\mathbb{R}_+)} \\ &\le \left\| G - G^N \right\|_{L^2(\mathbb{R}_+)} + \left\| G^N - G_l^N \right\|_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{4.41}$$

With regard to (4.37) and (4.40), we conclude that for all $\varepsilon > 0$, we can choose appropriate $N \in \mathbb{N}$ and $l > (N+1)/T$ such that

$$\left\| g - g_l^N \right\|_{L^2(\mathbb{R}_+)} < \varepsilon. \tag{4.42}$$

Let us prove that $g_l^N \in \mathbf{R}_T$. Put

$$u_l^n(\xi) = \begin{cases} (-1)^{n-j} \binom{n}{j} l^{n+1}, & \xi \in \left(\frac{j}{l}, \frac{j+1}{l}\right), \ j = \overline{0, n} \\ 0, & \xi \notin \left[0, \frac{n+1}{l}\right] \end{cases}, \quad l \in \mathbb{N}, \ n \in \mathbb{N}_0. \tag{4.43}$$

Note that $u_l^n \underset{l\to\infty}{\to} (-1)^n\delta^{(n)}$ in $H^{-1}(\mathbb{R})$ for each $n \in \mathbb{N}_0$. Taking into account (4.19) and (4.43), it is easy to obtain $\Phi\mathcal{Y}_{u_l^n}(\cdot,T) = -\frac{1}{\pi}\varphi_n^l$. Due to (4.39), we have

$$g_l^N = -\sqrt{2T}\pi\sum_{k=0}^{N}\frac{(-1)^k}{k!}(2T)^k d_k^N \mathcal{Y}_{u_l^k}(\cdot,T) = \mathcal{Y}_{\mathcal{U}_l^N}(\cdot,T), \tag{4.44}$$

where

$$\mathcal{U}_l^N(\xi) = -\sqrt{2T}\pi\sum_{k=0}^{N}\frac{(-1)^k}{k!}(2T)^k d_k^N u_l^k(\xi), \quad \xi \in \mathbb{R}_+. \tag{4.45}$$

In addition, due to (4.37), (4.40), and (4.41), we obtain

$$\begin{aligned}\left\|g - g_l^N\right\|_{L^2(\mathbb{R}_+)} &= \left\|g - \mathcal{Y}_{\mathcal{U}_l^N}(\cdot,T)\right\|_{L^2(\mathbb{R}_+)} \le \left(\sum_{n=N+1}^{\infty}|g_n|^2\right)^{1/2} \\ &+ \frac{\sqrt{T}}{4l}\sum_{k=0}^{N}\frac{T^k\sqrt{(2k+2)!}}{(T-(k+1)/l)^{k+3/2}}\frac{k+1}{k!}\left|d_k^N\right|, \quad N \in \mathbb{N},\ l > \frac{N+1}{T}.\end{aligned} \tag{4.46}$$

Evidently, $\mathcal{U}_l^N \in L^\infty(0,T)$. Thus, with regard to (3.8) and (4.44), we can see that $g_l^N \in \mathbf{R}_T$. Since we have considered an arbitrary $\varepsilon > 0$ in (4.42), we conclude that $g \in \overline{\mathbf{R}_T}$. □

5. Example

In this section, we give an example illustrating Theorem 3.14.

Example 5.1. Let

$$w^0(x) = \cosh\left(\frac{|x|^2}{12T}\right)e^{-\frac{|x|^2}{4T}}, \quad w^T(x) = \frac{3}{14}e^{-\frac{|x|^2}{7T}}, \quad x \in \mathbb{R}^2.$$

Consider the problem of approximate controllability for system (3.1), (3.2) with $W^0 = w^0$ and $W^T = w^T$.

We have $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ and $W^T \in \widehat{H}^0\left(\mathbb{R}^2\right)$. Moreover, $W^0 \in \mathcal{H}$ and $W^T \in \mathcal{H}$. With regard to (3.5) and (3.19), it is easy to see that

$$\begin{aligned}\mathcal{W}_0(x,T) &= \frac{3}{10}e^{-\frac{|x|^2}{10T}} + \frac{3}{14}e^{-\frac{|x|^2}{7T}}, && x \in \mathbb{R}^2,\\ W_0^T(x,T) &= -\frac{3}{10}e^{-\frac{|x|^2}{10T}}, && x \in \mathbb{R}^2.\end{aligned}$$

Evidently, $W_0^T \in \mathcal{H}$.

According to Theorem 3.14, the initial state W^0 is approximately controllable to the target state W^T in the given time T. Note that condition (3.16) does not hold for W_0^T.

To construct controls solving the approximate controllability problem for control system (3.1), (3.2), we use the method given in the proof of Theorem 4.8. Put $g = \Psi W_0^T$. Then

$$g(r) = -\frac{3}{10} e^{-\frac{r}{10T}}, \quad r \in \mathbb{R}_+.$$

With regard to (4.3), we get

$$\begin{aligned} g_n = \langle g, \psi_n \rangle_{L^2(\mathbb{R}_+)} &= -\frac{3}{10\sqrt{2T}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!(2T)^k} \int_0^\infty r^k e^{-\frac{7r}{20T}}\, dr \\ &= -\frac{3}{10\sqrt{2T}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!(2T)^k} k! \left(\frac{20T}{7}\right)^{k+1} \\ &= -\frac{3}{7}\sqrt{2T} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{10}{7}\right)^k \\ &= (-1)^{n+1} \left(\frac{3}{7}\right)^{n+1} \sqrt{2T}, \quad n \in \mathbb{N}_0. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\sum_{n=N+1}^{\infty} |g_n|^2\right)^{1/2} &= \sqrt{2T} \left(\sum_{n=N+1}^{\infty} \left(\frac{3}{7}\right)^{2n+2}\right)^{1/2} \\ = \sqrt{2T} \left(\frac{3}{7}\right)^{N+2} \left(\frac{1}{1-9/49}\right)^{1/2} &= \frac{3}{2}\sqrt{\frac{T}{5}} \left(\frac{3}{7}\right)^{N+1}, \quad N \in \mathbb{N}. \end{aligned} \tag{5.1}$$

Due to (4.38), we have

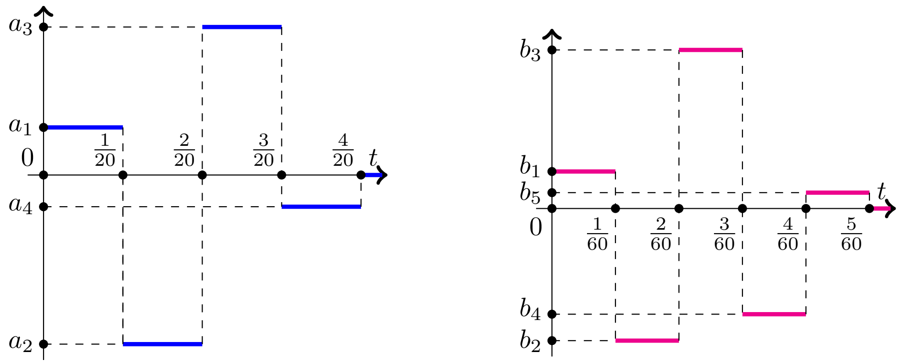
$$d_k^N = \sum_{n=k}^{N} \binom{n}{k} (-1)^n g_n = -\frac{3}{7}\sqrt{2T} \sum_{n=k}^{N} \binom{n}{k} \left(\frac{3}{7}\right)^n, \quad k = \overline{0, N}.$$

It follows from (4.45) that

$$\begin{aligned} \mathcal{U}_l^N &= -\sqrt{2T}\pi \sum_{k=0}^{N} \frac{(-1)^k}{k!} (2T)^k d_k^N u_l^k \\ &= 2T\pi \sum_{k=0}^{N} \sum_{n=k}^{N} \binom{n}{k} \left(\frac{3}{7}\right)^{n+1} \frac{(-1)^k}{k!} (2T)^k u_l^k. \end{aligned} \tag{5.2}$$

Taking into account (4.46) and (5.1), we obain

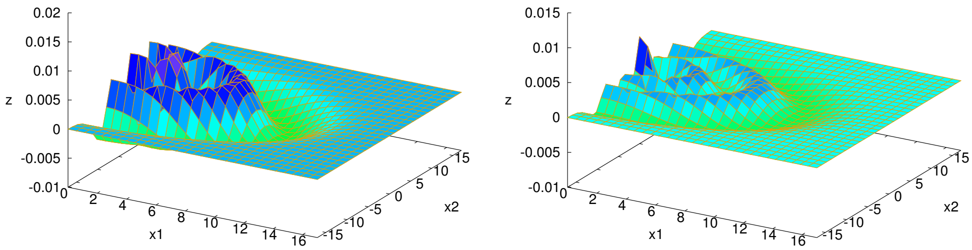
$$\begin{aligned} &\left\| g - y_{\mathcal{U}_l^N}(\cdot, T) \right\|_{L^2(\mathbb{R}_+)} \le \frac{3}{2}\sqrt{\frac{T}{5}} \left(\frac{3}{7}\right)^{N+1} \\ &+ \frac{T}{2\sqrt{2}l} \sum_{k=0}^{N} \frac{T^k \sqrt{(2k+2)!}}{(T-(k+1)/l)^{k+3/2}} \frac{k+1}{k!} \sum_{n=k}^{N} \binom{n}{k} \left(\frac{3}{7}\right)^{n+1}, \ N \in \mathbb{N}, \ l > \frac{N+1}{T}. \end{aligned}$$



(a) $N = 3$, $l = 20$,
$a_1 \approx 4171487.587754723$,
$a_2 \approx -11985246.36814925$,
$a_3 \approx 11476859.47814512$,
$a_4 \approx -3662827.493025041$.

(b) $N = 4$, $l = 60$,
$b_1 \approx 12268766670.45946$,
$b_2 \approx -48230066041.31739$,
$b_3 \approx 71097757825.27233$,
$b_4 \approx -46580177228.79937$,
$b_5 \approx 11443719610.35109$.

Fig. 5.1: The controls $\mathcal{U}_l^N$ defined by (5.2).



(a) $N = 3$, $l = 20$.

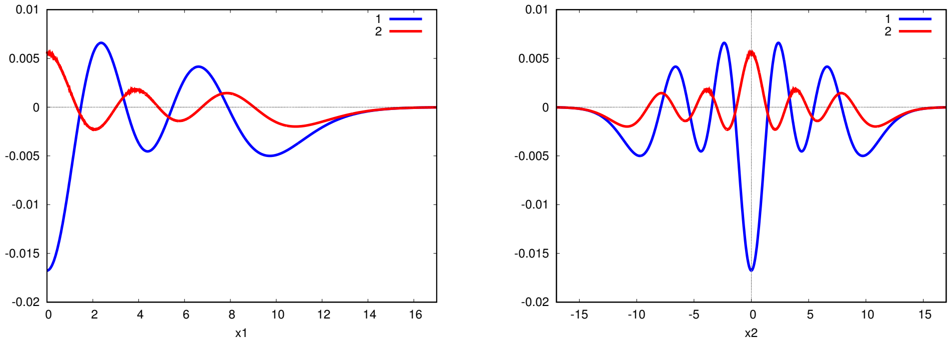
(b) $N = 4$, $l = 60$.

Fig. 5.2: The influence of the controls $\mathcal{U}_l^N$ on the difference $W^T - \Big(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{\mathcal{U}_l^N}(\cdot, T)\Big)$ with $T = 3$.

We have $\mathcal{W}_{\mathcal{U}_l^N}(\cdot, T) = \Psi^{-1}\mathcal{Y}_{\mathcal{U}_l^N}(\cdot, T)$ and $W_0^T = \Psi^{-1}g$. Taking into account Proposition 2.1 (iv), we get

$$\begin{aligned}\left\|W^T - \left(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{\mathcal{U}_l^N}(\cdot, T)\right)\right\|^0 &= \|W_0^T - \mathcal{W}_{\mathcal{U}_l^N}(\cdot, T)\|^0 \\ &= \sqrt{\pi}\left\|g - \mathcal{Y}_{\mathcal{U}_l^N}(\cdot, T)\right\|_{L^2(\mathbb{R}_+)}, \quad N \in \mathbb{N},\ l > \frac{N+1}{T}.\end{aligned}$$

The plots of the controls $\mathcal{U}_l^N$ are given in Fig. 5.1 for $T = 3$ with the cases of $N = 3$, $l = 20$ and $N = 4$, $l = 60$. Figs. 5.2 and 5.3 describe the influence of the control $\mathcal{U}_l^N$ on the difference $W^T - \Big(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{\mathcal{U}_l^N}(\cdot, T)\Big)$ with $T = 3$.



(a) 1) $N = 3$, $l = 20$; 2) $N = 4$, $l = 60$. (b) 1) $N = 3$, $l = 20$; 2) $N = 4$, $l = 60$.

Fig. 5.3: The influence of the controls $\mathcal{U}_l^N$ on the difference $W^T - \left(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{\mathcal{U}_l^N}(\cdot, T)\right)$ with $T = 3$ (vertical section for $x_2 = 0$ and horizontal section for $x_1 = 0$).

6. Proofs of theorems and auxiliary statements

Proof of Theorem 3.4. We prove the theorem using similar reasoning to those of the corresponding theorem in [17]. Put $V^0 = \mathscr{F} W^0$, $\mathcal{V}_0(\cdot, t) = \mathscr{F}_{x\to\sigma} \mathcal{W}_0(\cdot, t)$, $\mathcal{V}_u(\cdot, t) = \mathscr{F}_{x\to\sigma} \mathcal{W}_u(\cdot, t)$, $t \in [0, T]$. Evidently,

$$\mathcal{V}_0(\sigma, t) = e^{-t|\sigma|^2} V^0(\sigma), \qquad \sigma \in \mathbb{R}^2,\ t \in [0, T], \tag{6.1}$$

$$\mathcal{V}_u(\sigma, t) = -\frac{1}{\pi} \int_0^t e^{-\xi|\sigma|^2} u(t - \xi)\, d\xi, \qquad \sigma \in \mathbb{R}^2,\ t \in [0, T]. \tag{6.2}$$

Hence,

$$\|\mathcal{W}_0(\cdot, t)\|^0 = \|\mathcal{V}_0(\cdot, t)\|_0 \le \|V^0\|_0 = \|W^0\|^0, \quad t \in [0, T]. \tag{6.3}$$

Thus, (i) is proved.

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. Then

$$\begin{aligned} \|D^\alpha \mathcal{W}_0(\cdot, t)\|^0 &= \left\| (\cdot)_{[1]}^{\alpha_1} (\cdot)_{[2]}^{\alpha_2} \mathcal{V}_0(\cdot, t) \right\|_0 \le e^t \left(\frac{1 + |\alpha|}{2te} \right)^{(1+|\alpha|)/2} \|V^0\|_{-1} \\ &\le e^t \left(\frac{1 + |\alpha|}{2te} \right)^{(1+|\alpha|)/2} \|W^0\|^0, \quad t \in (0, T]. \end{aligned} \tag{6.4}$$

Here we used here the following estimates:

$$|\sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \mathcal{V}_0(\sigma, t)|^2 \le \left(1 + |\sigma|^2\right)^{1+|\alpha|} e^{-2t|\sigma|^2} \frac{|V^0(\sigma)|^2}{1 + |\sigma|^2}, \quad \sigma \in \mathbb{R}^2,\ t \in [0, T],$$

$$\xi^m e^{-\beta\xi} \le \left(\frac{m}{\beta e} \right)^m, \qquad m \in \mathbb{N},\ \beta > 0,\ \xi \ge 0,$$

$$\|V^0\|_{-1} \le \|V^0\|_0 = \|W^0\|^0.$$

Thus, (ii) is proved.

Suppose $W^0 \in \mathcal{H}$. Since $\mathcal{W}_0(\cdot, t) = \mathscr{F}_{\sigma\to x}^{-1} \mathcal{V}_0(\cdot, t)$, with regard to (6.1), we get $\mathcal{W}_0(\cdot, t) \in \mathcal{H}$, $t \in [0, T]$. Thus, (iii) holds.

It remains to prove (iv). Put

$$g(r,t)=\int_0^t e^{-\xi r^2}u(t-\xi)\,d\xi,\quad r\geq 0,\ t\in[0,T].$$

Taking into account, that

$$\frac{1-e^{-tr^2}}{r^2}\leq\frac{2(t+1)}{r^2+1},\quad r>0,\ t>0,$$

we obtain

$$|g(r,t)|\leq\|u\|_{L^\infty(0,T)}\frac{1-e^{-tr^2}}{r^2}\leq\|u\|_{L^\infty(0,T)}\frac{2(t+1)}{r^2+1},\quad r>0,\ t>0.$$

Hence,

$$\|\mathcal{W}_u(\cdot,t)\|^0=\|\mathcal{V}_u(\cdot,t)\|_0=\sqrt{\frac{2}{\pi}}\left(\int_0^\infty|g(r,t)|^2r\,dr\right)^{1/2}$$
$$\leq\frac{2}{\sqrt{\pi}}(t+1)\|u\|_{L^\infty(0,T)}\left(\int_0^\infty\frac{2r\,dr}{(1+r^2)^2}\right)^{1/2}=\frac{2}{\sqrt{\pi}}(t+1)\|u\|_{L^\infty(0,T)},\ t\in[0,T].$$

This completes the proof. □

Lemma 6.1. *Let $W^0\in\widehat{H}^0\left(\mathbb{R}^2\right)$, $t\in[0,T]$. Let W be a solution to (3.1), (3.2). Then (3.3) holds.*

Proof. We have from (3.4)

$$\frac{\partial}{\partial x_1}W(0^+,(\cdot)_{[2]},t)=\frac{\partial}{\partial x_1}\mathcal{W}_0(0^+,(\cdot)_{[2]},t)+\frac{\partial}{\partial x_1}\mathcal{W}_u(0^+,(\cdot)_{[2]},t),\ t\in(0,T].\quad(6.5)$$

According to Theorem 3.4 (ii), $\frac{\partial}{\partial x_1}\mathcal{W}_0(\cdot,t)$ is continuous on $\mathbb{R}^2$ for each $t\in(0,T]$. Moreover, $\frac{\partial}{\partial x_1}\mathcal{W}_0(\cdot,t)$ is odd with respect to x_1, $t\in[0,T]$. Hence,

$$\frac{\partial}{\partial x_1}\mathcal{W}_0(0^+,(\cdot)_{[2]},t)=0,\quad t\in(0,T].\tag{6.6}$$

For $\frac{\partial}{\partial x_1}\mathcal{W}_u(0^+,(\cdot)_{[2]},t)$, $t\in(0,T]$ we have

$$\frac{\partial}{\partial x_1}\mathcal{W}_u(x,t)=\frac{2}{\pi}\frac{x_1}{|x|^2}\int_{\frac{|x|}{2\sqrt{t}}}^\infty ye^{-y^2}u\left(t-\frac{|x|^2}{4y^2}\right)dy,\quad x\in\mathbb{R}^2,\ t\in(0,T].$$

Let $\varphi\in H^1(\mathbb{R})$, $\psi\in\mathcal{D}(\mathbb{R}_+)$, and $\alpha>0$. With regard to Definition 2.3, we consider (2.4) for $f=\frac{\partial}{\partial x_1}\mathcal{W}_u(\cdot,t)$, $t\in(0,T]$, changing additionally the variable x_j to the new variable αx_j in the integral with respect to x_j, $j=1,2$. We have

$$\left\langle\left\langle\frac{\partial}{\partial x_1}\mathcal{W}_u((\cdot)_{[1]},(\cdot)_{[2]}),\varphi((\cdot)_{[2]})\right\rangle_{[2]},\frac{1}{\alpha}\psi\left(\frac{(\cdot)_{[1]}}{\alpha}\right)\right\rangle_{[1]}$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x_1}{|x|^2} \int_{\frac{\alpha|x|}{2\sqrt{t}}}^{\infty} y e^{-y^2} u\left(t - \frac{\alpha^2 |x|^2}{4y^2}\right) dy\, \varphi(\alpha x_2)\, dx_2\, \psi(x_1)\, dx_1,$$

$$t \in (0, T]. \qquad (6.7)$$

Since

$$\int_{\frac{\alpha|x|}{2\sqrt{t}}}^{\infty} y e^{-y^2} \left| u\left(t - \frac{\alpha^2 |x|^2}{4y^2}\right)\right| dy \le \|u\|_{L^\infty(0,T)} \int_0^{\infty} y e^{-y^2}\, dy = \frac{1}{2} \|u\|_{L^\infty(0,T)}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x_1}{|x|^2} |\varphi(\alpha x_2)|\, dx_2\, |\psi(x_1)|\, dx_1$$

$$\le \sup_{\mu \in \mathbb{R}} |\varphi(\mu)| \int_0^{\infty} x_1 |\psi(x_1)| \int_{-\infty}^{\infty} \frac{dx_2}{x_1^2 + x_2^2}\, dx_1$$

$$= \pi \sup_{\mu \in \mathbb{R}} |\varphi(\mu)| \int_0^{\infty} |\psi(x_1)|\, dx_1 < \infty,$$

we can apply Lebesgue's dominated convergence theorem to (6.7) as $\alpha \to 0^+$:

$$\left\langle \left\langle \frac{\partial}{\partial x_1} \mathcal{W}_u((\cdot)_{[1]}, (\cdot)_{[2]}), \varphi((\cdot)_{[2]}) \right\rangle_{[2]}, \frac{1}{\alpha} \psi\left(\frac{(\cdot)_{[1]}}{\alpha}\right) \right\rangle_{[1]}$$

$$\to \frac{2}{\pi} u(t) \varphi(0) \int_0^{\infty} \int_{-\infty}^{\infty} \frac{x_1}{|x|^2} \int_0^{\infty} y e^{-y^2}\, dy\, dx_2\, \psi(x_1)\, dx_1$$

$$= \frac{1}{\pi} u(t) \varphi(0) \int_0^{\infty} x_1 \psi(x_1) \int_{-\infty}^{\infty} \frac{dx_2}{x_1^2 + x_2^2}\, dx_1 = u(t) \varphi(0) \int_{-\infty}^{\infty} \psi(x_1)\, dx_1$$

$$= \left\langle \left\langle u(t) \delta_{[2]}, \varphi \right\rangle_{[2]}, \psi \right\rangle_{[1]}, \quad t \in (0, T],$$

i.e.

$$\frac{\partial}{\partial x_1} \mathcal{W}_u\big(0^+, (\cdot)_{[2]}, t\big) = u(t) \delta_{[2]}, \quad t \in (0, T]. \qquad (6.8)$$

With regard to (6.5), (6.6), and (6.8), we conclude that (3.3) holds. □

Proof of Theorem 3.6. According to Theorem 3.4 (iv), we have $f \in \mathcal{H}$. Therefore, $F = \mathscr{F} f \in \mathcal{H}$ (see Theorem 2.1 (vi)). Set $g = \Psi f$ and $G = \Psi F$. It follows from Theorem 4.1 and (3.11) that Theorem 3.6 is true. □

Proof of Theorem 3.7. According to Theorem 3.4 (iv), we have $f \in \mathcal{H}$ (see Theorem 2.1 (vi)). Therefore, $F = \mathscr{F} f \in \mathcal{H}$. Set $g = \Psi f$ and $G = \Psi F$. It follows from Theorem 4.2 and (3.11) that Theorem 3.7 is true. □

Proof of Theorem 3.8. Set $g = \Psi f$ and $G = \Psi F$. It follows from Theorem 4.4 that Theorem 3.8 is true. □

Proof of Theorem 3.9. Set $g = \Psi W_0^T$. Let us prove that for $\{\omega_n\}_{n=0}^\infty$ defined by (3.20) and $\{\gamma_n\}_{n=0}^\infty$ defined by (4.15), we have

$$\omega_n = \gamma_n, \quad n \in \mathbb{N}_0. \tag{6.9}$$

Let $n \in \mathbb{N}_0$. Setting $x_1 = \sqrt{r}\cos\phi$ and $x_2 = \sqrt{r}\sin\phi$, $r \in \mathbb{R}_+$, $\phi \in [0, 2\pi)$, we have

$$\omega_n = -\frac{1}{2}\frac{n!}{(2n)!}\iint_{\mathbb{R}^2} x_1^{2n} W_0^T(x)\,dx = -\frac{1}{4}\frac{n!}{(2n)!}\int_0^{2\pi}\cos^{2n}\phi\,d\phi\int_0^\infty r^n g(r)\,dr.$$

Since

$$\begin{aligned}\frac{1}{4}\int_0^{2\pi}\cos^{2n}\phi\,d\phi &= \int_0^\infty \frac{dy}{(1+y^2)^{n+1}} = \frac{2n-1}{2n}\int_0^\infty \frac{dy}{(1+y^2)^n}\\ &= \frac{(2n-1)!!}{(2n)!!}\int_0^\infty \frac{dy}{1+y^2} = \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!} = \pi\frac{(2n)!}{2^{2n+1}(n!)^2},\end{aligned}$$

we have

$$\omega_n = -\pi\frac{(2n)!}{2^{2n+1}(n!)^2}\frac{n!}{(2n)!}\int_0^\infty r^n g(r)\,dr = -\frac{\pi}{2^{2n+1}n!}\int_0^\infty r^n g(r)\,dr = \gamma_n,$$

i.e. (6.9) holds. It follows from Theorems 4.5 and 3.5 (iv), (3.11), and (6.9) that Theorem 3.9 is true. □

Proof of Theorem 3.11. Set $g = \Psi W_0^T$. For $\{\omega_n\}_{n=0}^\infty$ defined by (3.20) and $\{\gamma_n\}_{n=0}^\infty$ defined by (4.15), it follows from Theorems 3.5 (iv) and 4.6, (3.11), (3.15), and (6.9) that Theorem 3.11 is true. □

Proof of Theorem 3.13. It follows from Theorems 2.1 (i) and 4.8, (3.11), (3.15) that Theorem 3.13 is true. □

Prof of Theorem 3.15. Let a state $W^0 \in \widehat{H}^0\left(\mathbb{R}^2\right)$ be controllable to the state $W^T = 0$. Then there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution W to system (3.1), (3.2) under this control and $W(\cdot,T) = 0$. It follows from (3.4) that

$$\left(\mathscr{F}W^0\right)(\sigma) = \frac{1}{\pi}\int_0^T e^{\xi|\sigma|^2}u(\xi)\,d\xi, \quad \sigma \in \mathbb{R}^2.$$

Evidently, $\mathscr{F}W^0 \in \mathcal{H}$. Setting $G = \Psi\mathscr{F}W^0$, we obtain

$$G(\rho) = \frac{1}{\pi}\int_0^T e^{\xi\rho}u(\xi)\,d\xi, \quad \rho \in \mathbb{R}_+. \tag{6.10}$$

Let $T^* > T$. Put

$$\widehat{\psi}_n^*(\rho) = (-1)^n\sqrt{2T^*}L_n(2T^*\rho)e^{-T^*\rho}, \qquad \rho \in \mathbb{R}_+, \qquad n \in \mathbb{N}_0, \tag{6.11}$$

$$\alpha_n = \langle G, \widehat{\psi}_n^*\rangle_{L^2(\mathbb{R}_+)}, \qquad n \in \mathbb{N}_0, \tag{6.12}$$

$$\beta_n(\xi) = \frac{1}{\pi}\langle e^{\xi(\cdot)}, \widehat{\psi}_n^*\rangle_{L^2(\mathbb{R}_+)}, \qquad \xi \in [0,T], \qquad n \in \mathbb{N}_0. \tag{6.13}$$

Obviously, the system $\{\widehat{\psi}_n^*\}_{n=0}^\infty$ is an orthonormal basis in $L^2(\mathbb{R}_+)$ (cf. (4.34)). Then, due to (6.10), we have

$$\sum_{n=0}^\infty \alpha_n \widehat{\psi}_n^* = \sum_{n=0}^\infty \left(\int_0^T \beta_n(\xi)u(\xi)\,d\xi\right)\widehat{\psi}_n^*.$$

Hence,

$$\int_0^T \beta_n(\xi)u(\xi)\,d\xi = \alpha_n, \quad n \in \mathbb{N}_0. \tag{6.14}$$

Let $n \in \mathbb{N}_0$ be fixed. Taking into account (4.32), we get

$$\begin{aligned}
\beta_n(\xi) &= \frac{(-1)^n}{\pi}\sqrt{2T^*}\sum_{k=0}^n \binom{n}{k}\frac{(-1)^k}{k!}(2T^*)^k \int_0^\infty \rho^k e^{-(T^*-\xi)\rho}\,d\rho \\
&= \frac{(-1)^n}{\pi}\sqrt{2T^*}\sum_{k=0}^n \binom{n}{k}\frac{(-1)^k}{k!}(2T^*)^k \frac{k!}{(T^*-\xi)^{k+1}} \\
&= \frac{(-1)^n}{\pi}\frac{\sqrt{2T^*}}{T^*-\xi}\sum_{k=0}^n \binom{n}{k}\left(-\frac{2T^*}{T^*-\xi}\right)^k \\
&= \frac{1}{\pi}\frac{\sqrt{2T^*}}{T^*-\xi}\left(\frac{T^*+\xi}{T^*-\xi}\right)^n, \quad \xi \in [0,T].
\end{aligned} \tag{6.15}$$

According to (6.14), we obtain

$$\alpha_n = \frac{\sqrt{2T^*}}{\pi}\int_0^T \left(\frac{T^*+\xi}{T^*-\xi}\right)^n \frac{u(\xi)}{T^*-\xi}\,d\xi = \frac{\sqrt{2T^*}}{\pi}\int_0^{T_1} e^{n\tau} u\left(T^*\frac{e^\tau-1}{e^\tau+1}\right)\frac{e^\tau\,d\tau}{e^\tau+1},$$

where $\frac{T^*+\xi}{T^*-\xi} = e^\tau$, $T_1 = \ln\frac{T^*+T}{T^*-T}$. Set

$$\alpha_n^* = \frac{\pi}{\sqrt{2T^*}}\alpha_n, \quad n \in \mathbb{N}_0, \quad u^*(\tau) = \frac{e^\tau}{e^\tau+1}u\left(T^*\frac{e^\tau-1}{e^\tau+1}\right), \quad \tau \in [0,T_1].$$

Then,

$$\int_0^{T_1} e^{n\tau}u^*(\tau)\,d\tau = \alpha_n^*, \quad n \in \mathbb{N}_0. \tag{6.16}$$

With regard to (6.12), we get

$$|\alpha_n^*| \le \frac{\pi}{\sqrt{2T^*}}\|G\|_{L^2(\mathbb{R}_+)}, \quad n \in \mathbb{N}_0.$$

Therefore, for all $\delta > 0$, there exists $C_\delta > 0$ such that

$$|\alpha_n^*| \le C_\delta e^{n\delta}, \quad n \in \mathbb{N}_0. \tag{6.17}$$

Obviously,

$$\|u^*\|_{L^2(0,T_1)} = \left(\frac{1}{2T^*}\int_0^T |u(\xi)|^2 \, d\xi\right)^{1/2} \le \left(\frac{T}{2T^*}\right)^{1/2} \|u\|_{L^\infty(0,T)}\,. \tag{6.18}$$

Taking into account (6.16)–(6.18), we conclude that all assertions of [31, Theorem 3.1, b)] hold. Thus, due to this theorem, $\alpha_n^* = 0$, $n \in \mathbb{N}_0$. Hence, $G = 0$. Therefore, $W^0 = 0$. □

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Проблеми керованості для рівняння теплопровідності на півплощині, керованого крайовою умовою Ноймана з точковим керуванням

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У роботі досліджено проблеми керованості та наближеної керованості для керованої системи $w_t = \Delta w$, $w_{x_1}(0, x_2, t) = u(t)\delta(x_2)$, $x_1 > 0$, $x_2 \in \mathbb{R}$, $t \in (0, T)$, де $u \in L^\infty(0, T)$ є керуванням. Для цього досліджено множину $\mathcal{R}_T(0) \subset L^2((0, +\infty) \times \mathbb{R})$ її кінцевих станів, які є досяжними з 0. Установлено, що функція $f \in \mathcal{R}_T(0)$ може бути подана у вигляді $f(x) = g(|x|^2)$ м.с. в $(0, +\infty) \times \mathbb{R}$, де $g \in L^2(0, +\infty)$. Фактично, ми зводимо задачу для функцій з $L^2((0, +\infty) \times \mathbb{R})$ до задачі для функцій з $L^2(0, +\infty)$. Необхідну і достатню умову керованості та достатню умову наближеної керованості за заданий час T за допомогою керувань u, обмежених заданою сталою, одержано в термінах розв'язності степеневої проблеми моментів Маркова. Застосовуючи функції Лаґерра (які утворюють ортонормований базис в $L^2(0, +\infty)$), одержано необхідні і достатні умови наближеної керованості та числові розв'язки проблеми наближеної керованості. Також показано, що не існує ненульового початкового стану системи, який був би нуль керованим за заданий час T. Результати проілюстровано прикладами.

Ключові слова: рівняння теплопровідності, керованість, наближена керованість, півплощина