Journal of Mathematical Physics, Analysis, Geometry, **21** (2025), No. 4, 380–405
doi: https://doi.org/10.15407/mag21.04.02

On New and Earlier Developments in Petrenko's Theory of Growth of Meromorphic Functions

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To Professor Ivan I. Marchenko on his 70th birthday

Petrenko's theory of growth of meromorphic functions lies within the wider spectrum of Nevanlinna theory and was initiated by V.P. Petrenko in 1960s. This paper is focused on the achievements of an outstanding student of V.P. Petrenko, I.I. Marchenko, and his contributions to the theory. An overview of some of I.I. Marchenko's main results (and their further generalizations and applications) concerning deviations, separated maximum modulus points and strong asymptotic values constitutes the body of the paper. The final part of the paper is devoted to a generalization of an early result of I.I. Marchenko and A.I. Shcherba on the sum of deviations of functions holomorphic in the unit disc.

Key words: meromorphic function, holomorphic function, deviation, maximum point, asymptotic value

Mathematical Subject Classification 2020: 30D35, 30D30

1. Introduction

We apply the standard notations of value distribution theory of meromorphic functions: $N(r,a,f)$, $N(r,f)$ for functions counting a-points and poles, $m(r,a,f)$ and $m(r,f)$ for mean proximity functions, $T(r,f)$ for characteristic function. We also use notations $\delta(a,f)$ for Nevanlinna's defect and $\Delta(a,f)$ for Valiron's defect of f at a value a [19, 25, 46].

Petrenko's theory of growth of meromorphic functions dates back to 1969, when the uniform metric was introduced into his research of meromorphic functions for the first time. Thus,

$$\mathcal{L}(r,a,f) = \begin{cases} \max\limits_{|z|=r} \log^+ |f(z)| & \text{for } a = \infty, \\ \max\limits_{|z|=r} \log^+ \left| \dfrac{1}{f(z)-a} \right| & \text{for } a \neq \infty \end{cases}$$

was called *the function of deviation* and

$$\beta(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)},$$

the magnitude of deviation. It can be easily seen that $\beta(a, f)$ is the uniform metric analogue of Nevanlinna's defect $\delta(a, f)$. Values with $\beta(a, f) > 0$ are called defective in the sense of V.P. Petrenko and the set of all such values is denoted as $\Omega(f)$.

It follows from respective definitions that for all $a \in \overline{\mathbb{C}}$ we have $\delta(a, f) \leq \beta(a, f)$ and $D(f) \subset \Omega(f)$, where $D(f)$ denotes the set of all values defective in the sense of R. Nevanlinna with respect to f.

Although $\beta(a, f) = \infty$ is possible for f of infinite order, for meromorphic functions of finite lower order $\lambda := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$ the properties of $\beta(a, f)$ and $\delta(a, f)$ are similar. Notwithstanding the lack of analogues of Nevanlinna's first and second theorems, analogues of Nevanlinna's defect relations are possible for deviations. V.P. Petrenko himself obtained the sharp upper estimate for the value $\beta(a, f)$ and the estimate for the sum $\sum_{a\in\overline{\mathbb{C}}} \beta(a, f)$ [48].

Theorem 1.1. *If $f(z)$ is a meromorphic function of finite lower order λ, then for all $a \in \overline{\mathbb{C}}$ we have*

$$\beta(a, f) \leq B(\lambda) := \begin{cases} \dfrac{\pi\lambda}{\sin \pi\lambda} & \text{if } \lambda \leq 0.5, \\ \pi\lambda & \text{if } \lambda > 0.5, \end{cases}$$

$$\sum_{a\in\overline{\mathbb{C}}} \beta(a, f) \leq 816\pi(\lambda + 1)^2.$$

The value $B(\lambda)$ appearing in Petrenko's theorem is called Paley's constant. In 1932, R. Paley in [47] stated a hypothesis that the inequality $\beta(\infty, g) \leq \pi\varrho$ holds for any entire function g of finite order ϱ, which was proved by N.V. Govorov in 1969 [22]. What is more, the estimate for meromorphic functions of finite lower order $\lambda > 0.5$ follows from a result of A.A. Gol'dberg and I.V. Ostrovskii [18] and the equality in this estimate is attained by the Mittag–Leffler function

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + \frac{n}{\rho})}.$$

It should be mentioned that M.N. Sheremeta in [52] proved that for any numbers $\lambda, \rho,\ 0 \leq \lambda \leq \rho$ there exists an entire function $E_{\lambda,\rho}(z)$ of finite lower order λ and order ρ such that

$$\beta(\infty, E_{\lambda,\rho}) = B(\lambda).$$

2. Exceptional values of meromorphic functions

In 1990, I.I. Marchenko together with his student A.I. Shcherba proved an analogue of the inequality $\sum_{a\in\overline{\mathbb{C}}} \delta(a, f) \leq 2$ for deviations, thus solving the problem formulated by Petrenko in his book [49]. The method applied by I.I. Marchenko and A.I. Shcherba in their estimate of the sum of deviations differed from Petrenko's method and involved application of Baernstein's T^*-function [2, 3].

Theorem 2.1. [45] *If $f(z)$ is a meromorphic function of finite lower order λ, then*

$$\sum_{a\in\overline{\mathbb{C}}} \beta(a,f) \leq 2B(\lambda).$$

In case of $\lambda = \frac{n}{2}$, where n is a natural number, the estimate in the theorem is exact. It is attained for the function constructed by R. Nevanlinna (see [19, p. 317]). This fact is not accidental, which is shown by the following result of A.E. Eremenko [14].

Theorem 2.2. *Let $f(z)$ be a meromorphic function of finite lower order λ and such that $\sum_{(a)} \beta(a,f) = 2B(\lambda)$. Then the order $\rho = \lambda = \frac{n}{2}$ for $(n = 2, 3, \ldots)$ and if $a \in \Omega(f)$, then $\beta(a,f) = \pi$.*

An interesting question concerned comparative structures of sets $D(f)$ and $\Omega(f)$, other than the inclusion $D(f) \subset \Omega(f)$. The first example of a meromorphic function of finite order such that $\beta(0,f) > \delta(0,f) = 0$ was given by A.F. Grishin in 1975 [23]. A comprehensive solution to this problem was reached by A.A. Gol'dberg, A.E. Eremenko, and M.L. Sodin [20, 21] in 1987.

Theorem 2.3. *Let $E_1 \subset E_2 \subset \overline{\mathbb{C}}$ be no more than countable sets and $\rho > 0$ be any positive number. There exists a meromorphic function of order ρ such that*

$$D(f) = E_1, \quad \Omega(f) = E_2.$$

The value

$$\Delta(a,f) = \limsup_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}$$

is called Valiron's defect and

$$V(f) = \{a \in \overline{\mathbb{C}} :\ \Delta(a,f) > 0\},$$

the set of Valiron's defective values. It is easy to notice that

$$D(f) \subset V(f).$$

G. Valiron proved that, contrary to the set of Nevanlinna's defects, $V(f)$ can be of cardinality of the continuum (see [19, p. 153]). Another interesting question involved a possible connection between the sets $V(f)$ and $\Omega(f)$. The answer was given by D.F. Shea (presented by W.H.J. Fuchs in [16], see also [49]).

Theorem 2.4. *Let $f(z)$ be a meromorphic function of finite lower order λ. Then for each $a \in \mathbb{C}$ we have*

$$\beta(a,f) \leq B(\lambda,\Delta) := \begin{cases} \pi\lambda\sqrt{\Delta(2-\Delta)} & \text{if } \lambda \notin \Lambda(\Delta), \\ \dfrac{\pi\lambda}{\sin\pi\lambda}(1-(1-\Delta)\cos\pi\lambda) & \text{if } \lambda \in \Lambda(\Delta), \end{cases} \tag{2.1}$$

where $\Lambda(\Delta) = \left\{\lambda :\ 0 \leq \lambda \leq 0.5,\ \sin\frac{\pi\lambda}{2} < \sqrt{\frac{\Delta}{2}}\right\}$, $\Delta = \Delta(a,f)$.

It follows from Theorem 2.4 that for meromorphic functions of finite lower order always $\Omega(f) \subset V(f)$. Moreover, the estimate in the theorem is exact, with an appropriate example given by M.A Ryshkov in [50].

If $\beta(a,f) > 0$ for a value $a \in \mathbb{C}$, then it is easy to imagine that $f(z)$ approaches a fast in appropriate components, which leads to the expectation that in these components the derivative $f'(z)$ tends to 0. A natural question arises whether it is possible to obtain an upper estimate of the sum $\sum_{a\neq\infty} \beta(a,f)$ involving $\Delta(0,f')$. A positive answer was given by I.I. Marchenko in 1999 [41].

Theorem 2.5. *For a meromorphic function of finite lower order λ the following inequality holds*:

$$\sum_{a\neq\infty} \beta(a,f) \leq 2B(\lambda, \Delta(0,f')),$$

where $B(\lambda,\Delta)$ is the value defined in (2.1).

Nevanlinna's first fundamental theorem implies the inequality

$$m(r,a,f) \leq T(r,f) + O(1) \quad \text{as } r \to \infty,$$

while Nevanlinna's second fundamental theorem, the inequality

$$\sum_{k=1}^{q} m(r,a_k,f) \leq 2T(r,f) + O(\log(rT(r,f))) \quad \text{as } r \to \infty,\ r \notin E,\ \operatorname{mes} E < \infty.$$

Formulation of analogues of these relationships for the uniform metrics involves the notions of upper and lower logarithmic densities of a set.

Let $E \subset (0,\infty)$ be a measurable set. The qualities

$$\overline{\operatorname{logdens}}\, E = \limsup_{R\to\infty} \frac{1}{\ln R} \int_{E\cap[1,R]} \frac{dt}{t},$$
$$\underline{\operatorname{logdens}}\, E = \liminf_{R\to\infty} \frac{1}{\ln R} \int_{E\cap[1,R]} \frac{dt}{t}$$

are called respectively upper and lower logarithmic densities of the set E. The following estimates were obtained by I.I. Marchenko in 1998 [39].

Theorem 2.6. *Let $f(z)$ be a meromorphic function of finite lower order λ and order ρ. Let also $0 < \gamma < \infty$ and a, $a_k \in \overline{\mathbb{C}}$, $1 \leq k \leq q$. We put*

$$E_1(\gamma) = \{r : \ \mathcal{L}(r,a,f) < B(\gamma)T(r,f)\},$$
$$E_2(\gamma) = \left\{ r : \ \sum_{k=1}^{q} \mathcal{L}(r,a_k,f) < 2B(\gamma)T(r,f) \right\}.$$

Then

$$\overline{\operatorname{logdens}}\, E_n(\gamma) \geq 1 - \frac{\lambda}{\gamma}, \quad \underline{\operatorname{logdens}}\, E_n(\gamma) \geq 1 - \frac{\rho}{\gamma}, \quad n = 1,2,$$

where $B(\gamma)$ is the Paley constant.

As it has already been mentioned, for meromorphic functions of infinite lower order $\beta(a, f)$ may be infinite. For instance, $\beta(\infty, \exp e^z) = \infty$. In 1994, W. Bergweiler and H. Bock [4] proved that for a meromorphic function of infinite lower order,

$$\liminf_{r\to\infty} \frac{\log^+ \max_{|z|=r} |f(z)|}{rT'_-(r, f)} \leq \pi,$$

where $T'_-(r, f)$ is the left derivative of the Nevanlinna characteristic function. This result opened a possibility to apply the function of deviation $\mathcal{L}(r, a, f)$ also for f of infinite order. In 1997, A.E. Eremenko [13] introduced the quality

$$b(a, f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r, a, f)}{A(r, f)},$$

where $A(r, f)\pi$ is the spherical area, counting multiplicities of the covering, of the image on the Riemann sphere of the disc $\{z : \ |z| \leq r\}$ under f. It follows directly from the estimate of W. Bergweiler and H. Bock that for all $a \in \overline{\mathbb{C}}$,

$$b(a, f) \leq \pi.$$

A.E. Eremenko also obtained an analogue of Theorem 2.1 for the infinite case.

Theorem 2.7. *For a meromorphic function such that the set*

$$\{a \in \overline{\mathbb{C}} : b(a, f) > 0\}$$

contains more than one point the following inequality holds:

$$\sum_{a\in\overline{\mathbb{C}}} b(a, f) \leq 2\pi.$$

Let us add that in 1998, I.I. Marchenko got the estimates of $b(a, f)$ and $\sum_{a\in\overline{\mathbb{C}}} b(a, f)$ involving Valiron's defect [40].

Theorem 2.8. *Let $f(z)$ be a meromorphic function of infinite lower order. Then for each $a \in \overline{\mathbb{C}}$,*

$$b(a, f) \leq \pi\sqrt{\Delta(a, f)(2 - \Delta(a, f))}$$

and also

$$\sum_{a\neq\infty} b(a, f) \leq 2\pi\sqrt{\Delta(0, f')(2 - \Delta(0, f'))}.$$

Other results concerning functions of infinite order can be found in [8, 40, 42].

If we replace constants with functions of relatively slow growth, it is still possible to obtain estimates similar to those in Nevanlinna's second main theorem. If f, a are meromorphic functions in $\mathbb{C}$, we say that *a is a small function of f* if $T(r, a) = S(r, f)$, which means that $T(r, a) = o(T(r, f))$ apart from an exceptional set of finite linear measure. The set of all small functions of f is denoted by $\mathcal{S}(f)$. In 1986, G. Frank and G. Weissenborn obtained an extension of the second main theorem for functions meromorphic in the plane, with rational functions replacing constants [15]. A result of N. Steinmetz concerning small defective functions in general soon followed [53]. The conclusion is that for functions meromorphic in the complex plane the set of their defective small functions is at most countable and

$$\delta(\infty, f) + \sum_{a \in \mathcal{S}(f)} \delta(a, f) \le 2.$$

The exact analogue of the second main theorem, including the ramification factor, was obtained by K. Yamanoi in 2004 [55].

These results inspired the efforts to find the estimates for deviations from small functions and the structure of respective sets of defective small functions. In an article from 2004 [9], for example, the following extension of Theorem 2.6 was given.

Let f, a be meromorphic functions. We put

$$\beta(a, f) = \liminf_{r \to \infty} \frac{\mathcal{L}(r, a, f)}{T(r, f)},$$

where $\mathcal{L}(r, a, f) = \mathcal{L}(r, \infty, \frac{1}{f-a})$ denotes the deviation of f at a.

Theorem 2.9. *Let $f(z)$ be a transcendental entire function of finite lower order λ, order ρ and let $0 < \gamma < \infty$. Let also $\{q_\nu(z)\}_{\nu=1}^k$ be distinct rational functions. We put $E(\gamma) = \{r : \sum_{\nu=1}^k \mathcal{L}(r, q_\nu, f) < B(\gamma)T(r, f)\}$. Then we have*

$$\overline{\operatorname{logdens}}\, E(\gamma) \ge 1 - \frac{\lambda}{\gamma} \quad \textit{and} \quad \underline{\operatorname{logdens}}\, E(\gamma) \ge 1 - \frac{\rho}{\gamma}.$$

As a result, for entire functions of finite lower order, it was possible to establish the structure of the set of rational functions defective in the sense of V.P. Petrenko.

Corollary 2.10. *Let $f(z)$ be a transcendental entire function of finite lower order λ and let $\mathfrak{M}$ denote the set of all rational functions.The set $\{q \in \mathfrak{M} : \beta(q, f) > 0\}$ is no more than countable. Moreover, for distinct rational functions $\{q_\nu(z)\}$ we have*

$$\sum_{(\nu)} \beta(q_\nu, f) \le B(\lambda).$$

Further results in this direction can be found in [10, 11].

3. Separated maximum modulus points of entire and meromorphic functions

Some of the problems which are most frequently attended to in the research of I.I. Marchenko concern the relationship between the number of so-called separated maximum modulus points and other values inhabiting the world of value distribution and growth theory.

Let $\nu(r)$ denote the number of maximum points of $|f(z)|$ on the circle $|z| = r$. In 1964, P. Erdös (see [1]) formulated a question whether it is possible to find an entire function other than cz^p with $\nu(r) \to \infty$ as $r \to \infty$. F. Herzog and G. Piranian in [26] gave an example of a function of infinite order with $\nu(r)$ unbounded, leaving the question open for functions of finite order.

In 1995, I.I. Marchenko, considering the problem in a wider context of meromorphic functions, introduced the notion of separated maximum modulus points. Initially, he considered the sets, where $|f(z)| > 1$ and two parameters concerning maximum modulus points: $p(r, \infty, f)$ — the number of component intervals of the set $\{\varphi : |f(re^{i\varphi})| > 1\}$, containing at least one point of maximum modulus of $f(z)$ and $p(\infty, f) = \liminf_{r\to\infty} p(r, \infty, f)$. As it has turned out, these values are closely related with other notions in the value distribution such as, for example, the deviation [38] .

Theorem 3.1. *If $f(z)$ is a meromorphic function of finite lower order λ, then*

$$\beta(\infty, f) \le \begin{cases} \dfrac{\pi\lambda}{p(\infty, f)} & \text{if } \dfrac{\lambda}{p(\infty, f)} \ge 0.5, \\ \dfrac{\pi\lambda}{\sin \pi\lambda} & \text{if } p(\infty, f) = 1 \text{ and } \lambda < 0.5, \\ \dfrac{\pi\lambda}{p(\infty, f)} \sin \dfrac{\pi\lambda}{p(\infty, f)} & \text{if } p(\infty, f) > 1 \text{ and } \dfrac{\lambda}{p(\infty, f)} < 0.5. \end{cases}$$

Corollary 3.2. *If $f(z)$ is a meromorphic function of finite lower order λ, then*

$$p(\infty, f) \le \max\left(\left[\frac{\pi\lambda}{\beta(\infty, f)}\right], 1\right),$$

while for an entire function of finite lower order λ,

$$p(\infty, f) \le \max([\pi\lambda], 1).$$

Here $[x]$ is the integer part of x.

The estimates in the theorem above are sharp. In the first two cases the equality holds for entire functions constructed on the basis of the Mittag–Leffler function, while in the third case it holds for a meromorphic function constructed on the basis of the function appearing in [19, Ex. 3, p. 282].

In [43], I.I. Marchenko was able to generalize Theorem 2.4 in the following way.

Theorem 3.3. *Let $f(z)$ be a meromorphic function of finite lower order λ and order $\rho, \gamma > 0$ be any positive number,*

$$E(\gamma) = \left\{ r > 0 : \mathcal{L}(r, \infty, f) < B(\frac{\gamma}{p(\infty, f)}, \Delta(\infty, f))T(r, f) \right\}.$$

Then

$$\overline{\text{logdens}}\, E(\gamma) \ge 1 - \frac{\lambda}{\gamma}, \quad \underline{\text{logdens}}\, E(\gamma) \ge 1 - \frac{\rho}{\gamma},$$

where $B(\gamma, \Delta)$ was defined in Theorem 2.4.

In later papers I.I. Marchenko applied more accurate means to separate maximum modulus points of meromorphic functions. Let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r > 0$ such that $\phi(r) = o(T(r, f))$. Then $\overline{p}_\phi(r, \infty, f)$ denotes the number of component intervals of the set

$$\left\{\theta : \log |f(re^{i\theta})| > \phi(r)\right\}$$

with at least one maximum modulus point of $f(z)$. Furthermore,

$$\overline{p}_\phi(\infty, f) = \liminf_{r\to\infty} \overline{p}_\phi(r, \infty, f),$$

and

$$\overline{p}(\infty, f) = \sup_{\{\phi\}} \overline{p}_\phi(\infty, f).$$

It is straightforward that $\overline{p}(\infty, f) \geq p(\infty, f)$. It was shown in [6] that Theorem 3.1 also holds if we replace $p(\infty, f)$ with $\overline{p}(\infty, f)$.

Another way to separate maximum modulus points is as follows. For $0 < \eta \leq 1$ and $r > 0$, let us denote by $\tilde{p}_\eta(r, \infty, f)$ the number of component intervals of the set

$$\left\{\theta : \log |f(re^{i\theta})| > (1-\eta)T(r, f)\right\}$$

with at least one maximum modulus point of the function $f(z)$. We set

$$\tilde{p}_\phi(\infty, f) = \liminf_{r\to\infty} \tilde{p}_\eta(r, \infty, f)$$

and

$$\tilde{p}(\infty, f) = \sup_{\eta} \tilde{p}_\eta(\infty, f).$$

It is easily seen that $\tilde{p}(\infty, f) \geq \overline{p}(\infty, f)$. The following estimate of $\tilde{p}(\infty, f)$ through the value of deviation appeared in [7].

Theorem 3.4. *For meromorphic functions $f(z)$ of finite lower order λ the following inequality is true:*

$$\tilde{p}(\infty, f) \leq \max\left(\left[\frac{2\pi\lambda}{\beta(\infty, f)}\right], 1\right),$$

while for $f(z)$ entire of finite lower order,

$$\tilde{p}(\infty, f) \leq \max([2\pi\lambda], 1).$$

Here $[x]$ is the integer part of x.

It should be mentioned here that recently A. Glücksam and L. Pardo-Simon [17] have constructed an example of an entire function with $\tilde{p}_\eta(r, \infty, f)$ tending to infinity, thus giving another argument in favor of a positive answer to the question of P. Erdös. Also, I.I. Marchenko and A. Kowalski [35] considered separated maximum modulus points of entire functions, estimated the Lebesgue measure of the set

$$\left\{\theta : \log |f(re^{i\theta})| > \alpha \log M(r, f)\right\}, \quad 0 \leq \alpha < 1,$$

and generalized the results of K. Arima and A. Baernstein.

4. Strong asymptotic values

According to the classical definition (see [19], p. 233), $a \in \overline{\mathbb{C}}$ is *an asymptotic value* of a meromorphic function f if there exists a continuous curve $\Gamma \subset \mathbb{C}$, $\Gamma :$ $z = z(t)$, $0 \le t < \infty$, $z(t) \to \infty$ as $t \to \infty$, such that

$$\lim_{z\to\infty, z\in\Gamma} f(z) = \lim_{t\to\infty} f(z(t)) = a.$$

A pair $\{a, \Gamma\}$, defined above, is called *an asymptotic spot of* f. Two asymptotic spots $\{a_1, \Gamma_1\}$ and $\{a_2, \Gamma_2\}$ are considered equal if $a_1 = a_2 = a$ and there exists a sequence of continuous curves γ_k with one end of each γ_k belonging to Γ_1 and the other to Γ_2, and

$$\lim_{k\to\infty} \min_{z\in\gamma_k} |z| = \infty, \quad \lim_{\substack{z\to\infty \\ z\in\bigcup_k \gamma_k}} f(z) = a.$$

The questions of relationship between the sets of deficient values and asymptotic values, and of the number of asymptotic values or asymptotic spots received a lot of attention earlier. It is easy to see that in general an asymptotic value does not have to be a deficient value in any sense, even in the sense of G. Valiron $\big($take, for example, $f(z) = \frac{\sin z}{z}$ and $a = 0\big)$. Iversen proved that if a is a E. Picard defective value of a meromorphic function f (f has only a finite number of a-points), then it is also an asymptotic value [19], while W.K. Hayman showed that a may not necessarily be an asymptotic value of f if a more general condition $N(r, a, f) = o(T(r, f))$ is fulfilled. In [25], he gave an example of a meromorphic function of order 0 with $\delta(\infty, f) = 1$ and ∞ not being an asymptotic value of f. As far as the number of asymptotic spots is concerned, a classical theorem of Denjoy–Carleman–Ahlfors states that an entire function of finite lower order λ cannot have more than $\max\{[2\lambda], 1\}$ different asymptotic spots ($[x]$ denotes here the integer part of x) [19]. As the example of $f(z) = e^{e^z}$ shows, the number of asymptotic spots of an entire function of infinite lower order may be infinite. In case of meromorphic functions, the set of asymptotic values may be infinite for functions of any order, which was shown by A.E. Eremenko in 1986 [12].

Theorem 4.1. *For every value ϱ, $0 \le \varrho \le \infty$, there exists a meromorphic function of order ϱ with the set of asymptotic values equal to $\overline{\mathbb{C}}$.*

In 2004, I.I. Marchenko introduced the definition of a strong asymptotic value, which stemmed from the conviction that if the speed of approach of a meromorphic function to an asymptotic value is high enough, then the number of such values must be limited.

Definition 4.2. A value $a \in \overline{\mathbb{C}}$ is called an α_0-strong asymptotic value of a meromorphic function f if there exists a continuous curve $\Gamma :$ $z = z(t)$, $0 \le t < \infty$, $z(t) \to \infty$ as $t \to \infty$, such that

$$\liminf_{t\to\infty} \frac{\log |f(z(t)) - a|^{-1}}{T(|z(t)|, f)} = \alpha(a) \ge \alpha_0 > 0 \quad \text{if } a \ne \infty,$$

$$\liminf_{t\to\infty} \frac{\log|f(z(t))|}{T(|z(t)|, f)} \geq \alpha_0 > 0 \quad \text{if } a = \infty.$$

An asymptotic spot $\{a, \Gamma\}$ is then called an α_0- strong asymptotic spot.

In other words, a is a strong asymptotic value of a meromorphic function f if on an asymptotic curve Γ the function tends to the value a with the speed comparable with characteristic $T(r, f)$.

It is easy to notice that if a is an α_0-strong asymptotic value of f, then the magnitude of the Petrenko deviation $\beta(a, f) \geq \alpha_0$. It means that a is also a defective value in the sense of Petrenko. Therefore I.I. Marchenko was able to show that the number of strong asymptotic spots is limited, at least for meromorphic functions of finite lower order [44].

Theorem 4.3. *Let f be a meromorphic function of finite lower order λ and $\{a_\nu, \Gamma_\nu\}$, $\nu = 1, 2, \ldots, k$, α_0 be strong asymptotic spots of f. Then* $k \leq \left[\frac{2B(\lambda)}{\alpha_0}\right]$.

The example of $f(z) = e^{e^z}$ and $a = \infty$ again shows that such an estimate cannot be made for functions of infinite order.

Later on, the notion of value which is strongly asymptotic was extended to include strongly asymptotic small functions. Results concerning the structure of the set of strong asymptotic rational functions appeared in 2011 [10]. More general (but less accurate) results concerning the structure of the set of strong asymptotic small functions appeared in 2017 [11].

5. Meromorphic minimal surfaces

Lately, scientific interests of I.I. Marchenko have been focused mainly on the theory of meromorphic minimal surfaces. The theory, introduced and developed in the 1960s and 1970s by E.F. Beckenbach and G.A. Huthinson, creates an interesting field of application of the Nevanlinna theory. The analogues of Nevanlinna's first and second main theorems function there with one notable distinction. The leading role played in classical value distribution theory by the counting function $N(r, a, f)$ is now held by the so-called visibility function denoted as $H(r, a, S)$. Thus, for example, the first main theorem for a meromorphic minimal surface S has the form of the equality

$$m(r, a, S) + N(r, a, S) + H(r, a, S) = T(r, S) + O(1).$$

The notions of Petrenko's theory appeared in the context of minimal surfaces as early as 1979, when I.I. Marchenko in [37] considered the deviation of meromorphic surfaces. Much later, he revisited this area of research. Working together with A. Kowalski, they obtained a number of results published in a series of papers. In [28], they presented upper estimates of deviations and the number of separated maximum modulus points of meromorphic minimal surfaces, while in [30], they estimated the spread of such a surface. The theorems are illustrated with examples showing their sharpness. Further results in this area included an analogue of Theorem 2.4 for minimal surfaces in [32] and analysis of the relationship between the number of separated maximum modulus points of minimal surfaces and the Baernstein T^*-function in [33].

Other recent results of I.I. Marchenko together with A. Kowalski concern entire curves [29, 34] and algebroid functions [31].

6. Functions meromorphic in the disc

In this section, we focus on functions meromorphic in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The order and lower order of a meromorphic in the unit disc function f are defined by

$$\varrho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}, \quad \lambda(f) := \liminf_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}.$$

Nevanlinna's first theorem states that for a function f meromorphic in the unit disc the equality

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(1), \tag{6.1}$$

holds for any value $a \in \mathbb{C}$ and $r \to 1^-$. By the second main theorem, for a set $\{a_\nu\}_{\nu=1}^n$ of pairwise distinct complex numbers the inequality

$$m(r,f) + \sum_{\nu=1}^{n} m(r,a_\nu,f) \le 2T(r,f) + O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right) \tag{6.2}$$

is true for $r \to 1^-$, possibly except for r in a set E such that $\int_E \frac{dr}{1-r} < +\infty$ [46]. A value $a \in \overline{\mathbb{C}}$ is called *a (Nevanlinna) defective value* of a meromorphic in the unit disc function f if

$$\delta(a,f) = \liminf_{r\to 1^-} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to 1^-} \frac{N(r,a,f)}{T(r,f)} > 0.$$

The defect relations following from Nevanlinna's theorems in the unit disc are

$$0 \le \delta(a,f) \le 1, \quad \sum_{a\in\overline{\mathbb{C}}} \delta(a,f) \le 2 + \frac{1}{A}, \tag{6.3}$$

where

$$A := \liminf_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)}$$

if $0 < A < \infty$ (see [46, 51]). It means that the set $D(f)$ of values defective in the sense of R. Nevanlinna is at most countable provided that $A \neq 0$. We say that a meromorphic in the unit disc function f is *admissible* if $A = +\infty$. In this case,

$$\sum_{a\in\overline{\mathbb{C}}} \delta(a,f) \le 2.$$

Proximity of a meromorphic in a disc function to a certain value a can also be measured in terms of deviation, which is defined here as

$$\beta(a,f) := \liminf_{r\to 1^-} \frac{\mathcal{L}(r,a,f)}{T(r,f)}.$$

For meromorphic in the unit disc functions the structure of the set $\Omega(f)$ of values with positive deviation may strongly differ from the set $D(f)$ of values defective in the sense of R. Nevanlinna, even for finite order functions. For instance, for $f(z) = \exp(\frac{1}{1-z})$, the deviation $\beta(\infty, f) = \infty$. In [49], V.P. Petrenko proved that for any value ρ, $0 \le \rho \le \infty$, there exists a meromorphic in a disc function of order ρ with $\Omega(f)$ of cardinality of the continuum.

For disc functions of finite lower order it is possible, however, to obtain upper estimates involving the quantity

$$\hat{\beta}(a,f) = \liminf_{r\to 1}(1-r)\frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

first introduced by A.V. Krytov in [36]. I.I. Marchenko and A.I. Shcherba in [45] obtained the following result concerning $\hat{\beta}(a,f)$.

Theorem 6.1. *Let f be an admissible meromorphic function of finite lower order in the unit disc. Then*

$$\sum_{a\in\overline{\mathbb{C}}} \hat{\beta}(a,f) \le 2\pi\lambda \cos^{-1-\lambda} \frac{\pi}{2(1+\lambda)}.$$

For other estimates involving $\hat{\beta}(a,f)$ see, for example, [27].

Let f, a be meromorphic functions. We apply the following notations:

$$m(r,a,f) := m\left(r, 0, \frac{1}{f-a}\right), \qquad \delta(a,f) := \delta\left(0, \frac{1}{f-a}\right),$$
$$\mathcal{L}(r,a,f) := \mathcal{L}\left(r, 0, \frac{1}{f-a}\right), \qquad \beta(a,f) := \beta\left(0, \frac{1}{f-a}\right).$$

Let us add the following definition.

Definition 6.2. Let f be a meromorphic function in the unit disc. We say that a function $s : [0,1) \to \mathbb{R}$ is an $S(r,f)$ (a small target of f) if

$$s(r) = o(T(r,f)) \quad \text{as } r \to 1^-,\ r \notin E,$$

for a set E such that $\int_E \frac{dr}{1-r} < +\infty$.

The following extension of the second main theorem for the unit disc was shown in [5].

Theorem 6.3. *Let f be a meromorphic function in the unit disc and let $a_1, \ldots, a_n$ be distinct meromorphic small functions with respect to f. For any $\varepsilon > 0$, the following inequality holds:*

$$m(r,f) + \sum_{\nu=1}^{n} m(r, a_\nu, f) \le (2+\varepsilon)T(r,f) + S(r,f) + O\left(\log \frac{1}{1-r}\right).$$

The estimates presented below refer back to the results of I.I. Marchenko and A.I. Shcherba from [45], while their proofs are based on a variety of techniques of the Nevanlinna theory applied in works of I.I. Marchenko. In both estimates the constants, which appeared in the original results are replaced with polynomials not intersecting on $|z| = 1$. The phrase "not intersecting on $|z| = 1$" refers to a situation when for a pair p_ν, p_η of distinct polynomials $p_\nu(z) - p_\eta(z) \neq 0$ for points z on the unit circle.

Theorem 6.4. *Let f be an admissible holomorphic function of lower order $0 < \lambda < \infty$ in the unit disc. Then, for any set of distinct polynomials $\{p_\nu\}_{\nu=1}^q$ not intersecting on $|z| = 1$,*

$$\sum_{\nu=1}^{q} \hat{\beta}(p_\nu, f) \leq \pi\lambda \cos^{-1-\lambda} \frac{\pi}{2(1+\lambda)}.$$

Theorem 6.5. *Let f be an admissible holomorphic function of zero lower order in the unit disc. Then, for any set of distinct polynomials $\{p_\nu\}_{\nu=1}^q$ not intersecting on $|z| = 1$ and of degree not exceeding d,*

$$\sum_{\nu=1}^{q} \hat{\beta}(p, f) \leq 2\Delta\left(0, f^{(d+1)}\right),$$

where $\Delta(0, f^{(d+1)})$ is Valiron's defect of $f^{(d+1)}$ at zero.

6.1. Auxiliary results. We start with a lemma on the logarithmic derivative as formulated in [25].

Lemma 6.6. *If f is a function meromorphic in the unit disc, $f(0) \neq 0, \infty$, then for $0 < r < R < 1$,*

$$m\left(r, \frac{f'}{f}\right) < 4\log^+ T(R, f) + 4\log^+ \log^+ \frac{1}{|f(0)|} + 5\log^+ R + 6\log^+ \frac{1}{R-r} + \log^+ \frac{1}{r} + 14.$$

Since we deal with disc functions of unbounded characteristic here, for our purposes we formulate the estimate

$$m\left(r, \frac{f'}{f}\right) < 5\log^+ T(R, f) + 6\log^+ \frac{1}{R-r}$$

holding for $r_0 \leq r < R < 1$. Setting $R = \frac{r+1}{2}$, we get

$$m\left(r, \frac{f'}{f}\right) < 5\log^+ T\left(1 - \frac{1-r}{2}, f\right) + 6\log^+ \frac{2}{1-r}.$$

It leads to the estimate for positive integers d and $r \to 1^-$,

$$m\left(r, \frac{f^{(d+1)}}{f}\right) < (5d+6)\log T\left(1 - \frac{1-r}{2^{d+1}}, f\right) + (6d+7)\log \frac{1}{1-r}. \quad (6.4)$$

The following result from [45] is a unit disc version of a theorem concerning the sequences of Polya peaks [49].

Lemma 6.7. *Let f be a function of finite lower order λ in the unit disc. For each fixed number $B > 1$ there exist two sequences of positive numbers $\{v_k\}$ and $\{R_k\}$ such that*

$$\lim_{k\to\infty} v_k = \lim_{k\to\infty} R_k = \lim_{k\to\infty} \frac{R_k}{v_k} = 0,$$

and for each $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that for $k > k_0(\varepsilon)$,

$$T\left(1 - \frac{R_k}{B}, f\right) R_k^{\lambda} + T\left(1 - \frac{v_k}{B}, f\right) v_k^{\lambda} < \varepsilon \int_{R_k}^{v_k} T(1-R,f) R^{\lambda-1}\, dR.$$

Let f be an admissible holomorphic function of finite lower order λ in the unit disc. For $1 \le \nu \le q$, let $\{p_\nu\}_{\nu=1}^q$ be a set of distinct polynomials such that $\deg(p_\nu) \le d$, $d \ge 1$, and $\hat\beta(p_\nu, f) > 0$, $1 \le \nu \le q$. Moreover, if $|z| = 1$, we have $p_\nu(z) \ne p_\eta(z)$, $\nu \ne \eta$. Let $S_0 > 0$ be chosen in such a way that if $S_0 \le |z| < 1$, then for all $1 \le \nu, \eta \le q$, $\nu \ne \eta$ we have $p_\nu(z) \ne p_\eta(z)$. We put for $\nu \ne \eta$,

$$c_{\nu,\eta} = \min_{|z|\ge S_0} |p_\nu(z) - p_\eta(z)| > 0,$$

$$c = \min_{\substack{1\le\nu\\ \eta\le q}} c_{\nu,\eta} > 0.$$

We apply notations from Lemma 6.7. We also denote by $A(r,R)$ the annulus $\{z : r < |z| < R\}$ and by $\overline{A}(r,R)$ its closure. Let $\varepsilon \in (0,1)$ be fixed. For $k \ge k_0(\varepsilon)$ and such that $1 - 2v_k \ge S_0$, we put

$$G_k := \left\{ z \in A\left(1 - 2v_k, 1 - \frac{2R_k}{B}\right) : \right.$$
$$\left. |f^{(d+1)}(z)| < \exp\left\{ -2\varepsilon T\left(1 - \frac{1-|z|}{2^{d+1}}, f\right)\right\}\right\},$$

Now, for $1 \le \nu \le q$, we put $G_{k,\nu}$ for the union of those connected components of G_k which contain a point z_0 such that

$$|f(z_0) - p_\nu(z_0)| < \frac{c}{4}$$

and points $z_1, z_2, \dots, z_d$ such that for $j = 1, \dots, d$,

$$|f^{(j)}(z_j) - p_\nu^{(j)}(z_j)| < \exp\{-2\varepsilon T(1 - \frac{1-|z_j|}{2^{d+1}}, f)\}.$$

Lemma 6.8. *The sets $G_{k,\nu}$ and $G_{k,\eta}$ are disjoint for $\nu \ne \eta$, $1 \le \nu$, $\eta \le q$.*

Proof. We show that the sets $G_{k,\nu}$ and $G_{k,\eta}$ are disjoint for $\nu \ne \eta$ applying the method introduced by A. Weitsman [54] and following the same lines as in [45] and [9].

Let $r_n = 1 - 2^{-n}$, $n \ge 1$. For a fixed k, we put $m_0(k)$, $M_0(k)$ for positive integers such that

$$r_{m_0(k)} \le 1 - 2v_k < r_{m_0(k)+1}, \quad r_{M_0(k)-1} < 1 - \frac{2R_k}{N} \le r_{M_0(k)}.$$

By Cartan's theorem, inequality (6.4) takes the form

$$\begin{aligned}\int_0^{2\pi} n\Big(1-\frac{1-r_n}{2}, \frac{1}{f^{(d+1)}-te^{i\varphi}}\Big) &= \int_0^{2\pi} n\left(1-\frac{1-r_n}{2}, e^{i\varphi}, \frac{f^{(d+1)}}{t}\right)\\ &\le \frac{4}{1-r_n}\int_0^{2\pi} N(1-\frac{1-r_n}{4}, e^{i\varphi}, \frac{f^{(d+1)}}{t})\\ &\le \frac{C}{1-r_n}\left(T(1-\frac{1-r_n}{4}, f^{(d+1)})+O(1)+\log^+\frac{1}{t}\right)\\ &\le \frac{C}{1-r_n}\left((1+o(1))T(1-\frac{1-r_n}{2^{d+1}}, f)+\log^+\frac{1}{t}\right) \quad \text{as } n\to\infty.\end{aligned}$$

Applying the length-area principle, we find that there exists a number α_n,

$$\varepsilon T\left(1-\frac{1-r_n}{2^{d+1}}, f\right) \le \alpha_n \le \varepsilon T\left(1-\frac{1-r_n}{2^{d+1}}, f\right)+\log 2$$

such that

$$l(e^{-\alpha_n}) \le C_2\sqrt{T\left(1-\frac{1-r_n}{2^{d+1}}, f\right)(1-r_n)^{-1}}, \tag{6.5}$$

where $l(t)$ denotes the total length of the level curves $|f^{d+1}(z)|=t$ in $|z|<r_{n+1}$. We put

$$G_k^n := A\left(1-2v_k, 1-\frac{2R_k}{B}\right)\cap\left\{z\in\overline{A}(r_n,r_{n+1}) : |f^{(d+1)}(z)|<e^{-\alpha_n}\right\}.$$

It follows that

$$G_{k,\nu}\subset G_k\subset\bigcup_{n=m_0(k)}^{M_0(k)} G_k^n. \tag{6.6}$$

Let $k\ge k_0(\varepsilon)$ and for a fixed ν let $z\in G_{k,\nu}$. Then there is a component G of $G_{k,\nu}$ such that $z\in G$ and there is a point $z_0\in G$ with $|f(z_0)-p_\nu(z_0)|<\frac{c}{4}$ and points $z_1,z_2,\dots,z_d$ such that for $j=1,\dots,d$,

$$|f^{(j)}(z_j)-p_\nu^{(j)}(z_j)| < \exp\left\{-2\varepsilon T\left(1-\frac{1-r_n}{2^{d+1}}, f\right)\right\}.$$

By (6.6), for $z\in G$ we can join z with the points z_j, $0\le j\le d$, by curves lying in G, each of them of the length not exceeding the sum of lengths of boundaries of G_k^n ($m_0(k)\le n\le M_0(k)$). By (6.5), the length of each of the boundaries does not exceed $C_2\sqrt{T\left(1-\frac{1-r_n}{2^{d+1}}, f\right)(1-r_n)^{-1}}$. Let us put $g(z):=f(z)-p_\nu(z)$ and notice that $g^{(d+1)}(z)=f^{(d+1)}(z)$. This way we obtain

$$\begin{aligned}|f(z)-p_\nu(z)| = |g(z)| \le |g(z_0)|+|g(z)-g(z_0)| &< \frac{c}{4}+\int_{z_0}^{z}|g'(\xi)|\,|d\xi|\\ &\le \frac{c}{4}+\int_{z_0}^{z}(|g'(z_1)|+|g'(\xi)-g'(z_1)|)\,|d\xi|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{c}{4} + |g'(z_1)|l_0 + \int_{z_0}^{z}\left(\int_{z_1}^{\xi}|g''(\xi_1)||d\xi_1|\right)|d\xi| \\
&\le \frac{c}{4} + |g'(z_1)|l_0 + |g''(z_2)|l_0 l_1 + \int_{z_0}^{z}\left(\int_{z_1}^{\xi}(|g''(\xi_1) - g''(z_2)||d\xi_1|\right)|d\xi| \\
&\le \frac{c}{4} + |g'(z_1)|l_0 + |g''(z_2)|l_0 l_1 + \int_{z_0}^{z}\left(\int_{z_1}^{\xi}\left(\int_{z_2}^{\xi_1}|g^{(3)}(\xi_2)||d\xi_2|\right)|d\xi_1|\right)|d\xi| \\
&\le \frac{c}{4} + |g'(z_1)|l_0 + |g''(z_2)|l_0 l_1 + \cdots + |g^{(d)}(z_d)|l_0 l_1 \cdots l_{d-1} \\
&\quad + \int_{z_0}^{z}\left(\int_{z_1}^{\xi}\cdots\int_{z_d}^{\xi_{d-1}}|f^{(d+1)}(\xi_d)||d\xi_d|\cdots|d\xi_1|\right)|d\xi| \\
&\le \frac{c}{4} + \exp\left\{-2\varepsilon T\left(1 - \frac{1-r_n}{8}, f\right)\right\}(l_0 + l_0 l_1 + \cdots + l_0 l_1 \cdots l_d),
\end{aligned}
$$

where $l_0, \ldots, l_d$ denote lengths of curves joining $z_0, \ldots, z_d$ with respective points of the component G. Thus ([45]),

$$
\begin{aligned}
|f(z) - p_\nu(z)| &\le \frac{c}{4} + C_3 \sum_{n=m_0(k)}^{M_0(k)} \exp\left\{-2\varepsilon T\left(1 - \frac{1-r_n}{2^{d+1}}, f\right)\right\} \\
&\qquad\qquad\qquad \times \sum_{j=1}^{d}\left(\frac{T\left(1 - \frac{1-r_n}{2^{d+1}}, f\right)}{1-r_n}\right)^{\frac{d}{2}} \\
&\le \frac{c}{4} + C_3 \sum_{n=m_0(k)}^{\infty} \exp\left\{-\varepsilon\frac{\log 2}{2}dn\right\} < \frac{c}{4} + \frac{c}{4} \quad \text{as } k \to \infty.
\end{aligned}
$$

It follows that $G_{k,\nu}$ and $G_{k,\eta}$ do not intersect for $\nu \ne \eta$. □

For $1 \le \nu \le q$ and $k \ge k_0$, we consider the functions

$$
u_{k,\nu}(z) := \begin{cases} \max\left\{\log\dfrac{1}{|f^{(d+1)}(z)|}, 4\varepsilon T(1 - \dfrac{1-|z|}{2^{d+1}}, f)\right\}, & z \in G_{k,\nu}, \\ 4\varepsilon T\left(1 - \dfrac{1-|z|}{2^{d+1}}, f\right), & z \notin G_{k,\nu}. \end{cases}
$$

The functions $u_{k,\nu}(z)$ are δ-subharmonic in $A(1 - v_k, 1 - (2R_k)/B)$, which can be shown by following the same lines as in the proof of Lemma 6 in [45]. Let us recall here the definition and basic properties of the Baernstein function T^*. For a complex number $z = re^{i\theta}$, we put [3]:

$$
\begin{aligned}
m^*(z, u_{k,\nu}) &= \sup_{|E|=2\theta}\frac{1}{2\pi}\int_E u_{k,\nu}(re^{i\varphi})d\varphi, \\
T^*(z, u_{k,\nu}) &= m^*(z, u_{k,\nu}) + N(r, u_{k,\nu}),
\end{aligned}
$$

where $\theta \in [0, \pi]$, $|E|$ is the Lebesgue measure of the set E and

$$
N(r, u_{k,\nu}) = \int_{1-2v_k}^{r}\frac{\mu_{k,\nu}(t)}{t}\,dt,
$$

where $\mu_{k,\nu}(r)$ is the number of zeros of $f^{(d+1)}(z)$ in $G_{k,\nu} \cap \{z : |z| < r\}$.

For a number t, $0 < t \leq +\infty$, consider the set

$$F_t = \left\{re^{i\theta} : u_{k,\nu}(re^{i\theta}) > t\right\},$$

and let

$$\tilde{u}_{k,\nu}\big(re^{i\theta}\big) = \sup\left\{t : re^{i\theta} \in F_t^*\right\},$$

where F_t^* is the symmetric rearrangement of F_t through the circular symmetrization with respect to the ray $\operatorname{Arg}(z) = 0$. [24]. The functions $\tilde{u}_{k,\nu}(re^{i\theta})$ are non-negative and non-increasing with respect to θ for $\theta \in [0, \pi]$ even in θ and, for a fixed r, equimeasureable with $u_{k,\nu}(re^{i\theta})$. Moreover,

$$\tilde{u}_{k,\nu}(r) = \max\left(\mathcal{L}(r, 0, f^{(d+1)}), 4\varepsilon T\left(1 - \frac{1-r}{2^{d+1}}, f\right)\right).$$

Let us also notice that

$$m^*(z, u_{k,\nu}) = \frac{1}{\pi}\int_0^\theta \tilde{u}_{k,\nu}\big(re^{i\varphi}\big)\, d\varphi.$$

The function $T^*(z, u_{k,\nu})$ is subharmonic in

$$D = \left\{z = re^{i\theta} : z \in A\left(1 - v_k, 1 - \frac{2R_k}{B}\right),\ 0 < \theta < \pi\right\},$$

continuous on $D \cup ((2R_k)/B) - 1, v_k - 1) \cup (1 - v_k, 1 - (2R_k)/B)$ and convex in $\log r$ for each fixed $\theta \in [0, \pi]$ [3]. What is more,

$$\begin{aligned} T^*(r, u_{k,\nu}) &= N(r, u_{k,\nu}), \\ \frac{\partial}{\partial\theta}T^*(re^{i\theta}, u_{k,\nu}) &= \frac{\tilde{u}_{k,\nu}(re^{i\theta})}{\pi} \quad \text{for } 0 < \theta < \pi. \end{aligned}$$

As in [45], we consider

$$T_0^*\big(re^{i\theta}, f\big) = \sum_{\nu=1}^{q} T^*\big(re^{i\theta}, u_{k,\nu}\big).$$

It follows that

$$\begin{aligned} T_0^*(z, f) = T_0^*\big(re^{i\theta}, f\big) &= \sum_{\nu=1}^{q} T^*(z, u_{k,\nu}) \\ &= \sum_{\nu=1}^{q} (m^*(z, u_{k,\nu}) + N(r, u_{k,\nu})) \\ &\leq m\left(r, \frac{1}{f^{(d+1)}}\right) + N\left(r, \frac{1}{f^{(d+1)}}\right) + 4q\varepsilon T\left(1 - \frac{1-r}{2^{d+1}}, f\right) \\ &= T\left(r, \frac{1}{f^{(d+1)}}\right) + 4q\varepsilon T\left(1 - \frac{1-r}{2^{d+1}}, f\right). \end{aligned}$$

By the first main theorem, the fact that f is holomorphic and inequality (6.4), we have

$$T_0^*(z,f) \le T(r,f) + \varepsilon_1 T\left(1 - \frac{1-r}{2^{d+1}}, f\right) + O\left(\log\frac{1}{1-r}\right), \tag{6.7}$$

Previous considerations lead to the following statement.

Lemma 6.9. *The function $T_0^*\big(re^{i\theta}, f\big)$ is subharmonic in*

$$D = \left\{z = re^{i\theta} : z \in A\left(1 - v_k, 1 - \frac{2R_k}{B}\right),\ 0 < \theta < \pi\right\},$$

continuous on $D \cup ((2R_k)/B) - 1, v_k - 1) \cup (1 - v_k, 1 - (2R_k)/B)$ and convex in $\log r$ for each fixed $\theta \in [0,\pi]$. Moreover,

$$\frac{\partial}{\partial\theta} T_0^*\big(re^{i\theta}, f\big) = \frac{1}{\pi}\sum_{\nu=1}^{q} \tilde{u}_{k,\nu}\big(re^{i\theta}\big).$$

In further considerations we assume that $T_0^*(z,f)$ is twice continuously differentiable. If not, it would be possible to approximate $T_0^*(z,f)$ by a monotone family of infinitely differentiable subharmonic functions uniformly converging to $T_0^*(z,f)$ (see [45]).

For $\lambda > 0$, we choose numbers α, ψ, α_1 for a fixed ε_1 such that

$$0 < \alpha = \alpha(\varepsilon_1) \le \min\left\{\frac{\pi}{2} - \varepsilon_1, \frac{\pi}{2\lambda}\right\},\ 0 < \alpha_1 \le \alpha/2,\ -\frac{\pi}{2\lambda} \le \psi \le \frac{\pi}{2\lambda} - \alpha. \tag{6.8}$$

Let us consider

$$\sigma(R) := \int_{\alpha_1}^{\alpha} T_0^*\big(1 - Re^{-i\varphi}, f\big)\cos\lambda(\psi + \varphi)\, d\varphi, \quad R \in [R_k, v_k],$$

where $v_k = v_k(B)$, $R_k = (B)$ with B such that $B\cos\alpha > 4$.

Lemma 6.10. *For a fixed $\varepsilon > 0$ and $k > k_0(\varepsilon)$,*

$$v_k^{\lambda+1}|\sigma'(v_k)| + R_k^{\lambda+1}|\sigma'(R_k)| + \lambda v_k^{\lambda}\sigma(v_k) + \lambda R_k^{\lambda}\sigma(R_k) < \varepsilon\int_{R_k}^{v_k} T(1-R,f)R^{\lambda-1}\, dR.$$

Proof. We conduct the proof in a similar way as the proof of Lemma 4 in [45]. We put

$$T_0^*(1 - Re^{-i\varphi}, f) = W(R,\varphi) = V(r,\theta),$$

where $z = re^{i\theta} = 1 - Re^{-i\varphi}$. Let $\theta \in (2\varepsilon, \pi/2 - \varepsilon)$. Then

$$\frac{\partial}{\partial R}W(R,\varphi) = -\frac{\partial}{\partial r}V(r,\theta)\cos(\theta+\varphi) + \frac{1}{r}\frac{\partial}{\partial\theta}V(r,\theta)\sin(\theta+\varphi), \tag{6.9}$$

$$\frac{\partial}{\partial\varphi}W(R,\varphi) = \frac{R}{r}\frac{\partial}{\partial\theta}V(r,\theta)\cos(\theta+\varphi) + R\frac{\partial}{\partial r}V(r,\theta)\sin(\theta+\varphi). \tag{6.10}$$

As $V(r,\theta)$ is a concave function of θ on $(2\varepsilon, \pi/2-\varepsilon)$ (see Lemma 2 in [45]), we get

$$-\frac{V(r,\theta)}{\theta} \le \frac{V(r,2\theta)-V(r,\theta)}{\theta} \le \frac{\partial}{\partial\theta}V(r,\theta) \le 2\frac{V(r,\theta)-V(r,\theta/2)}{\theta} \le 2\frac{V(r,\theta)}{\theta}.$$

It follows that

$$\left|\frac{\partial}{\partial\theta}V(r,\theta)\right| \le 2\frac{V(r,\theta)}{\theta}.$$

By the convexity of $V(r,\theta)$ in $\log r$, for a fixed $\theta \in (2\varepsilon, \pi-2\varepsilon)$, we have

$$r\frac{\partial}{\partial r}V(r,\theta) \le \frac{V(y,\theta)}{\log(y/r)}. \tag{6.11}$$

As $r = r(R,\varphi) = \sqrt{1-2R\cos\varphi+R^2}$ and $B\cos\varphi > 4$, setting $R = v_k$, $y = 1-\frac{3v_k}{B}$, $\varphi \in (\alpha_1,\alpha)$, we obtain

$$r = \sqrt{1-2\frac{v_k}{B}B\cos\varphi+v_k^2} \le \sqrt{1-\frac{8}{B}v_k+v_k^2} = 1-\frac{4}{B}v_k+o(v_k) \quad \text{as } k\to\infty,$$
$$\log\frac{y}{r} \ge \log\frac{1-(3v_k)/B}{1-(4v_k)/B}+o(v_k) = \frac{1}{B}v_k+o(v_k).$$

Applying (6.7) to inequality (6.11), we get

$$\begin{aligned} r\frac{\partial}{\partial r}V(r,\theta) &\le B\frac{T(1-(3v_k)/B,f)+T(1-(3v_k)/(2^{d+1}B),f)}{v_k} \\ &\qquad\qquad + \frac{O(\log(1/v_k))+O(1)}{v_k} \\ &\le 2B\frac{T(1-(3v_k)/(2^{d+1}B),f)}{v_k} \end{aligned}$$

and

$$\left|\frac{\partial}{\partial\theta}V(r,\theta)\right| \le 4\frac{T(1-(3v_k)/(2^{d+1}B),f)}{\theta}.$$

We can also derive similar inequalities with v_k replaced with R_k.

From the definition of $\sigma(R)$,

$$\begin{aligned} v_k^{\lambda+1}|\sigma'(v_k)| + R_k^{\lambda+1}|\sigma'(R_k)| &= v_k^{\lambda+1}\int_{\alpha_1}^{\alpha}\left|\frac{\partial}{\partial R}W(R,\varphi)\cos\lambda(\psi+\varphi)\,d\varphi\right|\Bigg|_{R=v_k} \\ &\quad + R_k^{\lambda+1}\int_{\alpha_1}^{\alpha}\left|\frac{\partial}{\partial R}W(R,\varphi)\cos\lambda(\psi+\varphi)d\varphi\right|\Bigg|_{R=R_k} \\ &\le v_k^{\lambda+1}\int_{\alpha_1}^{\alpha}\left[\frac{\partial}{\partial r}V(r,\theta)+\frac{1}{r}\frac{\partial}{\partial\theta}V(r,\theta)\sin(\theta+\varphi)\right]\Bigg|_{R=v_k}d\varphi \\ &\quad + R_k^{\lambda+1}\int_{\alpha_1}^{\alpha}\left[\frac{\partial}{\partial r}V(r,\theta)+\frac{1}{r}\frac{\partial}{\partial\theta}V(r,\theta)\sin(\theta+\varphi)\right]\Bigg|_{R=R_k}d\varphi. \end{aligned}$$

As $\theta = \arcsin\left(\frac{R}{r}\sin\varphi\right)$, we get

$$\int_{\alpha_1}^{\alpha} \frac{\sin(\theta+\varphi)}{\theta}\,d\varphi \le \int_{\alpha_1}^{\alpha}\left(1+\frac{\varphi}{\theta}\right)d\varphi \le \int_{\alpha_1}^{\alpha}\left(1+\frac{\varphi r}{R\sin\varphi}\right)d\varphi$$
$$\le \frac{\pi}{2}+\frac{r}{R}\int_0^{\pi/2}\frac{\varphi}{\sin\varphi}\,d\varphi \le \frac{c_1}{R}.$$

Finally, we obtain

$$v_k^{\lambda+1}|\sigma'(v_k)| + R_k^{\lambda+1}|\sigma'(R_k)| \le c_2 v_k^{\lambda} T(1-(3v_k)/(2^{d+1}B), f) \\ + c_3 R_k^{\lambda} T(1-(3R_k)/(2^{d+1}B), f)$$

and

$$v_k^{\lambda}\sigma(v_k) + R_k^{\lambda}\sigma(R_k) \le c_4 v_k^{\lambda} T(1-(3v_k)/(2^{d+1}B), f) \\ + c_5 R_k^{\lambda} T(1-(3R_k)/(2^{d+1}B), f),$$

Applying both inequalities and Lemma 6.7, we get the statement. □

We shall also need another lemma from [45].

Lemma 6.11. *For a fixed $\varepsilon > 0$ and $k > k_0(\varepsilon)$,*

$$\int_{R_k}^{v_k} T(|1-Re^{-i\alpha}|, f)R^{\lambda-1}\,dR \le \frac{1+\varepsilon}{\cos^{\lambda}\alpha}\int_{R_k}^{v_k} T(1-R, f)R^{\lambda-1}\,dR,$$
$$\int_{R_k}^{v_k} T(|1-Re^{-i\alpha}|, f)R^{\lambda-1}\,dR \ge \frac{1-\varepsilon}{\cos^{\lambda}\alpha}\int_{R_k}^{v_k} T(1-R, f)R^{\lambda-1}\,dR.$$

The following result is an extension of Lemma 9 from [45] and can be proved along the same lines with applications of Lemma 6.10 and inequality (6.4).

Lemma 6.12. *Let f be an admissible function of finite lower order λ in the unit disc. For each fixed $\varepsilon > 0$ and $k_0 = k_0(\varepsilon)$, for $k > k_0(\varepsilon)$ and $d \in \mathbb{N}_0$,*

$$\int_{R_k}^{v_k} \mathcal{L}(1-R, \infty, f^{(d+1)}/f)R^{\lambda}\,dR < \varepsilon\int_{R_k}^{v_k} T(1-R, f)R^{\lambda-1}\,dR.$$

6.2. Proof of Theorem 6.4. Let f be a function holomorphic in the unit disc of finite lower order $\lambda > 0$. We apply the differential operator $L = R\frac{d}{dR}(R\frac{d}{dR})$ to $\sigma(R)$. By subharmonicity of $W(R,\varphi) = T_0^*(1-Re^{-i\varphi}, f)$, we obtain

$$R\frac{d}{dR}(R\sigma'(R)) = \int_{\alpha_1}^{\alpha} LW(R,\varphi)\cos\lambda(\psi+\varphi)d\varphi$$
$$\ge \left[-\frac{\partial}{\partial\varphi}W(R,\varphi)\cos\lambda(\psi+\varphi)\right]\Big|_{\alpha_1}^{\alpha}$$
$$-\lambda\int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\varphi}W(R,\varphi)\sin\lambda(\psi+\varphi)d\varphi$$

$$
\begin{aligned}
&= \left[-\frac{\partial}{\partial \varphi} W(R,\varphi) \cos \lambda(\psi+\varphi) \right] \Big|_{\alpha_1}^{\alpha} \\
&\quad - [\lambda W(R,\varphi) \sin \lambda(\psi+\varphi)]_{\alpha_1}^{\alpha} + \lambda^2 \sigma(R) \\
&\equiv h(R) + \lambda^2 \sigma(R).
\end{aligned}
$$

It follows that

$$
R \frac{d}{dR} (R\sigma'(R)) \geq h(R) + \lambda^2 \sigma(R). \tag{6.12}
$$

We multiply (6.12) by $R^{\lambda-1}$ and integrate for $R \in [R_k, v_k]$ to obtain

$$
\begin{aligned}
\int_{R_k}^{v_k} h(R) R^{\lambda-1} dR + \lambda^2 \int_{R_k}^{v_k} \sigma(R) R^{\lambda-1} dR &\leq \int_{R_k}^{v_k} R^{\lambda} \frac{d}{dR} (R\sigma'(R)) dR \\
&= \left[R^{\lambda+1} \sigma'(R) - \lambda R^{\lambda} \sigma(R) \right] \Big|_{R_k}^{v_k} + \lambda^2 \int_{R_k}^{v_k} \sigma(R) R^{\lambda-1} dR.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_{R_k}^{v_k} h(R) R^{\lambda-1} dR = \int_{R_k}^{v_k} \Bigg\{ &\left[-\frac{\partial}{\partial \varphi} W(R,\varphi) \right] \Big|_{\varphi=\alpha} \cos \lambda(\psi+\alpha) \\
&+ \left[\frac{\partial}{\partial \varphi} W(R,\varphi) \right] \Big|_{\varphi=\alpha_1} \cos \lambda(\alpha_1+\psi) \\
&- \lambda W(R,\alpha) \sin \lambda(\psi+\alpha) \\
&+ \lambda W(R,\alpha_1) \sin \lambda(\psi+\alpha_1) \Bigg\} R^{\lambda-1} dR \\
\leq &\left[R^{\lambda+1} \sigma'(R) - \lambda R^{\lambda} \sigma(R) \right]_{R_k}^{v_k}.
\end{aligned}
$$

We take $\psi = \frac{\pi}{2\lambda} - \alpha$, pass to the limit with $\alpha_1 \to 0$ and apply Lemma 6.10. Hence,

$$
\begin{aligned}
\int_{R_k}^{v_k} \left\{ \left[\frac{\partial}{\partial \varphi} W(R,\varphi) \right] \Big|_{\varphi=0} \sin \lambda\alpha - \lambda W(R,\alpha) + \lambda W(R,0) \cos \lambda\alpha \right\} R^{\lambda-1} dR \\
< \varepsilon \int_{R_k}^{v_k} T(1-R,f) R^{\lambda-1} dR.
\end{aligned}
$$

As $re^{i\theta} = 1 - Re^{-i\varphi}$, from (6.10), we obtain

$$
\left[\frac{\partial}{\partial \varphi} W(R,\varphi) \right] \Big|_{\varphi=0} = \frac{R}{1-R} \left[\frac{\partial}{\partial \theta} T_0^*(re^{i\theta}, f) \right] \Big|_{\theta=0}.
$$

Thus, by Lemma 6.10, we have

$$
\int_{R_k}^{v_k} \frac{R}{1-R} \frac{\sin \lambda\alpha}{\pi} \sum_{\nu=1}^{q} \tilde{u}_{k,\nu}(1-R,0) R^{\lambda-1} dR
$$

$$- \lambda \int_{R_k}^{v_k} \left\{ T_0^*(1 - Re^{-i\alpha}, f) - T_0^*(1 - R, f) \cos \lambda\alpha \right\} R^{\lambda - 1} dR$$
$$< \varepsilon \int_{R_k}^{v_k} T(1 - R, f) R^{\lambda - 1} dR. \tag{6.13}$$

Applying (6.7), Lemma 6.11 and the fact that, by Lemma 6.10,

$$\int_{R_k}^{v_k} T(1 - \frac{1 - |1 - Re^{-i\alpha}|}{2^{d+1}}, f) R^{\lambda - 1} dR \leq (1 + \tilde{\varepsilon})\tilde{c} \int_{R_k}^{v_k} T(1 - R, f) R^{\lambda - 1} dR,$$

we obtain

$$\int_{R_k}^{v_k} \frac{R}{1 - R} \sum_{\nu=1}^{q} \tilde{u}_{k,\nu}(1 - R, 0) dR \leq \frac{\pi}{\sin \lambda\alpha} \left[\frac{\lambda(1 + \varepsilon)}{\cos^\lambda \alpha} + \varepsilon_1 \lambda (1 + \tilde{\varepsilon})\tilde{c} + \varepsilon \right]$$
$$\times \int_{R_k}^{v_k} T(1 - R, f) R^{\lambda - 1} dR.$$

Applying the inequality

$$\mathcal{L}\left(r, \infty, \frac{1}{f - p_\nu}\right) \leq \mathcal{L}\left(r, \infty, \frac{f^{(d+1)}}{f - p_\nu}\right) + \mathcal{L}\left(r, \infty, \frac{1}{f^{(d+1)}}\right)$$
$$\leq \tilde{u}_{k,\nu}(r, 0) + 4\varepsilon T\left(1 - \frac{1 - r}{2^{d+1}}, f\right)$$

and Lemma 6.12, we conclude that there exists a sequence $\{r_k\}$, $r_k \overset{k\to\infty}{\to} 1^-$ such that

$$\frac{1 - r_k}{r_k} \sum_{\nu=1}^{q} \mathcal{L}\left(r_k, \infty, \frac{1}{f - p_\nu}\right) \leq \frac{\pi}{\sin \lambda\alpha} \left[\frac{\lambda(1 + \hat{\varepsilon})}{\cos^\lambda \alpha} + \hat{\varepsilon} \right] T(r_k, f).$$

The statement follows from this inequality. □

6.3. Proof of Theorem 6.5. Let f be a function holomorphic in the unit disc of lower order $\lambda = 0$. All the considerations made before the definition of $\sigma(R)$ are equally true for this case. For any $\varepsilon > 0$, let us take a number α such that $0 < \alpha < \frac{\pi}{2} - \varepsilon$ and μ, $0 < \mu < \frac{1}{2}$ instead of λ in (6.8). Proceeding as before and putting $\lambda = 0$ in Lemmas 6.7 and 6.10, we arrive at the equivalent of (6.13) in the following form:

$$\int_{R_k}^{v_k} \frac{R}{1 - R} \frac{\sin \mu\alpha}{\pi} \sum_{\nu=1}^{q} \tilde{u}_{k,\nu}(1 - R, 0) \frac{dR}{R}$$
$$- \mu \int_{R_k}^{v_k} \left\{ T_0^*(1 - Re^{-i\alpha}, f) - T_0^*(1 - R, f) \cos \mu\alpha \right\} \frac{dR}{R}$$
$$< \varepsilon \int_{R_k}^{v_k} T(1 - R, f) \frac{dR}{R}.$$

It follows that

$$\sum_{\nu=1}^{q} \hat{\beta}(p_\nu, f) \leq \frac{\pi\mu}{\sin\mu\alpha}\left(\frac{1}{\cos^\mu\alpha} - \left(1 - \Delta\big(0, f^{(d+1)}\big)\right)\cos\mu\alpha\right).$$

Passing to the limit with $\mu \to 0$, we get

$$\sum_{\nu=1}^{q} \hat{\beta}(p_\nu, f) \leq \frac{\pi}{\alpha}\Delta\big(0, f^{(d+1)}\big).$$

Taking $\alpha \to \frac{\pi}{2} - \varepsilon.$ we have

$$\sum_{\nu=1}^{q} \hat{\beta}(p_\nu, f) \leq (2+\varepsilon)\Delta\big(0, f^{(d+1)}\big).$$

This leads to the statement in the case $\lambda = 0$. $\square$

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Received February 15, 2025, revised March 10, 2025.

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Про нові та попередні дослідження в теорії Петренка зростання мероморфних функцій

Ewa Ciechanowicz

Теорія Петренка зростання мероморфних функцій належить до ширшого спектра теорії Неванлінни і була започаткована В.П. Петренком у 1960-х роках. Ця стаття присвячена досягненням видатного учня В.П. Петренка, І.І. Марченка, та його внеску в теорію. Огляд деяких основних результатів І.І. Марченка (та їх подальших узагальнень і застосувань), що стосуються відхилень, відокремлених точок максимуму модуля та сильних асимптотичних значень, становить основну частину статті. Заключна частина статті присвячена узагальненню раннього результату І.І. Марченка та А.І. Щерби щодо суми відхилень голоморфних функцій в одиничному крузі.

Ключові слова: мероморфна функція, голоморфна функція, відхилення, точка максимуму, асимптотичне значення