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Finding a Positive Constrained Control for a Linear System to Reach a Given Point within a Finite Time

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In this paper, we consider a linear system with the control $u \in \Omega$, where Ω is a certain domain which does not contain the origin as an interior point. In particular, the origin may not belong to the set Ω. The synthesis problem is solved, i.e. the control $u(x) \in \Omega$ which transfers a point x that belongs to a neighbourhood $V(0)$ to 0 in a finite time is constructed by using the controllability function method. Moreover, this function can be found as the time of motion from a point $x \in V(0)$ to the origin. The case of the linear control system with a non-autonomous term is also considered.

Key words: control system, synthesis problem, controllability function, feedback control, positive constrained control

Mathematical Subject Classification 2020: 93B05, 93B50, 93B52, 93C28, 93D05, 93D40

1. Introduction

Consider the control system

$$\dot{x} = f(x,u), \quad x \in E_n,\ u \in E_r,$$

and suppose it is completely controllable.

The admissible local positional synthesis problem consists in finding a feedback control $u = u(x)$ satisfying the preassigned constraint ($u(x) \in \Omega$) such that the trajectory of the system

$$\dot{x} = f(x, u(x)) \tag{1.1}$$

starting at an arbitrary point x_0 of some neighbourhood of the origin ends in the origin in a time $T = T(x_0)$. If the neighbourhood V coincides with entire space E_n, then the problem is called global synthesis problem. In [8,9], the solution to the problem is considered for the set Ω containing origin as an interior point. In this work, we consider the case of $0 \notin \operatorname{int}\Omega$, in particular, $0 \notin \operatorname{co}\Omega$. We assume that the set Ω is in the positive octant of the set E_r. As an example, we consider the stopping problem for the mathematical pendulum by using a one-sided force applied to the pendulum. The case of an additional force $F(t)$ is also considered.

The solution to the admissible synthesis problem is based on construction of a controllability function [8] $\Theta(x)$ and a control $u(x) \in \Omega$ such that for $x \neq 0$ and

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some $\alpha \geq 1$ and $\beta > 0$ the following inequality holds:

$$\sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x), \tag{1.2}$$

where $f_i(x)$ is the i-th coordinate of the vector function $f(x)$.

In this case, the time of motion from a point x_0 to the origin with respect to system (1.1) satisfies the inequality $T(x) \leq \frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}(x_0)$. If $\alpha = \beta = 1$ and

$$\sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \equiv -1, \tag{1.3}$$

then the function $\Theta(x)$ is the time of motion from the point x_0 to the origin, $\Theta(x_0) = T(x_0)$ [4–9, 14]. If, in addition,

$$\min_{u \in \Omega} \sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u) = -1, \tag{1.4}$$

then the function Θ is the smallest time of motion from the point x_0 to the origin with respect to system (1.1). In this case, the time-optimal control problem in the neighbourhood V is solved. Equation (1.4) is called the Bellman equation. In the case of $\alpha = \infty$, $\beta = 0$, and $f(0,0) = 0$, inequality (1.2) has the form

$$\sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \leq 0.$$

In this case, the function $\Theta(x)$ is the Lyapunov function and the zero-solution to system (1.1) with this $u(x)$ is stable (asymptotically stable if $\sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) < 0$, $x \neq 0$).

The controllability function method is a development of the Lyapunov method for controllable systems.

The asymptotic stability in finite time for robust systems was considered in [5, 12, 13, 15]. The controllability of chaotic motion was considered in [2] using the controllability function, and it was considered for some nonlinear systems in [1, 4, 9].

2. Construction of a controllability function as the time of motion

Consider the linear control system

$$\dot{x} = Ax + Bu, \quad u \in \Omega, \tag{2.1}$$

where

$$\text{rang}(B, AB, \ldots, A^{(n-1)}B) = n.$$

First, we introduce the method used in [9,14] for constructing a controllability function and a control u solving the synthesis problem in the case of $\|u\| \leq d$. Then we modify it for the case when $0 \notin \operatorname{int}\Omega$, in particular, $0 \notin \operatorname{co}\Omega$.

Let $f(t)$ be a non-increasing non-negative function on the half-axis $[0,\infty)$ with at least m decreasing points such that

$$\int_0^\infty s^{2m-1} e^{-2\lambda_0' s\Theta} f(s)\,ds < \infty \quad \text{for } 0 \leq \Theta \leq c,$$

where m is a degree of the minimal polynomial of the matrix A, $\lambda_0' = \min(0, \lambda_0)$, λ_0 is a minimal real part of eigenvalues of the matrix A. Put

$$N = N(\Theta) = \int_0^\infty f\left(\frac{t}{\Theta}\right) e^{-At} BB^* e^{-A^*t}\,dt.$$

This matrix is positive definite. For $x \neq 0$, define the positive solution to the equation

$$2a_0\Theta = (N^{-1}x, x) \tag{2.2}$$

by $\Theta(x)$ (note that this equation has a unique positive solution). For $x = 0$, we put $\Theta(0) = 0$. The control

$$u(x) = -\frac{1}{2} B^* N^{-1}(\Theta(x))x \tag{2.3}$$

solves the synthesis problem. Put

$$f(t) = \begin{cases} 1 - t & \text{if} \quad 0 \leq t \leq 1, \\ 0 & \text{if} \quad t > 1. \end{cases}$$

Then

$$N = \int_0^\Theta \left(1 - \frac{t}{\Theta}\right) e^{-At} BB^* e^{-A^*t}\,dt, \tag{2.4}$$

and the controllability function defined by equation (2.2) is the time of motion from a point x_0 to the origin. In addition, equation (1.3) holds. For some $c > 0$, if

$$a_0 \leq \frac{2}{\sup\limits_{0 \leq \Theta \leq c} \Theta \sum\limits_{k=1}^{r} (N^{-1}b_k, b_k)},$$

then the control $u = -B^*N^{-1}x$ is bounded ($\|u\| \leq d$) in the domain $Q = \{x : \Theta(x) \leq c\}$ [5,9,14], where b_k is the k-th column of the matrix B. The trajectory $x(t)$ transferring the initial point to the origin can be found in the following way. For a given x_0, we find the positive solution Θ_0 to equation (2.2), which is unique. Then we find the solution to the Cauchy problem for the $(n+1)$-dimensional system:

$$\begin{cases} \dot{x} = Ax - \dfrac{1}{2} f(0) BB^* N^{-1} x, \\ \dot{\Theta} = -\dfrac{\Theta(\hat{N} N^{-1}x, N^{-1}x)}{(N^{-1}x, x) + \Theta(\hat{N} N^{-1}x, N^{-1}x)}, \\ x(0) = x_0, \\ \Theta(0) = \Theta_0. \end{cases}$$

We have $x(T(x_0)) = 0$ and $\Theta(T(x_0)) = 0$ for some $t = T(x_0)$. Here,

$$\widehat{N} = \int_0^\infty \frac{t}{\Theta} e^{-At} BB^* e^{-A^*t}\, d\left(-f\left(\frac{t}{\Theta}\right)\right),$$
$$\widetilde{N} = \int_0^\infty e^{-At} BB^* e^{-A^*t}\, d\left(-f\left(\frac{t}{\Theta}\right)\right).$$

If the function $\Theta(x)$ is a time of motion, we have $\dot{\Theta} = -1$.

We also note that the given above method is used in the synthesis problem for the class of systems which can be mapped to linear ones by a change of variables [19, 20]. The detailed description of this class is contained in [17, 18, 21].

System (2.1) must be locally controllable. In the case of $0 \notin \operatorname{int} \Omega$, the Kalman criterion is not sufficient. Starting from the works [3, 11] various controllability criteria were considered for different kinds of constraints. For constraints of the general form under the condition $0 \in \Omega$, the following criterion is true [11].

Proposition 2.1 ([11]). *System (2.1) is locally controllable if and only if there exists neither an eigenvector v corresponding to a real eigenvalue of the conjugate matrix A^* which is a support vector to the set $B\Omega$ (i.e. $(v, Bu) \geq 0$ for all $u \in \Omega$) nor an eigenvector $v = w_1 + iw_2$ corresponding to a complex eigenvalue of the matrix A^* orthogonal to the set $B\Omega$ (i.e. $(w_1, Bu) = 0$, $(w_2, Bu) = 0$ for all $u \in \Omega$).*

In the case of general constraints on the control without the assumption $0 \notin \Omega$, the necessary and sufficient conditions are given in [10]. In addition to the previous conditions, the return condition is required: the trajectory must return to the origin on some interval $[t_1, t_2]$.

Let us consider the constraints of the form $\Omega = \{u : 0 \leq u_i \leq c_i,\ i = 1, \dots, r\}$. If the matrix A has a real eigenvalue, then system (2.1) is not locally controllable.

Proposition 2.2. *Let the matrix A of system (2.1) have at least one real eigenvalue λ, and let v be an eigenvector of the matrix A^* corresponding to λ. Then, under the constraints $\Omega = \{u : 0 \leq u_i \leq c_i,\ i = 1, \dots, r\}$, this system is not locally controllable if $(v, b_i) \geq 0$, $i = 1, \dots, r$, or $(v, b_i) \leq 0$, $i = 1, \dots, r$.*

Proof. The set of points from which we can transfer to the origin in the time T is determined by the equality

$$x_0 = -\int_0^T e^{-A\tau} Bu(\tau)\, d\tau.$$

Let v be an eigenvector of the matrix A^* corresponding to an eigenvalue λ. We have

$$(v, x_0) = -\int_0^T (v, e^{-A\tau} Bu(\tau)) d\tau = -\int_0^T (e^{-A^*\tau} v, Bu(\tau))\, d\tau$$
$$= -\int_0^T e^{-\lambda\tau} (v, \sum_{i=1}^r b_i u_i(\tau)) d\tau = -\sum_{i=1}^r \int_0^T e^{-\lambda\tau} (v, b_j) u_j(\tau)\, d\tau \leq 0$$

if $(v, b_j) \geq 0$, $i = 1, \ldots, r$. If $(v, b_j) \leq 0$, $i = 1, \ldots, r$, we have $(v, x_0) \geq 0$. Therefore, the set of points from which it is possible to transfer to the origin in an arbitrary time T belongs to one of the subspaces $(v, x_0) \geq 0$ or $(v, x_0) \leq 0$. □

By K, denote the convex cone generated by the column vectors $b_1, b_2, \ldots, b_r$ of the matrix B. Let K^* be the dual cone

$$K^* = \{y : (y, x) \geq 0\}$$

for any $x \in K$. Then Proposition 2.2 can be reformulated in the following way:

Proposition 2.3. *Let the matrix A have at least one real eigenvalue λ. Let v be an eigenvector of the matrix A^* corresponding to λ and $\Omega = \{u : 0 \leq u_i \leq c_i,\ i = 1, \ldots, r\}$. If the eigenvector v belongs to the dual cone K^* ($v \in K^*$), then the system is not locally controllable.*

3. Construction of a positive control

Consider the solution to the synthesis problem for the linear control system

$$\dot{x} = Ax + Bu, \quad u \in \Omega, \tag{3.1}$$

where $\Omega = \{u : 0 \leq c_i \leq u_i \leq d_i,\ i = 1, \ldots, r\}$, $c_i < d_i$, $i = 1, \ldots, r$.

Let control system (3.1) be globally controllable. The system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u \end{cases}$$

is an example of the system of such kind (see [10, 16]). The control (2.3) is not appropriate for this case because its components may change their signs. So we have to operate in another way.

Consider a new control $v = (v_1, \ldots, v_r)^*$ such that

$$v_i = \frac{1}{d_i - c_i}(2u_i - d_i - c_i), \quad i = 1, \ldots, r. \tag{3.2}$$

Then $u_i = \big((d_i - c_i)v_i + (d_i + c_i)\big)/2$, $i = 1, \ldots, r$. If $c_i \leq u_i \leq d_i$, then $-1 \leq v_i \leq 1$, $i = 1, \ldots, r$. Put

$$B_1 = \left(\frac{d_1 - c_1}{2} b_1, \ldots, \frac{d_r - c_r}{2} b_r\right),$$
$$B_2 = \left(\frac{d_1 + c_1}{2} b_1 + \ldots + \frac{d_r + c_r}{2} b_r\right),$$

where B_1 is a matrix, B_2 is a vector, $b_1, \ldots, b_r$ are the column vectors of the matrix B. Then we have $Bu = B_1 v + B_2$, and system (3.1) takes the form

$$\dot{x} = Ax + B_1 v + B_2, \quad v \in \Omega_1, \tag{3.3}$$

where $\Omega_1 = \{v : |v_i| \le 1, i = 1, \dots, r\}$. We have

$$x(t) = e^{At}\left(x_0 + \int_0^t e^{-A\tau} Bu(\tau)\, d\tau\right)$$
$$= e^{At}\left(x_0 + \int_0^t e^{-A\tau} B_1 v(\tau)\, d\tau + \int_0^T e^{-A\tau} B_2\, d\tau\right).$$

Let the control $u(t)$ transfer a point x_0 to the origin in some time $t = T$. Then

$$x_0 + \int_0^T e^{-A\tau} B_2\, d\tau = -\int_0^T e^{-A\tau} B_1 v(\tau)\, d\tau.$$

This equality means that the control $v(t)$ transfers the point

$$y_0 = x_0 + \int_0^T e^{-A\tau} B_2\, d\tau$$

to the origin with respect to the system

$$\dot{y} = Ay + B_1 v. \tag{3.4}$$

Since the origin is an interior point of the set Ω_1, this problem can be solved using the controllability function method if the time T is known.

First, consider the case where the controllability function is the time of motion. The controllability function $\Theta(x)$ is determined by the equation

$$2a_0\Theta = \left(N_1^{-1} y, y\right), \tag{3.5}$$

where

$$N_1(\Theta) = \int_0^\Theta \left(1 - \frac{t}{\Theta}\right) e^{-At} B_1 B_1^* e^{-A^* t}\, dt.$$

The control $v(t)$ is given by the equation

$$v = -\frac{1}{2} B_1^* N_1^{-1}(\Theta) y.$$

The coefficient a_0 is chosen according to the condition

$$\max_{1 \le i \le r} |v_i| \le 1.$$

To this aid, it is sufficient that a_0 satisfy the inequality

$$a_0 \le \max_{1 \le\ i \le r} \frac{2}{\sup\limits_{\Theta} \Theta\left(N_1^{-1} \tilde{b}_i, \tilde{b}_i\right)},$$

where $\tilde{b}_i$ is the i-th column of the matrix B_1.

Let us find the equation for determining the time T. Replace Θ by T and y by y_0 in equation (3.5) to obtain

$$2a_0T = \left(N_1^{-1}(T)(x_0 + \int_0^T e^{-A\tau}B_2\,d\tau), (x_0 + \int_0^T e^{-A\tau}B_2\,d\tau)\right), \tag{3.6}$$

where

$$N_1(T) = \int_0^T \left(1 - \frac{t}{\Theta}\right) e^{-At}B_1B_1^*e^{-A^*t}\,dt.$$

In general, this equation has a non-unique solution. For each time T, we obtain the control and the trajectory $y(t)$. After finding T, we get the trajectory $y(t)$ as the solution to the following Cauchy problem on the interval $[0,T]$:

$$\begin{cases} \dot{y} = Ay - \frac{1}{2}B_1B_1^*N_1^{-1}(\Theta(y))y, \\ \dot{\Theta} = -1, \\ y(0) = y_0 = x_0 + \int_0^T e^{-A\tau}B_2\,d\tau, \\ \Theta(0) = T. \end{cases}$$

After finding the trajectory $y(t)$, we obtain the trajectory $x(t)$ according to the equality

$$x(t) = y(t) - e^{At}\int_t^T e^{-A\tau}B_2\,d\tau. \tag{3.7}$$

Indeed, by subtracting equality (3.3) from equality (3.4), we get

$$\dot{x} - \dot{y} = A(x-y) + B_2.$$

Hence,

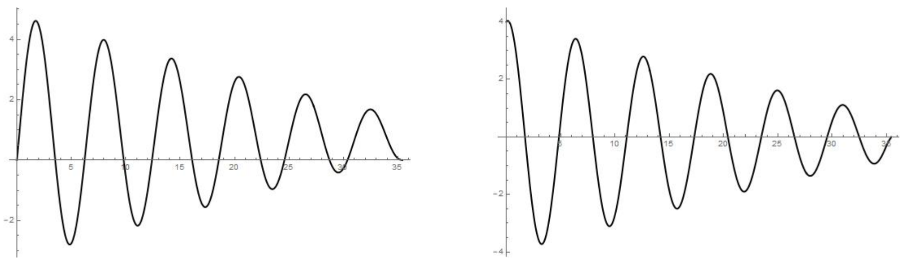
$$x - y = e^{At}\left(x_0 - y_0 + \int_0^t e^{-A\tau}B_2\,d\tau\right) =$$
$$= e^{At}\left(-\int_0^T e^{-A\tau}B_2\,d\tau + \int_0^t e^{-A\tau}B_2\,d\tau\right),$$

therefore, $x(t)$ is given by equality (3.7).

The trajectory $x(t)$ can also be found as a solution to the Cauchy problem on the segment $[0,T]$ for the system

$$\begin{cases} \dot{y} = Ay - \frac{1}{2}B_1B_1^*N_1^{-1}(\Theta(y))y, \\ \dot{\Theta} = -1, \\ \dot{x} = Ax - \frac{1}{2}B_1B_1^*N_1^{-1}(\Theta(y))y + B_2, \\ y(0) = x_0 + \int_0^T e^{-A\tau}B_2\,d\tau, \\ x(0) = x_0, \\ \Theta(0) = T, \end{cases} \tag{3.8}$$

where T is the found time of motion from the point x_0 to the origin.



(a) (b)

Fig. 3.1: Plots of (a) trajectory $x_1(t)$ and (b) trajectory $x_2(t)$. Horizontal axis means t, and vertical ones are $x_1(t)$ and $x_2(t)$.

3.1. The stopping problem for the mathematical pendulum. Let system (3.1) has the form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \quad \dfrac{1}{2} \le u \le 1, \\ x_1(0) = 0, \\ x_2(0) = 4. \end{cases} \tag{3.9}$$

Let us construct a control $u(t)$ transferring the point $x_0 = (0, 4)$ to the point $(0, 0)$. To this aid, change the control $u = v/4 + 3/4$. Then this system takes the form (3.3):

$$\dot{x} = Ax + B_1 v + B_2,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 3/4 \end{pmatrix}.$$

System (3.4) has the following form:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + \dfrac{1}{4} v, \quad -1 \le v \le 1. \end{cases} \tag{3.10}$$

Consider equation (3.5), where

$$\begin{aligned} N_1 =& \frac{1}{16} \int_0^\Theta \left(1 - \frac{t}{\Theta}\right) \begin{pmatrix} \sin^2 t & -\sin t \cos t \\ -\sin t \cos t & \cos^2 t \end{pmatrix} dt \\ =& \begin{pmatrix} \dfrac{-1 + 2\Theta^2 + \cos 2\Theta}{128\Theta} & \dfrac{-2\Theta + \sin 2\Theta}{128\Theta} \\ \dfrac{-2\Theta + \sin 2\Theta}{128\Theta} & \dfrac{1 + 2\Theta^2 - \cos 2\Theta}{128\Theta} \end{pmatrix}. \end{aligned}$$

Restriction on a_0 has the form $a_0 \le 1/3$. Further, we will assume $a_0 = 1/3$. Equation (3.5) for Θ has the form

$$\frac{\text{num}}{\text{den}} = 0, \tag{3.11}$$

where

$$\begin{aligned}\text{num} = {} & \frac{2}{3}\Theta(-1 - 2\Theta^2 + 2\Theta^4 + 2\Theta\sin 2\Theta + \cos 2\Theta) + 64\Theta(-1 \\ & - 2\Theta + \cos 2\Theta)y_1^2 + 128(-2\Theta + \sin 2\Theta)y_1 y_2 + \\ & + 64\Theta(1 - 2\Theta^2 + \cos 2\Theta)y_2^2, \\ \text{den} = {} & -1 - 2\Theta^2 + 2\Theta^4 + \cos 2\Theta + 2\Theta\sin 2\Theta.\end{aligned}$$

The initial state y_0 for system (3.10) is

$$\begin{aligned} y_0 &= x_0 + \int_0^T e^{-At} B_2\, dt \\ &= \left(\frac{3}{4}(-1 + \cos T), 4 + \frac{3\sin T}{4}\right) = (y_{10}, y_{20}).\end{aligned}$$

Substituting T for Θ and y_{10}, y_{20} for y_1, y_2 in (3.11), we obtain equation (3.6) for finding the time of motion from y_0 to the origin:

$$\begin{aligned} 2T^4 &- 3290T^2 + 1152T + 1535 - 1152T^2 \sin T \\ &+ 216T\sin T + 1152\sin T - 106T\sin 2T \\ &- 576\sin 2T + 72T(-16 + 3T)\cos T - 1535\cos 2T = 0.\end{aligned}$$

This equation has 5 positive roots: $\Theta_1 \approx 35.4201$, $\Theta_2 \approx 37.4601$, $\Theta_3 \approx 40.9275$, $\Theta_4 \approx 44.8576$, $\Theta_5 \approx 46.2312$. The control

$$v = v(t) = -\frac{1}{2}B_1^* N_1^{-1} y = \frac{-16\Theta(\Theta - \cos\Theta\sin\Theta)y_1 - 8\Theta(-1 + 2\Theta^2 + \cos 2\Theta)y_2}{-1 - 2\Theta^2 + 2\Theta^4 + \cos 2\Theta + 2\Theta\sin 2\Theta}$$

solves the problem for system (3.10). Further, for definiteness, we will assume $T = \Theta_1$. But each of the values $\Theta_1, \ldots, \Theta_5$ can be taken as T. Then we solve the system

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + \dfrac{1}{4}v, \\ \dot{\Theta} = -1, \\ y_1(0) = \dfrac{3}{4}(-1 + \cos T), \\ y_2(0) = 4 + \dfrac{3\sin T}{4}, \\ \Theta(0) = T \end{cases}$$

on the segment $[0, T]$. To obtain $x(t)$, we use equation (3.7):

$$\begin{aligned} x_1(t) &= y_1(t) - \frac{3}{4}\sin t(\sin T - \sin t) + \frac{3}{4}\cos t(\cos t - \cos T), \\ x_2(t) &= y_2(t) - \frac{3}{4}\sin t(\cos t - \cos T) - \frac{3}{4}\cos t(\sin T - \sin t).\end{aligned}$$

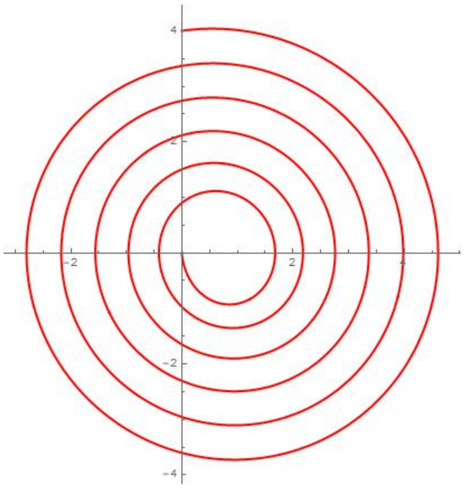


Fig. 3.2: The phase trajectory of trajectory of the system 3.9.

Plots of $x_1(t)$ and $x_2(t)$ are given in Figs. 3.1, the phase trajectory is given in Fig. 3.2.

The controllability problem for a pendulum with restrictions in the form $0 \leq u \leq 1$ was considered in [22].

Trajectories $x(t)$ can be found by solving the Cauchy problem (3.8) on the segment $[0, T]$. In this example, the problem takes the form

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + \dfrac{1}{4}v, \\ \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + \dfrac{1}{4}v + \dfrac{3}{4}, \\ \dot{\Theta} = -1, \\ y_1(0) = \dfrac{3}{4}(-1 + \cos T), \\ y_2(0) = 4 + \dfrac{3\sin T}{4}, \\ x_1(0) = 0, \\ x_2(0) = 4, \\ \Theta(0) = \Theta_1. \end{cases}$$

4. The case of an arbitrary controllability function

Consider the case where the controllability function is not the time of motion. E.g., the function $f(t/\Theta) = e^{-t/\Theta}$ allows us to find the time of motion. In this case, the main difference in finding the trajectory is in determining the time of motion T. This time can be found using an iterative approach, e.g. the half-division method. For a given y_0, we find Θ_0 as the only positive solution to equation (2.2). Then we solve the Cauchy problem (y, Θ) on some segment

$[0, T]$. We choose the value of T large enough to satisfy the inequality $\Theta(T) < 0$, then we divide the segment in half and solve the Cauchy problem (y, Θ) on the left part of the segment $[0, T_1]$, where $T_1 = T/2$. If it turns out that $\Theta(T_1) < 0$, then we divide the segment $[0, T_1]$ in half and solve the Cauchy problem on this segment. If it turns out that $\Theta(T_1) > 0$, then we divide the segment $[T_1, T]$ in half and solve the Cauchy problem on the segment $[T_1, T_2]$, where $T_2 = 3T/4$ etc. We continue the process until we find $\Theta(T_n)$ with the accuracy close to zero. We find the trajectory $x(t)$ in the same way as in the case where the function $\Theta(x)$ is the time of motion.

5. The case of a linear system with an additional term

The method of constructing the positive control given in Section 3 can be also applied to the system

$$\dot{x} = Ax + Bu + F(t). \tag{5.1}$$

Assume that

$$\left\| \int_0^T e^{-A\tau} F(\tau)\, d\tau \right\| < \infty \quad \text{for any } T > 0. \tag{5.2}$$

Changing the control u by the control v determined by (3.2) in control system (5.1), we obtain an analogue of system (3.3):

$$\dot{x} = Ax + B_1 v + B_2 + F(t).$$

Then we consider the problem of constructing a control transferring the trajectory of system (3.4) with initial condition

$$y(0) = x_0 + \int_0^T e^{-A\tau} (B_2 + F(\tau))\, d\tau \tag{5.3}$$

to the origin.

Example 5.1. Consider the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u + \sin t, \quad \dfrac{1}{2} \le u \le 1, \\ x_1(0) = 0, \\ x_2(0) = 4. \end{cases}$$

For this system, it is not possible to construct a control using the controllability function which is the time of motion because

$$\int_0^T e^{-A\tau} F(\tau)\, d\tau = \left(\frac{1}{4}(-2T + \sin T), \frac{\sin^2 T}{2} \right)$$

and condition (5.2) is true, however the equation for determining the time T has no positive roots.

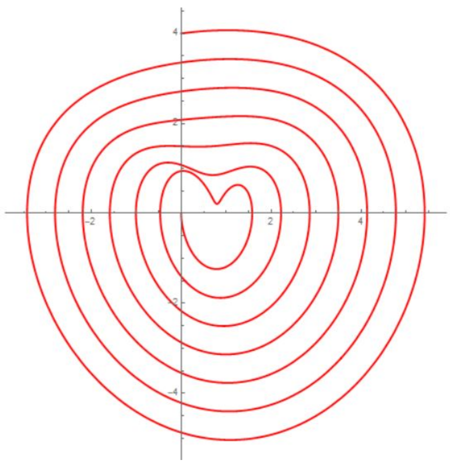


Fig. 5.1: The phase trajectory of example 5.2

Example 5.2. In the case of a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u + \sin 2t, \quad \frac{1}{2} \leq u \leq 1, \\ x_1(0) = 0, \\ x_2(0) = 4, \end{cases}$$

the initial condition (5.3) for system (3.4) has the form

$$y_1(0) = \frac{3}{4}(-1 + \cos T) - \frac{2}{3}\sin^3 T,$$
$$y_2(0) = 4 - \frac{2}{3}(-1 + \cos^3 T) + \frac{3}{4}\sin T$$

and condition (5.2) is true. The equation for determining the time of motion has positive solutions: $T_1 \approx 41.7791$, $T_2 \approx 44.4547$, $T_3 \approx 47.6571$, $T_4 \approx 51.2663$, $T_5 \approx 53.4776$. The type of control remains the same as in the case of system (3.9). The phase trajectory of the system is given in Fig. 5.1.

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Пошук позитивного обмеженого керування для лінійної системи для досягнення заданої точки за кінцевий час

Valerii Korobov and Katerina Sklyar

У цій роботі розглядається лінійна система з керуванням $u \in \Omega$, де Ω — деяка область, яка не містить початку координат, як внутрішньої точки. Зокрема, початок координат може не належати множині ω.

Розв'язано задачу синтезу, тобто за допомогою методу функції керованості побудоване керування $u(x) \in \Omega$, яке переводить точку x, що належить околу $V(0)$, до 0 за скінченний час. Крім того, цю функцію можна знайти, як час руху від точки $x \in V(0)$ до початку координат.

Також розглянуто задачу синтезу для керованої лінійної системи з неавтономним членом.

Ключові слова: керована система, задача синтезу, функція керованості, позиційне керування, додатне обмежене керування