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Local Rigidity of Convex Hypersurfaces in Spaces of Constant Curvature

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In this paper, we prove a local rigidity of convex hypersurfaces in the spaces of constant curvature of dimension $n \geq 4$. Namely, we show that two isometric convex hypersurfaces are congruent locally around their corresponding under the isometry points of strict convexity. This result extends the result of E.P. Senkin, who showed such rigidity under the additional assumption of C^1-smoothness of the hypersurfaces.

Key words: rigidity, convex hypersurface, space of constant curvature

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1. Introduction

In 1950, A.V. Pogorelov proved the unique determination of closed convex surfaces in the Euclidean space E^3 [2]. For this result, no other conditions on the closed surface except the convexity were assumed. Under additional regularity assumptions, the theorem had been proven earlier by S. Cohn-Vossen in 1924 [3] and by G. Herglotz in 1943 [4]. A.V. Pogorelov also established the unique determination of closed convex surfaces in the spherical space S^3. Using the result of A.V. Pogorelov and an idea of E.P. Senkin, A.D. Milka [5] proved the same result for closed convex surfaces in the hyperbolic (Lobachevsky) space $\mathbb{H}^3$. E.P. Senkin [6], [8] generalized Pogorelov's theorem to closed convex hypersurfaces in the Euclidean space E^{n+1} $(n \geq 3)$ under the additional assumption of C^1 regularity of the hypersurface. In [1], this theorem was established without any additional assumptions.

Furthermore, E.P. Senkin proved the following result on *local* unique determination of C^1-smooth convex hypersurfaces:

Theorem 1.1 ([6, 8]). *Let $F_1 \subset E^{n+1}$, $n \geq 3$, be a C^1-smooth convex hypersurface and $P_1 \in F_1$ be a point of strict convexity. Suppose $F_2 \subset E^{n+1}$ is a C^1-smooth convex hypersurface that is isometric to F_1, and let $P_2 \in F_2$ be the point corresponding to P_1 under this isometry. Then F_1 and F_2 are congruent in sufficiently small neighborhoods of P_1 and P_2.*

In the proof of this theorem, the assumption of smoothness was used only to prove that the isometric combination of F_1 and F_2 in a neighborhood of P_1 and P_2 is a convex hypersurface.

Since we proved this without the smoothness assumption, then Theorem 1.1 is now proven without this extra assumption.

Theorem 1.1′. *Let $F_1 \subset E^{n+1}$, $n \geq 3$, be a convex hypersurface and $P_1 \in F_1$ be a point of strict convexity. If $F_2 \subset E^{n+1}$ is a convex hypersurface that is isometric to F_1 and $P_2 \in F_2$ is the point corresponding to P_1 under the isometry, then F_1 and F_2 are congruent in sufficiently small neighborhoods of P_1 and P_2.*

Another result by E.P. Senkin is the following theorem:

Theorem 1.2 ([7, 8])**.** *Let $F_1, F_2 \subset E^{n+1}$, $n \geq 3$, be C^1-smooth isometric convex hypersurfaces. Let $P_1 \in F_1$ be a point that does not lie in any flat region of F_1 of dimension n, $(n-1)$, $(n-2)$, and let $P_2 \in F_2$ be the point corresponding to P_1 under the isometry. Then F_1 and F_2 are congruent in sufficiently small neighborhoods of P_1 and P_2.*

Again, in the proof of this theorem, the assumption of smoothness was used only to prove that the isometric combination of F_1 and F_2 in a neighborhood of P_1 and P_2 is a convex hypersurface. Since we proved that we can position F_1 and F_2 in such a way that their isometric combination in a neighborhood of the points P_1 and P_2 is a convex hypersurface, then Theorem 1.2 is true without the smoothness assumption.

Theorem 1.2′. *Let $F_1, F_2 \subset E^{n+1}$, $n \geq 3$, be isometric convex hypersurfaces, and let $P_1 \in F_1$ be a point that does not lie in any flat region of F_1 of dimension n, $(n-1)$, $(n-2)$, and $P_2 \in F_2$ be the point corresponding to P_1 under the isometry. Then sufficiently small neighborhoods of P_1 and P_2 in F_1 and F_2 are congruent.*

Our main result, which enables us to remove the regularity assumptions, thereby extending Theorems 1.1 and 1.2 to Theorems 1.1′ and 1.2′, is as follows.

Theorem 1.3. *Let $F_1, F_2 \subset E^{n+1}$, $n \geq 3$, be isometric convex hypersurfaces, $P_1 \in F_1$ and $P_2 \in F_2$ a pair of points corresponding under the isometry between F_1 and F_2. There exists a rigid motion of E^{n+1} which moves F_1 to $\tilde{F}_1$ so that the isometric combination of some neighborhoods of $\tilde{P}_1 \in \tilde{F}_1$ and $P_2 \in F_2$ is a convex hypersurface.*

The proof of this statement will consists of a sequence of steps which we will present in the following sections. At the end, for the sake of completeness, we will also briefly recall main ideas of proofs for E.P. Senkin's Theorems 1.1 and 1.2 leading to Theorems 1.1′ and 1.2′.

2. Isometric combination of convex curves

Let $F_1, F_2 \subset E^2$ be a pair of simple closed convex positively oriented curves of the same length and with a pair of marked points $P_1 \in F_1, P_2 \in F_2$. By a parallel translation of the curves, we can align the points P_1 and P_2. By a rotation around the point $P := P_1 = P_2$, we can assume that F_1 and F_2 have a common supporting line at P (so that F_1 and F_2 lie in the same half-plane with respect to this line). In this way, the regions bounded by the curves F_1 and F_2 have a non-empty intersection. Let $P_1(s), P_2(s)$ be the points on F_1, F_2 such that the lengths of the arcs $PP_1(s)$, $PP_2(s)$ along F_1, F_2 are both equal to s. Denote by $r_1(s), r_2(s)$ the position vectors of $P_1(s), P_2(s)$. Then the *isometric combination* of the curves F_1, F_2 is the set F given by the radius-vector

$$r(s) = r_1(s) + r_2(s).$$

We denote by $P(s)$ the isometric combination of $P_1(s)$ and $P_2(s)$.

Lemma 2.1. *Let $F_1, F_2 \subset E^2$ be a pair of simple closed convex positively oriented polygons of the same length and with a pair of marked points $P_1 \in F_1$ and $P_2 \in F_2$. Suppose the intersection of the domains bounded by F_1 and F_2 contains a disc of radius $r_0 > 0$. If for every s, the angle between the right half-tangents at the points $P_1(s), P_2(s)$ is less than π, then the isometric combination of F_1 and F_2 is a simple closed convex polygon.*

Proof. Only the following three cases are possible:

1. The points $P_1(s), P_2(s)$ lie in the interior of the sides of the polygons. Then the isometric combination of the neighborhoods of points will be a segment containing the point $P(s)$.
2. The point $P_1(s)$ lies in the interior of the side of F_1 and the point $P_2(s)$ is the vertex of F_2. Recall that the convex polygons in question have the same orientation. Let the right semi-tangent at $P_2(s)$ belong to the upper half-plane with respect to the side of F_1 containing $P_1(s)$, and let F_1 also belong to this half-plane. Let the left semi-tangent at $P_2(s)$ belong to the lower half-plane. Let α_i, respectively δ_i, be the angle between the right, respectively the left, semi-tangents at the points $P_1(s)$ and $P_2(s)$, $\beta_i^1 = \pi$ and $\beta_i^2 < \pi$ be the angles of the polygons at the vertices $P_1(s)$, $P_2(s)$, and let γ_i be the angle between the right semi-tangent at $P_2(s)$ and the left semi-tangent at $P_1(s)$. By construction, $P(s)$ is the vertex with angle $\beta_i := \alpha_i/2 + \delta_i/2 + \gamma_i$. But

$$\delta_i + \gamma_i = \beta_i^1 = \pi, \quad \alpha_i + \gamma_i = \beta_i^2 < \pi.$$

This gives that

$$\beta_i = \frac{\alpha_i}{2} + \frac{\beta_i}{2} + \gamma_i = \frac{\beta_i^1 + \beta_i^2}{2} < \pi.$$

We obtain the same formula for β_i for other dispositions of semi-tangents at $P_1(s)$ and $P_2(s)$. Observe that we have chosen the orientation of F_1 in such a way that F_1 lies in the upper half-plane with respect to the side of F_1 containing $P_1(s)$. It is therefore impossible that the both sides of F_2 at the vertex $P_2(s)$ lie in the lower half-plane (otherwise, the polygons F_1 and F_2 do not intersect, which is a contradiction to our choice above).
3. The points $P_1(s), P_2(s)$ are the vertices of their respective polygons. Let us make a parallel translation so that the vertices $P_1(s), P_2(s)$ coincide. Without loss of generality, we can assume that the distance of the vertex $P_1(s)$ to the supporting line at $P_1 = P_2$ is not greater than the distance of the vertex

$P_2(s)$. Then the parallel translation T of the point $P_1(s)$ to the point $P_2(s)$ will move the polygon F_1 to the polygon TF_1 that also intersects F_2. The case when the solid angles at the vertices $TP_1(s)$ and $P_2(s)$ do not intersect is impossible because otherwise the intersection of polygons is empty. We obtain that the angle at the vertex $P(s)$ of the isometric combination is $\beta_i = (\beta_i^1 + \beta_i^2)/2 < \pi$.

For the convex polygons F_1, F_2, we have that

$$\sum(\pi - \beta_i^1) = 2\pi, \quad \sum(\pi - \beta_i^2) = 2\pi.$$

Thus

$$\sum(\pi - \beta_i) = \beta_i = \sum \frac{\pi - \beta_i^1}{2} + \sum \frac{\pi - \beta_i^2}{2} = 2\pi.$$

Since $(\pi - \beta_i) \geq 0$, the closed polygonal line F will be a closed convex polygon. We obtain that F is the isometric combination of F_1 and F_2. □

This result generalizes easily to the case of arbitrary convex curves.

Lemma 2.2. *Let F_1, F_2 be closed convex curves of the same length s_0 and with the same orientation. Let $P_1 \in F_1$, $P_2 \in F_2$ be a pair of marked points with respect to which the length of arcs along the curve is measured. Assume that $P_1 = P_2$ and at this point the curves F_1, F_2 have a common supporting line. Further assume that the regions bounded by F_1, F_2 intersect. If at the corresponding with respect to the arc length points $P_1(s), P_2(s)$ the angle between the right semi-tangents is less than π, then the isometric combination of F_1, F_2 is a closed convex curve.*

Proof. Let F_1^n, F_2^n be a sequence of convex polygons circumscribed around F_1, F_2 such that

$$\lim_{n\to\infty} F_1^n = F_1, \quad \lim_{n\to\infty} F_2^n = F_2.$$

Then the lengths of F_1^n, F_2^n tend to s_0. Let us make a dilatation at a fixed point $P_1 = P_2 = P$ so that the length of image polygon $\tilde{F}_1^n$ coincides with the length of F_2^n. For sufficiently large n, the angles between the tangents to $\tilde{F}_1^n$, F_2^n at the corresponding under the isometry points is less than π. By Lemma 2.1, the isometric combination of the polygons $\tilde{F}_1^n$, F_2^n is a closed convex polygon F^n. The limit of the polygons F^n is the isometric combination of the curves F_1, F_2. This will be a convex curve F. □

Lemma 2.3. *Let F_1, F_2 be two closed nondegenerate convex curves with marked points P_1, P_2. Suppose that F_1, F_2 have the same orientation and the same length s_0. Then there exists an isometry of the plane that moves the curve F_1 to a curve $\tilde{F}_1$ so that the angles between the right semi-tangents at the points $\tilde{P}_1(s)$ and $P_2(s)$ is less than π, where s is the arc length measured starting from $P_1 = P_1(0)$, $P_2 = P_2(0)$.*

Proof. Let us align the points $P_1(0)$ and $P_2(0)$ using an appropriate isometry of the plane in such a way that these points coincide and that their right semi-tangents at these points also coincide. If the curve F_1 is strictly convex, then it is possible to choose as the point $P_1(0)$ any point of F_1. If on the curve F_1 there exists a line segment, then the beginning of the segment we choose as the point $P_1(0)$. The orientation on the segment is the same as the orientation on the curve F_1. If for every $s \in (0, s_0]$, the angle between the right semi-tangent at $P_1(s)$ and $P_2(s)$ is $< \pi$, then we are done. So suppose this is not the case. Then there is a pair of points $P_1(\sigma_0)$, $P_2(\sigma_0)$ such that the right semi-tangents at those points have the opposite direction. For definiteness, we can assume that the turning of the arc $P_1(0)P_1(\sigma_0)$ is bigger than the turning of the arc $P_2(0)P_2(\sigma_0)$. Then

$$\tau_1(\sigma_0) = \tau_2(\sigma_0) + \pi,$$

where for $i \in \{1, 2\}$, $\tau_i(s)$ is the turning of the arc of the curve F_i from the point $P_i(0)$ to the point $P_i(s)$. Observe that $\tau_2(\sigma_0) < \pi$ because the complete turning of a closed convex curve is equal to 2π and the point $P_1(\sigma_0)$ does not coincide with the point $P_1(0)$. Furthermore, both $\tau_1(s)$ and $\tau_2(s)$ are non-negative and non-decreasing functions.

For each $s \in [0, s_0]$, let us align the points $P_1(s)$ and $P_2(s)$ in the same way as we did above for $P_1(0)$ and $P_2(0)$, and assume that we have the same situation with the oppositely oriented right semi-tangents. That is, for the points $P_1(s)$, $P_2(s)$ we would find the points $P_1\big(f(s)\big)$, $P_2\big(f(s)\big)$ such that

$$\omega_1(s) = \omega_2(s) + \pi, \quad 0 < \omega_2(s) < \pi,$$

where $\omega_1(s)$, $\omega_2(s)$ are the turnings of the arcs $P_1(s)P_1\big(f(s)\big)$, $P_2(s)P_2\big(f(s)\big)$.

Let us pick such s so that

$$2\pi > \tau_2(s) + \omega_2\big(f(s)\big) = \tau_2\big(f(s)\big) > \pi. \tag{2.1}$$

Then $\tau_1\big(f(s)\big) > 2\pi$. But since $\tau_2\big(f(s)\big) < 2\pi$, then $f(s) < s_0$ and $\tau_1\big(f(s)\big) < 2\pi$. This is a contradiction. Therefore, there exists $\sigma_0 < s_0$ such that the isometry that maps $P_1(\sigma_0)$ to $P_2(\sigma_0)$, and the right semi-tangent at $P_1(\sigma_0)$ to the right semi-tangent at $P_2(\sigma_0)$ is the required isometry.

Let us explain it in detail. Let us take on F_2 the point $P_2(s)$ such that the turning $\tau(P_2(0)P_2(s)) = \pi$. The following cases are possible:

1) Such point $P_2(s)$ exists. If $\omega_2(s) > 0$, then inequality (2.1) is true.
2) Let us assume that $\omega_2(s) = 0$. Then

$$2\pi > \tau_2(s) + \omega_2\big(f(s)\big) = \pi \quad \text{and} \quad \tau_1\big(f(s)\big) = 2\pi.$$

But $\tau_2\big(f(s)\big) = \pi$ and it follows that $f(s) < s_0$. The point $P_1(f(s))$ does not coincide with the point $P_1\big(0\big)$, and the segment $P_1\big(f(s)\big)P_1(0)$ belongs to the curve F_1. If the point $P_1(0)$ is a corner point, then turning of the curve $F_1 > 2\pi$. It is impossible for a closed convex curve. If the point $P_1(0)$ is a smooth point, then we have a contradiction with the choice of the point $P_1(0)$.

3) The turning of the arc $P_2(0)P_2(s)$ is greater than π in the case when the point $P_2(s)$ is a corner point. The turning of the arc $P_2(s-\varepsilon)P_2(s)$ goes to $\pi - \alpha$ if $\varepsilon \to 0$, α is the angle at the point $P_2(s)$. Suppose that on the arc $P_2(s-\varepsilon)P_2(f(s-\varepsilon))$, $f(s-\varepsilon) > s$ the following holds:

$$\omega_1(s-\varepsilon) = \omega_2(s-\varepsilon) + \pi.$$

Then on the arc $P_1(s-\varepsilon)P_1(s)$ there exists a point with parallel support lines with the distance $\leq \varepsilon$. The curves F_1, F_2 are nongenerate and for $\varepsilon < c$, where $c > 0$ is a constant, it is impossible- Then for $\varepsilon < c$, strong inequalities

$$2\pi > \tau_2(s-\varepsilon) + \omega_2(s-\varepsilon) = \tau_2\big(f(s-\varepsilon)\big) > \pi,$$

are true, and we obtain a contradiction as in the case 1).

The lemma is proved. □

Lemmas 2.1, 2.2, and 2.3 yield the following theorem.

Theorem 2.4. *Let F_1, F_2 be two simple closed convex curves on the plane oriented in the same way and of the same length. Then there exists an orientation-preserving isometry of the plane that maps F_1, F_2 to a pair of curves whose isometric combination is also a simple closed convex curve.*

Thus, a pair of closed convex curves of the same length in E^2 can be positioned by a rigid motion of E^2 so that their isometric combination represents a closed convex curve as well, and no additional assumptions of regularity are needed.

3. Isometric combination of convex cones in E^3

The next step, Theorem 2.4 for convex curves in E^2 leads to the following statement for convex cones in E^3 which claims essentially the same: a pair of isometric convex cones in E^3 can be positioned by a rigid motion of E^3 so that their isometric combination represents a convex cone as well, without any additional assumptions of regularity.

Theorem 3.1. *Let F_1, F_2 be isometric convex cones in E^3. Then there exists an isometry of E^3 which moves F_1 to $\tilde{F}_1$ so that the isometric combination of $\tilde{F}_1$ and F_2 is a strictly convex cone.*

Proof. Let K_1, K_2 be isometric convex cones. We align those cones so that their vertices are at the origin and such that they share a common strictly supporting plane. We further suppose that they are given by the position vectors

$$R_1 = t \cdot r_1(s), \quad R_2 = t \cdot r_2(s), \quad \langle r_1, r_1\rangle = \langle r_2, r_2\rangle = 1.$$

The position vectors $r_1 = r_1(s)$, $r_2 = r_2(s)$ encode the curves M_1, M_2 at the intersection of the cones with the unit sphere S^2, $x_0^2+x_1^2+x_2^2 = 1$, where x_0, x_1, x_2 are the Cartesian coordinates in E^3 chosen so that the curves have $x_0 > 0$, and s is the arc-length parameter measured along those curves starting at points P_1, respectively P_2, that correspond each other under the cone isometry. The curves M_1, M_2 are convex curves in S^2 of the same length.

Using the Pogorelov transformation, let us map the curves M_1, M_2 to the (x_1, x_2)-plane:

$$\tilde{r}_1 = \frac{r_1 - \langle r_1, e_0\rangle}{\langle e_0, r_1 + r_2\rangle}, \quad \tilde{r}_2 = \frac{r_2 - \langle r_2, e_0\rangle}{\langle e_0, r_1 + r_2\rangle},$$

where e_0 is the unit vector in the direction of x_0 axes, $\tilde{r}_1, \tilde{r}_2$ are the orthogonal projections of the vectors r_1, r_2 onto the (x_1, x_2)-plane. By a theorem of Pogorelov [2, Ch.5, §3], the curves $\tilde{M}_1$, $\tilde{M}_2$ given by the position vectors $\tilde{r}_1$, $\tilde{r}_2$ are convex and isometric. Therefore, by Theorem 2.4, the curves $\tilde{M}_1$, $\tilde{M}_2$ can be rotated in the (x_1, x_2)-plane so that their isometric combination is a closed convex curve $\tilde{M}$.

Observe that if we rotate the curve $\tilde{M}_1$, then it corresponds to the rotation of the cone K_1 and the corresponding convex curve M_1 around the x_0-axis. Therefore, without loss of generality, we can assume that the isometric combination of curves $\tilde{M}_1$ and $\tilde{M}_2$ is a convex curve $\tilde{M}$ given by the position vector $\tilde{r} = \tilde{r}_1 + \tilde{r}_2$. This curve corresponds to a convex curve $M_0 \subset S^2$ with the position vector

$$r = \frac{(\tilde{r}_1 + \tilde{r}_2, 1)}{\sqrt{1 + \langle \tilde{r}_1 + \tilde{r}_2, \tilde{r}_1 + \tilde{r}_2\rangle}}.$$

We can express $\tilde{r}_1, \tilde{r}_2$ in another way as follows:

$$\tilde{r}_1 = \frac{\bar{r}_1}{x_1^0 + x_2^0}, \quad \tilde{r}_2 = \frac{\bar{r}_2}{x_1^0 + x_2^0},$$

where $\bar{r}_1$, $\bar{r}_2$ are the orthogonal projections of the position vectors r_1, r_2 onto the (x_1, x_2)-plane, and x_1^0, x_2^0 are the x_0-coordinates of r_1, respectively, r_2.

By direct computation, it follows that

$$r = \frac{\left(\frac{\bar{r}_1 + \bar{r}_2}{x_1^0 + x_2^0}, 1\right)}{\sqrt{1 + \frac{\langle \bar{r}_1 + \bar{r}_2, \bar{r}_1 + \bar{r}_2\rangle}{(x_1^0 + x_2^0)^2}}} = \frac{(\bar{r}_1 + \bar{r}_2, x_1^0 + x_2^0)}{\sqrt{\left(x_1^0 + x_2^0\right)^2 + \langle \bar{r}_1 + \bar{r}_2, \bar{r}_1 + \bar{r}_2\rangle}} = \frac{(\bar{r}_1 + \bar{r}_2, x_1^0 + x_2^0)}{\sqrt{2(1 + \langle r_1, r_2\rangle)}}.$$

The cones K_1, K_2 are given by the position vectors

$$R_1 = t \cdot r_1, \quad R_2 = t \cdot r_2.$$

Their isometric combination is a cone K with the position vector $R = t \cdot (r_1 + r_2)$. The curve M of the intersection of this cone with the unit sphere is given by

$$r = \frac{r_1 + r_2}{\sqrt{\langle r_1 + r_2, r_1 + r_2\rangle}} = \frac{(\bar{r}_1 + \bar{r}_2, x_1^0 + x_2^0)}{\sqrt{2(1 + \langle r_1, r_2\rangle)}}.$$

As we can see, the curves M and M_0 coincide. Hence, the isometric combination of the cones K_1 and K_2 will be the convex cone K. □

4. Isometric combinations of dihedral angles

For the sake of completeness, let us consider the exceptional case of dihedral angles as well.

If for a pair D_1, D_2 of dihedral angles their edges correspond under the isometry, then the isometric combination of D_1 and D_2 will be a dihedral angle D. If we put one dihedral angle inside the other so that their intersection with the unit sphere looks as shown in Figure 4.1 (left), then the angle between the corresponding under the isometry rays of dihedral angles is $< \pi - \epsilon_0$, $0 < \epsilon_0 < \pi$.

If the edges of D_1, D_2 do not correspond to each other under the isometry, we can still locate the angles so that their edges coincide and that their intersection with the unit sphere has the form as shown in Figure 4.1, left. In this case, however, the vertices of the digons do not correspond each other under the isometry. Nonetheless, in this case the angle between the corresponding under the isometry directions $< \pi - \epsilon_0$, $0 < \epsilon_0 < \pi$.

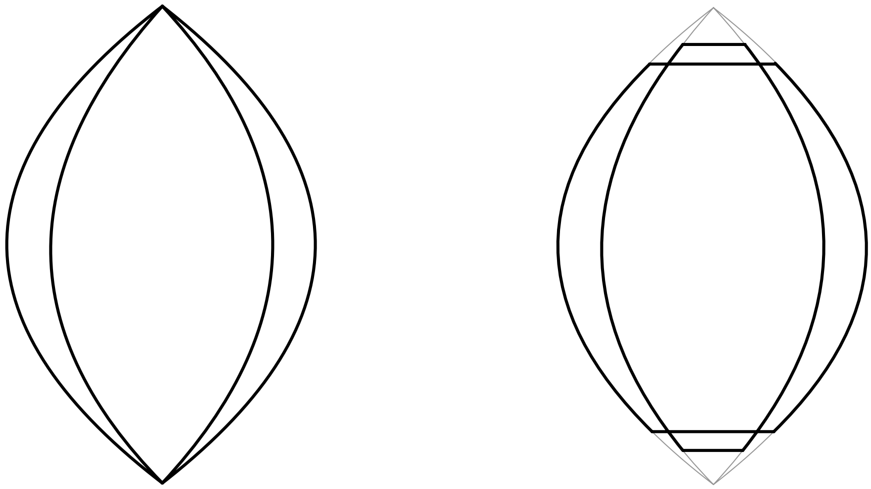

Fig. 4.1: Schematic illustrations of spherical digons generated by dihedral angles (left) and their truncated versions (right)

Let us substitute the digons with the sequence of the quadrilaterals D_1^n, D_2^n of the same length (see Figure 4.1, right). Their isometric combination will be a convex curve D^n. The limit of D^n's in the sphere will be a convex curve D which is the intersection with the sphere of the isometric combination of D_1 and D_2.

We can write dihedral angles in E^{n+1} as the metric Cartesian product of dihedral angles in E^3 with E^{n-2}:

$$K_1(P_1) = D_1 \times E_1^{n-2}, \quad K_1(P_2) = D_2 \times E_2^{n-2},$$

where E_1^{n-2}, E_2^{n-2} coincide and correspond to each other under the isometry. The isometric combination of such cones will be the Cartesian product of the isometric combination of D_1 and D_2 times E^{n-2}. Let us locate the dihedral angles so that they are nested. Then on the sphere they cut out digons that are also nested. The following cases are possible:

a) The edges of the dihedral angles correspond each other under the isometry. Then their isometric combination will be a dihedral angle.

b) The edges do not correspond under the isometry. In this case, let us substitute the digons in S^2 with the quadrilaterals D_1^n, D_2^n that are obtained by truncating the vertices of the digons so that the lengths of D_1^n and D_2^n are equal. By Theorem 2.4, the isometric combination of polyhedral angles that correspond to these quadrilaterals will be a convex polyhedral angle D^n. Moreover, for the polyhedral angles D_1, D_2, the angle between the isometric directions $< \pi - \epsilon_0$, $\epsilon_0 > 0$. The limit of D^n's will be a convex polyhedral angle D that is an isometric combination of dihedral angles in E^3.

5. Proof of Theorem 1.3

Let us place the tangent cones at the points $P_1 \in F_1$, $P_2 \in F_2$ so that their isometric combination is a convex cone. This can be done indeed:

1) Suppose that the tangent cone at $P_1 \in F_1$ has a form $K(P_1) = V_1^s \times E_1^{n-s}$, $s \geq 3$. Then, as was proven in [1], the cone $K(P_2)$ is congruent to $K(P_1)$, i.e., there is an isometry of E^{n+1} that sends one cone to the other. Hence, their isometric combination will be a cone that is homothetic to $K(P_1)$.

2) $K(P_1) = V_1^2 \times E_1^{n-2}$. Then $K(P_2) = V_2^2 \times E_2^{n-2}$, where E_1^{n-2}, E_2^{n-2} are isometric planes. We can align those plane by a proper motion, and the isometric cones V_1^2, V_2^2 will be in the common three-dimensional space E^3. From Theorem 2.4 and the Pogorelov transformation, it follows that there exists a motion in E^3 that moves the cones in the position so that their isometric combination is a convex cone.

3) $K(P_1) = V_1^1 \times E_1^{n-1}$ is a dihedral angle in E^{n+1} (here, V_1^1 is a cone in E^2), $K(P_2) = V_2^1 \times E_2^{n-1}$. We can represent them as a Cartesian product

$$K(P_1) = D_1 \times E_1^{n-2}, \quad K(P_2) = D_2 \times E_2^{n-2},$$

where E_1^{n-2}, E_2^{n-2} are the Euclidean subspaces that correspond each other under the isometry, and D_1, D_2 are the dihedral angles in E^3. They can be located so that their isometric combination is a convex cone.

4) $K(P_1) = E_1^n$, $K(P_2) = E_2^n$.

5) $K(P_1) = V_1^1 \times E_1^{n-1}$, $K(P_2) = E^n$. In those cases, their isometric combination will be 4) hyperplane, 5) dihedral angle.

Consider each of these cases separately.

1) Let $P_1^n \in F_1$, $P_2^n \in F_2$ be a sequence of corresponding under the isometry points that tend to $P_1 = P_2 = P_0$. The cones $K(P_1^n)$, $K(P_2^n)$ tend to the isometric cones K_1^0, K_2^0. Since the cones $K(P_1)$ and $K(P_2)$ coincide, then the convex cones K_1^0 and K_2^0 coincide and their isometric combination is a convex cone.

Let us show that for sufficiently large n, the isometric combination of the cones $K(P_1^n)$, $K(P_2^n)$ is a convex cone.

a) Suppose $K(P_1^n) = V_1^s \times E_1^{n-s}$, $s \geq 3$. Then the cones $K(P_1^n)$ and $K(P_2^n)$ are congruent and their isometric combination is a convex cone for sufficiently large n.

b) Suppose $K(P_1^n) = V_1^2 \times E_1^{n-2}$. Then $K(P_2^n) = V_2^2 \times E_2^{n-2}$, and the cones K_1^0, K_2^0 are either coinciding $V^2 \times E^{n-2}$, or $V^1 \times E^{n-1}$, or E^n. Suppose that on the cone γ_n, which is an isometric combination of cones $K(P_1^n)$, $K(P_2^n)$, there exists a generator t_n at which the local convexity is violated. Without loss of generality, we can assume that the corresponding generators t_n^1 on $K(P_1^n)$ converge to t_0 on K_1^0, where $t_0 \in V_1^s$.

Let A_n' be a point on t_n^1 at distance 1 from the subspace E_1^{n-2}, κ_n' be a tangent dihedral angle at this point, and κ_n'' be a tangent dihedral angle at the point $A_n'' \in K(P_2^n)$ that corresponds to A_n' under the isometry. Under the isometric combination of the angles κ_n', κ_n'' for sufficiently large n, we obtain a dihedral angle κ_n that contains a ball ω. The existence of the ball ω follows from the fact that κ_1^0 and κ_2^0 coincide and the limits of $K(P_1^n), K(P_2^n)$ are K_1^0, K_2^0. Passing through the edge of this dihedral angle, we can draw a supporting plane so that the inner normal ν to this plane, if placed starting at A_n', A_n'', directs inside the cones $K(P_1^n)$, $K(P_2^n)$. Let us connect a point B_n', that is close to A_n', with the geodesic γ' on $K(P_1^n)$. Let $r_1(s)$ be the position vector of a point on the curve γ' corresponding to the arc-length s (measured from the point A_n'), and $r_2(s)$ be the position vector of a point on $K(P_2^n)$ corresponding under the isometry. For $s = 0$, we have

$$\frac{d}{ds}\langle r_1 + r_2, \nu\rangle \geq 0.$$

By a theorem of Liberman about the convexity of geodesics γ' on $K(P_1^n)$ and the corresponding geodesic γ'' on $K(P_2^n)$, we conclude that

$$\frac{d}{ds}\langle r_1 + r_2, \nu\rangle \geq 0 \quad \text{for all } s \text{ along } \gamma'.$$

By this inequality, all points of the cone γ_n that are close to the image of the point A_n' are located on one side with respect to the supporting hyperplane with the inner normal ν. This yields the convexity of the cone. Here, for completeness, we have outlined the argument due to Pogorelov about the convexity of the isometric combination of isometric cones in E^3 [2].

2) $K(P_1) = V_1^2 \times E_1^{n-2}$, $K(P_2) = V_2^2 \times E_2^{n-2}$. We can align the vertices $P_1 = P_2 = P_0$ and place F_1 and F_2 so that the isometric combination of the cones $K(P_1)$, $K(P_2)$ is a convex cone. Let $P_1^n \in F_1$, $P_2^n \in F_2$ be a sequence of points that correspond each other under the isometry and $P_1^n \to P_0$, $P_2^n \to P_0$ as $n \to \infty$. Further, if $K(P_1^n)$, $K_2(P_2^n)$ are the tangent cones, $\lim_{n\to\infty} K(P_1^n) = K_1^0$, $\lim_{n\to\infty} K(P_2^n) = K_2^0$.

a) Let $K_1^0 = V_1^2 \times E^{n-2}$. Then $K_2^0 = V_2^2 \times E^{n-2}$. Then $K(P_1) = K_1^0$, $K(P_2) = K_2^0$, and the isometric combination of K_1^0, K_2^0 is a convex cone. Furthermore, there exists a ball ω that lies inside the cones K_1^0, K_2^0. The rest of the proof goes as in item 1) above.

b) K_1^0, K_2^0 are dihedral angles. Since those dihedral angles are supporting to the cones $K(P_1)$, $K(P_2)$, and their isometric combination is a convex cone, then there exists a ball ω inside these dihedral angles. The rest of the proof follows that of item 1) above.

Items 3)–5) are proven analogously.

Thus, we showed that in a neighborhood of the point $P_0 = P_1 = P_2$ the hypersurfaces F_1 and F_2 can be moved by a rigid motion in such a way that their isometric combination in a neighborhood of P_0 is a convex hypersurface. □

The proofs of Theorems 1.1′, 1.2′ follows the proofs of Theorems 1.1, 1.2 due to Senkin [7]. We briefly outline them here.

Proof of Theorem 1.1′. Let F_1 and F_2 be convex isometric hypersurfaces, $P_1 \in F_1$ be a point of strict convexity of the hypersurface F_1, and $P_2 \in F_2$ be the corresponding under the isometry point. By Theorem 1.3, using the isometry of the ambient space, we can align the points P_1 and P_2 so that their isometric combination given by the position vector $r = (r_1 + r_2)/2$ is a hypersurface Φ convex in a neighborhood of the point $P_0 = P_1 = P_2$, while the field $\tau = r_1 - r_2$ is an infinitesimal bending field of Φ.

It follows that almost everywhere

$$\langle dr, d\tau \rangle = 0.$$

Let us move the supporting hyperplane at P_0 by parallel translations so that it cut away a cap ω of the hypersurface Φ. Senkin [6] showed that the cap ω is rigid away from the hyperflat parts, i.e., the field τ is trivial away from such hyperflat regions.

Such hyperflat regions on Φ can happen only as isometric combinations of hyperflat regions on F_1 and F_2. Therefore, since the regions on F_1, F_2 that correspond to Φ coincide away from the flat regions, then they must coincide. □

Proof of Theorem 1.2′ coincides with the proof of Theorem 1.2 due to Senkin [7] after we established Theorem 1.3.

As a corollary to Theorem 1.2′, we obtain the following result: *Let F_1, F_2 be isometric convex hypersurfaces. If at a point $P_1 \in F_1$ the tangent cone has a form $K(P_1) = V_1^s \times E^{n-s}$, $s \geq 3$, then there exists a neighborhood of the point P_1 that is congruent to the corresponding isometric neighborhood on the hypersurface F_2.*

A.D. Alexandrov proved that at almost every point of a convex hypersurface there exists a second fundamental form. If the rank of the second fundamental form at such a point is ≥ 3, then at this point there exists a neighborhood on the hypersurface that is rigid up to proper isometry.

Theorems 1.1′ and 1.2′ are true in the spherical and the hyperbolic spaces. To see that, we use the Pogorelov transform that sends the isometric convex hypersurfaces in those spaces to the isometric convex hypersurfaces in E^{n+1} satisfying the assumptions of Euclidean Theorems 1.1′ and 1.2′. Now, using the inverse transformation, we obtain the congruent hypersurfaces in spaces of constant curvature.

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Локальна однозначна визначеність опуклих гіперповерхонь в просторах постійної кривини

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У цій роботі ми доводимо локальну однозначну визначеність опуклих гіперповерхонь в просторах постійної кривини розмірності $n \geq 4$. А саме, ми доводимо, що дві ізометричні опуклі гіперповерхні є конгруентними локально навколо їхніх відповідних згідно з ізометрією точок строгої опуклості. Цей результат розширює висновок Є.П. Сенькіна, який довів таку локальну однозначність за додаткового припущення про C^1-гладкість гіперповерхонь.

Ключові слова: однозначна визначеність, опукла гіперповерхня, простір постійної кривини