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Approximate Controllability Problems for the Heat Equation in a Half-Plane Controlled by the Dirichlet Boundary Condition with a Bounded Control

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In the paper, the problems of approximate controllability are studied for the control system $w_t = \Delta w$, $w(0, x_2, t) = u(x_2, t)$, $x_1 \in \mathbb{R}_+ = (0, +\infty)$, $x_2 \in \mathbb{R}$, $t \in (0, T)$, where u is a control belonging to a special subset of $L^\infty(\mathbb{R} \times (0, T)) \cap L^2(\mathbb{R} \times (0, T))$. It is proved that each initial state belonging to $L^2(\mathbb{R}_+ \times \mathbb{R})$ is approximately controllable to an arbitrary end state belonging to $L^2(\mathbb{R}_+ \times \mathbb{R})$ by applying these controls. A numerical algorithm of solving the approximate controllability problem for this system is given. The results are illustrated by an example.

Key words: heat equation, controllability, approximate controllability, half-plane

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1. Introduction

We consider the heat equation in a half-plane

$$w_t = \Delta w, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R},\ t \in (0, T), \tag{1.1}$$

controlled by the Dirichlet boundary condition

$$w\big(0, (\cdot)_{[2]}, t\big) = u((\cdot)_{[2]}, t), \qquad x_2 \in \mathbb{R},\ t \in (0, T), \tag{1.2}$$

under the initial condition

$$w\big((\cdot)_{[1]}, (\cdot)_{[2]}, 0\big) = w^0, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R}, \tag{1.3}$$

where $\mathbb{R}_+ = (0, +\infty)$, $T > 0$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$,

$$u \in U[0, T] \tag{1.4}$$

is a control,

$$U[0, T] = \left\{ \varphi \in L^\infty(\mathbb{R} \times (0, T)) \,\middle|\, \sup_{t \in [0, T]} |\varphi(\cdot, t)| \in L^2(\mathbb{R}) \right\}$$

is the set of admissible controls. The subscripts [1] and [2] associate with the variable numbers, e.g. $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. This problem is considered in spaces of the Sobolev type (see details in Section 2).

Controllability problems for the heat equation were studied both in bounded and unbounded domains. However, most of the papers studying these problems deal with domains bounded with respect to the spatial variables (see [1, 7, 8, 18–21, 23, 26–28, 30, 36] and the references therein). Note that each initial state of the heat equation in a bounded domain which belongs to a Sobolev space of non-positive order can be driven to the origin in an arbitrarily small time by an L^2 boundary control (see, e.g. [21, 27]). Although there are quite a few papers considering domains unbounded with respect to the spatial variables [2–6, 12–16, 24, 29, 31–33, 35, 39, 40], there are much less papers (see, e.g. [14, 17, 32]) where the boundary controllability of the heat equation was studied in a half-plane.

For a bounded domain $\Omega \subset \mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^2 (which is considered instead of the domain $\mathbb{R}_+ \times \mathbb{R}$), it is well-known that the control system of the form (1.1)–(1.3) is null-controllable for a given time $T > 0$. This result was obtained by using Carleman inequalities (see, e.g. [21, 27]).

For unbounded domains, the situation is essentially different. There exist pairs of initial and target states where the initial state can be driven to the end state by means of control system (1.1)–(1.3), and there exist those where the initial state cannot be driven to the end state by means of this system. For instance, there is no initial data in any negative Sobolev space that may be driven to zero in finite time (see [32]).

Although the statement of approximate null-controllability for control system (1.1)–(1.3) is considered known, there is no papers containing direct and clear proof of it. E.g. in [32], the authors assert that "one can easily prove the approximate controllability directly both in the case of bounded and unbounded domains", but they give neither a proof of this fact nor a reference for it.

In this paper we prove the approximate controllability for control system (1.1)–(1.3). In particular, we construct controls solving the approximate controllability problem and give a numerical algorithm of solving this problem.

In [32], the lack of the null-controllability property for the linear heat equation on the half-space $\mathbb{R}_+ \times \mathbb{R}^n$ with an L^2 Dirichlet boundary control is studied. By rewriting system on the similarity variables and separating of variables, the multi-dimensional control problem is reduced to an infinite family of one-dimensional control systems in weighted Sobolev spaces. Due to results obtained for one-dimensional control system (see [31]), it was shown that controllable data have Fourier coefficients that grow exponentially for large frequencies. So, any initial state from Sobolev spaces cannot be driven to the origin by an L^2 Dirichlet boundary control in a time $T > 0$. In [14], control system (1.1)–(1.3) was studied with a control of the form $u((\cdot)_{[2]}, t) = \delta((\cdot)_{[2]})v(t)$ (δ is the Dirac distribution, $v \in L^\infty(0, T)$), which is not bounded in contrast to the present paper. In fact, the control $\delta((\cdot)_{[2]})v(t)$ is a distribution of the class $H^{-1}(\mathbb{R})$ with respect to x_2. That is why the control system was considered in the Sobolev space $H^{-1}(\mathbb{R}_+ \times$

$\mathbb{R}$) in [14] in contrast to the space $H^0(\mathbb{R}_+ \times \mathbb{R}) = L^2(\mathbb{R}_+ \times \mathbb{R})$ in the present paper. In addition, the spaces $\mathcal{H}^s$ and $\mathcal{H}_s$ based on functions of the form $f(|x|)$ were constructed and studied to investigate the controllability problems in [14] because the states of the control system which could be approximately steered to the origin had the form $w^0(x) = \frac{\partial}{\partial x_1}\omega^0(|x|)$. So, under the control of the form $u((\cdot)_{[2]}, t) = \delta((\cdot)_{[2]})v(t)$, the set of states which can be approximately driven to the origin is not coincide with $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ in contrast to the case of bounded controls in the present paper, where this set is equal to $H^0(\mathbb{R}_+ \times \mathbb{R}) = L^2(\mathbb{R}_+ \times \mathbb{R})$. In [14], both necessary and sufficient conditions for controllability and sufficient conditions for approximate controllability in a given time T under a control u bounded by a given constant were obtained in terms of solvability of a Markov power moment problem. In addition, orthogonal bases were constructed in special spaces $\mathcal{H}^s$ and $\mathcal{H}_s$ of the Sobolev type in that paper. Using these bases, necessary and sufficient conditions for approximate controllability and numerical solutions to the approximate controllability problem were obtained in [14]. In [17], the results of [14] were extended to the case of the Neumann boundary control, i.e. to the case where condition (1.2) is replaced by the condition $w_{x_1}\big(0, (\cdot)_{[2]}, t\big) = \delta((\cdot)_{[2]})v(t)$. The boundary controllability of the wave equation in a half-plane $\mathbb{R}_+ \times \mathbb{R}$ with a pointwise control on the bound was studied in [9–11].

In the present paper, the approximate controllability problem for system (1.1)–(1.3) is studied in Sobolev spaces under controls from $U[0, T]$, in particular, $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ and $w(\cdot, t) \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $t \in [0, T]$. We show that $L^2(\mathbb{R} \times (0, T))$-controls are not appropriate to consider approximate controllability property for $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ and $w(\cdot, t) \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $t \in [0, T]$, because there exists a control $u \in L^2(\mathbb{R} \times (0, T))$ (with compact supports) for which the end state $w(\cdot, T)$ of the solution to (1.1)–(1.3) does not belong to $L^2(\mathbb{R}_+ \times \mathbb{R})$ for the initial state $w^0 = 0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ (see Example 3.4 in Section 3 below). This is why we consider the narrower set of controls $U[0, T]$, which provides the condition $w(\cdot, t) \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $t \in [0, T]$, for any $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ (see Theorem 3.3 in Section 3 below). Roughly speaking, we consider a specific subset of bounded controls in $L^2(\mathbb{R} \times (0, T))$. We prove that each initial state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ of system (1.1)–(1.3) can be driven to an arbitrary neighbourhood of any target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ by choosing an appropriate control $u \in U[0, T]$, in other words, a state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is approximately controllable to a target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ in a given time T (see Theorem 4.1 in Section 4 below). The method of proving this assertion is constructive. This allows to provide a numerical algorithm of solving the approximate controllability problem for system (1.1)–(1.3). To this aid, we consider the odd extension of w and w^0 with respect to x_1 and obtain control problem (3.1), (3.2) in Section 3. Then we develop the state and the control in this new system in the Fourier series with respect to a basis generated by Hermite functions that allows us to reduce the 2-d problem to a family of the 1-d ones. Since we consider the approximate controllability, we can solve only a finite number of these 1-d problems. To construct controls solving them, we apply the method introduced in [12] for solving the approximate controllability problem for the 1-d heat equation controlled by the Dirichlet boundary condition. We should note that similar development was used in [32] to reduce the 2-d problem to a family of the 1-d ones, but the basis of the eigenfunctions of the differential operator had been obtained after using similarity variables and weighted Sobolev spaces. In our paper, we use developing in the Fourier series directly in $L^2(\mathbb{R}^2)$ without using similarity variables that simplifies the numerical method. Moreover, we apply the Fourier transform and its inverse to analyse the solution to control problem.

The paper is organized as follows: in Section 2, some notations and definitions are given; in Section 3, the controllability problem is formulated for the control system (1.1)–(1.3), and preliminary results are given; in Section 4, the approximate controllability results are obtained; in Section 5, a numerical algorithm of solving the approximate controllability problem for system (1.1)–(1.3) is given; in Section 6, the results are illustrated by an example.

2. Notation

Let us introduce the spaces used in the paper. Let $n \in \mathbb{N}$. By $|\cdot|$, we denote the Euclidean norm in $\mathbb{R}^n$.

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions [37], $\mathscr{S}'(\mathbb{R}^n)$ be the dual space.

Let $D = \big(-i\partial/\partial x_1, \dots, -i\partial/\partial x_n\big)$, $D^\alpha = \big(-i(\partial/\partial x_1)^{\alpha_1}, \dots, -i(\partial/\partial x_n)^{\alpha_n}\big)$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $s = \overline{0,2}$, consider

$$H^s(\mathbb{R}^n) = \big\{\varphi \in L^2(\mathbb{R}^n) \,\big|\, \forall \alpha \in \mathbb{N}_0^n \quad \big(|\alpha| \le s \Rightarrow D^\alpha \varphi \in L^2(\mathbb{R}^n)\big)\big\}$$

with the norm

$$\|\varphi\|^s = \left(\sum_{|\alpha| \le s} \frac{s!}{(s-|\alpha|)!\alpha!} \Big(\|D^\alpha \varphi\|_{L^2(\mathbb{R}^n)} \Big)^2 \right)^{1/2}, \quad \varphi \in H^s(\mathbb{R}^n),$$

and $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^*$ with the norm $\|\cdot\|^{-s}$ associated with the strong topology of the adjoint space. We also have $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n) = \big(H^0(\mathbb{R}^n)\big)^*$.

For $m = \overline{-2,2}$, consider

$$H_m(\mathbb{R}^n) = \left\{ \psi \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \,\middle|\, \big(1 + |\cdot|^2\big)^{m/2} \psi \in L^2(\mathbb{R}^n) \right\}$$

with the norm

$$\|\psi\|_m = \left\| \big(1 + |\cdot|^2\big)^{m/2} \psi \right\|_{L^2(\mathbb{R}^n)}, \quad \psi \in H_m(\mathbb{R}^n).$$

Evidently, $H_{-m}(\mathbb{R}^n) = (H_m(\mathbb{R}^n))^*$.

Let $\langle f, \varphi \rangle$ be the value of a distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ on a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

By $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ denote the Fourier transform operator with the domain $\mathscr{S}'(\mathbb{R}^n)$. This operator is an extension of the classical Fourier transform operator which is an isometric isomorphism of $L^2(\mathbb{R}^n)$. The extension is given by the formula

$$\langle \mathscr{F} f, \varphi \rangle = \langle f, \mathscr{F}^{-1}\varphi \rangle, \quad f \in \mathscr{S}'(\mathbb{R}^n),\ \varphi \in \mathscr{S}(\mathbb{R}^n).$$

The operator $\mathscr{F}$ is an isometric isomorphism of $H^m(\mathbb{R}^n)$ and $H_m(\mathbb{R}^n)$, $m = \overline{-2, 2}$, [22, Chap. 1].

A distribution $f \in \mathscr{S}'\left(\mathbb{R}^2\right)$ is said to be *odd with respect to* x_1, if for all $\varphi \in \mathscr{S}(\mathbb{R}^2)$, we have $\left\langle f, \varphi\big((\cdot)_{[1]}, (\cdot)_{[2]}\big)\right\rangle = -\left\langle f, \varphi\big(-(\cdot)_{[1]}, (\cdot)_{[2]}\big)\right\rangle$.

Let $m = \overline{-2, 2}$. By $\widetilde{H}^m\left(\mathbb{R}^2\right)$ $\Big($or $\widetilde{H}_m\left(\mathbb{R}^2\right)\Big)$, denote the subspace of all distributions in $H^m\left(\mathbb{R}^2\right)$ $\Big($or $H_m\left(\mathbb{R}^2\right)$, respectively$\Big)$ that are odd with respect to x_1. Evidently, $\widetilde{H}^m\left(\mathbb{R}^2\right)$ $\Big($or $\widetilde{H}_m\left(\mathbb{R}^2\right)\Big)$ is a closed subspace of $H^m\left(\mathbb{R}^2\right)$ $\Big($or $H_m\left(\mathbb{R}^2\right)$, respectively$\Big)$.

For $s = \overline{0, 2}$, consider

$$H^s_{⓪} = \Big\{ \varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \,\Big|\, \Big(\forall \alpha \in \mathbb{N}_0^2 \quad (\alpha_1 + \alpha_2 \le s \Rightarrow D^\alpha \varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}))\Big) \\ \wedge \Big(\forall k = \overline{0, s-1} \quad D^{(k,0)}\varphi(0^+, (\cdot)_{[2]}) = 0\Big) \Big\}$$

with the norm

$$\|\varphi\|^s_{⓪} = \left(\sum_{\alpha_1 + \alpha_2 \le s} \frac{s!}{(s - (\alpha_1 + \alpha_2))!\alpha_1!\alpha_2!} \Big(\|D^\alpha \varphi\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \Big)^2 \right)^{1/2}, \quad \varphi \in H^s_{⓪},$$

and $H^{-s}_{⓪} = \left(H^s_{⓪}\right)^*$ with the norm $\|\cdot\|^{-s}_{⓪}$ associated with the strong topology of the adjoint space. We have

$$H^0_{⓪} = L^2(\mathbb{R}_+ \times \mathbb{R}).$$

Remark 2.1. Let $\varphi \in H^s_{⓪}$, $s = \overline{0, 2}$. Let $\widetilde{\varphi}$ be its odd extension with respect to x_1, i.e. $\widetilde{\varphi}(x_1, x_2) = \varphi(x_1, x_2)$ if $x_1 \ge 0$ and $\widetilde{\varphi}(x_1, x_2) = -\varphi(-x_1, x_2)$ otherwise. Then $\widetilde{\varphi} \in \widetilde{H}^s\left(\mathbb{R}^2\right)$, $s = \overline{0, 2}$. The converse assertion is true for $s = 0, 1$, and it is not true for $s = 2$. That is why the odd extension with respect to x_1 of a distribution $f \in H^{-2}_{⓪}$ may not belong to $\widetilde{H}^{-2}\left(\mathbb{R}^2\right)$. However, the following theorem holds.

Theorem 2.2. *Let $f \in H^0_{⓪}$ and there exists $f\big(0^+, (\cdot)_{[2]}\big) \in H^0(\mathbb{R})$. Then $f_{x_1x_1} \in H^{-2}_{⓪}$ can be extended to a distribution $F \in \widetilde{H}^{-2}\left(\mathbb{R}^2\right)$ such that F is odd with respect to x_1. This distribution is given by the formula*

$$F = \widetilde{f}_{x_1x_1} - 2f\big(0^+, (\cdot)_{[2]}\big)\delta\big((\cdot)_{[1]}\big)', \tag{2.1}$$

where $\widetilde{f}$ is the odd extension of f with respect to x_1, δ is the Dirac distribution.

In the case $f \in H^{-1/2}_{⓪}$, corresponding theorem has been proved in [11]. The proof of Theorem 2.2 is analogous to the proof of the mentioned theorem.

3. Problem formulation and preliminary results

We consider control system (1.1)–(1.3) in $H_{⓪}^{-l}$, $l = \overline{0,2}$, i.e. $\left(\frac{d}{dt}\right)^s w : [0,T] \to H_{⓪}^{-2s}$, $s = 0,1$, $w^0 \in H_{⓪}^0$.

Let $w^0, w(\cdot,t) \in H_{⓪}^0$, $t \in [0,T]$. Let W^0 and $W(\cdot,t)$ be the odd extensions of w^0 and $w(\cdot,t)$ with respect to x_1, respectively, $t \in [0,T]$. If w is a solution to control system (1.1)–(1.3), then W is a solution to control system

$$W_t = \Delta W - 2u((\cdot)_{[2]},t)\delta((\cdot)_{[1]})', \quad t \in (0,T), \tag{3.1}$$

$$W\big((\cdot)_{[1]},(\cdot)_{[2]},0\big) = W^0 \tag{3.2}$$

according to Theorem 2.2. Here $\left(\frac{d}{dt}\right)^s W : [0,T] \to \widetilde{H}^{-2s}\left(\mathbb{R}^2\right)$, $s = 0,1$, $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$. The converse assertion is also true. Let $W^0, W(\cdot,t) \in \widetilde{H}^0\left(\mathbb{R}^2\right)$, $t \in [0,T]$. Let w^0 and $w(\cdot,t)$ be the restrictions of W^0 and $W(\cdot,t)$ to $(0,+\infty)$ with respect to x_1, respectively, $t \in [0,T]$. If W is a solution to (3.1), (3.2), then w is a solution to (1.1)–(1.3) because

$$W\big(0^+,(\cdot)_{[2]},t\big) = u((\cdot)_{[2]},t) \quad \text{for almost all } t \in [0,T] \tag{3.3}$$

according to Lemma 3.6 (see below). Assume that $w^T \in H_{⓪}^0$. Evidently, $w\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = w^T$ iff $W\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = W^T$. Here W^T is the odd extension of w^T with respect to x_1 and $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$.

Thus, control systems (1.1)–(1.3) and (3.1), (3.2) are equivalent. Therefore, basing on this reason, we will further consider control system (3.1), (3.2) instead of original system (1.1)–(1.3).

Let $T > 0$, $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$. By $\mathscr{R}_T\left(W^0\right)$, denote the set of all states $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ for which there exists a control $u \in U[0,T]$ such that there exists a unique solution W to system (3.1), (3.2) such that $W\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = W^T$.

Definition 3.1. A state $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ is said to be controllable to a target state $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if $W^T \in \mathscr{R}_T\left(W^0\right)$.

In other words, a state $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ is controllable to a target state $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if there exists a control $u \in U[0,T]$ such that there exists a unique solution W to system (3.1), (3.2) and $W\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = W^T$.

Definition 3.2. A state $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ is said to be approximately controllable to a target state $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if $W^T \in \overline{\mathscr{R}_T\left(W^0\right)}$, where the closure is considered in the space $\widetilde{H}^0\left(\mathbb{R}^2\right)$.

In other words, a state $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ is approximately controllable to a target state $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$ if for each $\varepsilon > 0$, there exists $u_\varepsilon \in U[0,T]$ such that there exists a unique solution W_ε to system (3.1), (3.2) with $u = u_\varepsilon$ and $\left\|W_\varepsilon\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) - W^T\right\|^0 < \varepsilon$.

Using the fundamental solution to the heat operator (see, e.g. [38, Chapter 7]), we obtain the unique solution to system (3.1), (3.2)

$$W(x,t) = \mathcal{W}_0(x,t) + \mathcal{W}_u(x,t), \quad x \in \mathbb{R}^2,\ t \in [0,T], \tag{3.4}$$

where

$$\mathcal{W}_0(x,t) = \frac{1}{4\pi t} e^{-|x|^2/(4t)} * W^0(x), \quad x \in \mathbb{R}^2,\ t \in [0,T], \tag{3.5}$$

$$\mathcal{W}_u(x,t) = \frac{x_1}{\pi} \int_0^t \frac{1}{4\xi^2} e^{-|x|^2/(4\xi)} *_{[2]} u(x_2, t-\xi)\, d\xi, \quad x \in \mathbb{R}^2,\ t \in [0,T]. \tag{3.6}$$

Here $*$ is the convolution with respect to x and $*_{[2]}$ is the convolution with respect to x_2.

Theorem 3.3. *Let $u \in U[0,T]$, $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$. Then,*

(i) $\mathcal{W}_0(\cdot,t) \in \widetilde{H}^0\left(\mathbb{R}^2\right)$, $t \in [0,T]$;
(ii) $\mathcal{W}_0(\cdot,t) \in C^\infty\left(\mathbb{R}^2\right)$, $t \in (0,T]$;
(iii) $\|\mathcal{W}_u(\cdot,t)\|^0 \le 2t^{1/4} \left\| \sup_{t\in[0,T]} |u(\cdot,t)| \right\|_{L^2(\mathbb{R})}$, $t \in (0,T]$.

Proof. Denote $V^0 = \mathscr{F} W^0$, $\mathcal{V}_0(\cdot,t) = \mathscr{F}_{x\to\sigma} \mathcal{W}_0(\cdot,t)$, We have

$$\mathcal{V}_0(\sigma,t) = e^{-t|\sigma|^2} V^0(\sigma), \quad \sigma \in \mathbb{R}^2,\ t \in [0,T]. \tag{3.7}$$

Therefore,

$$\|\mathcal{W}_0(\cdot,t)\|^0 = \|\mathcal{V}_0(\cdot,t)\|_0 \le \|V^0\|_0 = \|W^0\|^0, \quad t \in [0,T], \tag{3.8}$$

i.e. (i) holds.

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. We have

$$|\sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \mathcal{V}_0(\sigma,t)|^2 \le (1+|\sigma|^2)^{1+\alpha_1+\alpha_2} e^{-2t|\sigma|^2} (1+|\sigma|^2)^{-1} |V^0(\sigma)|^2, \qquad \sigma \in \mathbb{R}^2,\ t \in (0,T].$$

Since

$$\xi^m e^{-\beta\xi} \le \left(\frac{m}{\beta e}\right)^m, \quad \xi \ge 0,\ \beta > 0,\ m \in \mathbb{N},$$

then

$$\begin{aligned} \|D^\alpha \mathcal{W}_0(\cdot,t)\|^0 &= \left\| (\cdot)_{[1]}^{\alpha_1} (\cdot)_{[2]}^{\alpha_2} \mathcal{V}_0(\cdot,t) \right\|_0 \le e^t \left(\frac{1+\alpha_1+\alpha_2}{2te}\right)^{(1+\alpha_1+\alpha_2)/2} \left\|V^0\right\|_{-1} \\ &\le e^t \left(\frac{1+\alpha_1+\alpha_2}{2te}\right)^{(1+\alpha_1+\alpha_2)/2} \|V^0\|_0 \\ &= e^t \left(\frac{1+\alpha_1+\alpha_2}{2te}\right)^{(1+\alpha_1+\alpha_2)/2} \left\|W^0\right\|^0, \quad t \in (0,T], \end{aligned} \tag{3.9}$$

i.e. (ii) holds.

Set

$$g(x_2) = \sup_{t\in[0,T]} |u(x_2,t)|, \quad x_2 \in \mathbb{R}.$$

Since $u \in U[0,T]$, we have $g \in L^2(\mathbb{R})$. Set $G = \mathscr{F} g$. Then we have

$$\left\| \sup_{t\in[0,T]} |u(\cdot,t)| \right\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})} = \|G\|_{L^2(\mathbb{R})}. \tag{3.10}$$

Taking into account (3.6), we get

$$\mathcal{W}_u(x,t) = \frac{x_1}{\pi} \int_0^t \frac{e^{-x_1^2/(4\xi)}}{4\xi^2} \int_{-\infty}^{\infty} e^{-(x_2-\mu)^2/(4\xi)} u(\mu, t-\xi)\, d\mu\, d\xi, \; x \in \mathbb{R}^2, \; t \in [0,T].$$

Therefore,

$$\begin{aligned} |\mathcal{W}_u(x,t)| &\le \frac{|x_1|}{\pi} \int_0^t \frac{e^{-x_1^2/(4\xi)}}{4\xi^2} \int_{-\infty}^{\infty} e^{-(x_2-\mu)^2/(4\xi)} g(\mu)\, d\mu\, d\xi \\ &= \frac{|x_1|}{\pi} \int_0^t \frac{1}{4\xi^2} e^{-|x|^2/(4\xi)} *_{[2]} g(x_2)\, d\xi = |\mathcal{W}_g(x,t)|, \quad x \in \mathbb{R}^2, \; t \in [0,T]. \end{aligned} \tag{3.11}$$

Here, we have again applied (3.6) to the last equality. Setting $\mathcal{V}_g(\cdot,t) = \mathscr{F}\mathcal{W}_g(\cdot,t)$, $t \in [0,T]$, we obtain

$$\mathcal{V}_g(\sigma,t) = -\sqrt{\frac{2}{\pi}} i\sigma_1 \int_0^t e^{-\xi|\sigma|^2} G(\sigma_2)\, d\xi, \quad \sigma \in \mathbb{R}^2, \; t \in [0,T].$$

Then, for $t \in [0,T]$, we have

$$(\|\mathcal{V}_g(\cdot,t)\|_0)^2 = \frac{2}{\pi} \int_0^t \int_0^t \int_{-\infty}^{\infty} \sigma_1^2 e^{-(\xi+\mu)\sigma_1^2}\, d\sigma_1 \int_{-\infty}^{\infty} e^{-(\xi+\mu)\sigma_2^2} |G(\sigma_2)|^2\, d\sigma_2\, d\xi\, d\mu.$$

Since

$$\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2}\, dy = -\frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\alpha y^2}\, dy = -\left(\sqrt{\frac{\pi}{\alpha}}\right)' = \frac{\sqrt{\pi}}{2\alpha^{3/2}}, \quad \alpha > 0,$$

we obtain

$$\begin{aligned} (\|\mathcal{V}_g(\cdot,t)\|_0)^2 &\le \frac{1}{\sqrt{\pi}} \left(\|G\|_{L^2(\mathbb{R})}\right)^2 \int_0^t \int_0^t \frac{1}{(\xi+\mu)^{3/2}}\, d\xi\, d\mu \\ &= \frac{2}{\sqrt{\pi}} \left(\|G\|_{L^2(\mathbb{R})}\right)^2 \int_0^t \left(\frac{1}{\sqrt{\mu}} - \frac{1}{\sqrt{\mu+t}} \right) d\mu \\ &= \frac{4\left(2-\sqrt{2}\right)}{\sqrt{\pi}} \sqrt{t} \left(\|G\|_{L^2(\mathbb{R})}\right)^2, \quad t \in [0,T]. \end{aligned}$$

Taking into account (3.10) and (3.11), we conclude that (iii) holds. □

It follows from Theorem 3.3(iii) that $\mathcal{W}_u(\cdot,T) \in \widetilde{H}^0\left(\mathbb{R}^2\right) \subset L^2\left(\mathbb{R}^2\right)$ if $U[0,T] \subset L^2\left(\mathbb{R}\times[0,T]\right) \cap L^\infty\left(\mathbb{R}\times[0,T]\right)$. The following example shows that the boundedness of a control u plays a significant role for $\mathcal{W}_u(\cdot,T)$ to belong to $L^2\left(\mathbb{R}^2\right)$.

Example 3.4. Consider

$$u(x_2,t) = \frac{1}{(T-t)^{5/8}}\Big(H\Big(x_2+2\sqrt{T-t}\Big)-H\Big(x_2-2\sqrt{T-t}\Big)\Big),\ x_2\in\mathbb{R},\ t\in(0,T],$$

where H is the Heaviside function ($H(\nu)=0$ if $\nu<0$ and $H(\nu)=1$ otherwise). Evidently, $u \notin L^\infty\left(\mathbb{R}\times[0,T]\right)$, but $u \in L^2\left(\mathbb{R}\times[0,T]\right)$ because

$$\left(\|u\|_{L^2(\mathbb{R}\times[0,T])}\right)^2 = \int_0^T \frac{1}{\xi^{5/4}}\int_{-2\sqrt{\xi}}^{2\sqrt{\xi}} dx_2\,d\xi = 4\int_0^T \xi^{-3/4}\,d\xi = 16T^{1/4} < \infty.$$

Let us show that $\mathcal{W}_u(\cdot,T) \notin L^2\left(\mathbb{R}^2\right)$. With regard to (3.6), we get

$$\mathcal{W}_u(x,T) = \frac{x_1}{\pi}\int_0^T \frac{e^{-|x|^2/(4\xi)}}{4\xi^{2+5/8}} *_{[2]} \Big(H\Big(x_2+2\sqrt{\xi}\Big)-H\Big(x_2-2\sqrt{\xi}\Big)\Big)\,d\xi,\quad x\in\mathbb{R}^2. \tag{3.12}$$

First, we calculate and estimate the convolution in the right-hand side of (3.12). We have

$$\begin{aligned} e^{-x_2^2/(4\xi)} *_{[2]} \Big(H\Big(x_2+2\sqrt{\xi}\Big)-H\Big(x_2-2\sqrt{\xi}\Big)\Big) &= \int_{x_2-2\sqrt{\xi}}^{x_2+2\sqrt{\xi}} e^{-\mu^2/(4\xi)}\,d\mu \\ &= 2\sqrt{\xi}\int_{x_2/(2\sqrt{\xi})-1}^{x_2/(2\sqrt{\xi})+1} e^{-y^2}\,dy,\quad x_2\in\mathbb{R},\xi\in(0,T], \end{aligned}$$

where $y=\mu/(2\sqrt{\xi})$. Since

$$\left|\frac{x_2}{2\sqrt{\xi}}\pm 1\right|^2 \le \left(\frac{|x_2|}{2\sqrt{\xi}}+1\right)^2 \le 2\left(\frac{x_2^2}{4\xi}+1\right) = \frac{x_2^2}{2\xi}+2,\quad x_2\in\mathbb{R},\xi\in(0,T],$$

we get

$$\left|e^{-x_2^2/(4\xi)} *_{[2]} \Big(H\Big(x_2+2\sqrt{\xi}\Big)-H\Big(x_2-2\sqrt{\xi}\Big)\Big)\right| \ge \frac{4\sqrt{\xi}}{e^2}e^{-x_2^2/(2\xi)},\quad x_2\in\mathbb{R},\xi\in(0,T].$$

Taking this into account, we obtain from (3.12) that

$$\left|\mathcal{W}_u(x,T)\right| \ge \frac{2|x_1|}{e^2\pi}\int_0^T \frac{e^{-(x_1^2+2x_2^2)/(4\xi)}}{\xi^{1/8}}\frac{d\xi}{2\xi^2} \ge \frac{2|x_1|}{e^2\pi}\int_0^T \frac{e^{-|x|^2/(2\xi)}}{\xi^{1/8}}\frac{d\xi}{2\xi^2},\quad x\in\mathbb{R}^2.$$

Replacing $|x|^2/(2\xi)$ by y in the integral, we get

$$\left|\mathcal{W}_u(x,T)\right| \ge \frac{2^{9/8}}{e^2\pi}\frac{|x_1|}{|x|^{9/4}}\int_{|x|^2/(2T)}^\infty e^{-y}y^{1/8}\,dy,\quad x\in\mathbb{R}^2. \tag{3.13}$$

Setting

$$F = \int_0^\infty e^{-y} y^{1/8}\, dy,$$

we conclude that there exists $\varepsilon > 0$ such that

$$\int_{|x|^2/(2T)}^\infty e^{-y} y^{1/8}\, dy \geq \frac{F}{2^{9/8}}, \quad |x| \leq \varepsilon.$$

Therefore (3.13) yields

$$\left|\mathcal{W}_u(x,T)\right| \geq \frac{F}{e^2\pi} \frac{|x_1|}{|x|^{9/4}}, \quad |x| \leq \varepsilon.$$

Hence,

$$\iint_{\mathbb{R}^2} \left|\mathcal{W}_u(x,T)\right|^2 dx \geq \frac{F^2}{e^4\pi^2} \iint_{|x|\leq\varepsilon} \frac{|x_1|^2}{|x|^{9/2}}\, dx = \infty,$$

i.e. $\mathcal{W}_u(\cdot,T) \notin L^2\left(\mathbb{R}^2\right)$.

This example demonstrates that the set of admissible controls $U[0,T]$ being considered in the present paper cannot be extended to $L^2\left(\mathbb{R}\times[0,T]\right)$ if we want each control from the set of admissible controls to generate the end state belonging to $L^2\left(\mathbb{R}^2\right)$.

According to (3.4), we have

$$\mathscr{R}_T(W^0) = \left\{W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right) \,\middle|\, \exists u \in U[0,T] \quad W^T = \mathcal{W}_0(\cdot,T) + \mathcal{W}_u(\cdot,T)\right\}, \tag{3.14}$$

in particular,

$$\mathscr{R}_T(0) = \left\{W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right) \,\middle|\, \exists u \in U[0,T] \quad W^T = \mathcal{W}_u(\cdot,T)\right\}. \tag{3.15}$$

Taking into account (3.4) and Theorem 3.3, we obtain the following theorem.

Theorem 3.5. *Let $T > 0$. We have $f \in \mathscr{R}_T(W_0)$ iff $(f - \mathcal{W}_0(\cdot,T)) \in \mathscr{R}_T(0)$. We also have $f \in \overline{\mathscr{R}_T(W_0)}$ iff $(f - \mathcal{W}_0(\cdot,T)) \in \overline{\mathscr{R}_T(0)}$.*

Lemma 3.6. *Let $u \in U[0,T]$ and $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$, $t \in [0,T]$. Let W be a solution to (3.1), (3.2). Then (3.3) holds.*

Proof. According to Theorem 3.3(ii), $\mathcal{W}_0(\cdot,t)$ is continuous on $\mathbb{R}^2$ for each $t \in (0,T]$. Moreover, $\mathcal{W}_0(\cdot,t)$ is odd with respect to x_1, $t \in [0,T]$. Hence,

$$\mathcal{W}_0(0^+,(\cdot)_{[2]},t) = 0, \quad t \in (0,T]. \tag{3.16}$$

Let us calculate $\mathcal{W}_u(0^+,(\cdot)_{[2]},t)$, $t \in [0,T]$. It follows from (3.6) that

$$\mathcal{W}_u(x,t) = \frac{x_1}{\pi} \int_{-\infty}^\infty \int_0^t \frac{1}{4\xi^2} e^{-(x_1^2+\mu_2^2)/(4\xi)} u(x_2-\mu_2,t-\xi)\, d\xi\, d\mu_2,\ x \in \mathbb{R}^2,\ t \in [0,T].$$

Changing variables in the integral: $\xi = \frac{x_1^2}{4\zeta^2}(1+\eta^2)$, $\mu_2 = |x_1|\eta$, we get

$$\mathcal{W}_u(x,t) = \frac{\operatorname{sgn} x_1}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} F(x,t,\zeta,\eta)\, d\zeta\, d\eta, \tag{3.17}$$

where

$$F(x,t,\zeta,\eta) = H\left(\zeta - \frac{|x_1|}{2\sqrt{t}}\sqrt{1+\eta^2}\right) \frac{2\zeta}{1+\eta^2} e^{-\zeta^2} u\left(x_2 - |x_1|\eta, t - \frac{x_1^2}{4\zeta^2}\left(1+\eta^2\right)\right),$$
$$x \in \mathbb{R}^2,\ t \in (0,T],\ \zeta \in \mathbb{R}_+,\ \eta \in \mathbb{R}.$$

With regard to (1.4), we obtain

$$|F(x,t,\zeta,\eta)| \le \frac{2\zeta}{1+\eta^2} e^{-\zeta^2} \|u\|_{L^\infty(\mathbb{R}\times(0,T))}$$
$$\text{for a.a. } x \in \mathbb{R}^2,\ t \in (0,T],\ \zeta \in \mathbb{R}_+,\ \eta \in \mathbb{R}. \tag{3.18}$$

On the other hand, for a.a. $x_2 \in \mathbb{R}$, $t \in (0,T]$, $\zeta \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, we have

$$F(x,t,\zeta,\eta) \to \frac{2\zeta}{1+\eta^2} e^{-\zeta^2} u(x_2,t) \quad \text{as } x_1 \to 0^+. \tag{3.19}$$

Taking into account (3.18) and (3.19), passing to the limit as $x_1 \to 0^+$, and applying Lebesgue's dominated convergence theorem to the right-hand side of (3.17), we obtain

$$\begin{aligned}\mathcal{W}_u\left(0^+, (\cdot)_{[2]}, t\right) &= \frac{1}{\pi} u(x_2,t) \int_{-\infty}^{\infty} \frac{d\eta}{1+\eta^2} \int_0^{\infty} 2\zeta e^{-\zeta^2}\, d\zeta \\ &= u((\cdot)_{[2]}, t), \quad \text{for a.a. } x_2 \in \mathbb{R},\ t \in [0,T]. \end{aligned} \tag{3.20}$$

Taking into account (3.4), (3.16), and (3.20), we obtain (3.3). □

4. Approximate controllability

In this section we prove the main theorem.

Theorem 4.1. *Each state $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ is approximately controllable to a state $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ in a given time $T > 0$, i.e. $\widetilde{H}^0\left(\mathbb{R}^2\right) = \overline{\mathscr{R}_T(W^0)}$.*

First, consider the Hermite polynomials [34, 18.5.5 (and Tables 18.3.1, 18.5.1), 18.5.13]

$$\mathcal{H}_n(\mu) = (-1)^n e^{\mu^2} \left(\frac{d}{d\mu}\right)^n e^{-\mu^2} = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{(n-2k)!k!} (2\mu)^{n-2k}, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $\lfloor\cdot\rfloor$ denotes the floor of a number (i.e. the integral part of a number).

Let $\psi_n(\mu) = \mathcal{H}_n(\mu) e^{-\mu^2/2}$, $\mu \in \mathbb{R}$, $n \in \mathbb{N}_0$. It is well known [34, 18.2.1, 18.2.5 (and Table 18.3.1)] that

$$\int_{-\infty}^{\infty} \psi_n(\mu)\psi_m(\mu)\, d\mu = \sqrt{\pi} 2^n n! \delta_{mn}, \quad n \in \mathbb{N}_0,\ m \in \mathbb{N}_0, \tag{4.2}$$

where δ_{mn} is the Kronecker delta, and $\{\psi_n\}_{n=0}^{\infty}$ is an orthogonal basis in $L^2(\mathbb{R})$. It is easy to see that

$$\widehat{\psi}_n = \mathscr{F}\psi_n = (-i)^n\psi_n, \qquad n \in \mathbb{N}_0. \tag{4.3}$$

Let $\alpha > 0$. Set

$$\psi_n^\alpha(\mu) = \frac{1}{\left(\sqrt{2\alpha\pi}2^n n!\right)^{1/2}}\psi_n\left(\frac{\mu}{\sqrt{2\alpha}}\right), \quad \mu \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{4.4}$$

$$\widehat{\psi}_n^\alpha(\lambda) = \left(\mathscr{F}\psi_n^\alpha\right)(\lambda), \quad \lambda \in \mathbb{R},\ n \in \mathbb{N}_0. \tag{4.5}$$

Then

$$\widehat{\psi}_n^\alpha(\lambda) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \frac{(-i)^n}{\left(2^n n!\right)^{1/2}}\psi_n(\sqrt{2\alpha}\lambda), \quad \lambda \in \mathbb{R},\ n \in \mathbb{N}_0. \tag{4.6}$$

According to (4.2), we get

$$\left\langle \psi_n^\alpha, \psi_m^\alpha \right\rangle = \left\langle \widehat{\psi}_n^\alpha, \widehat{\psi}_m^\alpha \right\rangle = \delta_{mn}, \quad n \in \mathbb{N}_0,\ m \in \mathbb{N}_0. \tag{4.7}$$

Thus, $\left\{\psi_n^\alpha\right\}_{n=0}^{\infty}$ and $\left\{\widehat{\psi}_n^\alpha\right\}_{n=0}^{\infty}$ are orthonormal bases in $H^0(\mathbb{R}) = H_0(\mathbb{R}) = L^2(\mathbb{R})$. In addition, $\left\{\psi_{2p+1}^\alpha\right\}_{p=0}^{\infty}$ and $\left\{\widehat{\psi}_{2p+1}^\alpha\right\}_{p=0}^{\infty}$ are orthonormal bases in $\widetilde{H}^0(\mathbb{R}) = \widetilde{H}_0(\mathbb{R})$. These bases were considered in [12].

Let $T^* > T$ be fixed. Set

$$\Theta_{nm}^{TT^*}\left((\cdot)_{[1]}, (\cdot)_{[2]}\right) = \psi_{2n+1}^T\left((\cdot)_{[1]}\right)\psi_m^{T*}\left((\cdot)_{[2]}\right), \quad n \in \mathbb{N}_0,\ m \in \mathbb{N}_0. \tag{4.8}$$

Since $\{\psi_p^\alpha\}_{p=0}^{\infty}$ is an orthonormal basis in $H^0(\mathbb{R}) = L^2(\mathbb{R})$ (for any $\alpha > 0$), we obtain that $\{\Theta_{nm}^{TT^*}\}_{n,m=0}^{\infty}$ is an orthonormal basis in $\widetilde{H}^0\left(\mathbb{R}^2\right) \subset L^2\left(\mathbb{R}^2\right)$.

Consider an auxiliary control system for the heat equation on $\mathbb{R}$:

$$y_t = y_{x_1x_1} - 2\upsilon\delta((\cdot)_{[1]})', \quad t \in [0, T], \tag{4.9}$$

$$y(\cdot, 0) = 0, \tag{4.10}$$

where $\left(\frac{d}{dt}\right)^s y : [0, T] \to \widetilde{H}^{-2s}(\mathbb{R})$, $s = 0, 1$, $\upsilon : [0, T] \to \mathbb{R}$ is a control, $\upsilon \in L^\infty(0, T)$.

Set $z(\cdot, t) = \mathscr{F}_{x_1\to\sigma_1}y(\cdot, t)$, $t \in [0, T]$. Then problem (4.9), (4.10) is equivalent to the problem

$$z_t = -\sigma_1^2 z - \sqrt{\frac{2}{\pi}}i\sigma_1\upsilon, \quad \sigma_1 \in \mathbb{R},\ t \in [0, T], \tag{4.11}$$

$$z(\cdot, 0) = 0, \tag{4.12}$$

where $\left(\frac{d}{dt}\right)^s z : [0, T] \to \widetilde{H}_{-2s}(\mathbb{R})$, $s = 0, 1$. Set

$$z(\sigma_1, t) = -\sqrt{\frac{2}{\pi}}i\sigma_1 \int_0^t e^{-(t-\xi)\sigma_1^2}\upsilon(\xi)\,d\xi, \quad \sigma_1 \in \mathbb{R},\ t \in [0, T]. \tag{4.13}$$

Then $z(\cdot,t) \in \widetilde{H}_0(\mathbb{R})$, and it is a unique solution to (4.11), (4.12). Let q be a polynomial. Set

$$f(\sigma_2,\xi) = e^{-(T^*-T+\xi)\sigma_2^2} q(\sigma_2), \quad \sigma_2 \in \mathbb{R},\ \xi \in [0,T], \tag{4.14}$$

$$\widehat{u}(\sigma_2,\xi) = \upsilon(\xi) f(\sigma_2,\xi) = \upsilon(\xi) e^{-(T^*-T+\xi)\sigma_2^2} q(\sigma_2), \quad \sigma_2 \in \mathbb{R},\ \xi \in [0,T], \tag{4.15}$$

$$V(\sigma,t) = -\sqrt{\frac{2}{\pi}} i\sigma_1 \int_0^t e^{-(t-\xi)|\sigma|^2} \widehat{u}(\sigma_2,\xi)\, d\xi, \quad \sigma \in \mathbb{R}^2,\ t \in [0,T]. \tag{4.16}$$

With regard to (4.13)–(4.16), we have

$$\begin{aligned} V(\sigma,t) &= \left(-\sqrt{\frac{2}{\pi}} i\sigma_1 \int_0^t e^{-(t-\xi)\sigma_1^2} \upsilon(\xi)\, d\xi \right) e^{-(T^*-T+t)\sigma_2^2} q(\sigma_2) \\ &= z(\sigma_1,t) f(\sigma_2,t), \quad \sigma \in \mathbb{R}^2,\ t \in [0,T]. \end{aligned} \tag{4.17}$$

Since $z(\cdot,t) \in \widetilde{H}_0\big(\mathbb{R}\big)$ and $f(\cdot,t) \in L^2(\mathbb{R})$, we have $V(\cdot,t) \in \widetilde{H}_0\left(\mathbb{R}^2\right)$, $t \in [0,T]$. Moreover,

$$V(\sigma,T) = z(\sigma_1,T) f(\sigma_2,T) = z(\sigma_1,T) e^{-T^*\sigma_2^2} q(\sigma_2), \quad \sigma \in \mathbb{R}^2. \tag{4.18}$$

It follows from (3.6) and (4.16) that

$$\mathscr{F}^{-1}_{\sigma \to x} V(\cdot,t) = \mathcal{W}_u(\cdot,t), \quad t \in [0,T], \tag{4.19}$$

where $u(\cdot,t) = \mathscr{F}^{-1}_{\sigma_2 \to x_2} \widehat{u}(\cdot,t)$ and $u \in U[0,T]$.

Theorem 4.2. *Let $T^* > T > 0$, $n \in \mathbb{N}_0$, and $m \in \mathbb{N}_0$. Then $\Theta_{nm}^{TT^*} \in \overline{\mathscr{R}_T(0)}$.*

Proof. It follows from [12, Theorem 7.2] that there exists a sequence $\{\upsilon_l^n\}_{l=1}^\infty \subset L^\infty(0,T)$ such that the solutions y_l^n to (4.9), (4.10) with $\upsilon = \upsilon_l^n$ satisfy the condition

$$\left\| y_l^n(\cdot,T) - \psi_{2n+1}^T \right\|^0 \to 0 \quad \text{as } l \to \infty. \tag{4.20}$$

Setting $z_l^n(\cdot,t) = \mathscr{F}_{x_1 \to \sigma_1} y_l^n(\cdot,t)$, $t \in [0,T]$, we conclude that $z_l^n(\cdot,t) \in \widetilde{H}_0(\mathbb{R})$, $t \in [0,T]$, and it is the unique solution to (4.11), (4.12) with $\upsilon = \upsilon_l^n$, $l \in \mathbb{N}$. Taking into account (4.5) and (4.20), we get

$$\left\| z_l^n(\cdot,T) - \widehat{\psi}_{2n+1}^T \right\|_0 \to 0 \quad \text{as } l \to \infty. \tag{4.21}$$

Set

$$f_m(\sigma_2,\xi) = e^{(T-\xi)\sigma_2^2} \widehat{\psi}_m^{T^*}(\sigma_2), \quad \sigma_2 \in \mathbb{R},\ \xi \in [0,T], \tag{4.22}$$

$$\widehat{u}_{nm}^l(\sigma_2,\xi) = \upsilon_l^n(\xi) f_m(\sigma_2,\xi), \quad \sigma_2 \in \mathbb{R},\ \xi \in [0,T],\ l \in \mathbb{N}, \tag{4.23}$$

$$V_{nm}^l(\sigma,t) = -\sqrt{\frac{2}{\pi}} i\sigma_1 \int_0^t e^{-(t-\xi)|\sigma|^2} \widehat{u}_{nm}^l(\sigma_2,\xi)\, d\xi,\ \sigma \in \mathbb{R}^2,\ t \in [0,T],\ l \in \mathbb{N}. \tag{4.24}$$

Note that $f_m(\cdot,\xi)$ is a polynomial multiplied by $e^{-(T^*-T+\xi)(\cdot)^2}$, $\xi\in[0,T]$. With regard to (4.15)–(4.17), we have $V^l_{nm}(\cdot,t)\in\widetilde{H}_0(\mathbb{R}^2)$ and

$$V^l_{nm}(\sigma,t)=z^n_l(\sigma_1,t)f_m(\sigma_2,t),\quad \sigma\in\mathbb{R}^2,\ t\in[0,T],\ l\in\mathbb{N}. \tag{4.25}$$

According to (4.7) and (4.21), we get

$$\begin{aligned}\left\|\widehat{\Theta}^{TT*}_{nm}-V^l_{nm}(\cdot,T)\right\|_0&=\left\|z^n_l(\cdot,T)-\widehat{\psi}^T_{2n+1}\right\|_0\left\|\widehat{\psi}^{T^*}_m\right\|_0\\&=\left\|z^n_l(\cdot,T)-\widehat{\psi}^T_{2n+1}\right\|_0\to 0\quad\text{as } l\to\infty,\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}\widehat{\Theta}^{TT^*}_{nm}=\mathscr{F}\Theta^{TT^*}_{nm}&=\left(\mathscr{F}_{x_1\to\sigma_1}\psi^T_{2n+1}\right)((\cdot)_{[1]})\left(\mathscr{F}_{x_2\to\sigma_2}\psi^{T^*}_m\right)((\cdot)_{[2]})\\&=\widehat{\psi}^T_{2n+1}((\cdot)_{[1]})\widehat{\psi}^{T^*}_m((\cdot)_{[2]}).\end{aligned} \tag{4.27}$$

Here (4.5) and (4.8) are taken into account. Due to (4.19), we have $\mathscr{F}^{-1}_{\sigma\to x}V^l_{nm}(\cdot,t)=\mathcal{W}_{u^l_{nm}}(\cdot,t)$, $t\in[0,T]$, where $u^l_{nm}(\cdot,t)=\mathscr{F}^{-1}_{\sigma_2\to x_2}\widehat{u}^l_{nm}(\cdot,t)$, $l\in\mathbb{N}$. Therefore, (4.26) yields

$$\left\|\Theta^{TT*}_{nm}-\mathcal{W}_{u^l_{nm}}(\cdot,T)\right\|^0\to 0\quad\text{as } l\to\infty. \tag{4.28}$$

Finally, let us calculate u^l_{nm} to ensure $u^l_{nm}\in U[0,T]$, $l\in\mathbb{N}$. Let $l\in\mathbb{N}$ be fixed. Set $\Delta T=T^*-T$. We have

$$\begin{aligned}&\mathscr{F}^{-1}_{\sigma_2\to x_2}\left(e^{(T-\xi)(\cdot)^2}\widehat{\psi}^{T^*}_m\right)(x_2)\\&=\left(\frac{2T^*}{\pi}\right)^{1/4}\frac{(-i)^m}{(2^m m!)^{1/2}}\mathscr{F}^{-1}_{\sigma_2\to x_2}\left(\mathcal{H}_m\left(\sqrt{2T^*}(\cdot)\right)e^{-(\Delta T+\xi)(\cdot)^2}\right)(x_2)\\&=\left(\frac{2T^*}{\pi}\right)^{1/4}\frac{(-i)^m}{(2^m m!)^{1/2}}\mathcal{H}_m\left(-i\sqrt{2T^*}\frac{\partial}{\partial x_2}\right)\frac{e^{-x_2^2/(4(\Delta T+\xi))}}{\sqrt{2(\Delta T+\xi)}}\\&=\left(\frac{2T^*}{\pi}\right)^{1/4}\frac{(-1)^m m!}{(2^m m!)^{1/2}}\frac{1}{\sqrt{2(\Delta T+\xi)}}\\&\quad\times\sum_{k=0}^{\lfloor m/2\rfloor}\frac{(2\sqrt{2T^*})^{m-2k}}{(m-2k)!k!}\left(\frac{\partial}{\partial x_2}\right)^{m-2k}e^{-x_2^2/(4(\Delta T+\xi))},\\&\qquad\qquad x_2\in\mathbb{R},\ \xi\in[0,T].\end{aligned} \tag{4.29}$$

From (4.1), it follows that

$$\left(\frac{d}{d\mu}\right)^p e^{-\mu^2}=(-1)^p\mathcal{H}_p(\mu)e^{-\mu^2},\quad \mu\in\mathbb{R},\ p\in\mathbb{N}_0. \tag{4.30}$$

Therefore,

$$\left(\frac{\partial}{\partial x_2}\right)^p e^{-x_2^2/(4(\Delta T+\xi))}=\frac{(-1)^p}{(2\sqrt{\Delta T+\xi})^p}\mathcal{H}_p\left(\frac{x_2}{2\sqrt{\Delta T+\xi}}\right)e^{-x_2^2/(4(\Delta T+\xi))},$$

$$x_2 \in \mathbb{R},\ \xi \in [0,T],\ p \in \mathbb{N}_0. \quad (4.31)$$

With regard to (4.29), we obtain

$$\begin{aligned}
&\mathscr{F}^{-1}_{\sigma_2\to x_2}\left(e^{(T-\xi)(\cdot)^2}\widehat{\psi}^{T^*}_m\right)(x_2)\\
&\quad= \left(\frac{2T^*}{\pi}\right)^{1/4}\left(\frac{m!}{2^m}\right)^{1/2}\frac{1}{\sqrt{2(\Delta T+\xi)}}\\
&\qquad\times\sum_{k=0}^{\lfloor m/2\rfloor}\frac{e^{-x_2^2/(4(\Delta T+\xi))}}{(m-2k)!k!}\mathcal{H}_{m-2k}\left(\frac{x_2}{2\sqrt{\Delta T+\xi}}\right)\left(\frac{2T^*}{\Delta T+\xi}\right)^{m/2-k}\\
&\quad= \left(\frac{2T^*}{\pi}\right)^{1/4}\frac{1}{(2^m m!)^{1/2}}\frac{e^{-x_2^2/(4(\Delta T+\xi))}}{\sqrt{2(\Delta T+\xi)}}\left(\frac{T^*+T-\xi}{\Delta T+\xi}\right)^{m/2}\left(\frac{2T^*}{T^*+T-\xi}\right)^{m/2}\\
&\qquad\times\sum_{k=0}^{\lfloor m/2\rfloor}\left(1-\frac{T^*+T-\xi}{2T^*}\right)^k\frac{m!}{k!(m-2k)!}\mathcal{H}_{m-2k}\left(\frac{x_2}{2\sqrt{\Delta T+\xi}}\right),\\
&\qquad\qquad x_2\in\mathbb{R},\ \xi\in[0,T].
\end{aligned}$$

Taking into account the multiplication theorem for Hermite polynomials (see, e.g. [34, 18.18.13]):

$$\mathcal{H}_m(\lambda\mu) = \lambda^m\sum_{k=0}^{\lfloor m/2\rfloor}\frac{m!}{k!(m-2k)!}\left(1-\frac{1}{\lambda^2}\right)^k\mathcal{H}_{m-2k}(\mu),\quad \mu\in\mathbb{R},\ \lambda\neq 0,$$

and setting $\lambda = \sqrt{\frac{2T^*}{T^*+T-\xi}}$, $\mu = \frac{x_2}{2\sqrt{\Delta T+\xi}}$, we get

$$\begin{aligned}
&\mathscr{F}^{-1}_{\sigma_2\to x_2}\left(e^{(T-\xi)(\cdot)^2}\widehat{\psi}^{T^*}_m\right)(x_2)\\
&\qquad= \left(\frac{2T^*}{\pi}\right)^{1/4}\frac{1}{(2^m m!)^{1/2}}\frac{e^{-x_2^2/(4(\Delta T+\xi))}}{\sqrt{2(\Delta T+\xi)}}\left(\frac{T^*+T-\xi}{\Delta T+\xi}\right)^{m/2}\\
&\qquad\quad\times\mathcal{H}_m\left(x_2\sqrt{\frac{T^*}{2(\Delta T+\xi)(T^*+T-\xi)}}\right),\quad x_2\in\mathbb{R},\ \xi\in[0,T]. \quad (4.32)
\end{aligned}$$

Due to (4.23), we have

$$\begin{aligned}
u^l_{nm}(x_2,\xi) &= \mathscr{F}^{-1}_{\sigma_2\to x_2}\left(e^{(T-\xi)(\cdot)^2}\widehat{\psi}^{T^*}_m\right)(x_2)v^n_l(\xi)\\
&= \left(\frac{2T^*}{\pi}\right)^{1/4}\frac{v^n_l(\xi)}{(2^m m!)^{1/2}}\frac{e^{-x_2^2/(4(\Delta T+\xi))}}{\sqrt{2(\Delta T+\xi)}}\left(\frac{T^*+T-\xi}{\Delta T+\xi}\right)^{m/2}\\
&\quad\times\mathcal{H}_m\left(x_2\sqrt{\frac{T^*}{2(\Delta T+\xi)(T^*+T-\xi)}}\right),\ x_2\in\mathbb{R},\ \xi\in[0,T]. \quad (4.33)
\end{aligned}$$

Evidently, $u^l_{nm}\in U[0,T]$, $l\in\mathbb{N}$. The theorem is proved. □

Proof of Theorem 4.1. With regard to Definition 3.2 and Theorems 3.3, 3.5, it is sufficient to show that $\widetilde{H}^0\left(\mathbb{R}^2\right) = \overline{\mathscr{R}_T(0)}$ to prove the theorem. Set $T^* > T$. Since $\{\Theta_{nm}^{TT^*}\}_{n,m=0}^{\infty}$ is an orthonormal basis in $\widetilde{H}^0\left(\mathbb{R}^2\right)$, each $f \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ can be approximated in this space by the sums

$$\sum_{n=0}^{N}\sum_{m=0}^{M} f_{nm}\Theta_{nm}^{TT^*},$$

where $N \in \mathbb{N}$, $M \in \mathbb{N}$. With regard to Theorem 4.2, we conclude that $f \in \overline{\mathscr{R}_T(0)}$ that was to be proved. □

5. Numerical solution to the approximate controllability problem

In this section, we construct controls approximately steering to a state $W^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ from a state $W^0 \in \widetilde{H}^0\left(\mathbb{R}^2\right)$ by using the proof of Theorem 4.2.

Set $W_0^T = W^T - \mathcal{W}_0(\cdot,T)$, $V_0^T = \mathscr{F}W_0^T$. According to Theorem 3.3, $W_0^T \in \widetilde{H}^0\left(\mathbb{R}^2\right)$, hence $V_0^T \in \widetilde{H}_0\left(\mathbb{R}^2\right)$. Due to Theorem 3.5, we have to construct controls $\{u_k\}_{k=1}^{\infty} \subset U[0,T]$ such that

$$\left\|W_0^T - \mathcal{W}_{u_k}(\cdot,T)\right\|^0 \to 0 \quad \text{as } k \to \infty. \tag{5.1}$$

Set $T^* > T$. Since $\left\{\Theta_{nm}^{TT^*}\right\}_{n,m=0}^{\infty}$ is an orthonormal basis in $\widetilde{H}^0\left(\mathbb{R}^2\right)$, we have

$$\left\|W_0^T - W_{NM}\right\|^0 \to 0 \quad \text{as } (N,M) \to \infty \tag{5.2}$$

for

$$W_{NM} = \sum_{n=0}^{N}\sum_{m=0}^{M} W_{nm}\Theta_{nm}^{TT^*}, \quad N \in \mathbb{N},\ M \in \mathbb{N}, \tag{5.3}$$

where $W_{nm} = \left\langle W_0^T, \Theta_{nm}^{TT^*}\right\rangle$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$.

Let an arbitrary $\varepsilon > 0$ be fixed. Then there exists $N \in \mathbb{N}$ and $M \in \mathbb{N}$ such that

$$\left\|W_0^T - W_{NM}\right\|^0 < \varepsilon/2. \tag{5.4}$$

Let us construct controls $\{u_k\}_{k=1}^{\infty} \subset U[0,T]$ for which

$$\|W_{NM} - \mathcal{W}_{u_k}(\cdot,T)\|^0 \to 0 \quad \text{as } k \to \infty. \tag{5.5}$$

To this aid, we use the method proposed in [12] to construct controls for the heat equation on a half-axis. With regard to (4.27) and (5.3), we have

$$\begin{aligned} W_{NM} &= \sum_{n=0}^{N}\sum_{m=0}^{M} W_{nm}\psi_{2n+1}^{T}\big((\cdot)_{[1]}\big)\psi_m^{T^*}\big((\cdot)_{[2]}\big) \\ &= \sum_{m=0}^{M} \omega_m^N\big((\cdot)_{[1]}\big)\psi_m^{T^*}\big((\cdot)_{[2]}\big), \quad N \in \mathbb{N},\ M \in \mathbb{N}, \end{aligned} \tag{5.6}$$

where

$$\omega_m^N = \sum_{n=0}^{N} W_{nm} \psi_{2n+1}^T, \quad m = \overline{0, M}. \tag{5.7}$$

Set

$$\varphi_{2p+1}(\lambda) = i\lambda^{2p+1} e^{-T\lambda^2}, \quad \lambda \in \mathbb{R},\ p \in \mathbb{N}_0, \tag{5.8}$$

$$\varphi_{2p+1}^l(\lambda) = i\lambda^{2p+1} e^{-T\lambda^2} \left(\frac{e^{\lambda^2/l} - 1}{\lambda^2/l} \right)^{p+1}, \quad \lambda \in \mathbb{R},\ p \in \mathbb{N}_0,\ l \in \mathbb{N},$$

$$v_l^p(\xi) = \begin{cases} (-1)^{p-j} \binom{p}{j} l^{p+1}, & \xi \in \left(\frac{j}{l}, \frac{j+1}{l} \right),\ j = \overline{0, p} \\ 0, & \xi \notin \left(0, \frac{p+1}{l} \right) \end{cases}, \quad p \in \mathbb{N}_0,\ l \in \mathbb{N}. \tag{5.9}$$

Note that $v_l^p \to (-1)^p \delta^{(p)}$ as $l \to \infty$ in $H^{-1}(\mathbb{R})$ for each $p = \overline{0, \infty}$. Due to [12], we get

$$\forall p \in \mathbb{N}_0 \quad \left\| \varphi_{2p+1} - \varphi_{2p+1}^l \right\|_0 \to 0 \quad \text{as } l \to \infty. \tag{5.10}$$

Let $p \in \mathbb{N}_0$, $l \in \mathbb{N}$, and let $\mathbf{y}_l^p$ be the solution to (4.9), (4.10) with $v = v_l^p$. Set $\mathbf{z}_l^p(\cdot, t) = \mathscr{F} \mathbf{y}_l^p(\cdot, t)$, $t \in [0, T]$. Then $\mathbf{z}_l^p$ is the solution to (4.11), (4.12), and (4.13) holds. It is easy to see that

$$\mathbf{z}_l^p(\sigma_1, T) = -\sqrt{\frac{2}{\pi}} \varphi_{2p+1}^l(\sigma_1), \quad \sigma_1 \in \mathbb{R}. \tag{5.11}$$

Taking into account (5.10), we get

$$\left\| \mathbf{z}_l^p(\cdot, T) + \sqrt{\frac{2}{\pi}} \varphi_{2p+1} \right\|_0 \to 0 \quad \text{as } l \to \infty.$$

Set $V_{NM} = \mathscr{F}_{x \to \sigma} W_{NM}$. It follows from (5.6) and (5.7) that

$$V_{NM} = \sum_{m=0}^{M} \widehat{\omega}_m^N \big((\cdot)_{[1]} \big) \widehat{\psi}_m^{T*} \big((\cdot)_{[2]} \big), \quad N \in \mathbb{N},\ M \in \mathbb{N}, \tag{5.12}$$

where

$$\widehat{\omega}_m^N = \sum_{n=0}^{N} W_{nm} \widehat{\psi}_{2n+1}^T, \quad m = \overline{0, M}. \tag{5.13}$$

Using (4.1), (4.5), (4.6), and (5.8), we obtain

$$\begin{aligned} \widehat{\omega}_m^N(\sigma_1) &= \sum_{n=0}^{N} W_{nm} (-i)^{2n+1} \left(\frac{2T}{\pi} \right)^{1/4} \frac{e^{-T\sigma_1^2} \mathcal{H}_{2n+1}(\sqrt{2T}\sigma_1)}{\sqrt{2^{2n+1}(2n+1)!}} \\ &= -i \left(\frac{2T}{\pi} \right)^{1/4} e^{-T\sigma_1^2} \end{aligned}$$

$$\times \sum_{n=0}^{N} W_{nm}(-1)^n \sqrt{\frac{(2n+1)!}{2^{2n+1}}} \sum_{j=0}^{n} \frac{(-1)^j (2\sqrt{2T}\sigma_1)^{2n+1-2j}}{j!(2n+1-2j)!}$$

$$= -\left(\frac{2T}{\pi}\right)^{1/4} \sum_{n=0}^{N} W_{nm} \sqrt{\frac{(2n+1)!}{2^{2n+1}}} \sum_{p=0}^{n} \frac{(-1)^p (2\sqrt{2T})^{2p+1}}{(n-p)!(2p+1)!} i\sigma_1^{2p+1} e^{-T\sigma_1^2}$$

$$= \sum_{p=0}^{N} \varphi_{2p+1}(\sigma_1) \sum_{n=p}^{N} W_{nm} h_p^n = \sum_{p=0}^{N} g_{pm}^N \varphi_{2p+1}(\sigma_1), \quad \sigma_1 \in \mathbb{R},\ m = \overline{0, M},$$

where

$$g_{pm}^N = \sum_{n=p}^{N} W_{nm} h_p^n, \qquad p = \overline{0, N},\ m = \overline{0, M}, \tag{5.14}$$

$$h_p^n = \left(\frac{2T}{\pi}\right)^{1/4} \frac{(-1)^{p+1}(2\sqrt{2T})^{2p+1}}{(n-p)!(2p+1)!} \sqrt{\frac{(2n+1)!}{2^{2n+1}}}, \quad p = \overline{0, N},\ n = \overline{p, N}. \tag{5.15}$$

Thus,

$$V_{NM} = \sum_{m=0}^{M} \sum_{p=0}^{N} g_{pm}^N \varphi_{2p+1}\big((\cdot)_{[1]}\big) \widehat{\psi}_m^{T^*}\big((\cdot)_{[2]}\big). \tag{5.16}$$

Set

$$V_{NM}^l = \sum_{m=0}^{M} \sum_{p=0}^{N} g_{pm}^N \varphi_{2p+1}^l\big((\cdot)_{[1]}\big) \widehat{\psi}_m^{T^*}\big((\cdot)_{[2]}\big), \quad l \in \mathbb{N}. \tag{5.17}$$

Taking into account (5.10), we see that there exists $l \in \mathbb{N}$ such that

$$\left\| V_{NM} - V_{NM}^l \right\|_0 \le \sum_{m=0}^{M} \sum_{p=0}^{N} \left| g_{pm}^N \right| \left\| \varphi_{2p+1} - \varphi_{2p+1}^l \right\|_0 \left\| \widehat{\psi}_m^{T^*} \right\|_0 < \varepsilon/2. \tag{5.18}$$

Using (5.11), we get

$$V_{NM}^l(\sigma) = -\sqrt{\frac{\pi}{2}} \sum_{m=0}^{M} \sum_{p=0}^{N} g_{pm}^N \mathbf{z}_l^p(\sigma_1, T) \widehat{\psi}_m^{T^*}(\sigma_2), \quad \sigma \in \mathbb{R}^2. \tag{5.19}$$

Set

$$\widehat{u}_{NM}^l(\cdot, \xi) = -\sqrt{\frac{\pi}{2}} \sum_{m=0}^{M} \sum_{p=0}^{N} g_{pm}^N v_l^p(\xi) f_m(\cdot, \xi), \quad \xi \in [0, T], \tag{5.20}$$

$$u_{NM}^l(\cdot, \xi) = \mathscr{F}_{\sigma_2 \to x_2}^{-1} \widehat{u}_{NM}^l(\cdot, \xi), \quad \xi \in [0, T], \tag{5.21}$$

where f_m is defined by (4.22). With regard to (4.14), (4.15), (4.18), (4.19), and (4.22), we conclude that

$$\mathscr{F}^{-1} V_{NM}^l = \mathcal{W}_{u_{NM}^l}(\cdot, T). \tag{5.22}$$

Taking into account (5.18), we have

$$\left\|W_{NM}-\mathcal{W}_{u^l_{NM}}(\cdot,T)\right\|^0=\left\|V_{NM}-V^l_{NM}(\cdot,T)\right\|_0<\varepsilon/2. \tag{5.23}$$

With regard to (4.32) and (5.21), we obtain

$$\begin{aligned} &u^l_{NM}(x_2,\xi)\\ &\quad=-\left(\frac{\pi T^*}{2}\right)^{1/4}\sum_{m=0}^{M}\sum_{p=0}^{N}g^N_{pm}\frac{v^p_l(\xi)}{(2^m m!)^{1/2}}\frac{e^{-x_2^2/(4(\Delta T+\xi))}}{\sqrt{2(\Delta T+\xi)}}\left(\frac{T^*+T-\xi}{\Delta T+\xi}\right)^{m/2}\\ &\quad\quad\times\mathcal{H}_m\left(x_2\sqrt{\frac{T^*}{2(\Delta T+\xi)(T^*+T-\xi)}}\right),\quad x_2\in\mathbb{R},\ \xi\in[0,T], \end{aligned} \tag{5.24}$$

where $\Delta T=T^*-T$, g^N_{pm} is defined by (5.14), v^p_l is defined by (5.9), $p=\overline{0,N}$, $m=\overline{0,M}$, $l\in\mathbb{N}$, $N\in\mathbb{N}$, $M\in\mathbb{N}$. Due to (5.4) and (5.23), we conclude that

$$\left\|W^T-\left(\mathcal{W}_0(\cdot,T)+\mathcal{W}_{u^l_{NM}}(\cdot,T)\right)\right\|^0=\left\|W^T_0-\mathcal{W}_{u^l_{NM}}(\cdot,T)\right\|^0<\varepsilon, \tag{5.25}$$

therefore the controls u^l_{NM}, $N\in\mathbb{N}$, $M\in\mathbb{N}$, $l\in\mathbb{N}$, solves the approximate controllability problem for system (3.1), (3.2).

Note that we have found the approximate end state in the form

$$\mathcal{W}_0(\cdot,T)+\mathcal{W}_{u^l_{NM}}(\cdot,T),\quad l\in\mathbb{N},\ N\in\mathbb{N},\ M\in\mathbb{N}, \tag{5.26}$$

and the control u^l_{NM} solving the approximate controllability problem for considered system in the form (5.24).

6. Example

Example 6.1. Let $T=2$, $x_1>0$, $x_2\in\mathbb{R}$,

$$w^0(x)=-\frac{9}{8}e^{-5/8}\left(\frac{1}{8\pi T^3}\right)^{1/4}x_1e^{-|x|^2/(8T)},$$

$$w^T(x)=-e^{-5/8}\left(\frac{1}{8\pi T^3}\right)^{1/4}\left(e^{-|x|^2/(4T)}e^{-x_2/\sqrt{T}}\sinh\left(\frac{x_1}{2\sqrt{T}}\right)+\frac{x_1}{2}e^{-|x|^2/(12T)}\right).$$

Let W^0 and W^T be the odd extensions of w^0 and w^T with respect to x_1. Consider system (3.1), (3.2). We can see that $W^0\in\widetilde{H}^0\left(\mathbb{R}^2\right)$ and $W^T\in\widetilde{H}^0\left(\mathbb{R}^2\right)$. Due to Theorem 4.1, we conclude that the state W^0 is approximately controllable to the state W^T in the time $T=2$. Let us find controls u^l_{NM} approximately steering the state W^0 to the state W^T and the approximate end states associated with these controls.

Set $W^T_0=W^T-\mathcal{W}_0(\cdot,T)$ (see (3.5)), $V^T_0=\mathscr{F}_{x\to\sigma}W^T_0$. One can easily obtain

$$\mathcal{W}_0(x,T)=-e^{-5/8}\left(\frac{1}{8\pi T^3}\right)^{1/4}\frac{x_1}{2}e^{-|x|^2/(12T)},\qquad x\in\mathbb{R}^2, \tag{6.1}$$

$$W_0^T(x) = -e^{-5/8}\left(\frac{1}{8\pi T^3}\right)^{1/4} e^{-|x|^2/(4T)} e^{-x_2/\sqrt{T}} \sinh\left(\frac{x_1}{2\sqrt{T}}\right), \quad x \in \mathbb{R}^2.$$

Consider any $T^* > T$. Let $N \in \mathbb{N}$ and $M \in \mathbb{N}$ be fixed, and let W_{NM} be determined by (5.3), where $W_{nm} = \langle W_0^T, \Theta_{nm}^{TT^*}\rangle$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$. Let us find the coefficients. We have

$$\begin{aligned} W_{nm} = -e^{-5/8}\left(\frac{1}{8\pi T^3}\right)^{1/4} \int_{-\infty}^{\infty} e^{-x_1^2/(4T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) \psi_{2n+1}^T(x_1)\, dx_1 \\ \times \int_{-\infty}^{\infty} e^{-x_2^2/(4T)} e^{-x_2/\sqrt{T}} \psi_m^{T^*}(x_2)\, dx_2, \quad n \in \mathbb{N}_0,\ m \in \mathbb{N}_0. \end{aligned} \tag{6.2}$$

Using (4.1) and (4.4), we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-x_1^2/(4T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) \psi_{2n+1}^T(x_1)\, dx_1 \\ &= \left(\frac{1}{2\pi T}\right)^{1/4} \sqrt{\frac{(2n+1)!}{2^{2n+1}}} \int_{-\infty}^{\infty} e^{-x_1^2/(2T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) \\ &\quad \times \sum_{k=0}^{n} \frac{(-1)^k}{(2n-2k+1)!k!} \left(\sqrt{\frac{2}{T}} x_1\right)^{2n-2k+1} dx_1 \\ &= \left(\frac{1}{2\pi T}\right)^{1/4} \sqrt{\frac{(2n+1)!}{2^{2n+1}}} \sqrt{\frac{2}{T}} \sum_{p=0}^{n} \frac{(-1)^{n-p}}{(2p+1)!(n-p)!} \left(\frac{2}{T}\right)^p \\ &\quad \times \int_{-\infty}^{\infty} x_1^{2p+1} e^{-x_1^2/(2T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) dx_1, \quad n \in \mathbb{N}_0. \end{aligned} \tag{6.3}$$

Taking into account the integral representation ([34, 18.10.10]):

$$\mathcal{H}_j(x) = \frac{(-2i)^j e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^j e^{2ixt}\, dt, \quad x \in \mathbb{R},\ j \in \mathbb{N}_0, \tag{6.4}$$

for $j = 2p+1$, we get

$$\begin{aligned} &\int_{-\infty}^{\infty} x_1^{2p+1} e^{-x_1^2/(2T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) dx_1 \\ &\qquad = i\sqrt{\pi} e^{1/8} \frac{(-1)^{p+1} T^{p+1}}{2^p} \mathcal{H}_{2p+1}\left(\frac{i}{2\sqrt{2}}\right), \quad p = \overline{0,n},\ n \in \mathbb{N}_0. \end{aligned}$$

Continuing (6.3), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-x_1^2/(4T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) \psi_{2n+1}^T(x_1)\, dx_1 \\ &\qquad = \left(\frac{1}{2\pi T}\right)^{1/4} \sqrt{2\pi T} i e^{1/8} (-1)^{n+1} \sqrt{\frac{(2n+1)!}{2^{2n+1}}} \\ &\qquad\quad \times \sum_{p=0}^{n} \frac{1}{(2p+1)!(n-p)!} \mathcal{H}_{2p+1}\left(\frac{i}{2\sqrt{2}}\right) \end{aligned}$$

$$= (2\pi T)^{1/4} i e^{1/8} (-1)^{n+1} \sqrt{\frac{(2n+1)!}{2^{2n+1}}}$$
$$\times \sum_{k=0}^{n} \frac{1}{k!(2n+1-2k)!} \mathcal{H}_{2n+1-2k}\left(\frac{i}{2\sqrt{2}}\right), \quad n \in \mathbb{N}_0.$$

Using the connection formula ([34, 18.18.20])

$$(2x)^n = \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{n!}{l!(n-2l)!} \mathcal{H}_{n-2l}(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0,$$

we get

$$\int_{-\infty}^{\infty} e^{-x_1^2/(4T)} \sinh\left(\frac{x_1}{2\sqrt{T}}\right) \psi_{2n+1}^T(x_1)\, dx_1$$
$$= (2\pi T)^{1/4} i e^{1/8} (-1)^{n+1} \sqrt{\frac{(2n+1)!}{2^{2n+1}}} \frac{2^{2n+1}}{(2n+1)!} \left(\frac{i}{2\sqrt{2}}\right)^{2n+1}$$
$$= \frac{(2\pi T)^{1/4} e^{1/8}}{2^{2n+1}\sqrt{(2n+1)!}}, \quad n \in \mathbb{N}_0. \tag{6.5}$$

Using again (4.1) and (4.4), we obtain

$$\int_{-\infty}^{\infty} e^{-x_2^2/(4T)} e^{-x_2/\sqrt{T}} \psi_m^{T^*}(x_2)\, dx_2$$
$$= \left(\frac{1}{2\pi T^*}\right)^{1/4} \sqrt{\frac{m!}{2^m}} \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k}{k!(m-2k)!} \left(\sqrt{\frac{2}{T^*}}\right)^{m-2k}$$
$$\times \int_{-\infty}^{\infty} x_2^{m-2k} e^{-x_2^2(T^*+T)/(4TT^*) - x_2/\sqrt{T}}\, dx_2, \quad m \in \mathbb{N}_0. \tag{6.6}$$

Taking again into account (6.4), we have

$$\int_{-\infty}^{\infty} x_2^{m-2k} e^{-x_2^2(T^*+T)/(4TT^*) - x_2/\sqrt{T}}\, dx_2$$
$$= e^{T^*/(T^*+T)} \left(2\sqrt{\frac{TT^*}{T^*+T}}\right)^{m-2k+1} \frac{\sqrt{\pi}}{(-2i)^{m-2k}} \mathcal{H}_{m-2k}\left(i\sqrt{\frac{T^*}{T^*+T}}\right)$$
$$= 2\sqrt{\pi} e^{T^*/(T^*+T)} \frac{(-1)^{m+k}}{i^m} \left(\sqrt{\frac{TT^*}{T^*+T}}\right)^{m-2k+1} \mathcal{H}_{m-2k}\left(i\sqrt{\frac{T^*}{T^*+T}}\right),$$
$$k = \overline{0, \lfloor m/2 \rfloor},\ m \in \mathbb{N}_0.$$

Continuing (6.6), we get

$$\int_{-\infty}^{\infty} e^{-x_2^2/(4T)} e^{-x_2/\sqrt{T}} \psi_m^{T^*}(x_2)\, dx_2$$
$$= (8\pi T^*)^{1/4} e^{T^*/(T^*+T)} \sqrt{\frac{T}{T^*+T}} \frac{(-1)^m}{i^m} \sqrt{\frac{m!}{2^m}}$$

$$\times \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{1}{k!(m-2k)!} \left(\sqrt{\frac{2T}{T^*+T}} \right)^{m-2k} \mathcal{H}_{m-2k} \left(i \sqrt{\frac{T^*}{T^*+T}} \right), \quad m \in \mathbb{N}_0. \tag{6.7}$$

Using the multiplication theorem ([34, 18.18.13])

$$\mathcal{H}_n(\lambda\mu) = \lambda^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} (1-\lambda^{-2})^j \mathcal{H}_{n-2j}(\mu), \quad \lambda \in \mathbb{R},\ \mu \in \mathbb{R},\ n \in \mathbb{N}_0,$$

and setting $\lambda = i\sqrt{\frac{2T}{\Delta T}}$ and $\mu = i\sqrt{\frac{T^*}{T^*+T}}$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\lfloor m/2 \rfloor} \frac{1}{k!(m-2k)!} \left(\sqrt{\frac{2T}{T^*+T}} \right)^{m-2k} \mathcal{H}_{m-2k} \left(i \sqrt{\frac{T^*}{T^*+T}} \right) \\
&\quad = \left(\frac{2T}{T^*+T} \right)^{m/2} \frac{1}{m! i^m} \left(\frac{\Delta T}{2T} \right)^{m/2} \left(i \sqrt{\frac{2T}{\Delta T}} \right)^m \\
&\qquad \times \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!}{k!(m-2k)!} \left(1 - \left(i \sqrt{\frac{2T}{\Delta T}} \right)^{-2} \right)^k \mathcal{H}_{m-2k} \left(i \sqrt{\frac{T^*}{T^*+T}} \right) \\
&\quad = \left(\frac{\Delta T}{T^*+T} \right)^{m/2} \frac{i^{-m}}{m!} \mathcal{H}_m \left(\sqrt{\frac{2TT^*}{(T^*)^2 - T^2}} \right), \quad m \in \mathbb{N}_0.
\end{aligned}$$

Continuing (6.7), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-x_2^2/(4T)} e^{-x_2/\sqrt{T}} \psi_m^{T^*}(x_2)\, dx_2 \\
&\qquad = (8\pi T^*)^{1/4} e^{T^*/(T^*+T)} \sqrt{\frac{T}{T^*+T}} \frac{(-1)^m}{\sqrt{2^m m!}} \left(\frac{\Delta T}{T^*+T} \right)^{m/2} \\
&\qquad\quad \times \mathcal{H}_m \left(\sqrt{\frac{2TT^*}{(T^*)^2 - T^2}} \right), \quad m \in \mathbb{N}_0.
\end{aligned} \tag{6.8}$$

Thus, using (6.5) and (6.8) and continuing (6.2), we get

$$\begin{aligned}
W_{nm} &= \left(\frac{\sqrt{2\pi T^*}}{e(T^*+T)} \right)^{1/2} \frac{e^{T^*/(T^*+T)}}{2^{2n+1}\sqrt{(2n+1)!}} \frac{(-1)^{m+1}}{\sqrt{2^m m!}} \\
&\quad \times \left(\frac{\Delta T}{T^*+T} \right)^{m/2} \mathcal{H}_m \left(\sqrt{\frac{2TT^*}{(T^*)^2 - T^2}} \right), \quad n \in \mathbb{N}_0,\ m \in \mathbb{N}_0.
\end{aligned} \tag{6.9}$$

Taking into account (5.14), (5.15), and (6.9), we obtain

$$g_{pm}^N = \left(\frac{2\sqrt{T^* T^3}}{e(T^*+T)} \right)^{1/2} e^{T^*/(T^*+T)} \frac{(-1)^m}{\sqrt{2^m m!}} \left(\frac{\Delta T}{T^*+T} \right)^{m/2}$$

$$\times \mathcal{H}_m\left(\sqrt{\frac{2TT^*}{(T^*)^2-T^2}}\right)\frac{(-1)^p(8T)^p}{(2p+1)!}\sum_{n=p}^{N}\frac{1}{2^{3n}(n-p)!},$$
$$p=\overline{0,N},\ m=\overline{0,M}. \tag{6.10}$$

For $l\in\mathbb{N}$, let us construct a control u^l_{NM} in the form (5.24) with g^N_{pm} defined by (6.10). Then we obtain the end state of the solution W to (3.1), (3.2) with $u=u^l_{NM}$:

$$W(\cdot,T)=\mathcal{W}_0(\cdot,T)+\mathcal{W}_{u^l_{NM}}(\cdot,T), \tag{6.11}$$

where $\mathcal{W}_0(\cdot,T)$ is given by (6.1), $\mathcal{W}_{u^l_{NM}}$ is determined by (3.6).

Let us obtain estimate (5.25) in explicit form. We have

$$V_0^T=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}W_{nm}\widehat{\Theta}^{TT^*}_{nm}=\sum_{m=0}^{\infty}\widehat{\omega}_m\big((\cdot)_{[1]}\big)\widehat{\psi}^{T^*}_m\big((\cdot)_{[2]}\big), \tag{6.12}$$

where

$$\widehat{\omega}_m\big((\cdot)_{[1]}\big)=\sum_{n=0}^{\infty}W_{nm}\widehat{\psi}^T_{2n+1}\big((\cdot)_{[1]}\big),\quad m\in\mathbb{N}_0. \tag{6.13}$$

Substituting (4.1), (4.6), and (6.9) into (6.13), we obtain

$$\widehat{\omega}_m(\sigma_1)=i\left(\frac{2\sqrt{TT^*}}{T^*+T}\right)^{1/2}e^{T^*/(T^*+T)-3/8}\frac{(-1)^m}{\sqrt{2^m m!}}\left(\frac{\Delta T}{T^*+T}\right)^{m/2}$$
$$\times\mathcal{H}_m\left(\sqrt{\frac{2TT^*}{(T^*)^2-T^2}}\right)e^{-T\sigma_1^2}\sin\left(\sqrt{T}\sigma_1\right),\quad m\in\mathbb{N}_0,\ \sigma_1\in\mathbb{R}.$$

Since

$$\int_{-\infty}^{\infty}\left|e^{-T\sigma_1^2}\sin\left(\sqrt{T}\sigma_1\right)\right|^2d\sigma_1\le\sqrt{\frac{\pi}{2T}},$$

we have

$$\|\widehat{\omega}_m\|_0\le\left(\frac{\sqrt{2\pi T^*}}{T^*+T}\right)^{1/2}e^{T^*/(T^*+T)-3/8}\frac{1}{\sqrt{2^m m!}}\left(\frac{\Delta T}{T^*+T}\right)^{m/2}$$
$$\times\mathcal{H}_m\left(\sqrt{\frac{2TT^*}{(T^*)^2-T^2}}\right),\quad m\in\mathbb{N}_0. \tag{6.14}$$

Let V_{NM} be determined by (5.12). Taking into account (6.12), we get

$$\left\|V_0^T-V_{NM}\right\|_0\le\left\|\sum_{m=M+1}^{\infty}\widehat{\omega}_m\big((\cdot)_{[1]}\big)\widehat{\psi}^{T^*}_m\big((\cdot)_{[2]}\big)\right\|_0$$
$$+\left\|\sum_{m=0}^{M}\left(\widehat{\omega}_m\big((\cdot)_{[1]}\big)-\widehat{\omega}^N_m\big((\cdot)_{[1]}\big)\right)\widehat{\psi}^{T^*}_m\big((\cdot)_{[2]}\big)\right\|_0, \tag{6.15}$$

where $\widehat{\omega}_m^N$ is determined by (5.13). Taking into account the following estimate ([34, 18.14.9])

$$\frac{1}{\sqrt{2^n n!}} e^{-x^2/2} \left|\mathcal{H}_n(x)\right| \le 1, \quad x \in \mathbb{R},\ n \in \mathbb{N}_0,$$

we conclude that

$$\left|\mathcal{H}_m\left(\sqrt{\frac{2TT^*}{(T^*)^2 - T^2}}\right)\right| \le \sqrt{2^m m!}\, e^{TT^*/\left((T^*)^2 - T^2\right)}, \quad m \in \mathbb{N}_0. \tag{6.16}$$

Using (6.14) and (6.16) we obtain

$$\begin{aligned}
&\left\|\sum_{m=M+1}^{\infty} \widehat{\omega}_m\big((\cdot)_{[1]}\big) \widehat{\psi}_m^{T^*}\big((\cdot)_{[2]}\big)\right\|_0 \\
&\quad\le \left(\frac{\sqrt{2\pi T^*}}{T^*+T}\right)^{1/2} e^{T^*/(T^*+T) - 3/8} \\
&\qquad\times \left(\sum_{m=M+1}^{\infty} \frac{1}{2^m m!} \left(\frac{\Delta T}{T^*+T}\right)^m \left(\mathcal{H}_m\left(\sqrt{\frac{2TT^*}{(T^*)^2 - T^2}}\right)\right)^2\right)^{1/2} \\
&\quad\le \left(\frac{\sqrt{2\pi T^*}}{T^*+T}\right)^{1/2} e^{(T^*)^2/\left((T^*)^2 - T^2\right) - 3/8} \left(\sum_{m=M+1}^{\infty} \left(\frac{\Delta T}{T^*+T}\right)^m\right)^{1/2} \\
&\quad= \left(\frac{\sqrt{2\pi T^*}}{T^*+T}\right)^{1/2} e^{(T^*)^2/\left((T^*)^2 - T^2\right) - 3/8} \\
&\qquad\times \left(\frac{\Delta T}{T^*+T}\right)^{(M+1)/2} \left(\frac{1}{1 - \Delta T/(T^*+T)}\right)^{1/2} \\
&\quad= \left(\frac{\pi T^*}{2}\right)^{1/4} \frac{e^{(T^*)^2/\left((T^*)^2 - T^2\right) - 3/8}}{\sqrt{T}} \left(\frac{\Delta T}{T^*+T}\right)^{(M+1)/2}.
\end{aligned} \tag{6.17}$$

We have taken into account that $T^* > T$, hence $\Delta T/(T^*+T) < 1$.

With regard to (5.13), (6.9), (6.13), and (6.16), we have

$$\begin{aligned}
&\left\|\sum_{m=0}^{M} \left(\widehat{\omega}_m\big((\cdot)_{[1]}\big) - \widehat{\omega}_m^N\big((\cdot)_{[1]}\big)\right) \widehat{\psi}_m^T\big((\cdot)_{[2]}\big)\right\|_0 \le \left(\sum_{m=0}^{M} \sum_{n=N+1}^{\infty} |W_{nm}|^2\right)^{1/2} \\
&= \left(\frac{\sqrt{2\pi T^*}}{e(T^*+T)}\right)^{1/2} e^{T^*/(T^*+T)} \left(\sum_{m=0}^{M} \frac{1}{2^m m!} \left(\frac{\Delta T}{T^*+T}\right)^m\right. \\
&\quad\times \left.\left(\mathcal{H}_m\left(\sqrt{\frac{2TT^*}{(T^*)^2 - T^2}}\right)\right)^2 \sum_{n=N+1}^{\infty} \frac{1}{4^{2n+1}(2n+1)!}\right)^{1/2} \\
&\le \left(\frac{\sqrt{2\pi T^*}}{e(T^*+T)}\right)^{1/2} e^{(T^*)^2/\left((T^*)^2 - T^2\right)} \left(\sum_{m=0}^{\infty} \left(\frac{\Delta T}{T^*+T}\right)^m\right)^{1/2}
\end{aligned}$$

$$\times\left(\sinh(1/4)-\sum_{n=0}^{N}\frac{1}{4^{2n+1}(2n+1)!}\right)^{1/2}$$
$$\le\left(\frac{\sqrt{2\pi T^*}}{e(T^*+T)}\right)^{1/2}e^{(T^*)^2/\left((T^*)^2-T^2\right)}$$
$$\times\left(\frac{1}{1-\Delta T/(T^*+T)}\right)^{1/2}\left(\frac{\cosh(1/4)}{4^{2N+3}(2N+3)!}\right)^{1/2}$$
$$=\left(\frac{\pi T^*}{2}\right)^{1/4}e^{(T^*)^2/((T^*)^2-T^2)}\sqrt{\frac{\cosh(1/4)}{eT}}\,\frac{1}{2^{2N+3}\sqrt{(2N+3)!}}. \tag{6.18}$$

Then, continuing (6.15) and taking into account (6.17) and (6.18), we get

$$\left\|V_0^T-V_{NM}\right\|_0\le\left(\frac{\pi T^*}{2}\right)^{1/4}\frac{e^{(T^*)^2/\left((T^*)^2-T^2\right)}}{\sqrt{T}}$$
$$\times\left(e^{-3/8}\left(\frac{\Delta T}{T^*+T}\right)^{(M+1)/2}+\sqrt{\frac{\cosh(1/4)}{e}}\,\frac{1}{2^{2N+3}\sqrt{(2N+3)!}}\right). \tag{6.19}$$

Let V_{NM}^l be determined by (5.17). Then (5.22) holds. Taking into account (5.16) and (5.17), we have

$$\left\|V_{NM}-V_{NM}^l\right\|_0\le\sum_{m=0}^{M}\sum_{p=0}^{N}\left|g_{pm}^N\right|\left\|\varphi_{2p+1}-\varphi_{2p+1}^l\right\|_0. \tag{6.20}$$

Let us estimate $\left\|\varphi_{2p+1}-\varphi_{2p+1}^l\right\|_0$ for $p\in\mathbb{N}_0$. Applying the Tailor formula for the function $((e^x-1)/x)^{p+1}$ at $x=0$, we get

$$\left(\frac{e^x-1}{x}\right)^{p+1}=1+(p+1)x\left(\frac{e^y-1}{y}\right)^p\frac{ye^y-e^y+1}{y^2},\quad x\ge0, \tag{6.21}$$

for some $y\in(0,x)$. Using again the Tailor formula for the function ye^y-e^y at $y=0$, we obtain

$$ye^y-e^y=-1+\frac{y^2}{2}(1+z)e^z,\quad y\ge0,$$

for some $z\in(0,y)$. Since $|(1+z)e^{-z}|\le1$, $z\ge0$, we have

$$\left|\frac{ye^y-e^y+1}{y^2}\right|\le\frac{1}{2}e^{2y},\quad y\ge0.$$

Since $(e^y-1)/y\le e^y\le e^x$, $x\ge0$, continuing (6.21), we get

$$\left|\left(\frac{e^x-1}{x}\right)^{p+1}-1\right|\le\frac{1}{2}(p+1)xe^{(p+2)x},\quad x\ge0.$$

Then, for $l\ge2(p+2)/T$, we have

$$\left(\left\|\varphi_{2p+1}-\varphi_{2p+1}^l\right\|_0\right)^2\le2\left(\frac{p+1}{2l}\right)^2\int_0^\infty\left(\lambda^2\right)^{2p+3}e^{-2(T-(p+2)/l)\lambda^2}d\lambda$$

$$\begin{aligned}
&\le \left(\frac{p+1}{2l}\right)^2 2\int_0^\infty \left(\lambda^2\right)^{2p+3} e^{-T\lambda^2} d\lambda \\
&= -2\left(\frac{p+1}{2l}\right)^2 \left(\frac{d}{dT}\right)^{2p+3} \int_0^\infty e^{-T\lambda^2} d\lambda \\
&= \sqrt{\pi}\left(\frac{p+1}{2l}\right)^2 \frac{(4p+5)!!}{2^{2p+3}T^{2p+7/2}}.
\end{aligned} \tag{6.22}$$

Let us estimate $\sqrt{(4p+5)!!}$. Taking into account the Stirling formula (see, e.g. [25, 6.1.38]):

$$\sqrt{2\pi}n^{n+1/2}e^{-n} \le n! \le e n^{n+1/2}e^{-n}, \quad n \in \mathbb{N}_0, \tag{6.23}$$

we obtain

$$\begin{aligned}
(4p+5)!! &= \frac{(4p+5)!!(4p+6)!!}{(4p+6)!!} = \frac{(4p+6)!}{2^{2p+3}(2p+3)!} \\
&\le \frac{e2^{-(2p+3)}}{\sqrt{\pi}} \frac{(4p+6)^{4p+6}e^{-(4p+6)}}{(2p+3)^{2p+3}e^{-(2p+3)}} = \frac{e}{\sqrt{\pi}} 2^{2p+3}(2p+3)^{2p+3}e^{-(2p+3)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sqrt{(4p+5)!!} &\le \frac{e^{1/2}}{\pi^{1/4}} 2^{p+3/2}(2p+3)^{p+3/2}e^{-(p+3/2)} \\
&= \frac{2^{2p+3}}{\pi^{1/4}}(p+1)^{p+1+1/2}e^{-(p+1)}\left(1+\frac{1}{2p+2}\right)^{p+3/2}.
\end{aligned}$$

Applying again the Stirling formula (see (6.23)), we get

$$\sqrt{(4p+5)!!} \le \left(\frac{e}{\pi}\right)^{3/4} 2^{2p+5/2}(p+1)! \tag{6.24}$$

because

$$\begin{aligned}
\left(1+\frac{1}{2p+2}\right)^{p+3/2} &= \exp\left(\left(p+\frac{3}{2}\right)\ln\left(1+\frac{1}{2p+2}\right)\right) \\
&\le \exp\left(\frac{p+3/2}{2(p+1)}\right) \le e^{3/4}.
\end{aligned}$$

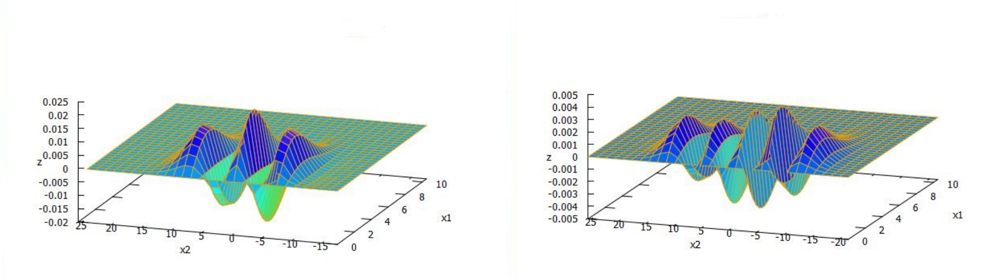
Continuing (6.22) and taking into account (6.24), we obtain

$$\left\|\varphi_{2p+1} - \varphi_{2p+1}^l\right\|_0 \le \frac{e^{3/4}}{\pi^{1/2}} \frac{2^p(p+1)(p+1)!}{lT^{p+7/4}}, \quad p \in \mathbb{N}_0. \tag{6.25}$$

Using (6.10) and (6.16), we get

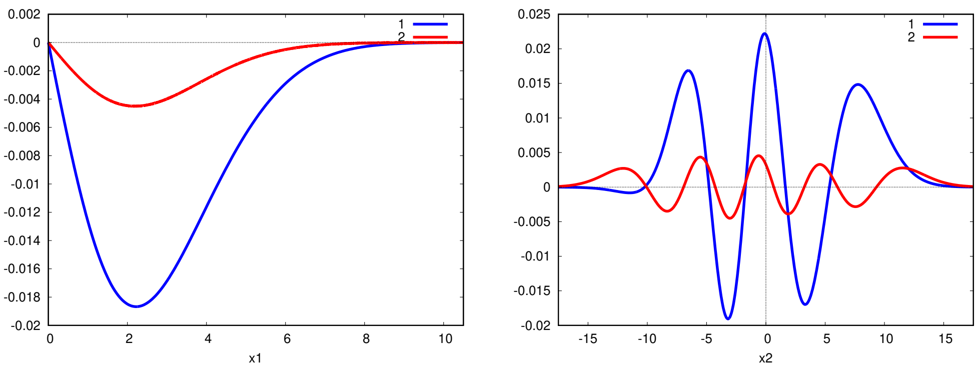
$$\begin{aligned}
\left|g_{pm}^N\right| \le &\left(\frac{2\sqrt{T^*T^3}}{e(T^*+T)}\right)^{1/2} e^{T^*/(T^*+T)} \left(\frac{\Delta T}{T^*+T}\right)^{m/2} \\
&\times e^{TT^*/\left((T^*)^2-T^2\right)} \frac{(8T)^p}{(2p+1)!} \sum_{j=0}^{\infty} \frac{1}{8^{p+j}j!}
\end{aligned}$$

$$= \left(\frac{2\sqrt{T^* T^3}}{T^* + T} \right)^{1/2} e^{(T^*)^2/\left((T^*)^2 - T^2\right) - 3/8}$$
$$\times \left(\frac{\Delta T}{T^* + T} \right)^{m/2} \frac{T^p}{(2p+1)!}, \quad p = \overline{0, N}, \ m = \overline{0, M}. \tag{6.26}$$



(a) $N = M = 3$, $l = 10$. (b) $N = M = 6$, $l = 200$.

Fig. 6.1: The difference $W^T - \left(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{u^l_{NM}}(\cdot, T) \right)$ with $T = 2$ and $T^* = 6$.

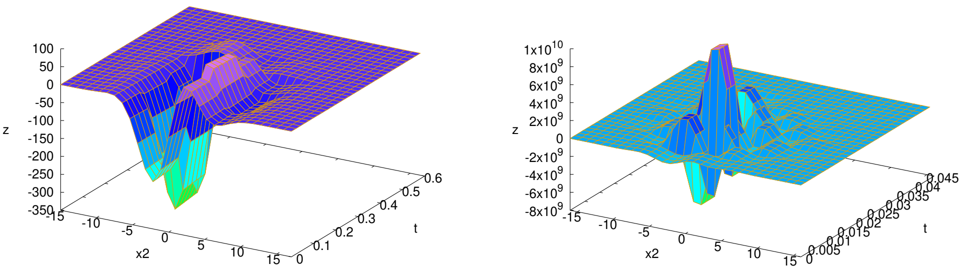


(a) 1) $N = M = 3$, $l = 10$; 2) $N = M = 6$, $l = 200$. (b) 1) $N = M = 3$, $l = 10$; 2) $N = M = 6$, $l = 200$.

Fig. 6.2: The difference $W^T - \left(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{u^l_{NM}}(\cdot, T) \right)$ with $T = 2$ and $T^* = 6$ (vertical section for $x_2 = -3$ and horizontal section for $x_1 = 2.2$).

Therefore for $l \geq 2(N+2)/T$, continuing (6.20) and taking into account (6.25) and (6.26), we obtain

$$\left\| V_{NM} - V^l_{NM} \right\|_0 \leq \sqrt{\frac{2}{\pi(T^* + T)}} \frac{(T^*)^{1/4} e^{3/8}}{Tl} e^{(T^*)^2/\left((T^*)^2 - T^2\right)}$$
$$\times \sum_{m=0}^{M} \left(\frac{\Delta T}{T^* + T} \right)^{m/2} \sum_{p=0}^{N} \frac{2^p (p+1)(p+1)!}{(2p+1)!}. \tag{6.27}$$



(a) $N = M = 3$, $l = 10$ (b) $N = M = 6$, $l = 200$

Fig. 6.3: The controls u^l_{NM} defined by (5.24) and (6.10) with $T = 2$ and $T^* = 6$.

For the first sum in (6.27), we have

$$\sum_{m=0}^{M} \left(\frac{\Delta T}{T^* + T} \right)^{m/2} \leq \sum_{m=0}^{\infty} \left(\sqrt{\frac{\Delta T}{T^* + T}} \right)^{m} = \frac{\sqrt{T^* + T}}{\sqrt{T^* + T} - \sqrt{\Delta T}}. \tag{6.28}$$

For the second sum in (6.27), taking into account that $(2p+1)! \geq \Big((2p)!!\Big)^2 = \Big(2^p p!\Big)^2$, we have

$$\begin{aligned} \sum_{p=0}^{N} \frac{2^p (p+1)(p+1)!}{(2p+1)!} &\leq \sum_{p=0}^{\infty} \frac{(p+1)^2}{2^p p!} = \sum_{p=0}^{\infty} \frac{(p-1)p + 3p + 1}{2^p p!} \\ &= \frac{1}{4} \sum_{p=2}^{\infty} \frac{1}{2^{p-2}(p-2)!} + \frac{3}{2} \sum_{p=1}^{\infty} \frac{1}{2^{p-1}(p-1)!} + \sum_{p=0}^{\infty} \frac{1}{2^p p!} = \frac{11}{4} e^{1/2}. \end{aligned} \tag{6.29}$$

Substituting (6.28) and (6.29) in (6.27), we get

$$\left\| V_{NM} - V^l_{NM} \right\|_0 \leq \frac{1}{l} \frac{11 (T^*)^{1/4} e^{7/8}}{2T\sqrt{2\pi}} \frac{e^{(T^*)^2/\left((T^*)^2 - T^2\right)}}{\sqrt{T^* + T} - \sqrt{\Delta T}}, \quad l \geq \frac{2(N+2)}{T}. \tag{6.30}$$

Taking into account (6.19), (5.22), and (6.30) for $l \geq 2(N+2)/T$, we have

$$\begin{aligned} &\left\| W^T - \Big(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{u^l_{NM}}(\cdot, T) \Big) \right\|^0 \leq \left\| W^T_0 - \mathcal{W}_{u^l_{NM}}(\cdot, T) \right\|^0 \\ &\quad \leq \left\| V^T_0 - V_{NM} \right\|_0 + \left\| V_{NM} - V^l_{NM} \right\|_0 \\ &\quad \leq \left(\frac{\pi T^*}{2} \right)^{1/4} \frac{e^{(T^*)^2/\left((T^*)^2 - T^2\right)}}{\sqrt{T}} \left(e^{-3/8} \left(\frac{\Delta T}{T^* + T} \right)^{(M+1)/2} \right. \\ &\qquad \left. + \sqrt{\frac{\cosh(1/4)}{e}} \frac{1}{2^{2N+3}\sqrt{(2N+3)!}} + \frac{1}{l} \frac{11 e^{7/8}}{\sqrt{T}(2^5\pi^3)^{1/4} \left(\sqrt{T^* + T} - \sqrt{\Delta T} \right)} \right). \end{aligned}$$

Thus, we have constructed the control u^l_{NM}, determined by (5.24) and (6.10), and the approximate end state, determined by (6.11). Let us recall that $T = 2$. Put $T^* = 6$. Then, we have

$$\begin{aligned}
&\left\| W^T - \Big(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{u^l_{NM}}(\cdot, T)\Big) \right\|^0 \\
&\le \frac{(3\pi)^{1/4}}{\sqrt{2}} \left(e^{3/4} \left(\frac{1}{2}\right)^{M/2} + \frac{\sqrt{\cosh(1/4)} e^{5/8}}{2^{2N+3}\sqrt{(2N+3)!}} + \frac{1}{l} \frac{11e^2}{(2^{11}\pi^3)^{1/4}\left(\sqrt{2}-1\right)} \right) \\
&= 2.622851155438146 \left(\frac{1}{2}\right)^{M/2} + \frac{2.350732202502537}{2^{2N+3}\sqrt{(2N+3)!}} + \frac{15.31493739172921}{l}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l \ge \frac{2(N+2)}{T}.
\end{aligned}$$

The difference $W^T - \Big(\mathcal{W}_0(\cdot, T) + \mathcal{W}_{u^l_{NM}}(\cdot, T)\Big)$ is shown in Fig. 6.1 and Fig. 6.2 for the cases of $N = M = 3$, $l = 10$ and $N = M = 6$, $l = 200$. The controls u^l_{NM} are given in Fig. 6.3 for the same cases.

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Проблеми наближеної керованості для рівняння теплопровідності на півплощині керованого крайовою умовою Діріхле з обмеженим керуванням

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У роботі досліджено проблеми наближеної керованості для керованої системи $w_t = \Delta w$, $w(0, x_2, t) = u(x_2, t)$, $x_1 \in \mathbb{R}_+ = (0, +\infty)$, $x_2 \in \mathbb{R}$, $t \in (0, T)$, де u є керуванням, що належить до деякої спеціальної підмножини $L^\infty(\mathbb{R} \times (0, T)) \cap L^2(\mathbb{R} \times (0, T))$. Доведено, що кожен початковий стан, що належить $L^2(\mathbb{R}_+ \times \mathbb{R})$, є наближено керованим за допомогою таких керувань у будь-який кінцевий стан, що належить $L^2(\mathbb{R}_+ \times \mathbb{R})$. Побудовано числовий алгоритм розв'язання проблеми наближеної керованості для цієї системи. Результати проілюстровано прикладом.

Ключові слова: рівняння теплопровідності, керованість, наближена керованість, півплощина