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On the Restrictions of Weakly Demicompact Operators and Generalized Fredholm Theory

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The purpose of this paper is to explore the weak demicompactness of the operator T_n, which is the restriction of a linear operator T to its range $\mathcal{R}(T^n)$, T_n are considered as linear operators from $\mathcal{R}(T^n)$ to itself, $n \in \mathbb{N}$. As well, we present several results on upper generalized semi-Fredholm operators, focusing on the concept of weak demicompact operators. We specify conditions on certain ranges that ensure the persistence of the weak demicompactness property under restrictions. Moreover, our study provides perturbation results concerning the generalized Gustafson essential spectrum for 2×2 operator matrices.

Key words: weakly S-demicompact operators, generalized Fredholm operators, generalized essential spectrum

Mathematical Subject Classification 2020: 47A53, 47A10

1. Introduction

The notion of demicompactness was developed in the literature as a mean to explore fixed points, initially introduced by Petryshyn in 1966. This class is employed by Petryshyn [16] and Akashi [1] to provide several results in Fredholm theory. Chaker et al. [8] have pursued this study to investigate the essential spectra of densely defined linear operators. In 2014, Krichen [13] introduced the relative demicompactness class concerning a specific closed linear operator, as a generalization of the demicompactness notion. In 2016, Krichen and O'Regan [14] explored certain Fredholm and perturbation results involving the class of weakly demicompact linear operators. Additionally, they investigated the relationship between this class and the measures of weak noncompactness of linear operators.

As an extension of the classical Fredholm theory, Yang [20] initiated the concept of generalized Fredholm operators and gave a characterization using weakly compact sets. Accordingly, Azzouz et al. [3] elaborated on certain results presented in [20] and provided sufficient conditions for an operator to be generalized semi-Fredholm. They formulated their results concerning certain quantities associated with measures of weak noncompactness. In spite of that, these studies were limited to bounded linear operators. Recently, Ferjani et al. [10] explored some spectral properties of unbounded generalized Fredholm operators acting on a non-reflexive Banach space under a certain density condition. To state these results, needed in the sequel, we first give some specific notations. Let X and Y be two Banach spaces. We denote by $\mathcal{C}(X, Y)$ the set of all closed linear operators from X into Y having dense domains and by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X into Y. If $X = Y$, then $\mathcal{C}(X,Y)$ and $\mathcal{L}(X,Y)$ are respectively replaced by $\mathcal{C}(X)$ and $\mathcal{L}(X)$. For $T \in \mathcal{C}(X,Y)$, the sets $\Delta(T)$, $\mathcal{R}(T)$, $Y/\mathcal{R}(T)$ and $\mathcal{N}(T)$ respectively refer to the domain, the range, the co-kernel and the kernel of T. Following the notation used in [4], the classes of upper generalized semi-Fredholm operators, lower generalized semi-Fredholm operators and generalized Fredholm operators are respectively denoted by

$$\Phi_{g+}(X,Y) := \{T \in \mathcal{C}(X,Y) \mid \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\};$$
$$\Phi_{g-}(X,Y) := \{T \in \mathcal{C}(X,Y) \mid Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\};$$

and

$$\Phi_g(X,Y) := \Phi_{g+}(X,Y) \cap \Phi_{g-}(X,Y).$$

The linear spaces X^* denote the topological dual of the Banach space X. When $X = Y$, all the sets $\Phi_g(X,Y)$, $\Phi_{g+}(X,Y)$ and $\Phi_{g-}(X,Y)$ are simply replaced by $\Phi_g(X)$, $\Phi_{g+}(X)$ and $\Phi_{g-}(X)$ respectively. A complex number λ is in $\Phi_{g+,T}(X)$ if $\lambda - T$ is in $\Phi_{g+}(X)$. If $T \in \mathcal{C}(X)$, then $\Delta(T)$ equipped with the graph norm $\|x\|_T = \|x\| + \|Tx\|$ is a Banach space denoted by X_T. Note that the restriction $\widehat{T}$ of the closed operator T on $\Delta(T)$ is bounded from X_T into X. Furthermore, we have the obvious relations

$$\begin{cases} \mathcal{N}(\widehat{T}) = \mathcal{N}(T); \\ \mathcal{R}(\widehat{T}) = \mathcal{R}(T). \end{cases}$$

Then we can see that $T \in \Phi_g(X)$ if, and only if, $\widehat{T} \in \Phi_g(X_T, X)$.

Now, we recall that an operator $T \in \mathcal{L}(X,Y)$ is said weakly compact if $T(M)$ is relatively weakly compact in Y for every bounded subset $M \subset X$. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, then $\mathcal{W}(X,X) := \mathcal{W}(X)$.

Let $S \in \mathcal{L}(X,Y)$ such that $S \neq 0$. For $T \in \mathcal{C}(X,Y)$, we define the S-resolvent set of T by

$$\rho_S(T) := \{\lambda \in \mathbb{C} \mid \lambda S - T \text{ has a bounded inverse}\};$$

and the S-spectrum of T by

$$\sigma_S(T) := \mathbb{C} \backslash \rho_S(T).$$

Note that for $S = I$, $\rho_S(T)$ will be denoted by $\rho(T)$. Recently, Azzouz, Beghdadi and Krichen [3] introduced the generalized essential spectrum and provided a characterization of the generalized essential spectral radius via the concept of measure of weak noncompactness. The same authors [4] further investigated the generalized essential spectra of the sum of two bounded linear operators under some specific conditions.

In this work, we are interested in the generalized Gustafson and the generalized Wolf essential spectrum of a closed densely defined linear operator T. They are defined as follows:

$$\sigma_{e_1,g}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \notin \Phi_{g+}(X)\};$$
$$\sigma_{e_4,g}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \notin \Phi_{g}(X)\}.$$

The generalized resolvent $\rho_g(T)$ is $\mathbb{C}\backslash\sigma_{e_4,g}(T)$.

For each integer $n \in \mathbb{N}$, define T_n as the restriction of a linear operator T to its range $\mathcal{R}(T^n)$, viewed as a map from $\mathcal{R}(T^n)$ into itself (we set $T_0 = T$).

This work aims to explore the weak demicompactness of the operator T_n and its relation to the class of upper generalized semi-Fredholm operators. As well, we investigate the relationship between the class of weakly demicompact operators and the generalized essential spectral radius of T_n. Furthermore, we develop several significant results related to the class of generalized semi-Fredholm operators, focusing on the concept of weak demicompactness.

The theory of block operator matrices appears in many areas of mathematics and its applications. Their spectral properties are important because they determine the time evolution and stability of physical systems. We highlight [18] as one of the most important works on the spectral theory of block operator matrices. In the second part of our paper, we provide some perturbation results regarding the generalized Gustafson essential spectrum for 2×2 operator matrices involving the class of weak demicompactness.

The paper is organized as follows. In section 2, we start with a review of the definitions and results that will be used throughout the paper. In section 3, we establish several results concerning generalized semi-Fredholm operators and their relation to weak demicompact operators. Through our investigations under some specific conditions we establish the persistence of the demicompactness property under restrictions. In section 4, we explore the relationships between the generalized Gustafson essential spectrum of the sum of two linear operators and the generalized Gustafson essential spectrum of each these operators, considering the class of weakly demicompact operators.

2. Preliminary Results

In this section, we outline important definitions and crucial results. Let X be a Banach space and $B_r = B(0, r)$ denote the open ball in X centered at 0_X with radius $r > 0$. We denote $\mathcal{M}_X$ as the set of bounded subsets of X and $\mathcal{W}_X$ as the set of weakly compact subsets of X. Additionally, $\text{conv}(A)$ denotes the convex hull of a subset $A \subset X$.

The concept of a measure of weak noncompactness was introduced by F. S. De Blasi [9] and has been applied in topology, functional analysis, and integral equations. The De Blasi measure of noncompactness of a non empty bounded subset $A \subset X$, denoted by $\omega : \mathcal{M}_X \to [0, +\infty[$ is defined as follows:

$$\omega(A) = \inf\{r > 0 \mid \exists N \in \mathcal{W}_X \quad A \subset N + \overline{B}_r\}.$$

An axiomatic definition was given in [5, 6] as follows:

Definition 2.1. Let X be a Banach space. A function $\upsilon : \mathcal{M}_X \to [0, +\infty[$ is said to be a measure of weak noncompactness in X (in short, MWNC) if, for all $A, B \in \mathcal{M}_X$, it satisfies the following conditions:

(i) $\mathcal{N}(\upsilon) = \{A \in \mathcal{M}_X \text{ such that } \upsilon(A) = 0\}$ is nonempty and $\mathcal{N}(\upsilon) \subset \mathcal{W}_X$;
(ii) if $A \subset B$, then $\upsilon(A) \le \upsilon(B)$;
(iii) $\upsilon(\overline{\text{conv}(A)}) = \upsilon(A)$;
(iv) $\upsilon(\lambda A + (1-\varsigma)B) \le \varsigma\upsilon(A) + (1-\varsigma)\upsilon(B)$ for $\varsigma \in [0, 1]$;
(v) if $A_n \in \mathcal{M}_X$ are such that A_n is weakly closed in X with $A_{n+1} \subset A_n$ for $n = 1, 2, \ldots$ and if $\lim\limits_{n\to+\infty} \upsilon(A_n) = 0$, then $A_\infty := \bigcap\limits_{i=1}^{+\infty} A_n \neq \varnothing$.

Definition 2.2 ([5, 6])**.** Let X be a Banach space and υ be a MWNC in X. Then,

(i) υ satisfies the maximum property if $\upsilon(A \cup B) = \max\{\upsilon(A), \upsilon(B)\}$, for any $A, B \in \mathcal{M}_X$;
(ii) υ is called homogeneous if $\upsilon(\varsigma A) = |\varsigma|\upsilon(A)$, for any $A \in \mathcal{M}_X$ and $\varsigma \in \mathbb{C}$;
(iii) υ is called subadditive if $\upsilon(A + B) \le \upsilon(A) + \upsilon(B)$, for any $A, B \in \mathcal{M}_X$;
(iv) υ is called sublinear if it is homogeneous and subadditive;
(v) υ is called regular if it is sublinear, has the maximum property and $\mathcal{N}(\upsilon) = \mathcal{W}_X$.

Definition 2.3. Let X be a Banach space and $T \in \mathcal{C}(X)$. The measure of weak noncompactness of T, represented by $\overline{\omega}(T)$, is defined as follows:

$$\overline{\omega}(T) = \inf\{\beta > 0 \mid \forall A \in \mathcal{M}_X \quad \omega(T(A)) \le \beta\omega(A)\}.$$

Proposition 2.4. *Let X be a Banach space and T, $S \in \mathcal{L}(X)$ and let $B \in \mathcal{M}_X$. Then, we have the following properties:*

(i) $\overline{\omega}(T) = 0$ *if and only if T is weakly compact*;
(ii) $\omega(T(B)) \le \overline{\omega}(T)\omega(B)$;
(iii) $\overline{\omega}(TS) \le \overline{\omega}(T)\overline{\omega}(S)$;
(iv) $\overline{\omega}(T + S) \le \overline{\omega}(T) + \overline{\omega}(S)$;
(v) $\overline{\omega}(\lambda T) = |\lambda|\overline{\omega}(T)$ *for* $\lambda \in \mathbb{C}$.

Definition 2.5. Let X be a Banach space. We say that X has the property $(\mathbf{H_1})$ (respectively, $(\mathbf{H_2})$) if every closed reflexive subspace admits a closed complementary subspace (respectively, if every closed subspace with reflexive quotient space admits a closed complementary subspace). We say that X has the property $(\mathbf{H})$, if it satisfies the both properties $(\mathbf{H_1})$ and $(\mathbf{H_2})$.

Observe that the Hilbert space $L_2(0, 1)$ has the property $(\mathbf{H_1})$. However, $L_1(0, 1)$ and $\mathcal{C}(S)$ the space of all bounded (real or complex valued) continuous functions on an infinite compact Hausdorff space S do not have it [11].

Theorem 2.6 ([20]). *Let X and Y be Banach spaces satisfying the properties* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *respectively and let $T \in \mathcal{L}(X,Y)$. Then, the following assertions are equivalent:*

(i) *T is a generalized Fredholm operator;*

(ii) *there exist weakly compact operators $W_1 \in \mathcal{W}(X), W_2 \in \mathcal{W}(Y)$ and an operator $T_0 \in \mathcal{L}(Y,X)$ such that $T_0T = I + W_1$, $TT_0 = I + W_2$ and $\mathcal{R}(T)$ is closed.*

Lemma 2.7. *[15] Let X be a Banach space and $T, S, U, V \in \mathcal{L}(X)$ be commuting operators satisfying $TU + SV = I$ and let $n \geq 0$. Then*

(i) $\mathcal{N}(T^nS^n) = \mathcal{N}(T^n) + \mathcal{N}(S^n)$, $\mathcal{R}(T^nS^n) = \mathcal{R}(T^n) \cap \mathcal{R}(S^n)$;

(ii) $\mathcal{N}^\infty(TS) = \mathcal{N}^\infty(T) + \mathcal{N}^\infty(S)$, $\mathcal{R}^\infty(TS) = \mathcal{R}^\infty(T) \cap \mathcal{R}^\infty(S)$;

(iii) $\mathcal{N}^\infty(T) \subset \mathcal{R}^\infty(S)$, $\mathcal{N}^\infty(S) \subset \mathcal{R}^\infty(T)$;

(iv) *$\mathcal{R}(T^nS^n)$ is closed if and only if $\mathcal{R}(T^n), \mathcal{R}(S^n)$ are closed.*

3. Restrictions of weakly demicompact operators

This section begins with the following definition:

Definition 3.1. Let X be a Banach space and $T, S \in \mathcal{C}(X)$ with $\Delta(T) \subset \Delta(S)$. We say that T is weakly S-demicompact (or weakly relative demicompact with respect to S), if for every bounded sequence $(x_n)_n$ in $\Delta(T)$ such that $Sx_n - Tx_n \rightharpoonup x$, for some $x \in X$, there exists a weakly convergent subsequence of $(x_n)_n$.

Remark 3.2.

(i) We define $\mathcal{WDC}_S(X)$ as the class of weakly S-demicompact linear operators on X. For $S = I$, we recover the usual definition of weakly demicompact operator. In this case, the class $\mathcal{WDC}_S(X)$ is simply denoted by $\mathcal{WDC}(X)$;

(ii) Every weakly compact operator is a weakly demicompact operator.

Proposition 3.3. *Let X be a Banach space and $T, S \in \mathcal{C}(X)$ with $\Delta(T) \subset \Delta(S)$ such that $S - T$ is closed operator. If $T \in \mathcal{WDC}_S(X)$, then $\mathcal{N}(S - T)$ is reflexive. In addition, if we assume that $\mathcal{N}(S - T)$ is complemented in X with a complement X_0 such that $(X_0 \cap \Delta(T), \|\cdot\|_T)$ is compactly embedded with an isometric embedding i in $(Y, \|\cdot\|_T)$, where Y is a closed subspace of X, then the operator $S - T$ is upper generalized semi-Fredholm.*

Proof. Let $(z_n)_n$ be bounded sequence in $\mathcal{M} = S(0,1) \cap \mathcal{N}(S - T)$, where $S(0,1) = \{z \in X, \|z\| \leq 1\}$ then $Sz_n - Tz_n = 0$ and $\|z_n\| \leq 1, z_n \in \Delta(T)$ for all $n \in \mathbb{N}$. Hence, $(z_n)_n$ is bounded and $Sz_n - Tz_n \rightharpoonup 0$. Since $T \in \mathcal{WDC}_S(X)$, we deduce that $(z_n)_n$ has a subsequence $(z_{\varphi(n)})_n$ such that $z_{\varphi(n)} \rightharpoonup z_0$, $z_0 \in X$. We have $S - T$ is closed, then $z_0 \in \Delta(T)$ and $(S - T)z_0 = 0$. Moreover, since $\|z_0\| \leq \liminf\|z_{\varphi(n)}\| \leq 1$ it follows that, $z_0 \in S(0,1) \cap \mathcal{N}(S - T)$. Which implies that $\mathcal{M}$ is weakly compact. Thus, $\mathcal{N}(S-T)$ is reflexive. Now, it remains to show that $\mathcal{R}(S - T)$ is closed. To see this, since

$$\mathcal{N}(S - T) \oplus X_0 \cap \Delta(T) = \Delta(T),$$

it follows that $(S-T)_{|X_0\cap\Delta(T)}$ is injective. Therefore, to prove that $\mathcal{R}(S-T) = \mathcal{R}((S-T)_{|X_0\cap\Delta(T)})$ is closed, it is sufficient to show that $(S-T)_{|X_0\cap\Delta(T)}$ is bounded below. To this end, assume the contrary then for all $n\in\mathbb{N}^*$, there exists $x_n\in X_0\cap\Delta(T)$ such that $\|(S-T)_{|X_0\cap\Delta(T)}x_n\|\leq\frac{1}{n}\|x_n\|_T$. Without loss of generality, we suppose that $\|x_n\|_T=1$, then $(S-T)x_n\to 0$ as $n\to\infty$ and hence $(S-T)x_n\rightharpoonup 0$, $n\to\infty$. Using the fact that $T\in\mathcal{WDC}_S(X)$, there exists a subsequence $(x_{\phi(n)})_n\subset X_0\cap\Delta(T)$ such that $x_{\phi(n)}\rightharpoonup x$. As $(S-T)x_{\phi(n)}\rightharpoonup 0$ and $S-T$ is weakly closed, we deduce that $x\in X_0\cap\Delta(T)$ and $(S-T)x=0$. Which implies that $x\in\mathcal{N}(S-T)\cap X_0\cap\Delta(T)=\{0\}$. Hence, $x=0$. Now, since $(X_0\cap\Delta(T),\|\cdot\|_T)$ is compactly embedded it follows that there exists a subsequence still denoted $(x_{\phi(n)})_n$ such that $i(x_{\phi(n)})\to 0$ in $(Y,\|\cdot\|_T)$. Moreover, since i is isometric then $1=\|x_{\phi(n)}\|_T=\|i(x_{\phi(n)})\|_T\to 0$. Which contradicts the continuity of the norm. Therefore, $\mathcal{R}(S-T)$ is closed. We conclude that $S-T$ is an upper generalized semi-Fredholm operator. □

An immediate consequence of Proposition 3.3 follows.

Corollary 3.4. *Let X be a Banach space and $T,S\in\mathcal{C}(X)$ with $\Delta(T)\subset\Delta(S)$. Assume that $S-T$ is a closed operator and has a closed range. If $T\in\mathcal{WDC}_S(X)$, then $S-T$ is an upper generalized semi-Fredholm.*

In the following proposition, we focus on the weakly S_n-demicompacteness of the operator T_n. If S is a commuting operator with T, then $\mathcal{R}(T^n)$ is invariant under S and we denote by S_n the restriction of S to $\mathcal{R}(T^n)$, viewed as an operator from $\mathcal{R}(T^n)$ to $\mathcal{R}(T^n)$.

Proposition 3.5. *Let X be a Banach space and $T,S\in\mathcal{L}(X)$ be two commuting operators. Assume that there exists $n\in\mathbb{N}$ such that $S-T$ and T^n are prime, and S_n-T_n has a closed range. If $\mathcal{R}(T^n)$ is closed and T_n is weakly S_n-demicompact operator, then $\mathcal{R}((S-T)T^n)$ is closed and $\mathcal{N}(S-T)$ is reflexive.*

Proof. Since T_n is weakly S_n-demicompact operator then from Corollary 3.4, it follows that S_n-T_n is an upper generalized semi-Fredholm operator. Taking into account that $\mathcal{R}((S-T)T^n)=\mathcal{R}(S_n-T_n)$ then, $\mathcal{R}((S-T)T^n)$ is closed. Furthermore, as $S-T$ and T^n are prime, there exists $L,M\in\mathcal{L}(X)$, commuting with S and T, such that $L(S-T)+MT^n=I$. Given that $x\in\mathcal{N}(S-T)$, so $x=T^n(M(x))$. Thus, we get $\mathcal{N}(S-T)\subset\mathcal{R}(T^n)$, therefore $\mathcal{N}(S-T)=\mathcal{N}(S_n-T_n)$. Consequently, $\mathcal{N}(S-T)$ is reflexive. This proves the result. □

By setting $S=I$, it follows that for any $n\in\mathbb{N}$, both $I-T$ and T^n are prime.

Theorem 3.6. *Let X be a Banach space and $T\in\mathcal{C}(X)$ such that $T(\Delta(T))\subset\Delta(T)$, $I-T$ has a closed range. If T is weakly demicompact and $\mathcal{R}(T^n)$ is closed for some $n\in\mathbb{N}^*$, then I_n-T_n is an upper generalized semi-Fredholm operator.*

Proof. Taking into account that $T\in\mathcal{WDC}(X)$, we infer by Corollary 3.4 that $I-T$ is an upper generalized semi-Fredholm operator. It is straightforward to see that $\mathcal{R}(T^n)\subset\Delta(T)$. Given that both $\mathcal{R}(T^n)$ and T are closed, it follows that

T_n is a closed operator from $\mathcal{R}(T^n)$ into itself. Obviously, $\mathcal{N}(I_n - T_n) = \mathcal{N}(I - T)$ is reflexive. Now we prove that $\mathcal{R}(I_n - T_n)$ is closed. Therefore, let $y \in \mathcal{R}(T^n)$ such that y in the closure of $\mathcal{R}(I_n - T_n)$. Then, there exist a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathcal{R}(T^n)$ such that

$$(I - T)x_k \to y.$$

Since $\mathcal{R}(I - T)$ is closed in X then, $\xi = (I - T)(z), z \in X$. Hence, $z = \xi + T(z)$. On the other hand, we have $\xi \in \mathcal{R}(T^n)$, then there exists $h \in \Delta(T)$ such that $\xi = T^n(h)$ then, $z = T^n(h) + T(z) = T(T^{n-1}(h) + z)$. Hence, $z \in \mathcal{R}(T)$ and by recurrence it follows that $z \in \mathcal{R}(T^n)$ and hence $\xi \in \mathcal{R}(I_n - T_n)$. Consequently, $\mathcal{R}(I_n - T_n)$ is closed in $\mathcal{R}(T^n)$. This combined with the fact that $\mathcal{N}(I_n - T_n)$ is reflexive allows us to deduce that $I_n - T_n$ is an upper generalized Fredholm operator. □

Theorem 3.7 ([7]). *Let X be a Banach space satisfying the property* $(\mathbf{H_1})$ *and $T \in \mathcal{C}(X)$. If $I - T$ is an upper generalized semi-Fredholm, then $T \in \mathcal{WDC}(X)$.*

In the next proposition, we specify conditions that ensure the persistence of the weak demicompactness property under restrictions.

Proposition 3.8. *Let X be a Banach space and $T \in \mathcal{L}(X)$ and let $n \in \mathbb{N}^*$ such that $\mathcal{R}(T^n)$ is closed satisfying the property* $(\mathbf{H_1})$ *and $I - T$ has a closed range. If T is weakly demicompact, then T_n is weakly demicompact.*

Proof. Since T is weakly demicompact, then Theorem 3.6 implies that $I_n - T_n$ is an upper generalized Fredholm operator. Furthermore, taking into consideration the assumption $\mathcal{R}(T^n)$ satisfies property $(\mathbf{H_1})$, we apply Theorem 3.7 to conclude that T_n is weakly demicompact. This completes the proof. □

Proposition 3.9. *Let X be a Banach space, and $T \in \mathcal{L}(X)$ such that $I - T$ has a closed range. Let m and n be two integers and suppose that $\mathcal{R}(T^n)$ and $\mathcal{R}(T^m)$ are closed having the property* $(\mathbf{H_1})$. *Then, T_n is weakly demicompact if and only if T_m is weakly demicompact.*

Proof. Since $I - T$ is prime with respect to T^n and T^m, it follows that

$$\mathcal{N}(I_n - T_n) = \mathcal{N}(I - T) = \mathcal{N}(I_m - T_m).$$

Taking into account that T_n is weakly demicompact operator, then Proposition 3.3 allows us to conclude that $\mathcal{N}(I_m - T_m)$ is reflexive. Since $\mathcal{R}(T^m)$ and $\mathcal{R}(I - T)$ are closed then using Lemma 2.7 leads to conclude that $\mathcal{R}((I - T)T^m) = \mathcal{R}(I_m - T_m)$ is closed. Furthermore, using Theorem 3.7 it follows that T_m is weakly demicompact operator. Conversely, we obtain the requested result by using the reasoning above and replacing m with n. □

Remark 3.10. The hypothesis of Proposition 3.9, which assumes that both $\mathcal{R}(T^m)$ and $\mathcal{R}(T^n)$ are closed, is crucial. In fact, there is no implication between the closedness of $\mathcal{R}(T^n)$ and $\mathcal{R}(T^m)$. We illustrate this with two counterexamples.

Let X be a Banach space and $T \in \mathcal{L}(X)$ such that $\mathcal{R}(T)$ is not closed and consider

$$A = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X \times X).$$

Clearly, $\mathcal{R}(A) = \mathcal{R}(T) \times \{0\}$ is not closed. However,

$$A^2 = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{R}(A^2) = \{(0,0)\}$ is closed.

Now, consider $X := C([0,1])$, and let T denote the Volterra operator defined by

$$(Tg)(t) := \int_0^t g(s)\,ds \quad \text{for all } g \in X \text{ and } t \in [0,1]\,.$$

And $S : X \times X \to X \times X$ given by

$$S(x,y) := (0, x + Ty) \quad \text{for all } x, y \in X.$$

Then, $\mathcal{R}(S) = \{0\} \times X$ is closed. However, $\mathcal{R}(S^2) = \{0\} \times \mathcal{R}(T)$ is not closed in $X \times X$. See [2, Page 280].

Remark 3.11. Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ satisfies the property $(\mathbf{H_1})$, then so does $\mathcal{R}(T^m)$ for each $m > n$. Indeed, let N be a closed reflexive subspace of $\mathcal{R}(T^m)$. Since $\mathcal{R}(T^m) \subset \mathcal{R}(T^n)$ and using the fact that $\mathcal{R}(T^n)$ satisfies the property $(\mathbf{H_1})$ we infer that there exists a closed subspace X_0 of $\mathcal{R}(T^n)$ such that

$$\mathcal{R}(T^n) = N \oplus X_0$$

and so

$$\mathcal{R}(T^m) = N \oplus X_0 \cap \mathcal{R}(T^m).$$

Which proves the result.

Theorem 3.12. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ satisfies the property $(\mathbf{H_1})$. Then the following assumptions are equivalent:*

(i) *$\mathcal{R}(T^n)$ is closed, $I_n - T_n$ has a closed range and T_n is weakly demicompact operator*;

(ii) *$\mathcal{R}((I-T)T^n)$ is closed and $\mathcal{N}(I-T)$ is reflexive.*

Proof. Since T_n is weakly demicompact operator, it follows that $I_n - T_n$ is an upper generalized semi-Fredholm operator. Taking into account that $\mathcal{R}((I-T)T^n) = \mathcal{R}(I_n - T_n)$ then, $\mathcal{R}((I-T)T^n)$ is closed. Now, since $I-T$ and T^n are prime we infer that $\mathcal{N}(I_n - T_n) = \mathcal{N}(I-T)$. Consequently, $\mathcal{N}(I-T)$ is reflexive. Conversely, since $I-T$ and T^n are prime and $\mathcal{R}((I-T)T^n)$ is closed then using

Lemma 2.7, we obtain that $\mathcal{R}(T^n)$ is closed. Moreover, since $\mathcal{R}((I-T)T^n) = \mathcal{R}(I_n - T_n)$, we infer that $\mathcal{R}(I_n - T_n)$ is closed. On the other hand, we have

$$\mathcal{N}(I_n - T_n) = \mathcal{N}(I - T),$$

then we deduce that $\mathcal{N}(I_n - T_n)$ is reflexive. This combined with the fact that $\mathcal{R}(I_n - T_n)$ is closed and $\mathcal{R}(T^n)$ satisfies the property $(\mathbf{H_1})$, and by applying Theorem 3.7 allows us to infer that T_n is weakly demicompact operator. □

Definition 3.13 ([4])**.** Let X be a Banach space and $T \in \mathcal{L}(X)$. The generalized essential spectral radius of T is given by:

$$r_{e,g}(T) = \sup\{|\lambda| \text{ such that } \lambda \in \sigma_{e_4,g}(T)\} = \lim_{n\to\infty} \overline{\omega}(T^n)^{\frac{1}{n}}.$$

Theorem 3.14. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed. If $r_{e,g}(T_n) < 1$, then T_n is weakly demicompact operator.*

Proof. Since $r_{e,g}(T_n) < 1$, it follows that there exists a positive integer m such that $\overline{\omega}(T_n^m) < 1$. Now, let $(y_k)_k$ be a bounded sequence of X such that $(I_n - T_n)y_k \rightharpoonup y$. We can write

$$I_n - T_n^m = \sum_{j=0}^{m-1} T_n^{m-1-j}(I_n - T_n).$$

Consider $K = \sum_{j=0}^{m-1} T_n^{m-1-j}$, then we have $z_k := (I_n - T_n^m)y_k \rightharpoonup Ky$. Taking into account that

$$\{y_k, k \in \mathbb{N}\} \subset \{z_k, k \in \mathbb{N}\} + \{T_n^m y_k, k \in \mathbb{N}\}.$$

It follows that,

$$\omega(\{y_k, k \in \mathbb{N}\}) \leqslant \omega(\{z_k, k \in \mathbb{N}\}) + \overline{\omega}(T_n^m)\omega(\{y_k, k \in \mathbb{N}\}).$$

Thus, we get $(1 - \overline{\omega}(T_n^m))\omega(\{y_k, k \in \mathbb{N}\}) \leqslant 0$. Since $\overline{\omega}(T_n^m) < 1$, we infer that $\omega(\{y_k, k \in \mathbb{N}\}) = 0$. Therefore, $\{y_k, k \in \mathbb{N}\}$ is relatively weakly compact. We conclude that the sequence $(y_k)_k$ has a convergent subsequence in $\mathcal{R}(T^n)$, then it follows that T_n is weakly demicompact operator. Which completes the proof. □

Lemma 3.15. *Let X and Y be Banach spaces and $T, S \in \mathcal{L}(X)$. If T is invertible and $S \in \Phi_g(X)$, then $TS \in \Phi_g(X)$.*

Proof. Since $\mathcal{N}(S)$ is reflexive, it follows that $\mathcal{N}(TS)$ is reflexive. Taking into account that T is invertible and $\mathcal{R}(S)$ is closed then $\mathcal{R}(TS)$ is closed. The map f defined by

$$\begin{array}{cccc} f & : & X/T\mathcal{R}(S) & \to & X/\mathcal{R}(S) \\ & & x^{\cdot} & \mapsto & \overline{T^{-1}x} \end{array}$$

is an isomorphism. Indeed, let us check that f is well defined. So, let $x, y \in X$ such that $x^{\cdot} = y^{\cdot}$, then $x - y \in T\mathcal{R}(S)$. Thus, $x - y = T(\alpha)$, for all $\alpha \in \mathcal{R}(S)$. Since T is invertible we infer that $T^{-1}x - T^{-1}y = \alpha \in \mathcal{R}(S)$. Thus $\overline{T^{-1}x} = \overline{T^{-1}y}$. Hence f is well defined. Moreover, if $f(x^{\cdot}) = \overline{0}$, thus $\overline{T^{-1}x} = \overline{0}$, $x \in T\mathcal{R}(S)$ it follows that $x^{\cdot} = 0^{\cdot}$. Then f is injective. Now, we will prove that f is surjective. For this purpose, let $\overline{y} \in X/\mathcal{R}(S)$, Take $\alpha \in \mathcal{R}(S)$ and $x = Ty - T\alpha$ such that $f(x^{\cdot}) = \overline{y}$. Hence, f is surjective. Consequently, Since $X/\mathcal{R}(S)$ is reflexive and f is an isomorphism, we deduce that $X/T\mathcal{R}(S)$ is reflexive. This establishes the result. □

Theorem 3.16. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that $\Omega \neq \varnothing$ is a connected open subset of $\mathbb{C}$ such that $\sigma(T) \subset \Omega$ and let $\Upsilon : \Omega \to \mathbb{C}$ be an analytic function such that $\Upsilon(0) = 0$. If there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed, $I_n - \Upsilon(I_n - T_n)$ has a closed range and $I_n - \Upsilon(I_n - T_n)$ is weakly demicompact, then $I - T \in \Phi_{g+}(X)$.*

Proof. Since $I_n - \Upsilon(I_n - T_n)$ is weakly demicompact, then by applying Corollary 3.4 it follows that $\Upsilon(I_n - T_n)$ is an upper generalized semi-Fredholm. Given that Υ is analytic on the compact set $\sigma(T_n)$ thus they are in finite number $\{\alpha_1, ..., \alpha_n\}$. It follows that

$$\Upsilon(y) = \prod_{k=1}^{n} (y - \alpha_k)^{m_k} g(y).$$

Where m_k denotes the multiplicity of α_k and g is an analytic function defined in a neighborhood of $\sigma(T_n)$ such that $g(y)$ has no zeros in $\sigma(T_n)$. Define $\nu(y) = \frac{1}{g(y)}$, where ν is an analytic function in a neighborhood of $\sigma(T_n)$. Consider

$$\prod_{k=1}^{n} (y - \alpha_k)^{m_k} = \Upsilon(y)g(y),$$

this fact together with Lemma 6.15 [17], shows that $\prod_{k=1}^{n}(I_n - T_n - \alpha_k)^{m_k} = \Upsilon(I_n - T_n)g(I_n - T_n)$. Since $\Upsilon(0) = 0$, it follows that there exists a complex polynomial P such that

$$P(I_n - T_n)(I_n - T_n) = \Upsilon(I_n - T_n)\nu(I_n - T_n).$$

Where $P(I_n - T_n) = \prod_{k=1, k \neq k_0}^{n}(I_n - T_n - \alpha_k)^{m_k}$. Since $\Upsilon(I_n - T_n)$ is an upper generalized semi-Fredholm it follows from Lemma 3.15 that $P(I_n - T_n)(I_n - T_n)$ is an upper generalized semi-Fredholm. Accordingly to [3, Theorem 3.8], we conclude that $I_n - T_n$ is an upper generalized semi-Fredholm. Hence, $\mathcal{R}((I - T)T^n) = \mathcal{R}(I_n - T_n)$ is closed. As $I - T$ and T^n are prime, then applying Lemma 2.7 we conclude that $\mathcal{R}(I - T)$ is closed. Again, since $I - T$ and T^n are prime, then $\mathcal{N}(I - T) \subset \mathcal{R}(T^n)$. Thus, $\mathcal{N}(I - T) = \mathcal{N}(I_n - T_n)$ is reflexive. Consequently, $I - T \in \Phi_{g+}(X)$. Which gives the desired result. □

4. Generalized Fredholm of matrix operators

Let X be a Banach space, $T \in \mathcal{C}(X)$ and υ be a MWNC in X. We define

$$\mathcal{H}_\upsilon(X) := \Big\{ T \in \mathcal{C}(X) \;\Big|\; A_n \in \mathcal{M}_{X_T} \text{ weakly closed in } X_T \text{ with } A_{n+1} \subset A_n \text{ and, if } \lim_{n\to+\infty} \upsilon(A_n) = 0 \text{ and } \lim_{n\to+\infty} \upsilon(T(A_n)) = 0, \text{ then } \bigcap_{i=0}^{+\infty} A_n \neq \varnothing. \Big\}$$

In what follows, let X and X_T be Banach spaces and $T \in \mathcal{H}_\upsilon(X)$, we make the following assumption

$$(\mathcal{P}_1) : X^* + X^* \circ T \text{ is dense in } (X_T)^*.$$

Theorem 4.1 ([10]). *Let X be a non-reflexive Banach space, υ be a regular MWNC in X and $T \in \mathcal{H}_\upsilon(X)$ satisfying the hypothesis $(\mathcal{P}_1)$. Let $S \in \mathcal{L}(X)$ and X_T satisfies the property $(\mathbf{H_1})$. The following statements hold:*

(i) *If $T \in \Phi_{g+}(X)$, $S \in \Phi_{g+}(X)$, and ST has a closed range, then $ST \in \Phi_{g+}(X)$;*
(ii) *If $T \in \Phi_{g+}(X)$ and $W \in \mathcal{W}(X)$, then $(T + W) \in \Phi_{g+}(X)$.*

Proposition 4.2 ([12]). *Let X and Y be Banach spaces. Assume that $X \times Y$ is non reflexive space. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and consider the 2×2 operator matrices*

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(Y, X)$. Then,

(i) *if $X \times Y$ has the property $(\mathbf{H})$, and $A \in \Phi_g(X)$ and $B \in \Phi_g(Y)$, then $M_C \in \Phi_g(X \times Y)$. For every $C \in \mathcal{L}(Y, X)$;*
(ii) *if $X \times Y$ has the property $(\mathbf{H_1})$, and $A \in \Phi_{g+}(X)$ and $B \in \Phi_{g+}(Y)$, then $M_C \in \Phi_{g+}(X \times Y)$. For every $C \in \mathcal{L}(Y, X)$;*
(iii) *if $(X \times Y)^*$ has the property $(\mathbf{H_1})$, and $A \in \Phi_{g-}(X)$ and $B \in \Phi_{g-}(Y)$, then $M_C \in \Phi_{g-}(X \times Y)$. For every $C \in \mathcal{L}(Y, X)$.*

Definition 4.3. Let X be a Banach spaces. An operator $T \in \mathcal{C}(X)$ is said to have a left generalized Fredholm inverse if there exists $A \in \mathcal{L}(X)$ such that $I - AT \in \mathcal{W}(X)$.

Theorem 4.4. *Let X be a non-reflexive Banach space, υ be a regular MWNC in X and $T \in \mathcal{H}_\upsilon(X)$ satisfying the hypothesis $(\mathcal{P}_1)$. Let $S \in \mathcal{L}(X)$ and X_T satisfies the property $(\mathbf{H_1})$. If for every $\lambda \in \Phi_{g+,T}(X)$, there exists a left generalized Fredholm inverse T_0 of $\lambda - T$ such that ST_0 is weakly demicompact and $(I - ST_0)(\lambda - T)$has a closed range. Then,*

$$\sigma_{e_1,g}(T + S) \subset \sigma_{e_1,g}(T).$$

Proof. Let $\lambda \notin \sigma_{e1,g}(T)$, then $\lambda - T \in \Phi_{g+}(X)$. Taking into account that T_0 is a left generalized Fredholm inverse of T. Then, there exists $W \in \mathcal{W}(X)$ such that $T_0(\lambda - T) = I - W$. Moreover, we have

$$\lambda - T - S = (I - ST_0)(\lambda - T) - SW$$

Using the fact that ST_0 is weakly demicompact we deduce from Theorem 3.4 that

$$(I - ST_0) \in \Phi_{g+}(X).$$

Furthermore, since $\lambda - T \in \Phi_{g+}(X)$ then in light of Theorem 4.1 we find that

$$(I - ST_0)(\lambda - T) \in \Phi_{g+}(X).$$

Now, by applying Theorem 4.1 and using the fact that $SW \in W(X)$ we conclude that

$$\lambda - T - S \in \Phi_{g+}(X).$$

Thus, we get $\lambda \notin \sigma_{e1,g}(T + S)$. □

In what follows, we are concerned with an operator which is formally defined by the block operator matrix

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

that acts on a product Banach space $X \times X$.

Let consider the following sum decomposition of the block operator matrix $\mathcal{A}$:

$$\mathcal{A} := \mathcal{T} + \mathcal{S},$$

where

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{and } \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{4.1}$$

Lemma 4.5. *Let X be a Banach space and $A, D \in \mathcal{C}(X)$ satisfying the hypothesis $(\mathcal{P}_1)$ then, $\mathcal{T}$ satisfying the hypothesis $(\mathcal{P}_1)$.*

Proof. Let $h \in ((X \times X)_{\mathcal{T}})^*$, that is $h : (\Delta(A) \times \Delta(D), \|\cdot\|_{\mathcal{T}}) \to \mathbb{R}$ a bounded linear form, then there exists $\kappa : X_A \to \mathbb{R}$ and $\chi : X_D \to \mathbb{R}$ two bounded linears forms such that $h(x, y) = \kappa(x) + \chi(y)$, where $\kappa(x) = h(x, 0)$ and $\chi(y) = h(0, y)$. Taking into account that A and D satisfying the assumption $(\mathcal{P}_1)$, then there exist $(\kappa_{1,n} + \kappa_{2,n} \circ A)_n \subset X^* + X^* \circ A$ such that $\kappa_{1,n} + \kappa_{2,n} \circ A \to \kappa$ and $(\chi_{1,n} + \chi_{2,n} \circ D)_n \subset X^* + X^* \circ D$ such that $\chi_{1,n} + \chi_{2,n} \circ D \to \chi$. Now, set $R_n(x, y) = U_n(x, y) + V_n \circ \mathcal{T}(x, y)$ where $U_n(x, y) = \kappa_{1,n}(x) + \kappa_{2,n}(y)$ and $V_n(x, y) = \chi_{1,n}(x) + \chi_{2,n}(y)$ for all $(x, y) \in X \times X$. Obviously, R_n, U_n and V_n are linears. Furthermore, $|U_n(x, y)| \leq (\|\kappa_{1,n}\| + \|\kappa_{2,n}\|)\|(x, y)\|$ and $|V_n(x, y)| \leq (\|\chi_{1,n}\| + \|\chi_{2,n}\|)\|(x, y)\|$. Therefore, $R_n(x, y) = \kappa_{1,n}(x) + \chi_{1,n}(y) + \kappa_{2,n}(Ax) + \chi_{2,n}(Dy) \to \kappa(x) + \chi(y) = h(x, y)$. We immediately conclude that

$$(X \times X)^* + (X \times X)^* \circ \mathcal{T} \text{ is dense in } ((X \times X)_{\mathcal{T}})^*.$$

Which proves the result. □

Lemma 4.6 ([7]). *Let X and Y be Banach spaces having the property* $(\mathbf{H_1})$*. Then, $X \times Y$ has the property* $(\mathbf{H_1})$*.*

Theorem 4.7. *Let X be a non-reflexive Banach space, υ be a regular MWNC in X and $A, D \in \mathcal{H}_\upsilon(X)$ satisfying the hypothesis $(\mathcal{P}_1)$. Let $\mathcal{S} \in \mathcal{L}(X)$ and X_A, X_D satisfies the property $(\mathbf{H_1})$. If for every $\lambda \in \Phi_{g+,\mathcal{T}}(X)$, there exists a left generalized Fredholm inverse $\mathcal{T}_0$ of $\lambda - \mathcal{T}$ such that $\mathcal{S}\mathcal{T}_0$ is weakly demicompact. Then,*

$$\sigma_{e_1,g}(\mathcal{T} + \mathcal{S}) \subset \sigma_{e_1,g}(A) \cup \sigma_{e_1,g}(D).$$

Proof. Let $\lambda \notin \sigma_{e_1,g}(A) \cup \sigma_{e_1,g}(D)$, then it follows that $\lambda - A \in \Phi_{g+}(X)$ and $\lambda - D \in \Phi_{g+}(X)$. Hence, $\lambda - \widehat{A} \in \Phi_{g+}(X_A, X)$ and $\lambda - \widehat{D} \in \Phi_{g+}(X_D, X)$. Now, the use of Lemma 4.6 we deduce that $X_A \times X_D$ has the property $(\mathbf{H_1})$ together with proposition 4.2 enables us to conclude that $\lambda - \mathcal{T}$ is an upper generalized Fredholm operator. Thus, from the fact that there exists a left generalized Fredholm inverse $\mathcal{T}_0$ of $\lambda - \mathcal{T}$ such that $\mathcal{S}\mathcal{T}_0$ is weakly demicompact and in view of Theorem 4.4 and Lemma 4.5, we infer that $\lambda - (\mathcal{T} + \mathcal{S})$ is an upper generalized Fredholm operator. Consequently, $\lambda \notin \sigma_{e_1,g}(\mathcal{T} + \mathcal{S})$. □

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Про обмеження слабко напівкомпактних операторів та узагальнену теорію Фредгольма

Omaima Kchaou and Bilel Krichen

Метою цієї статті є дослідження слабкої напівкомпактності оператора T_n, який є обмеженням лінійного оператора T на його область значень $\mathcal{R}(T^n)$, де T_n розглядаються як лінійні оператори з $\mathcal{R}(T^n)$ у себе, $n \in \mathbb{N}$. Також ми представляємо декілька результатів щодо верхніх узагальнених напівфредгольмових операторів, зосереджуючись на концепції слабко напівкомпактних операторів. Ми визначаємо умови на певні області значень, які забезпечують збереження властивості слабкої напівкомпактності при обмеженнях. Крім того, наше дослідження надає результати про збурення, що стосуються узагальненого істотного спектра Густафсона для операторних матриць 2×2.

Ключові слова: слабко S-напівкомпактні оператори, узагальнені оператори Фредгольма, узагальнений істотний спектр