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Simple Closed Geodesics on Regular Spherical Polyhedra

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This article provides a complete characterization of all simple closed geodesics on regular spherical octahedra and cubes. Additionally, estimates for the number of such geodesics on regular spherical tetrahedra are presented.

Key words: simple closed geodesic, regular tetrahedron, octahedron, cube, spherical space

Mathematical Subject Classification 2020: 52B05, 53C22, 52B10

1. Introduction

The characterization of simple closed geodesics on convex surfaces in Euclidean space constitutes a longstanding problem in geometry. A key distinction between smooth and non-smooth surfaces arises from the fact that geodesics cannot traverse singular points. Specifically, on convex polyhedra, geodesics cannot pass through vertices. The first results on this topic were presented by Alexandrov [1, 2], Pogorelov [11], and Toponogov [14].

For any simple closed geodesic γ on a polyhedron in Euclidean space, there exists an infinite family of geodesics that intersect the polyhedron edges in the same sequence as γ. These geodesics are considered equivalent to γ. Inside the development of the polyhedron, equivalent geodesics are parallel. Geodesics on the Euclidean polyhedron are called distinct if they are not equivalent to each other and no symmetry of the polyhedron maps one geodesic onto the other.

Fuchs and Fuchs [7, 8] provided a comprehensive and systematic analysis of all closed geodesics on regular polyhedra in Euclidean space. In particular, they showed that, excluding the tetrahedron, all regular polyhedra possess a finite number of distinct simple closed geodesics. Conversely, a regular tetrahedron in Euclidean space has an infinite number of distinct simple closed geodesics. Furthermore, Akopyan and Petrunin [3] showed that isosceles tetrahedra are the unique convex surfaces in Euclidean space that admits arbitrarily long simple closed geodesics.

In this paper, we study simple closed geodesics on regular spherical tetrahedra, octahedra, and cubes. In this case, the facets of polyhedra have constant curvature 1, and the intrinsic geometry of polyhedra depends on the value of the planar angle of the facets.

On a spherical polyhedron, if a simple closed geodesic γ exists, then γ is unique within its class of equivalents. Two geodesics on the spherical polyhedron are considered different if they cannot be mapped onto each other by any symmetry of the surface.

In joint work with Borisenko [6], we studied the behavior of simple closed geodesics on regular spherical tetrahedra. We established conditions for the existence of these geodesics in terms of bounds on the tetrahedron planar angle $\alpha \in (\pi/3, 2\pi/3)$. Notably, we demonstrated that, unlike in the Euclidean case, only a finite number of geodesics exists on any regular spherical tetrahedron. Borisenko [5] refined these results presenting a necessary and sufficient condition for the existence of a simple closed geodesic. For a comprehensive overview, we refer the reader to [13].

To ensure completeness, this paper presents a corollary of these results that provides an asymptotic of the number of different simple closed geodesics on a spherical tetrahedron with a planar angle α when $\alpha \to \pi/3$. Furthermore, we correct an inaccuracy in the result of the paper [6] concerning the number of different simple closed geodesics on the tetrahedron with the planar angle greater than $\pi/2$, as it was pointed out by an anonymous reviewer.

In contrast to a regular Euclidean tetrahedron, regular Euclidean octahedra and cubes exhibit two and respectively three simple closed geodesics. This follows from the tiling of the Euclidean plane. The key fact of the proof is that any geodesic intersecting a facet of the octahedron or cube twice must necessarily have a self-intersection.

In this work, we show that the aforementioned result holds true for regular spherical octahedra and spherical cubes. It is worth noting that there is no tiling of a unit two-dimensional sphere with regular triangles or squares for any planar angle α. Furthermore, a direct analogue of the key fact, established in Euclidean space, is not known in spherical settings. To prove our result, we analyze the domain enclosed by a simple closed geodesic on a spherical polyhedron. Convexity plays a crucial role in the proof as it ensures that geodesics cannot pass through the vertices of spherical polyhedra.

2. Main definitions

A *spherical triangle* is a convex polygon on a unit sphere bounded by three shortest lines. A triangle is regular if the lengths of its edges are equal. A regular triangle is uniquely determined by the value α of its inner angles. The length of its edges is equal to

$$a_t = \arccos\left(\frac{\cos\alpha}{1-\cos\alpha}\right). \tag{2.1}$$

The function a_t is monotonically increasing function on α, and

$$\lim_{\alpha\to\pi/3} a_t = 0, \qquad \lim_{\alpha\to 2\pi/3} a_t = \arccos(-1/3), \quad a_t(\pi/2) = \pi/2.$$

A *regular spherical tetrahedron* $A_1A_2A_3A_4$ is a closed convex polyhedron that consists of four regular spherical triangles glued by three in each vertex. A planar angle α of a regular spherical tetrahedron satisfies the conditions $\pi/3 < \alpha \leq 2\pi/3$.

If $\alpha = 2\pi/3$, then the tetrahedron is a unit two-dimensional sphere. There are infinitely many simple closed geodesics on it. In the following, we suppose that the angle α satisfies $\pi/3 < \alpha < 2\pi/3$ on the tetrahedron.

A *regular spherical octahedron* is a closed convex polyhedron that consists of eight spherical triangles glued by four in each vertex. A planar angle α of the octahedron satisfies $\pi/3 < \alpha \leq \pi/2$. If $\alpha = \pi/2$, then the octahedron is a unit two-dimensional sphere with infinitely many simple closed geodesics on it. In the following, we assume that the planar angle α of the octahedron satisfies $\pi/3 < \alpha < \pi/2$.

A *spherical square* is a convex polygon on a unit sphere that is bounded by the four shortest lines of the same length and has four equal angles α. The length of the edges of the spherical square is equal to

$$a_s = \arccos\left(\cot^2(\alpha/2)\right), \tag{2.2}$$

$$\lim_{\alpha\to\pi/2} a_s = 0, \qquad \lim_{\alpha\to 2\pi/3} a_s = \arccos(1/3).$$

For more convenience, we write the following formula for the length of the square diagonal d, which we will need later on:

$$d = \arccos\left(\frac{\cos^4(\alpha/2) - \cos^2\alpha}{\sin^4(\alpha/2)}\right), \tag{2.3}$$

$$\lim_{\alpha\to\pi/2} d = 0, \qquad \lim_{\alpha\to 2\pi/3} d = \arccos(-1/3).$$

A *spherical cube* is a closed convex polyhedron that consists of six spherical squares glued by three in each vertex. A planar angle α of a cube satisfies the conditions $\pi/2 < \alpha \leq 2\pi/3$. If $\alpha = 2\pi/3$, then the cube is a unit two-dimensional sphere and has infinitely many simple closed geodesics on it. In the following, we assume that the planar angle α satisfies $\pi/2 < \alpha < 2\pi/3$ on the cube.

By the gluing theorem, the regular spherical tetrahedron, octahedron and cube are Alexandrov spaces of curvature ≥ 1 (in sense of Alexandrov [2]). For all regular spherical polyhedra the inner geometry depends on the planar angle α.

A *geodesic* is a locally shortest curve $\gamma : [0,1] \to M$. The geodesic is called *closed* if $\gamma(0) = \gamma(1)$ and $\gamma'(0) = \gamma'(1)$. The geodesic is called *simple* if it has no points of self intersection, i.e., the map $\gamma : (0,1) \to M$ is injective.

A geodesic has the following properties on a convex polyhedron:

1) on the facets, the segments of the geodesic are the locally shortest curves between its points on the edges;
2) crossing an edge, the geodesic forms equal angles with this edge on the adjacent facets;
3) the geodesic cannot pass through a vertex of the convex polyhedron [1].

In what follows, the words *tetrahedron*, *octahedron* or *cube* refer to a regular spherical tetrahedron, octahedron, or spherical cube respectively unless the opposite is specified. The word *geodesic* refers to a simple closed geodesic.

In Euclidean space, the facets of a polyhedron can be unfolded onto the plane sequentially in the same order in which they are traversed by a geodesic. The resulting polygon is called the *development* of the polyhedron. The geodesic itself unfolds into a straight line segment inside the development.

Similarly, spherical polyhedra can be unfolded onto a unit sphere creating a spherical development. The geodesic in this case unfolds locally into an arc of a great circle on the sphere. However, in this context, careful consideration should be given to the global embedding of the development on the sphere.

3. Properties

Borisenko [4] proved a generalization of the Toponogov theorem [14] to the case of two-dimensional Alexandrov space.

Let G be a domain homeomorphic to a disc and bounded by a rectifiable curve γ in a two-dimansional Alexandrov space of curvature $\geq c$ (in sense of Alexandrov). A curve γ is called λ-convex with $\lambda > 0$ if any subarc γ_0 of γ satisfies

$$\tau(\gamma_0)/s(\gamma_0) \geq \lambda > 0,$$

where $\tau(\gamma_0)$ is the integral geodesic curvature (the swerve) of subarc γ_0 and $s(\gamma_0)$ is the length of γ_0.

Theorem 3.1 (Borisenko [4])**.** *Let G be a domain homeomorphic to a disc and G lies in a two-dimansional Alexandrov space of curvature $\geq c$ (in sense of Alexandrov). If the boundary curve γ of G is λ-convex, $\lambda > 0$ and $c + \lambda^2 > 0$, then the length $s(\gamma)$ of γ satisfies*

$$s(\gamma) \leq \frac{2\pi\sqrt{|c|}}{\sqrt{c+\lambda^2}}.$$

The equality holds if and only if the domain G is a disc on the plane of constant curvature c.

A convex spherical polyhedron with unit-curvature facets is a two-dimensional Alexandrov space of curvature ≥ 1 in sense of Alexandrov. A simple closed geodesic γ encloses a domain homeomorphic to a disc with $\lambda = 0$. In this case, we invoke the following corollary from Theorem 3.1

Lemma 3.2 (The length of a geodesic)**.** *The length of a simple closed geodesic on a convex spherical polyhedron is $< 2\pi$.*

Two geodesics are called *equivalent* if they cross the edges of a polyhedron in the same sequence. The developments of the polyhedron along these geodesics are equal to polygons with equal labelings.

Lemma 3.3 (Uniqueness of a geodesic)**.** *Assume there are two simple closed geodesics γ_1 and γ_2 on a spherical polyhedron. If γ_1 and γ_2 are equivalent, then they coincide.*

Proof. Initially, let us assume that the geodesics γ_1 and γ_2 do not intersect. In this case, they enclose an annular region Ω that does not contain any vertex of the polyhedron. Consequently, Ω is locally isometric to a unit sphere. From the Gauss-Bonnet theorem, it follows that the integral of the curvature over the area of Ω equals zero. Hence, γ_1 and γ_2 must coincide.

Conversely, if γ_1 and γ_2 intersect, they must have at least two points of intersection Z_k, $k = 1, 2, \dots$. Between any two consecutive intersection points, Z_k and Z_{k+1}, geodesics bound a region Ω_k which remains free of any polyhedron vertices. Furthermore, Ω_k is isometric to a lune on a unit sphere, $k = 1, 2, \dots$ (see Fig. 3.1). This implies that the length of the geodesic segment belonging to the boundary of Ω_k for both γ_1 and γ_2 equals π. Consequently, the total length of the geodesic γ_i is $\geq 2\pi$, which contradicts Lemma 3.2. □

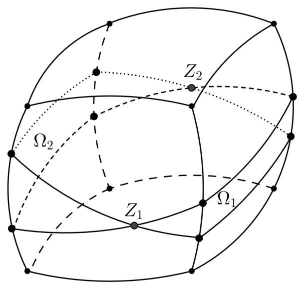


Fig. 3.1

4. Tetrahedron

Any simple closed geodesic γ on a tetrahedron has p vertices on each of two opposite edges of the tetrahedron, q vertices on each of other two opposite edges, and $(p+q)$ vertices on each of the remaining two opposite edges. The integers (p, q) are coprime and satisfy $0 \leq p < q$. The pair (p, q) is called the *type* of the geodesic and it uniquely defines one [7, 12].

In [6], for $1 \leq p < q$, the following necessary condition for the existence of a geodesic of type (p, q) was proved.

Theorem 4.1 (Borisenko and Sukhorebska [6])**.** *If the planar angle α of a regular spherical tetrahedron satisfies*

$$\alpha > 2\sin^{-1}\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}}, \tag{4.1}$$

then there is no simple closed geodesic of type (p, q) on the tetrahedron.

In [6], there was also found a small $\varepsilon > 0$ depending on (p, q) such that a simple closed geodesic of type (p, q) exists on the tetrahedron with a planar angle $\alpha < \pi/3 + \varepsilon$. Using this result, Borisenko [5] proved the necessary and sufficient condition for the existence of a geodesic on a spherical tetrahedron. This result then yields a sufficient condition for the edge length of the tetrahedron to contain a geodesic of type (p,q).

Theorem 4.2 (Borisenko [5])**.** *If the length a_t of the edge of a regular spherical tetrahedron satisfies the inequality*

$$a_t < 2\sin^{-1}\frac{\pi}{\sqrt{p^2+pq+q^2}+\sqrt{(p^2+pq+q^2)+2\pi^2}}, \tag{4.2}$$

then this tetrahedron has a simple closed geodesic of type (p,q).

Theorem 4.1 implies that the number of simple closed geodesics on a regular spherical tetrahedron is finite. From Theorem 4.2, it follows that if $p,q\to\infty$, then α tends to $\pi/3$.

The geodesics of types $(0,1)$ and $(1,1)$ were considered separately in [6]. It was demonstrated that the geodesic $\gamma_{(0,1)}$ of type $(0,1)$ exists on the tetrahedron for the entire range of the planar angle $\alpha\in(\pi/3,2\pi/3)$. A geodesic $\gamma_{(1,1)}$ of type $(1,1)$ exists on the tetrahedron with the planar angle satisfying $\alpha\in(\pi/3,\pi/2)$. When $\alpha\to\pi/2$ from below, the distance between the geodesic $\gamma_{(1,1)}$ and the tetrahedron vertices converges to zero. At the limit $\alpha=\pi/2$, the geodesic merges with the edges of the tetrahedron.

Lemma 4 from [6] states that the geodesic $\gamma_{(0,1)}$ of type $(0,1)$ is the unique simple closed geodesic on a regular tetrahedron with the planar angle $\alpha\in[\pi/2,2\pi/3)$. However, the reviewer pointed out an inaccuracy in this result. Specifically, there exists another simple closed geodesic σ different from $\gamma_{(0,1)}$ on the tetrahedron with the planar angle $\alpha>\pi/2$. To construct σ, let us denote the vertices of the tetrahedron as $A_1,\dots,A_4$. Since $\alpha>\pi/2$, the length of the tetrahedron edge $a_t>\pi/2$. Let B_1, B_2 and B_3 be the points on the edges A_4A_1, A_4A_2 and A_4A_3, respectively, such that the length A_4B_i is equal to $\pi/2$ for $i=1,2,3$. The closed curve $B_1B_2B_3B_1$ intersects the tetrahedron edges orthogonally and constitutes a simple closed geodesic σ on it. If $\alpha=\pi/2+\varepsilon$ and $\varepsilon\to0$, then σ converges to the boundary of the facet $A_1A_2A_3$ and coincides with it when $\alpha=\pi/2$.

The following theorem supplements and summarizes the results on the number of different simple closed geodesics on a regular spherical tetrahedron.

Theorem 4.3. *Let $N(\alpha)$ be a number of different simple closed geodesics on a regular spherical tetrahedron with a planar angle α.*

1. *If $\alpha=\pi/3+\delta$, then*

$$\frac{\sqrt{3}}{8}\frac{1}{\delta}+O\left(\frac{1}{\sqrt{\delta}}\ln\frac{1}{\sqrt{\delta}}\right)<N\left(\frac{\pi}{3}+\delta\right)<\frac{1}{2\sqrt{3}}\frac{1}{\delta}+O\left(\frac{1}{\sqrt{\delta}}\ln\frac{1}{\sqrt{\delta}}\right)$$

   when $\delta\to0$.
2. *If $\alpha=\pi/2-\varepsilon$, then $N(\pi/2-\varepsilon)=2$ for a small $\varepsilon>0$.*
3. *If $\alpha=\pi/2$, then $N(\pi/2)=1$.*
4. *If $\alpha=\pi/2+\varepsilon$, $0<\varepsilon<\pi/6$, then $N(\pi/2+\varepsilon)=2$.*

Proof. The proof of 2–4 was previously considered in this text. Let us show 1 now.

Fix the angle $\alpha \in (\pi/3, \pi/2)$ and consider a geodesic $\gamma_{p,q}$ of type (p,q) on a tetrahedron with the planar angle α. It follows from Theorem 4.2 that the geodesic of type (p,q) exists if

$$p^2 + pq + q^2 < \frac{\pi^2}{4} \frac{\left(2\sin^2(a_t/2) - 1\right)^2}{\sin^2(a_t/2)}. \tag{4.3}$$

Using (2.1), we can calculate

$$\sin(a_t/2) = \frac{4\sin^2(\alpha/2) - 1}{4\sin^2(\alpha/2)}.$$

Thus, equation (4.3) can be written as follows:

$$p^2 + pq + q^2 < \frac{\pi^2 \cos^2\alpha}{4\sin^2(\alpha/2)\left(4\sin^2(\alpha/2) - 1\right)} =: f(\alpha). \tag{4.4}$$

Let $\psi_1(\alpha)$ be the number of pairs (p,q) such that p,q are coprime, $0 < p \le q$ and

$$p^2 + pq + q^2 < f(\alpha).$$

Hence,

$$\psi_1(\alpha) < N(\alpha). \tag{4.5}$$

Let $c_1(\alpha)$ be the number of pairs (p,q) such that p,q are coprime, $0 < p \le q$ and

$$p + q < \sqrt{f(\alpha)}.$$

Consider (p,q) as the coordinates of the Euclidean plane. The curve $p^2+pq+q^2 = f(\alpha)$ is an ellipse with the focal points $\left(\mp\sqrt{2f(\alpha)/3}, \pm\sqrt{2f(\alpha)/3}\right)$. The ellipse intersects axes $p = 0$, $q = 0$ at the points $(\pm\sqrt{f(\alpha)}, 0)$, $(0, \pm\sqrt{f(\alpha)})$. Since the domain $p + q \le \sqrt{f(\alpha)}$ and $p, q, \ge 0$ lies inside the ellipse, it follows that

$$c_1(\alpha) < \psi_1(\alpha) < N(\alpha). \tag{4.6}$$

Euler's function $\varphi(n)$ is equal to the number of positive integers not greater than n and prime to $n \in \mathbb{N}$. From [9, Theorem 330], it is known that

$$\sum_{n=1}^{x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \ln x). \tag{4.7}$$

The error term $O(x \ln x) < Cx \ln x$ when $x \to +\infty$. In [10], there is an improvement on the error term.

If $(p,q) = 1$ and $p + q = n$, then $(p,n) = 1$ and $(q,n) = 1$. Thus the set of integers not greater than and prime to n are separated into the pairs of coprime integers (p,q) such that $p < q$ and $p + q = n$. It follows that $\varphi(n)$ is even and

$$c_1(\alpha) = \frac{1}{2} \sum_{n=1}^{\sqrt{f(\alpha)}} \varphi(n). \tag{4.8}$$

To determine the asymptotic behavior of the function $c_1(\alpha)$ as $\alpha \to \pi/3$, we employ a Taylor series expansion of the function $f(\alpha)$ at the point $\alpha = \pi/3$. Consider a small parameter $\delta > 0$ such that $\alpha = \pi/3 + \delta$, then

$$f\left(\frac{\pi}{3} + \delta\right) = \frac{\pi^2}{4\sqrt{3}}\frac{1}{\delta} - \frac{19\,\pi^2}{24} + O(\delta) \quad \text{when } \delta \to 0.$$

Using equations (4.7) and (4.8), we have

$$\begin{aligned} c_1\left(\frac{\pi}{3} + \delta\right) &= \frac{3}{2\pi^2} f\left(\frac{\pi}{3} + \delta\right) + O\left(\sqrt{f\left(\frac{\pi}{3} + \delta\right)} \ln \sqrt{f\left(\frac{\pi}{3} + \delta\right)}\right) \\ &= \frac{\sqrt{3}}{8}\frac{1}{\delta} + O\left(\frac{1}{\sqrt{\delta}} \ln \frac{1}{\sqrt{\delta}}\right) \quad \text{when } \delta \to 0. \end{aligned}$$

From inequality (4.6), we get the estimation

$$\frac{\sqrt{3}}{8}\,\frac{1}{\delta} + O\left(\frac{1}{\sqrt{\delta}} \ln \frac{1}{\sqrt{\delta}}\right) < N\left(\frac{\pi}{3} + \delta\right) \quad \text{when } \delta \to 0.$$

To find an estimation on $N(\alpha)$ from above, we use the estimation from Theorem 4.1. If geodesic $\gamma_{p,q}$ on a tetrahedron exists, then the pair of coprime integers (p, q) satisfies

$$p^2 + pq + q^2 < \frac{\pi^2 \sin^2(\alpha/2)}{4\sin^2(\alpha/2) - 1} =: g(\alpha). \tag{4.9}$$

Let $\psi_2(\alpha)$ be the number of pairs (p, q) such that p, q are coprime, $0 < p \le q$ and

$$p^2 + pq + q^2 < g(\alpha).$$

Hence,

$$N(\alpha) < \psi_2(\alpha) + 1. \tag{4.10}$$

Notice that $\psi(\alpha)$ does not include the pair $(0, 1)$. Since the geodesic of type $(0, 1)$ exists on the tetrahedron with any planar angle $\pi/3 < \alpha < 2\pi/3$, we add 1 on the right side of (4.10). The tangent line to ellipse at the point $\left(\sqrt{g(\alpha)/3}, \sqrt{g(\alpha)/3}\right)$ satisfies

$$p + q = 2\sqrt{g(\alpha)/3}.$$

Let $c_2(\alpha)$ be the number of the pairs (p, q) such that p, q are coprime, $0 < p \le q$ and

$$p + q < 2\sqrt{g(\alpha)/3}.$$

Since the ellipse lies inside the half-space $p + q \le 2\sqrt{g(\alpha)/3}$, it follows that

$$N(\alpha) < \psi_2(\alpha) + 1 < c_2(\alpha) + 1. \tag{4.11}$$

Using (4.7), we have

$$c_2(\alpha) = \frac{1}{2} \sum_{n=1}^{2\sqrt{g(\alpha)/3}} \varphi(n). \tag{4.12}$$

By analogy to the preceding case, we consider a Taylor series expansion of the function $g(\alpha)$ at the point $\alpha = \pi/3$ to find the asymptotic of the function $c_2(\alpha)$ as $\alpha \to \pi/3$,

Consider a small parameter $\delta > 0$ such that $\alpha = \pi/3 + \delta$, then

$$g\left(\frac{\pi}{3} + \delta\right) = \frac{\pi^2}{4\sqrt{3}}\,\frac{1}{\delta} - \frac{5\,\pi^2}{24} + O(\delta) \quad \text{when } \delta \to 0.$$

Combining with (4.7), (4.11) and (4.12), we have

$$N\left(\frac{\pi}{3} + \delta\right) < \frac{1}{2\sqrt{3}}\frac{1}{\delta} + O\left(\frac{1}{\sqrt{\delta}}\ln\frac{1}{\sqrt{\delta}}\right) \quad \text{when } \delta \to 0. \qquad \square$$

5. Octahedron

Consider a regular octahedron with a planar angle $\alpha \in (\pi/3, \pi/2)$ and vertices labeled from A_1 to A_6, where $A_1A_2A_3A_4$ form a square facet. Let A_5 and A_6 represent the vertices opposite to this square facet. A geodesic γ divides the surface of the octahedron into two closed domains D_1 and D_2. Since γ does not pass through any vertex of the octahedron, each vertex belongs to either domain D_1 or D_2 and not to both.

Lemma 5.1. *If a domain D_i, $i = 1, 2$, contains a vertex of the octahedron, then it must necessarily include at least one edge emanating from that vertex.*

Proof. Given that the planar angle of the octahedron $\alpha \in (\pi/3, \pi/2)$, and the length of its edge is $< \pi/2$, the geodesic γ cannot sequentially intersect more than three edges emanating from one vertex. Consequently, each domain D_i contains at least two vertices of the octahedron. Without loss of generality, we can assume that the domain D_1 contains at most three vertices of the octahedron. If it is not true, we can consider the domain D_2 instead. Suppose D_1 encloses a vertex A_5 inside and does not contain any edge emanating from it. In this case, the geodesic γ must intersect all edges A_5A_i, $i = 1, \dots, 4$ at least once. Let X_i denote the closest to A_5 point of γ on the edge A_5A_i, $i = 1, \dots, 4$.

First, we assume that there is no segments X_iX_{i+1} of γ within the facet $A_5A_iA_{i+1}$, where $i = 1, \dots, 4$ (if $i + 1 > 4$, then take $i + 1 \mod 5$). On the facet $A_5A_1A_2$, the geodesic γ traverses from the point X_1 to Y_1 and from X_2 to Y_2. If Y_1 belongs to the edge A_5A_2, it follows that $\text{dist}(A_5, Y_1) > \text{dist}(A_5X_2)$, as X_2 represents the closest to A_5 point of γ on A_5A_2. This, however, necessitates an intersection between the segments X_2Y_2 and X_1Y_1, which leads to a contradiction. Thus Y_1 and Y_2 lie on the edge A_1A_2 and

$$\text{dist}(A_1, Y_1) < \text{dist}(A_1, Y_2). \tag{5.1}$$

Similarly, on the facet $A_5A_2A_3$, the segment of γ starting at X_2 traverses to a point Q_2 on the edge A_2A_3, and on the facet $A_5A_4A_1$, the segment of γ starting at X_1 traverses to a point Q_1 on the edge A_1A_4. Now, consider the facet $A_1A_2A_6$. The segment of γ traverses from the point Y_1 to the point Z_1 that must lie on the edge A_6A_2. Otherwise, γ would intersect all four edges emanating from the vertex A_1 sequentially at the points Q_1, X_1, Y_1, Z_1, resulting in a contradiction. By the same argument, the segment of γ goes from Y_2 to the point Z_2 at the edge A_1A_6. Equation (5.1) implies that the segments Y_1Z_1 and Y_2Z_2 intersect, which also leads to a contradiction.

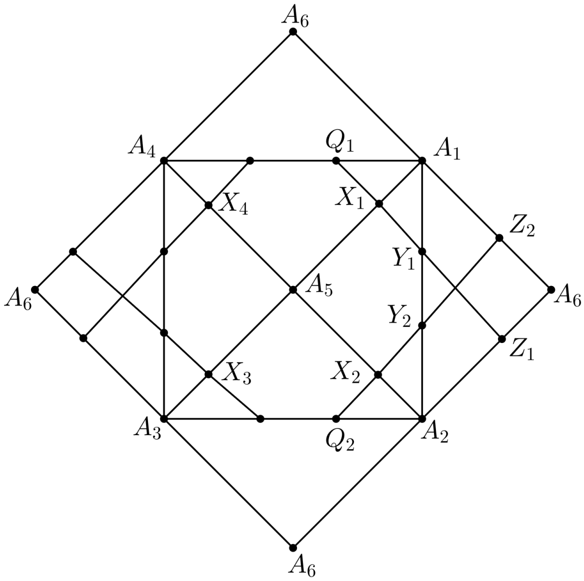


Fig. 5.1

Consequently, we assume that there exists a segment X_1X_2 of γ inside $A_5A_1A_2$. Let us first assume that there are no geodesic segments X_2X_3, X_3X_4 or X_4X_1 within their facets. Applying the same argument as before, it can be demonstrated that the geodesic γ traverses from the point X_4 to the edge A_3A_4 and subsequently to the edge A_3A_6, and from the point X_3 to the edge A_3A_4 and subsequently the edge A_4A_6. This leads to a contradiction (see Fig. 5.1).

Therefore, at least one of the segments X_2X_3, X_3X_4 or X_4X_1 must exist. If two of them are present, then γ would necessarily intersect all four edges emanating from A_5, which is not possible.

Case 1. Assume there exists the segment X_3X_4 of γ. Then, on the facet $A_5A_2A_3$, the geodesic γ traverses from the point X_2 to the point Y_2 at the edge A_2A_3 and from the point X_3 to the point Y_3 on A_2A_3, where $\mathrm{dist}(A_2, Y_2) < \mathrm{dist}(A_2, Y_3)$. Notice that no other points of γ intersect the edge A_2A_3 between Y_2 and Y_3. If it is not true and there exists a point Q of γ on the edge A_2A_3 between Y_2 and Y_3, then the geodesic must have a segment QP emanating from the point Q within the facet $A_2A_3A_5$. As X_2 and X_3 represent the points of γ on their respective edges that are closest to the vertex A_5, and since QP cannot intersect X_2Y_2 or X_3Y_3, a contradiction arises.

Denote by γ_1 a part of the geodesic that starts at Y_2 on the facet $A_6A_2A_3$, traverses the octahedron surface and ends at the point Y_3. Let σ_1 denote the segment Y_2Y_3 of the edge A_2A_3. Consequently, γ_1 and σ_1 define the boundary of the domain G_1 on the octahedron.

In a similar way, on the facet $A_5A_4A_1$, the geodesic goes from the point X_1 to the point Y_1 on the edge A_1A_4 and from the point X_4 to the point Y_4 also on A_1A_4, and $\text{dist}(A_4, Y_4) < \text{dist}(A_4, Y_1)$. As before, it can be shown that no other points of the geodesic intersect the edge A_1A_4 between Y_1 and Y_4. Let γ_2 denote a part of the geodesic that starts at Y_1 on A_1A_4, traverses the surface in the direction opposite to X_1 and ends at Y_4. Let σ_2 denote the segment Y_1Y_4 on the edge A_1A_4. The segments γ_2 and σ_2 bound the domain G_2 on the octahedron. (see Fig. 5.2(a)).

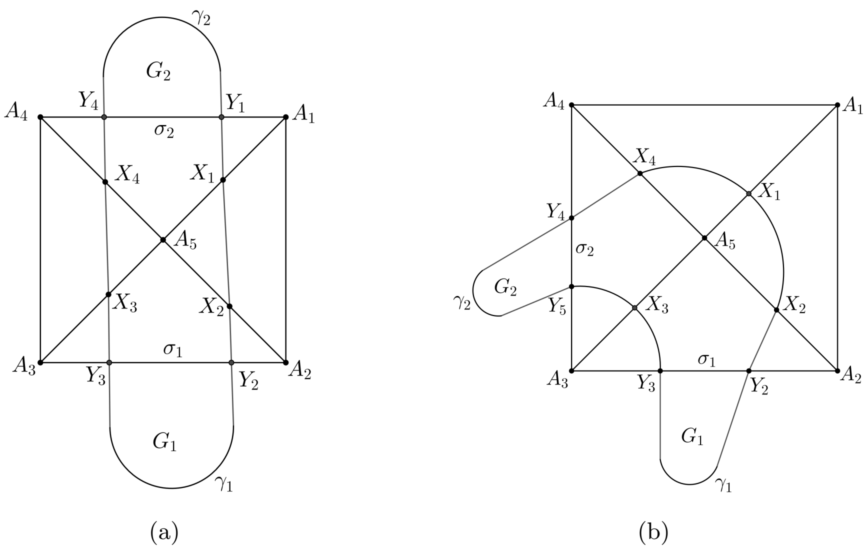


Fig. 5.2

Case 2. Consider now the case, where the segment X_1X_4 exists on the facet $A_5A_4A_1$. If, instead, the segment X_3X_4 exists on $A_5A_3A_4$, the argument is similar.

Analogously to Case 1, on the facet $A_5A_2A_3$, the geodesic γ goes from the point X_2 to a point Y_2 at the edge A_2A_3, and from the point X_3 to a point Y_3 on A_2A_3, and $\text{dist}(A_2, Y_2) < \text{dist}(A_2, Y_3)$. As before, let γ_1 be a part of the geodesic that starts at Y_2 on the facet $A_6A_2A_3$, goes along the surface of the octahedron and comes to the point Y_3. And let σ_1 be the segment Y_2Y_3 of the edge A_2A_3. Then γ_1 and σ_1 bound the domain G_1 on the octahedron.

On a facet $A_5A_4A_3$, the geodesic has the segment X_4Y_4. If Y_4 lies on the edge A_5A_3, then $\text{dist}(A_5, Y_4) > \text{dist}(A_5, X_3)$. But then the segment of γ emanating from X_3 intersects X_4Y_4 since X_4 is the closest to A_5 point of γ on A_5A_4. Hence Y_4 lies on A_4A_3. The geodesic goes from the point X_3 to a point Y_5 on the edge A_4A_3 and $\text{dist}(A_4, Y_4) < \text{dist}(A_4, Y_5)$. We denote by γ_2 the part of the geodesic that starts at Y_4 on A_4A_3 in opposite direction to X_4, traverses the surface of the octahedron and ends at Y_5. Denote by σ_2 the segment Y_4Y_5 on the edge A_4A_3. The segments γ_2 and σ_2 bound the domain G_2 on the octahedron (see Fig. 5.2(b)).

In both cases, we obtain two domains G_i, $i = 1, 2$, bounded by the segment γ_i of the geodesic γ and the segment of the octahedron edge σ_i. The domains G_i,

$i = 1, 2$, are locally convex domains homeomorphic to a disc. On the octahedron, the domains G_1 and G_2 do not intersect since γ_1 and γ_2 do not intersect.

If G_i does not contain any vertex of the octahedron, then G_i forms a lune on a unit sphere. In this case, the length of both σ_i and γ_i are equal to π.

However, since σ_i is a part of the octahedron edge, from equation (2.1) it follows that its length is less than $\pi/2$. Thus we get a contradiction.

Hence, G_i should have at least one octahedron vertex, $i = 1, 2$. Since G_i is a part of the domain D_1, and D_1 has at most three octahedron vertices, it follows that G_i contains exactly one vertex A_{s_i}, $s_i \in \{1, 2, 3, 4, 6\}$ of the octahedron and has no edges emanating from A_{s_i}. Since there is no points of the geodesic on σ_i, then A_{s_1} and A_{s_2} are different vertices of the octahedron, otherwise γ_1 and γ_2 intersect.

Consider the point A_{s_1} inside G_1. The behavior of the geodesic around A_{s_1} is similar to the behavior around A_5. Hence, the domain G_1 is divided into three parts G'_1, G'_2 and $G_1 \backslash (G'_1 \cup G'_2)$ with two segments σ'_1 and σ'_2 of the edges. The domain $G_1 \backslash (G'_1 \cup G'_2)$ contains the vertex A_{s_1}, and G'_1, G'_2 do not intersect and do not have any vertex of the octahedron. Since there is no points of geodesic on σ_1, then σ_1 belongs to the boundary of one of the domains G'_1 or G'_2, for example, to G'_2 (see Fig. 5.3). Consequently, the perimeter of G'_1 is a subarc γ'_1 of γ_1 and the subarc σ'_1 of the octahedron edge. Since G'_1 does not have any vertex of octahedron, then G'_1 forms a lune on a unit sphere. Hence, the length of σ'_1 equals π, which leads to a contradiction. $\square$

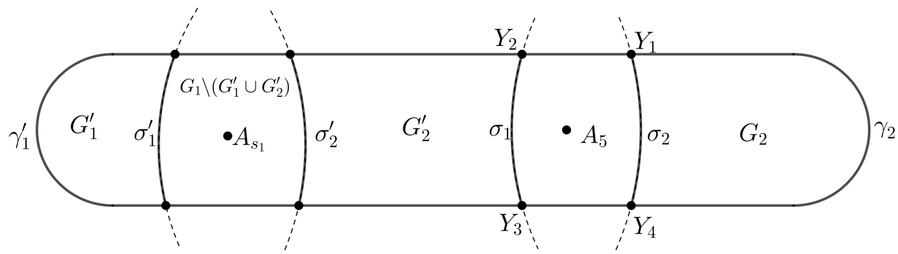


Fig. 5.3

Lemma 5.2. *A domain D_i, $i = 1, 2$, contains three vertices connected with two edges on the octahedron.*

Proof. By Lemma 5.1, if the domain D_i contains only two vertices of the octahedron, then D_i has an edge connecting these vertices.

Suppose the domain D_1 contains just one edge of the octahedron, for example, A_4A_5. Denote the vertices of γ as X_1, X_2, X_3, X_4, X_5, and X_6 on the edges A_5A_1, A_5A_2, A_5A_3, A_3A_4, A_4A_6, and A_1A_4, respectively.

Consider an isometry r of the octahedron defined as a rotation on the angle π around a line A_1A_3. Under this isometry, the geodesic γ is mapped to a geodesic σ which encloses a domain containing the edge $r(A_4A_5) = A_2A_6$. Denote by Y_k the vertices of σ such that $Y_k = r(X_k)$ (see Fig. 5.4).

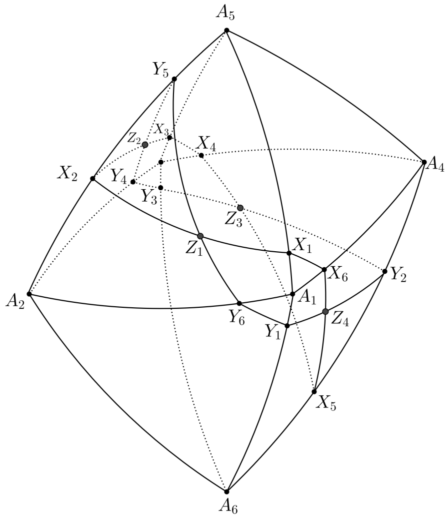


Fig. 5.4

Given that $r(A_6A_4) = A_2A_5$, $r(A_1A_6) = A_1A_5$ and $r(A_3A_6) = A_3A_5$, it follows that $\operatorname{dist}(A_5, X_2) = \operatorname{dist}(A_6, Y_2)$ and $\operatorname{dist}(A_4, X_5) = \operatorname{dist}(A_2, Y_5)$. Consequently, if γ and σ intersect, they must also intersect at four distinct points: Z_1, Z_2, Z_3, and Z_4 located on the facets $A_1A_2A_5$, $A_2A_3A_5$, $A_3A_4A_6$, and $A_1A_4A_6$, respectively.

The segments $Z_1Y_5Z_2$ and $Z_1X_2Z_2$ bound a lune that does not enclose any vertex inside. Since the facets of the octahedron are the domains of the unit sphere, the length of $Z_1X_2Z_2$ is equal to π. The same holds for the segments $Z_4Y_2Z_3$ and $Z_4X_5Z_3$. Consequently, the length of the geodesic γ exceeds 2π, which contradicts Lemma 3.2.

Therefore, γ and σ must not intersect. From the Gauss-Bonnet theorem, it follows that the integral of the curvature over the area enclosed by γ and σ equals zero. However, since the octahedron has positive curvature at every point, it follows that the geodesics γ and σ must coincide after the rotation r. This leads to a contradiction as γ cannot traverse through the vertex of the octahedron.

Consequently, the domain D_1 contains exactly three vertices of the octahedron. According to Lemma 5.1, at least one edge emanating from each vertex must lie within D_1. Hence, D_1 contains at least two edges of the octahedron connecting three vertices. The same is true for the domain D_2. □

Theorem 5.3. *There are only two different simple closed geodesics on regular spherical octahedra.*

Proof. Consider a regular spherical octahedron with a planar angle $\alpha \in (\pi/3, \pi/2)$. A simple closed geodesic γ divides the surface of the octahedron into two closed domains D_1 and D_2. Lemma 5.2 states that each domain D_i contains three vertices of the octahedron connected by two edges. Two distinct configurations are possible for two edges emanating from a single vertex:

1) the endpoints of these edges are connected by another edge forming a facet,

2) the endpoints of these edges are are not connected with an edge.

Type 1. Let $X_1, \dots, X_6$ denote the midpoint of the edges A_1A_2, A_2A_5, A_5A_3, A_3A_4, A_4A_6, and A_6A_1, respectively. These points are then sequentially connected by line segments traversing the facets forming a closed curve with X_6 connected to X_1. As X_i, $i = 1, \dots, 6$, are the midpoints of their edges, the broken line $X_1X_2X_3X_4X_5X_6$ constitutes a simple closed geodesic γ on the octahedron with a planar angle $\alpha \in (\pi/3, \pi/2)$ (see Fig. 5.5). The domain D_1 contains the facet $A_1A_4A_5$ and D_2 encloses the facet $A_2A_3A_6$.

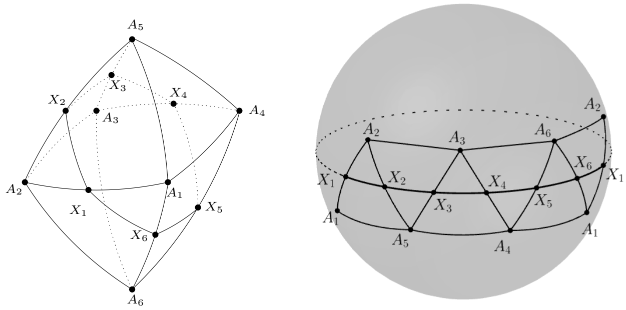


Fig. 5.5

If we consider any simple closed broken line σ on the octahedron that does not intersect the facets $A_1A_4A_5$ and $A_2A_3A_6$, then this broken line is equivalent to γ. In fact, assume σ starts at the edge A_1A_2. Then σ can intersect only the edge A_2A_5 and cannot go to A_1A_5. After it, σ traverses to the edge A_3A_5 since it cannot go to A_2A_3. In a similar way, one can deduce that σ should intersect the same edges as γ and in the same order. Therefore, if there is another geodesic that encloses the domain with $A_1A_4A_5$ from one side and with $A_2A_3A_6$ from another, then, by Lemma 3.3, this geodesic coincides with γ.

In general, there are four geodesics that bound a domain with a facet of an octahedron. They can be constructed in a similar way if we fix other two non-connected facets of the octahedron. These geodesics can be mapped into each other with the symmetries of the octahedron.

Type 2. Let X_1, X_2, X_3, and X_4 denote the midpoints of the edges A_1A_2, A_2A_3, A_3A_4, and A_4A_1, respectively. Unfold two adjacent facets $A_1A_2A_6$ and $A_2A_6A_3$ onto the sphere. Construct a geodesic line segment connecting X_1 and X_2 within this unfolded region. Since $\alpha < \pi/2$, the segment X_1X_2 remains entirely within the development and intersects the edge A_2A_6 at the right angle. Similarly as above, unfold another two adjacent facets $A_2A_3A_5$ and $A_3A_5A_4$ and construct the segment X_2X_3. Following the same procedure, connect the points X_3 and X_4 via the edge A_4A_6 and subsequently connect X_4 and X_1 via the edge A_1A_5. The broken line $X_1X_2X_3X_4$ forms a simple closed geodesic on the spherical octahedron with a planar angle $\alpha \in (\pi/3, \pi/2)$ (see Fig. 5.6). The domain D_1 contains the edges A_2A_5 and A_5A_4 that do not enclose a facet. The complementary domain D_2 contains the edges A_1A_6 and A_6A_3.

Any simple closed broken line σ on the octahedron, that encloses the domain containing the edges A_2A_5 and A_5A_4 from one side and the domain containing the edges A_1A_6 and A_6A_3 from the other, is equivalent to γ. To demonstrate this, consider the following: if σ starts at the edge A_1A_2, it can only cross the edge A_2A_6 and cannot intersect A_1A_6. Subsequently, σ must traverse to the edge A_2A_3 since it cannot intersect A_6A_3. Continuing this analysis, σ intersects A_3A_5 avoiding A_2A_5, and so on. This implies that σ intersects the same sequence of edges as γ. By Lemma 3.3, γ is a unique geodesic that encloses the domain containing A_2A_5 and A_5A_4 on one side and the domain containing A_1A_6 and A_6A_3 from the other.

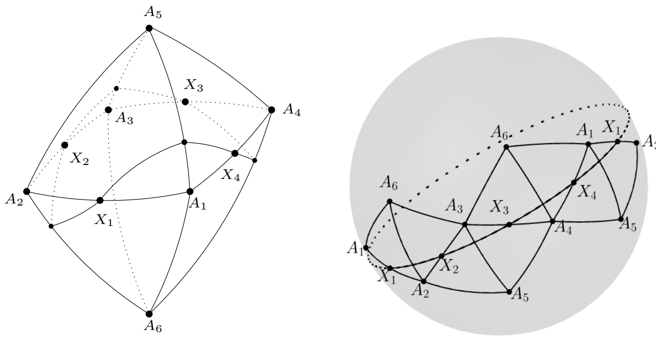


Fig. 5.6

In general, there are six geodesics of such type and they coincide up to the octahedral symmetry. □

6. Cube

A simple closed geodesic γ divides the surface of a cube with the planar angle $\alpha \in (\pi/2, 2\pi/3)$ into two domains D_1 and D_2.

Lemma 6.1. *If the domain D_i, $i = 1, 2$, encompasses a vertex of a cube, it must necessarily include at least one edge emanating from that vertex.*

Proof. Given that the planar angle $\alpha \in (\pi/2, 2\pi/3)$, and the length of its edge is $< \pi/2$, a simple geodesic cannot traverse sequentially all three edges emanating from one vertex of a cube. Consequently, a domain D_i contains at least two vertices of the cube.

Without loss of generality, we can assume that the domain D_1 has at most four vertices of the cube. If not, we can instead consider the domain D_2. We will use the following labeling of the vertices of the cube for the proof of this lemma. Let the vertex A_4 be arbitrarily fixed. Then choose vertices A_1, A_2, and A_3 such that the edge A_4A_i exists on the cube. Let A_4' be the vertex such that A_4 and A_4' do not belong to any facet of the cube. Subsequently, we label vertices A_1', A_2', and A_3' as the endpoints of the edges emanating from A_4'. For certainty, choose A_1' such that A_1' forms a facet on the cube together with the vertices A_1, A_3 and A_4.

Suppose D_1 contains the vertex A_4 and does not have any edge emanating from it. In this case, the geodesic γ must necessarily intersect all edges A_4A_i, $i =$

$1,2,3$, at least once. Let X_i denote the point of γ on the edge A_4A_i, $i = 1,2,3$, such that X_i is closest to A_4 on its edge.

Case 1. Assume there is no segments X_iX_{i+1} of γ within the facets $A_4A_iA'_{i+1}A_{i+1}$, $i = 1,2,3$, and $i+1$ is calculated modulo 4. The geodesic γ traverses from the point X_1 to the point Y_1 on the facet $A_4A_1A'_2A_2$. If Y_1 lies on the edge A_2A_4, it follows that $\mathrm{dist}(A_4, X_2) < \mathrm{dist}(A_4, Y_1)$. Consequently, the segment of γ originating from X_2 must intersect the segment X_1Y_1, resulting in a contradiction. Therefore, Y_1 must necessarily belong to either edge A'_2A_2 or to the edge $A_1A'_2$.

Subcase 1.1. If Y_1 belongs to the edge A'_2A_2, it follows that a segment of γ emanating from X_2 on this facet ends at the point Y_2 on the edge $A_2A'_2$ and $\mathrm{dist}(A_2, Y_2) < \mathrm{dist}(A_2, Y_1)$.

Now consider the facet $A_4A_2A'_3A_3$. The segment of γ traverses from the point X_2 to the point Z_2, and from the point X_3 to the point Y_3. If the point Z_2 lies on the edge A_4A_3, it follows that $\mathrm{dist}(A_4, X_3) < \mathrm{dist}(A_4, Z_2)$. In this case, the segment X_3Y_3 must intersect the segment X_2Z_2, resulting in a contradiction. If the point Z_2 lies on the edge $A_2A'_3$, it follows that γ intersects all three edges emanating from the vertex A_2, which also leads to a contradiction. Therefore, the point Z_2 must necessarily lie on the edge $A_3A'_3$. Furthermore, the point Y_3 also belongs to the edge $A_3A'_3$ and $\mathrm{dist}(A_3, Y_3) < \mathrm{dist}(A_3, Z_2)$. Within the facet $A_4A_3A'_1A_1$, the segment of γ traverses from X_3 to Z_3 and from X_1 to Z_1. As demonstrated previously, Z_3 cannot belong to the edge A_1A_4 or to the edge A'_1A_3. Consequently, both points Z_3 and Z_1 must reside on the edge $A_1A'_1$ and $\mathrm{dist}(A_1, Z_1) < \mathrm{dist}(A_1, Z_3)$.

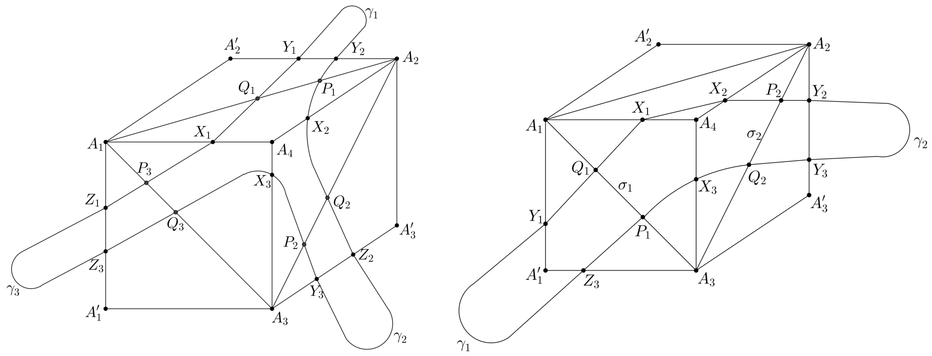
Subcase 1.2. If Y_1 belongs to the edge $A_1A'_2$, it follows that on the facet $A_4A_3A'_1A_1$ the point Z_1 can be only on the edge A'_1A_3. Consequently, the point Z_3 can only reside on edge A'_1A_3. From this, it can be deduced that on the facet $A_4A_2A'_3A_3$ the point Y_3 lies on $A_2A'_3$ and therefore, Z_2 must also lie on $A_2A'_3$, and Y_2 can be located only on $A_1A'_2$.

Notice that the case, where Y_1 and Y_2 do not belong to the same edge, is not possible. In fact, if Y_1 lies on $A_1A'_2$ and Y_2 lies on $A_2A'_2$, it follows that Z_1 should belong to A'_1A_3 and Z_2 should belong to $A_3A'_3$. Consequently, Z_3 must also reside on the edge A'_1A_3 and Q_3 must reside on the edge $A_3A'_3$. This implies that the segment $Z_3X_3Y_3$ of γ intersects all three edges emanating from A_3, which is a contradiction.

In both Subcases 1.1 and 1.2, the geodesic segments X_1Y_1 and X_2Y_2 intersect the diagonal A_1A_2 at the points Q_1 and P_1. The segments X_2Z_2 and X_3Y_3 intersect the diagonal A_2A_3 at the points Q_2 and P_2. And the segments X_3Z_3 and X_1Z_1 intersect the diagonal A_1A_3 at the points Q_3 and P_3.

It should be noticed that no points of γ exist between Q_i and P_i, $i = 1,2,3$. To illustrate this, let us assume the contrary, i.e., that the point R exists on the segment P_1Q_1. Since there is no points of γ on the segments A_4X_i, then the part of the geodesic originating at R and traversing within the triangle $A_1A_2A_4$ must inevitably intersect the segment P_1Q_1 again at a distinct point S. Consequently, the segment RS does not intersect any edge of the cube. It means that the segment RS and the segment of the facet diagonal form a lune on a sphere. Therefore, the length of the diagonal segment between R and S equals π, which contradicts to (2.3).

Consider the domain of the cube outside the triangular regions $A_4A_iA_{i+1}$, $i = 1, 2, 3$, and $i + 1 \mod 4$. Let γ_i be a segment of the geodesic γ emanating from P_i. Given that the domain D_1 is homeomorphic to a disc, then γ_i must terminate at the point Q_i and not intersect any other points P_j, Q_k, $j, k \neq i$ along its path. Let σ_i denote a part of the diagonal of the facet between the points Q_i and P_i. The segments γ_i and σ_i together enclose a domain G_i inside D_1. Each domain G_i is locally convex and homeomorphic to a disc, $i = 1, 2, 3$.



(a)

(b)

Fig. 6.1

Case 2. Assume that only one of the three geodesic segments X_iX_{i+1}, $i = 1, 2, 3$, of γ exists, for example, X_1X_2. The existence of two of such segments would necessitate the geodesic to intersect three edges emanating from vertex A_4 sequentially. This leads to a contradiction.

On the facet $A_4A_2A'_3A_3$, the geodesic γ traverses from the point X_2 to the point Y_2 and from the point X_3 to Y_3. On the facet $A_4A_3A'_1A_1$, γ traverses from the point X_1 to the point Y_1 and from the point X_3 to Z_3.

If Y_2 is on the edge $A_3A'_3$, then Y_3 should also be on this edge. In this case, Z_3 and Y_1 should be on the edge $A_1A'_1$.

If Y_2 resides on the edge $A_2A'_3$, then Y_3 can be located either on $A_2A'_3$ or on $A_3A'_3$, provided that X_2Y_2 and X_3Y_3 do not intersect. If Y_3 is located on $A_2A'_3$, then Z_3 and Y_1 can be either on $A_1A'_1$ or on A'_1A_3, provided that X_1Y_1 and X_3Z_3 do not intersect. If Y_3 is located on $A_3A'_3$, then Z_3 and Y_1 should be on the edge $A_1A'_1$.

In any case, the geodesic segment X_2Y_2 intersects the diagonal A_2A_3 at the point P_2, and the segment X_3Y_3 intersects A_2A_3 at Q_2. On the facet $A_4A_3A'_1A_1$, the segment X_3Z_3 intersects A_1A_3 at P_1, and X_1Y_1 intersects A_1A_3 at Q_1. There is no other points of the geodesic between Q_i and P_i, $i = 1, 2$.

Again, let us consider the domain of the cube outside the triangles $A_4A_iA_{i+1}$, $i = 1, 2, 3$ and $i + 1 \mod 4$. Let γ_i be a segment of γ that emanates from P_i, $i = 1, 2$. Since the domain D_i is homeomorphic to a disc, then it is easy to show that

γ_i terminates at the point Q_i and does not intersect other points P_j, Q_j, $j \neq i$. Denote by σ_i the parts of the diagonals of the facets between the points Q_i and P_i. The segments γ_i and σ_i enclose a domain G_i inside D_1. Each G_i is locally convex and homeomorphic to a disc, $i = 1, 2$.

Let $l = 2$ or 3. In both Case 1 and Case 2, we obtain l distinct domains G_i, each bounded by the geodesic segment γ_i, and the segment σ_i of the diagonal of the corresponding cube facet. These domains do not intersect within the cube. If G_i does not contain any vertex of the cube, it necessarily forms a lune on the unit sphere. This implies that the length of the segment σ_i is equal to π, which contradicts (2.3).

Consequently, each domain G_i should enclose at least one vertex of the cube. In Case 1, since there are three domains G_i around A_4, then each domain must necessarily enclose exactly one vertex A_{s_i} of the cube. Moreover, the geodesic should intersect all edges emanating from A_{s_i}. Otherwise the domain D_1 has more than four vertices, which is in contradiction to the initial assumption. Since domains G_i do not intersect, then all A_{s_i} must be distinct vertices of the cube.

In Case 2, only two domains G_1 and G_2 exist. Given that the domain D_1 has at most four vertices of the cube, the domains G_1 and G_2 together have up to three vertices of the cube. Consequently, one of the domains, for example, G_1, encloses exactly one vertex of a cube A_{s_1}, and A_{s_1} does not belong to G_2. In this case, the geodesic γ must also intersect all edges emanating from A_{s_i}.

In both cases, let us apply again the aforementioned construction to the vertex A_{s_1} inside G_1. This process subdivides the domain G_1 into $l+1$ parts: G'_k, $k = 1, \ldots, l$ and $G_1 \backslash (\cup_{k=1}^{l} G'_k)$ located between them. This subdivision is facilitated by the segments σ'_1 and σ'_2 of the diagonals. The domain $G_1 \backslash (\cup_{k=1}^{l} G'_k)$ contains the vertex A_{s_1} and the domain G'_k does not contain a vertex of the cube. Since there is no points of the geodesic on σ_1, then σ_1 belongs to the boundary of one of the domains G'_k. Without loss of generality, let us assume it belongs to the boundary of G'_2. In this case, the perimeter of G'_1 consists of a geodesic subarc γ'_1 and the part σ'_1 of the diagonal of the cube facet. Given G'_1 does not contain any vertex of the cube, it forms a lune on a unit sphere. This implies that the length of σ_i is equal to π, which again leads to a contradiction. □

In what follows, we label the front facet of the cube as $A_1A_2A_3A_4$ and the back one as $A'_1A'_2A'_3A'_4$.

Lemma 6.2. *A domain D_i, $i = 1, 2$, contains four vertices of the cube connected with at least three edges.*

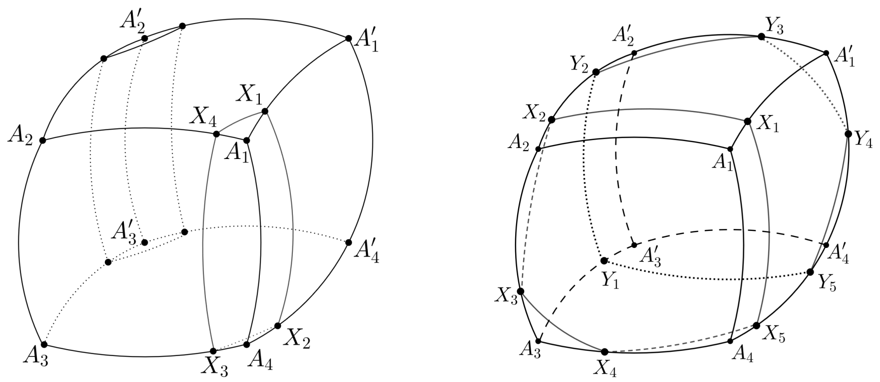
Proof. By Lemma 6.1, the domain D_i contains the vertex of a cube together with at least one edge emanating from this vertex.

Case 1. Let us first consider the case, where the domain D_1 encloses only one edge, for instance, A_1A_4. In this case, the geodesic γ intersects the edges $A_1A'_1$, $A_4A'_4$, A_4A_3 and A_1A_2 at the points X_1, X_2, X_3 and X_4, respectively.

A rotation r_0 on the angle π around the line passing through the midpoints of the edges A_1A_4 and $A'_2A'_3$ maps γ to a geodesic $\tilde{\gamma}$. Given that the edge A_1A_4

is mapped to itself under r_0 and that $r_0(A_1A_1') = A_3A_4$, $r_0(A_1A_2) = A_4A_4'$, it follows that $\tilde{\gamma}$ is equivalent to γ. By Lemma 3.3, γ and $\tilde{\gamma}$ must coincide. Consequently, $\operatorname{dist}(A_1, X_4) = \operatorname{dist}(A_4, X_2)$ and $\operatorname{dist}(A_1, X_1) = \operatorname{dist}(A_4, X_3)$.

Consider an isometry r of a cube such that $r(A_1A_4) = A_2'A_3'$. This isometry r is a rotation on the angle π around the line passing through the centers of the facets $A_1'A_1A_2A_2'$ and $A_4'A_4A_3A_3'$. The image of the geodesic γ under r is a geodesic σ which bounds a domain containing the edge $A_2'A_3'$ (see Fig. 6.2(a)). It can be easily verified that σ is also invariant under the action r_0.



(a) (b)

Fig. 6.2

If γ and σ intersect, this intersection would necessarily occur within the facets $A_1'A_1A_2A_2'$ and $A_4'A_4A_3A_3'$. However, this would imply that the length of γ is greater than 2π, which contradicts Lemma 3.2. Therefore, γ and σ cannot intersect. In this case, the Gauss-Bonnet theorem implies that the integral of the curvature over the area enclosed by γ and σ equals zero. This also leads to a contradiction.

Case 2. Now, let us consider the case, where the domain D_1 contains three vertices of a cube. Since each vertex within a domain must have at least one edge emanating from it, these three vertices must be connected by two edges. For instance, let us assume D_1 contains A_1A_2 and A_1A_4. We denote the points, where the geodesic γ intersects the edges A_1A_1', A_2A_2', A_2A_3, A_3A_4 and A_4A_4', as $X_1, \dots, X_5$.

A reflection s through the plane $A_1A_1'A_3'A_3$ maps the geodesic γ to a geodesic $\tilde{\gamma}$. Because the edges A_1A_1' and A_3A_3' are invariant under s and $s(A_2A_2') = A_4A_4'$, it follows that γ and $\tilde{\gamma}$ are equivalent geodesics. Therefore, by Lemma 3.3, they must coincide. Consequently,

$$\operatorname{dist}(A_4, X_5) = \operatorname{dist}(A_2, X_2) \tag{6.1}$$

and $\angle(X_5X_1A_1) = \angle(X_2X_1A_1) = \pi/2$.

Consider two isometries of the cube, namely, the rotation r_1 on an angle π around the line passing through the midpoints of the edges A_1A_1' and A_3A_3', and the rotation r_2 on the angle π along the line passing through the centers of the facets $A_1A_2A_3A_4$ and $A'_1A'_2A'_3A'_4$. The composite isometry $r_1 \circ r_2$ maps γ to a geodesic σ with the vertices $Y_k = (r_1 \circ r_2)(X_k)$, $k = 1, \ldots, 5$ located on the edges $A_3A'_3$, $A_2A'_2$, $A'_1A'_2$, $A'_1A'_4$ and $A_4A'_4$, respectively (see Fig. 6.2(b).). If γ and σ do not intersect, then the Gauss–Bonnet theorem implies that the integral of the curvature over the area enclosed by γ and σ must be zero. This, however, yields a contradiction. Therefore, γ and σ must intersect.

The intersection is possible only if $\mathrm{dist}(A'_2, X_2) \leq \mathrm{dist}(A_2, X_2)$. If the strict inequality holds, then γ and σ intersect at the points Z_1 and Z_2 on the facets $A_1A_2A'_2A'_1$ and $A_3A_2A'_2A'_3$, respectively. Consequently, the length of the segment $Z_1X_2Z_2$ of γ equals π. It follows from (6.1) that γ and σ also intersect at the points Z_3 and Z_4 on the facets $A_1A_4A'_4A'_1$ and $A_3A_4A'_4A'_3$, respectively. The length of the segment $Z_3X_5Z_4$ of γ is therefore also π. Hence, the total length of γ is greater than 2π, which yields a contradiction.

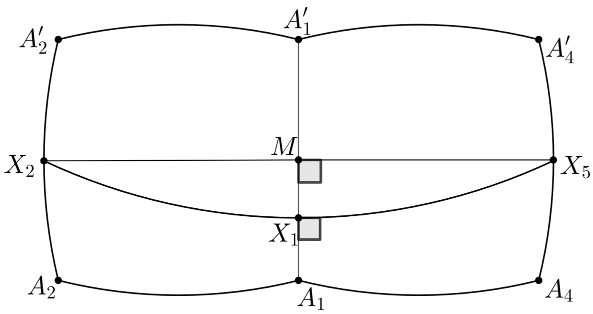


Fig. 6.3

If $\mathrm{dist}(A'_2, X_2) = \mathrm{dist}(A_2, X_2)$, then it follows from (6.1) that $\mathrm{dist}(A'_4, X_5) = \mathrm{dist}(A_4, X_5)$ as well. Consider the geodesic segment X_2MX_5, where M is a middle point of the edge $A_1A'_1$. The segments $X_2X_1X_5$ and X_2MX_5 form a geodesic lune on a sphere implying that the length of X_2MX_5 equals π. However, the length of X_2MX_5 is also equal to $2h$, where h is the length of arc connecting the midpoints of opposite edges of a spherical square. To find h, consider the quadrilateral $A_1A_2X_2M$. From the triangle $A_1A_2X_2$, we have

$$\cos A_1X_2 = \cos a_s \, \cos(a_s/2) + \sin a_s \, \sin(a_s/2) \, \cos\alpha.$$

From the triangle A_1X_2M, it follows that

$$\cos A_1X_2 = \cos(a_s/2) \, \cos h.$$

Combining these two formulas, we obtain

$$\cos h = \cos a_s + 2\sin^2(a_s/2) \, \cos\alpha. \tag{6.2}$$

From (2.2), one can compute

$$\sin(a_s/2) = \frac{\sqrt{-\cos\alpha}}{\sqrt{2}\sin(\alpha/2)}.$$

Substituting this into equation (6.2), we have

$$\cos h = \cot^2(\alpha/2) - \frac{\cos^2\alpha}{\sin^2(\alpha/2)}$$
$$= \frac{\cos^2(\alpha/2) - \cos^2\alpha}{\sin^2(\alpha/2)} = \frac{\cos\alpha - \cos 2\alpha}{2\sin^2(\alpha/2)} = \frac{2\sin(\alpha/2)\sin(3\alpha/2)}{2\sin^2(\alpha/2)}.$$

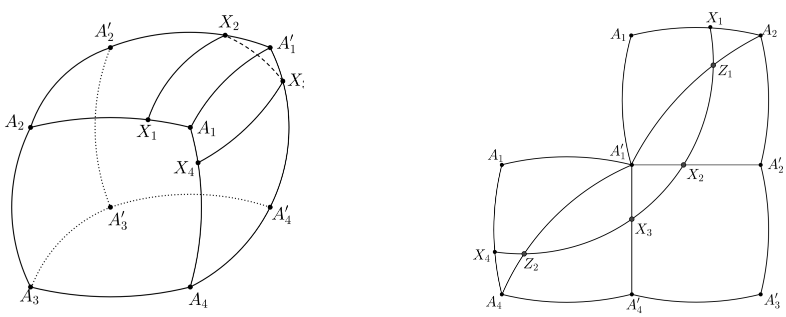
Therefore,

$$h = \arccos\left(\frac{\sin(3\alpha/2)}{\sin(\alpha/2)}\right). \tag{6.3}$$

From equation (6.3), it follows that the length of X_2MX_5 equals π if and only if $\alpha = 2\pi/3$, which leads to a contradiction.

Case 3. Suppose the domain D_1 contains four vertices of a cube with only two edges lying within D_1. Recall that by Lemma 6.1, D_1 cannot have a vertex without an edge emanating from it. Since D_1 has two edges and four vertices, these edges cannot share a common vertex. For example, let us assume D_1 contains the edge A_1A_1' and any other edge of a cube that does not have A_1 or A_1' as its endpoint. Because γ bounds a domain containing A_1A_1', there exists a segment of γ that intersects the edges A_1A_2, $A_1'A_2'$, $A_1'A_4'$ and A_1A_4 at the points X_1, X_2, X_3 and X_4, respectively. Note that the segment $X_1X_2X_3X_4$ is just a part of the geodesic. The points X_1 and X_4 are not connected within the facet $A_1A_2A_3A_4$ (see Fig. 6.4(a)).

Consider the development of the facets $A_1A_2A_2'A_1'$, $A_1'A_2'A_3'A_4'$ and $A_1'A_4'A_4A_1$ onto a unit sphere and consider the diagonals $A_1'A_2$ and $A_1'A_4$. Then X_1X_2 intersects $A_1'A_2$ at Z_1, and X_3X_4 intersects $A_1'A_4$ at Z_2 (see Fig. 6.4(b)). The angle $\angle Z_1A_1'Z_2 = 2\alpha > \pi$. This implies that the shortest path connecting Z_1 and Z_2 lies on the opposite side of the vertex A_1' on a sphere. Consequently, the segment γ_1 of the geodesic γ between the points Z_1 and Z_2 does not lie within the development unless the length of $A_1'Z_1$ and the length of $A_1'Z_2$ are each equal to $\pi/2$, and the length of Z_1Z_2 is greater than π. Then, the length of $A_1'X_3$ and the length of $A_1'X_2$ also equal $\pi/2$. Since $A_1'X_3$ and $A_1'X_2$ are parts of the cube's edges, we get a contradiction with (2.2). □



(a) (b)

Fig. 6.4

Theorem 6.3. *There are only three different simple closed geodesics on spherical cubes.*

Proof. Consider a spherical cube with a planar angle $\alpha \in (\pi/2, 2\pi/3)$. A simple closed geodesic γ divides the surface of the cube into two closed domains D_1 and D_2. By Lemma 6.2, each domain D_i contains four vertices of the cube connected with at least three edges. There are three possible configurations for four vertices of a cube that are sequentially connected by edges:

1) they form a facet of the cube;
2) three edges share a common vertex of the cube;
3) three edges form a broken line that does not define a facet.

Type 1. Let X_k be the midpoint of the edges $A_kA'_k$, $k = 1, \dots 4$. Connect these midpoints within their facets (see Fig. 6.5). The broken line $X_1X_2X_3X_4$ forms a simple closed geodesic γ on the spherical cube with the planar angle $\alpha \in (\pi/2, 2\pi/3)$. The geodesic γ encloses the facet $A_1A_2A_3A_4$ on one side and the facet $A'_1A'_2A'_3A'_4$ on the other.

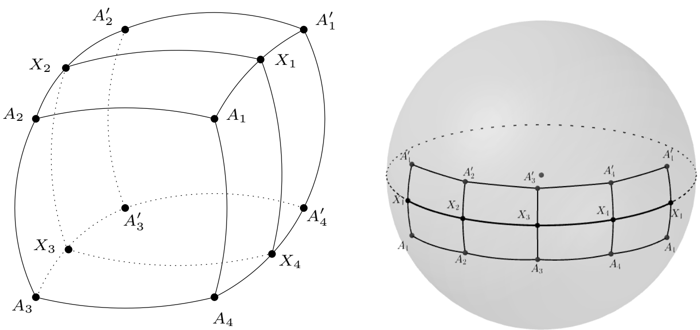


Fig. 6.5

If there exists another broken line σ on the cube that does not intersect the facets $A_1A_2A_3A_4$ and $A'_1A'_2A'_3A'_4$, then σ must be equivalent to γ. Indeed, suppose σ starts at the edge $A_1A'_1$. Then it must proceed to the edge $A_2A'_2$ as it cannot intersect A_1A_2 or $A'_1A'_2$. Subsequently, σ must go to $A_3A'_3$ as it cannot go to A_2A_3 or $A'_2A'_3$. Then σ crosses $A_4A'_4$ and returns again to $A_1A'_1$. By Lemma 3.3, it follows that γ is the unique geodesic that bounds the domain containing the facet of the cube.

In general, there are three geodesics of such type on the cube with the planar angle $\alpha \in (\pi/2, 2\pi/3)$. These geodesics lie on the planes of symmetry of the cube and can be mapped to each other with the cube symmetries.

Type 2. Let $X_1, \dots, X_6$ be the midpoints of the edges $A'_1A'_2$, A'_2A_2, A_2A_3, A_3A_4, $A_4A'_4$, and $A'_4A'_1$, respectively. Connect these points within their facets (see Fig. 6.6). Since the triangles $X_2A'_2X_1$ and $X_2A_2X_3$ are equal, it follows that $\angle A'_2X_2X_1 = \angle A_2X_2X_3$. The same reasoning can be applied to any pair of adjacent segments of the closed broken line $X_1X_2X_3X_4X_5X_6$. Therefore, this broken line forms a simple closed geodesic γ on a cube with a planar angle $\alpha \in (\pi/2, 2\pi/3)$.

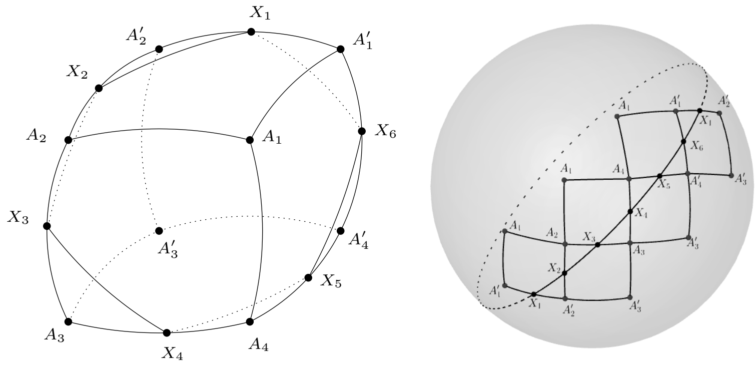


Fig. 6.6

This geodesic γ bounds a domain containing three edges emanating from the vertex A_1 on one side, and the domain containing three edges emanating from A_3' on the other. If there exists another broken line σ that bounds the same domains, then σ must be equivalent to γ. Therefore, by Lemma 3.3, γ is a unique simple closed geodesic of such type on the cube.

In general, there are four geodesics that bound the domains containing three edges that share a vertex on the spherical cube. These geodesics lie in the planes passing through the center of the cube orthogonal to the diagonal. All of them can be mapped to each other with the symmetries of the cube.

Type 3. Let X_1 and X_2 be the midpoints of the edges A_2A_3 and $A_1'A_4'$. First, develop the facets $A_3A_2A_2'A_3'$, $A_2A_2'A_1'A_1$ and $A_1A_1'A_4'A_4$ onto a unit sphere and connect the points X_1, X_2 within the development. Then develop the facets $A_1'A_2'A_3'A_4'$, $A_3'A_4'A_3A_4$, $A_3A_4A_1A_2$ and connect the points X_1 and X_2 within this development (see Fig. 6.7). Since the developments are equal to spherical polygons, it follows that $\angle X_1X_2A_1' = \angle X_1X_2A_4'$ and $\angle X_2X_1A_2 = \angle X_2X_1A_3$. Therefore, the closed broken line X_1X_2 forms a simple closed geodesic γ on the spherical cube with the planar angle $\alpha \in (\pi/2, 2\pi/3)$.

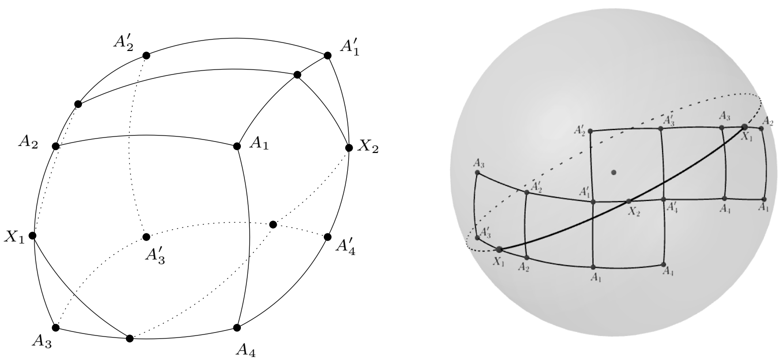


Fig. 6.7

By construction, γ encloses the domain D_1 containing the edges A_2A_1, A_1A_4 and A_4A_4' on one side and the domain D_2 with the edges $A_1'A_2'$, $A_2'A_3'$ and $A_3'A_3$ on the other. These edges form a broken line that does not define a facet. Any broken line σ that bounds the same domains D_1 and D_2 is equivalent to γ. Indeed, assume σ starts on the edge A_2A_3. Then, on the facet $A_2A_3A_3'A_2'$, σ can not go to the edges $A_2'A_3'$ and $A_3'A_3$. Therefore, σ must go to A_2A_2'. Because σ cannot cross $A_1'A_2'$ and A_1A_2, it must go to A_1A_1'. Subsequently, σ can only proceed to $A_1'A_4'$. Continuing this reasoning, we find that σ intersects the same edges as γ and in the same order. Therefore, by Lemma 3.3, γ is a unique geodesic of such type.

In general, on a cube with the planar angle $\alpha \in (\pi/2, 2\pi/3)$, there are 12 simple closed geodesics that enclose domains with three edges that form a broken line. These geodesics can be mapped into each other with the symmetries of the cube. □

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References

[1] A.D. Alexandrov, *Convex Polyhedra,* Springer, New York, 2005.

[2] A.D. Alexandrov, *Intrinsic Geometry of Convex Surfaces,* Selected Works, **II**, Taylor and Francis Group, LLC, 2006.

[3] A. Akopyan and A. Petrunin, *Long geodesics on convex surfaces,* Math. Intelligencer **40** (2018), 26–31.

[4] A.A. Borisenko, *An estimation of the length of a convex curve in two-dimensional Aleksandrov spaces,* J. Math. Phys. Anal. Geom. **16** (2020), 221–227.

[5] A.A. Borisenko, *A necessary and sufficient condition for the existence of simple closed geodesics on regular tetrahedra in spherical space,* Sb. Math. **213** (2022), 161–172.

[6] A.A. Borisenko and D.D. Sukhorebska, *Simple closed geodesics on regular tetrahedra in spherical space,* Sb. Math. **212** (2021), 1040–1067.

[7] D.B. Fuchs and E. Fuchs, *Closed geodesics on regular polyhedra,* Mosc. Math. J. **7** (2007), 265–279.

[8] D.B. Fuchs, *Geodesics on a regular dodecahedron,* Preprint No. 91, Max Planck Inst. Math., Bonn, 2009, 14 pp. Available at `http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2010/2009_91.pdf`.

[9] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers,* Oxford Univ. Press, 1975.

[10] H.L. Montgomery, *Fluctuations in the mean of Euler's phi function,* Proc. Indian Acad. Sci. Math. Sci. **97** (1987), 239–245.

[11] A.V. Pogorelov, *Quasi-geodesic lines on a convex surface,* Mat. Sb. (N.S.) **25** (1949), 275–306.

[12] V.Yu. Protasov, *Closed geodesics on the surface of a simplex,* Sb. Math. **198** (2007), 243–260.

[13] D.D. Sukhorebska, *Simple closed geodesics on regular tetrahedra in spaces of constant curvature,* J. Math. Phys. Anal. Geom. **4** (2022), 562–610.

[14] V.A. Toponogov, *A bound for the length of a convex curve on a two-dimensional surface,* Sibirsk. Mat. Zh. **4** (1963), 1189–1193.

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Прості замкнені геодезичні на правильних сферичних поліедрах

Darya Sukhorebska

У цій роботі було знайдено усі прості замкнені геодезичні на правильних сферичних октаедрах та сферичних кубах. Також була знайдена оцінка числа простих замкнених геодезичних на правильних сферичних тетраедрах.

Ключові слова: прості замкнені геодезичні, правильний тетраедр, октаедр, куб, сферичний простір