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A Nonlinear PDE with Two Hardy–Sobolev Critical Exponents with One-Dimensional Singularity

Abdourahmane Diatta and El Hadji Abdoulaye Thiam

For $N \geq 4$, we let Ω be a bounded domain of $\mathbb{R}^N$ and Γ be a closed curve contained in Ω. We study the existence of positive solutions $u \in H_0^1(\Omega)$ to the equation

$$-\Delta u + hu = \lambda \rho_\Gamma^{-s_1} u^{2^*_{s_1}-1} + \rho_\Gamma^{-s_2} u^{2^*_{s_2}-1} \quad \text{in } \Omega, \tag{1}$$

where $h : \Omega \longrightarrow \mathbb{R}$ is a continuous function, λ is a positive real parameter, $0 \leq s_2 < s_1 < 2$, and ρ_Γ is the distance function to Γ. In this paper, we prove the existence of mountain pass solutions for the Euler–Lagrange equation (1) depending on the local geometry of the curve and the potential h. We also study the existence, symmetry and decay estimates of the positive global solutions of (1) with $\Omega = \mathbb{R}^N$ and Γ being the real line.

Key words: two Hardy–Sobolev exponents, curvature, mountain pass solution, curve singularity

Mathematical Subject Classification 2020: 35J60, 35B33, 35A15, 35R45, 35B40

1. Introduction

For $N \geq 3$, the famous Caffarelli–Kohn–Nirenberg inequality asserts that there exists a positive constant $C_{N,a,b}$ depending only on N, a and b such that

$$C_{N,a,b} \left(\int_{\mathbb{R}^N} |x|^{-bq} |u|^q dx \right)^{2/q} \leq \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx, \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^N), \tag{1.1}$$

where $-\infty < a < \frac{N-2}{2}, 0 \leq b-a \leq 1$ and $q = \frac{2N}{N-2+2(b-a)}$. We refer to Caffarelli–Kohn–Nirenberg [5]. Note that in the case $b = a+1$ and $q = 2$, (1.1) corresponds to the following classical Hardy inequality:

$$\left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} |x|^{-2} |u|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$u \longmapsto \sqrt{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}.$$

The constant $\left(\frac{N-2}{2}\right)^2$ is sharp and never achieved in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The case $a = b = 0$ and $q = \frac{2N}{N-2}$ corresponds to the famous Sobolev inequality

$$S_{N,0} \left(\int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{2/2^*} \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where the best constant

$$S_{N,0} = \frac{N(N-2)}{4} \omega_N^{2/N}$$

is achieved in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Here $\omega_N = |S^{N-1}|$ is the volume of the N-sphere and $2^* := 2_0^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

By the Hölder inequality, we get the interpolation between the Hardy and the Sobolev inequalities, called the Hardy–Sobolev inequality given by

$$S_{N,s} \left(\int_{\mathbb{R}^N} |x|^{-s} |u|^{2_s^*} \, dx \right)^{2/2_s^*} \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.2}$$

where, for $s \in [0, 2]$, we have that $2_s^* = \frac{2(N-s)}{N-2}$ is the critical Hardy–Sobolev exponent. We refer to [11] for more details about the Hardy–Sobolev inequality. The value of the best constant is

$$S_{N,s} := (N-2)(N-s) \left[\frac{w_{N-1}}{2-s} \frac{\Gamma^2(N - \frac{s}{2-s})}{\Gamma(\frac{2(N-s)}{2-s})} \right]^{\frac{2-s}{N-s}},$$

where Γ is the Euler gamma function. It was computed by Lieb [22] when $s \in (0, 2)$. The ground state solution is given, up to dilation, by

$$w(x) = C_{N,s} \left(1 + |x|^{2-s}\right)^{\frac{2-N}{2-s}}$$

for some positive known constant $C_{N,s}$ depending on N and s.

The Caffarelli–Kohn–Nirenberg inequality on domains and related problems have been studied these last years. For instance, we let Ω be a domain of $\mathbb{R}^N$ and consider the equation

$$\begin{cases} -\operatorname{div}(|x|^{-2a} \nabla u) = |x|^{-bq} u^{q-1}, & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

To study (1.3), one could let $w(x) = |x|^{-a} u(x)$. Direct computations show that

$$\int_\Omega |x|^{-2a} |\nabla u|^2 \, dx = \int_\Omega |\nabla w|^2 dx - a(N-2-a) \int_\Omega |x|^{-2} w^2 \, dx.$$

Then the solutions of (1.3) can be obtained by minimizing the following quotient:

$$\mathcal{L}(u) := \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx - a(N-2-a) \int_\Omega |x|^{-2} w^2 \, dx}{\left(\displaystyle\int_\Omega |x|^{-bq} |u|^q \, dx \right)^{2/q}} \tag{1.4}$$

in $u \in \mathcal{D}_a^{1,2}(\Omega) \setminus \{0\}$, where $\mathcal{D}_a^{1,2}(\Omega)$ is the completion of $\mathcal{C}_c^\infty(\Omega)$ with respect to the norm

$$u \longmapsto \sqrt{\int_\Omega |x|^{-2a} |\nabla u|^2 dx}.$$

The question related to the attainability of the best constant $S_{a,b}^N(\Omega) := \inf \mathcal{L}(u)$ in (1.4) has been studied by many authors. For more developments related to that, we refer the readers to [2–4,6,7,9,11,12,17,22,24,25] and references therein.

For $0 \in \partial\Omega$, the existence of minimizers for $S_{a,b}^N(\Omega)$ was first studied by Ghoussoub and Kang [11] and Ghoussoub and Robert [12]. Later, Chern, and Lin [6] proved the existence of minimizer provided the mean curvature of the boundary at the origin is negative and ($a < b < a+1$ and $N \geq 3$) or ($b = a > 0$ and $N \geq 4$). The case $a = 0$ and $0 < b < 1$ was first studied in [12] before the generalization given in [6]. More generally the questions related to Partial Differential Equations (PDE) involving multiple Hardy–Sobolev critical exponents have been investigated these last decades. In particular, we let Ω be a domain of $\mathbb{R}^N$ such that $0 \in \partial\Omega$ and consider the equation

$$\begin{cases} -\Delta u(x) = \lambda \dfrac{u^{2^*_{s_1}-1}(x)}{|x|^{s_1}} + \dfrac{u^{2^*_{s_2}-1}(x)}{|x|^{s_2}} & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $0 \leq s_2 < s_1 < 2$, $\lambda \in \mathbb{R}$ and for $i = 1,2$, the $2^*_{s_i} := \frac{2(N-s_i)}{N-2}$ are two critical Hardy–Sobolev exponents. If $s_2 = 0$ and $\lambda < 0$, then equation (1.5) has no nontrivial solution. For $\lambda > 0$, $0 < s_1 < 2$ and $s_2 = 0$, by using variational methods, Hsia, Lin, and Wadade [18] proved the existence of solutions provided $N \geq 4$ and the mean curvature at the origin is negative. For the case $N = 3$, $\lambda \in \mathbb{R}$ and $0 < s_2 < s_1 < 2$, equation (1.5) has a least-energy solution provided the mean curvature at the origin is negative, see [23].

Concerning the existence and nonexistence of a solution related to equation (1.5) in the half-space $\Omega = \mathbb{R}^N_+$, we refer to Bartsch, Peng, and Zhang [2] for the case $0 < s_2 < s_1 = 2$ and $\lambda < \left(\frac{N-2}{2}\right)^2$; to Musina [26] when $N \geq 4$, $s_2 = 0$, $s_1 = 2$ and $0 < \lambda < \left(\frac{N-2}{2}\right)^2$ and to Hsia, Lin, and Wadade [18] when $s_2 = 0$, $0 < s_1 < 2$ and $\lambda > 0$.

In this paper, we are concerned with the effect of the local geometry of the singularity Γ on the existence of solutions of the following nonlinear partial differential equation involving two Hardy–Sobolev critical exponents. More precisely, letting h be a continuous function and λ be a positive real parameter, we consider

$$\begin{cases} -\Delta u(x) + hu(x) = \lambda \dfrac{u^{2^*_{s_1}-1}(x)}{\rho_\Gamma^{s_1}(x)} + \dfrac{u^{2^*_{s_2}-1}(x)}{\rho_\Gamma^{s_2}(x)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\rho_\Gamma(x) := \inf_{y \in \Gamma} |y - x|$ is the distance function to the curve Γ, $0 \leq s_2 < s_1 < 2$, $2^*_{s_1} := \frac{2(N-s_1)}{N-2}$ and $2^*_{s_2} := \frac{2(N-s_2)}{N-2}$ are two critical Hardy–Sobolev exponents.

To study equation (1.6), we consider its associated functional $J : H^1_0(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega h(x)u^2\, dx - \frac{\lambda}{2^*_{s_1}}\int_\Omega \frac{|u|^{2^*_{s_1}}}{\rho_\Gamma^{s_1}(x)}\, dx - \frac{1}{2^*_{s_2}}\int_\Omega \frac{|u|^{2^*_{s_2}}}{\rho_\Gamma^{s_2}(x)}\, dx.$$

It is easy to verify that there exists a positive constant $r > 0$ and $u_0 \in H^1_0(\Omega)$ such that $\|u_0\|_{H^1_0(\Omega)} > r$ and

$$\inf_{\|u\|_{H^1_0(\Omega)}=r} J(u) > J(0) \geq J(u_0),$$

see Lemma 4.4 below. Then the point $(0, J(0))$ is separated from the point $(u_0, J(u_0))$ by a ring of mountains. Set

$$c^* := \inf_{P\in\mathcal{P}} \max_{v\in P} J(v),$$

where $\mathcal{P}$ is the class of continuous paths in $H^1_0(\Omega)$ connecting 0 to u_0. Since $2^*_{s_2} > 2^*_{s_1}$, the function $t \longmapsto J(tv)$ has the unique maximum for $t \geq 0$. Furthermore, we have

$$c^* := \inf_{\substack{u\in H^1_0(\Omega)\\ u>0}} \max_{t\geq 0} J(tu).$$

Due to the fact that the embedding of $H^1_0(\Omega)$ into the weighted Lebesgue spaces $L^{2^*_{si}}(\Omega, \rho_\Gamma^{-si}dx)$ is not compact, the functional J does not satisfy the Palais–Smale condition. Therefore, in general, c^* might not be a critical value for J.

To recover compactness, we study the following nonlinear problem:

Let $x = (y, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and consider

$$\begin{cases} -\Delta u(x) = \lambda \dfrac{u^{2^*_{s_1}-1}(x)}{|z|^{s_1}} + \dfrac{u^{2^*_{s_2}-1}(x)}{|z|^{s_2}} & \text{in } \mathbb{R}^N, \\ u(x) > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.7}$$

To obtain solutions of (1.7), we consider the functional $\Pi : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Pi(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^{-s_1}|u|^{2^*_{s_1}}\, dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N} |z|^{-s_2}|u|^{2^*_{s_2}}\, dx$$

and set

$$\beta^* := \inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\\ u>0}} \max_{t\geq 0} \Pi(tu).$$

Then we get compactness provided

$$c^* < \beta^*,$$

see Proposition 4.2 below. So it is important to study the existence, symmetry and decay estimates of the nontrivial solution $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ of (1.7). Then we have the following results.

Theorem 1.1. *Let $N \geq 3$, $0 \leq s_2 < s_1 < 2$, $\lambda \in \mathbb{R}^*_+$. Then the equation*

$$\begin{cases} -\Delta u = \lambda \dfrac{u^{2^*_{s_1}-1}(x)}{|z|^{s_1}} + \dfrac{u^{2^*_{s_2}-1}}{|z|^{s_2}} & \text{in } \mathbb{R}^N, \\ u(x) > 0 & \text{in } \mathbb{R}^N \end{cases} \tag{1.8}$$

has a solution $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, w depends only on $|y|$ and $|z|$. In other words, there exists a function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$w(x) = \theta(|y|, |z|).$$

Next, we have the following decay estimates of the solution w and its higher order derivatives.

Theorem 1.2. *Let w be a solution of the Euler–Lagrange equation (1.8). Then*

(i) *there exist two positive constants $c_1 < c_2$ such that*

$$\frac{c_1}{1 + |x|^{N-2}} \leq w(x) \leq \frac{c_2}{1 + |x|^{N-2}}, \quad x \in \mathbb{R}^N;$$

(ii) *for $|x| = |(t, z)| \leq 1$,*

$$|\nabla w(x)| + |x||D^2 w(x)| \leq C_2 |z|^{1-s_1};$$

(iii) *for $|x| = |(t, z)| \geq 1$,*

$$|\nabla w(x)| + |x||D^2 w(x)| \leq C_2 \max(1, |z|^{-s_1})|x|^{1-N}.$$

These two theorems will play a crucial role in the following which is our main result.

Theorem 1.3. *Let $N \geq 4$, $0 \leq s_2 < s_1 < 2$, and Ω be a bounded domain of $\mathbb{R}^N$. Consider a smooth closed curve Γ contained in Ω. Let h be a continuous function such that the linear operator $-\Delta + h$ is coercive. Then there exists a positive constant $A^N_{s_1,s_2}$, depending only on N, s_1 and s_2 with the property that if there exists $y_0 \in \Gamma$ such that*

$$A^N_{s_1,s_2} |\kappa(y_0)|^2 + h(y_0) < 0 \quad \text{for } N \geq 4,$$

then

$$c^* < \beta^*,$$

where $\kappa : \Gamma \to \mathbb{R}^N$ is the curvature vector of Γ. Moreover, there exists $u \in H^1_0(\Omega) \setminus \{0\}$, a nonnegative solution of

$$-\Delta u(x) + hu(x) = \lambda \frac{u^{2^*_{s_1}-1}(x)}{\rho^{s_1}_\Gamma(x)} + \frac{u^{2^*_{s_2}-1}(x)}{\rho^{s_2}_\Gamma(x)} \quad \text{in } \Omega.$$

The effect of the curvature in the Hardy–Sobolev inequalities has been intensively studied in the recent years. For each of these works, the sign of the curvature at the point of singularity plays an important role for the existence of a solution. The first paper, to our knowledge, is the one of Ghoussoub and Kang [11] who considered the Hardy–Sobolev inequality with singularity at the boundary. For more results in this direction, see the works of Ghoussoub and Robert in [13,14], Demyanov and Nazarov [8], Chern and Lin [6], Lin and Li [23], Fall, Minlend, and Thiam in [10], Ciss, Diatta, and Thiam [31] and references therein. The Hardy–Sobolev inequality with interior singularity on Riemannian manifolds was studied by Jaber [20] and Thiam [28]. Here also the impact of the scalar curvature at the point singularity plays an important role for the existence of minimizers in higher dimensions $N \geq 4$. The paper [20] contains also the existence result under positive mass condition for $N = 3$. We point out that the 3-dimensional version of this paper is presented in [30]. The existence of solutions does not depend on the local geometry of the singularity but on the regular part of the Green function of the operator $-\Delta + h$.

The proof of Theorem 1.3 relies on test function methods. More precisely, on building appropriate test functions allowing to compare c^* and β^*. While it always holds that $c^* \leq \beta^*$, our main task is to find a function for which $c^* < \beta^*$, see Section 5. This then allows to recover compactness and thus every minimizing sequence for c^* converges to a minimizer up to a subsequence. Building of these approximate solutions requires sharp decay estimates of a minimizer w for β^*, see Section 2. Section 3 is devoted to the local parametrization and computation of the local metric.

2. Proofs of Theorems 1.1 and 1.2

Let $N \geq 3$, $1 \leq k \leq N-1$. Consider the problem

$$\begin{cases} -\Delta u = \lambda \dfrac{u^{2^*_{s_1}-1}}{|z|^{s_1}} + \dfrac{u^{2^*_{s_2}-1}}{|z|^{s_2}}, & x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}^*_+$, $0 \leq s_2 < s_1 < 2$, and for $i = 1, 2$, the $2^*_{s_i} = \frac{2(N-s_i)}{N-2}$ are two Hardy–Sobolev critical exponents. In this section, we establish the existence, symmetry and decay estimates (when $k = 1$) of solutions to problem (2.1). The argument relies on the existence of solutions to an approximating problem (introduced below in (2.2)) which converges, as the regularization parameter tends to zero, to a solution of (2.1). The overall approach and computations are largely inspired by those in [32, Section 3]. For readers convenience, we provide a brief outline of the proof.

2.1. Existence of a solution to the associated approximating problem. We let $N \geq 3$, $1 \leq k \leq N-1$, $0 \leq s_2 < s_1 < 2$, $\epsilon \in (0, s_1)$, and λ be a positive parameter. Consider the problem

$$\begin{cases} -\Delta u = \lambda a_\epsilon(z) u^{2^*_{s_1}-1} + u^{2^*_{s_2}-1}|z|^{-s_2} & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.2}$$

where

$$a_\epsilon(z) = \frac{1_{\{|z|<1\}}}{|z|^{s_1-\epsilon}} + \frac{1_{\{|z|\geq 1\}}}{|z|^{s_1+\epsilon}} \tag{2.3}$$

and $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Note that the sequence $(a_\epsilon)_{0\leq\epsilon<s_1}$ is well defined and strictly decreasing, where we have set

$$a_0(z) = \frac{1}{|z|^{-s_1}}.$$

The energy functional corresponding to problem (2.2) is given by

$$\Psi_\epsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s_2}}}{|z|^{s_2}}\,dx. \tag{2.4}$$

We consider the Nehari manifold

$$\mathcal{N}_\epsilon := \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\} : \langle \Psi'_\epsilon(u), u\rangle = 0\right\}.$$

Then we start by the following.

Lemma 2.1. *For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}$, there exists $t_{\epsilon,u}$ positive, depending only on ϵ and u, such that $t_{\epsilon,u}u \in \mathcal{N}_\epsilon$. Further, $t_{\epsilon,u}$ is strictly increasing with respect to $\epsilon \in (0, s_1)$ and $\mathcal{N}_\epsilon$ is bounded away from 0.*

Proof. Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. Then we have

$$\Psi_\epsilon(tu) = \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \lambda\frac{t^{2^*_{s_1}}}{2^*_{s_1}}\int_{\mathbb{R}^N} a_\epsilon(z)u^{2^*_{s_1}}\,dx - \frac{t^{2^*_{s_2}}}{2^*_{s_2}}\int_{\mathbb{R}^N} \frac{u^{2^*_{s_2}}}{|z|^{s_2}}\,dx.$$

By direct computation and using the fact that $t > 0$, we obtain

$$\frac{d}{dt}\Psi_\epsilon(tu) = 0 \quad \text{iff}$$

$$g_\epsilon(t) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - t^{2^*_{s_1}-2}\lambda\int_{\mathbb{R}^N} a_\epsilon(z)u^{2^*_{s_1}}\,dx - t^{2^*_{s_2}-2}\int_{\mathbb{R}^N} \frac{u^{2^*_{s_2}}}{|z|^{s_2}} dx = 0. \tag{2.5}$$

Since the map $t \to g_\epsilon(t)$ is continuous, strictly decreasing and

$$\lim_{t\to 0} g_\epsilon(t) > 0 \quad \text{and} \quad \lim_{t\to\infty} g_\epsilon(t) = -\infty,$$

we get the existence and uniqueness of $t_{\epsilon,u} > 0$ thanks to the Intermediate Value Theorem. Further, by the classical Sobolev inequality and (2.5), for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, we obtain that

$$C_N\left(\int_{\mathbb{R}^N} |u|^{2^*_0}\,dx\right)^{\frac{2}{2^*_0}} \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx$$

$$= t_{\epsilon,u}^{2^*_{s_1}-2}\int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}-1}dx + t_{\epsilon,u}^{2^*_{s_2}-2}\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s_2}-1}}{|z|^{s_2}}\,dx,$$

for some positive constant C_N depending on N. Therefore there exists $\beta_\epsilon > 0$ such that

$$t_{\epsilon,u} \geq \beta_\epsilon,$$

and hence $\mathcal{N}_\epsilon$ is bounded away from the origin. For the monotonicity, we let $0 \leq \epsilon_1 < \epsilon_2 < s_1$. Then there exist $t_{\epsilon_1,u}$ and $t_{\epsilon_2,u}$ positive such that

$$g_{\epsilon_1}(t_{\epsilon_1,u}) = g_{\epsilon_2}(t_{\epsilon_2,u}) = 0.$$

Since the sequence $(a_\epsilon)_\epsilon$ is strictly decreasing, it easily follows that

$$g_{\epsilon_2}(t_{\epsilon_2,u}) = 0 = g_{\epsilon_1}(t_{\epsilon_1,u}) < g_{\epsilon_2}(t_{\epsilon_1,u}).$$

By the fact that the function $t \longmapsto g_\epsilon(t)$ is strictly decreasing, we obtain

$$t_{\epsilon_1,u} < t_{\epsilon_2,u}.$$

Hence the sequence $(t_{\epsilon,u})_\epsilon$ is strictly increasing. □

Lemma 2.2. *Any Palais–Smale sequence for Ψ_ϵ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Proof. Let $(u_n)_n \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for Ψ_ϵ. Then, as $n \to \infty$, we have

$$c_\epsilon = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} a_\epsilon(z)|u_n|^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s_2}}}{|z|^{s_2}}\,dx + o(1)$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx - \lambda\int_{\mathbb{R}^N} a_\epsilon(z)|u_n|^{2^*_{s_1}}\,dx - \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s_2}}}{|z|^{s_2}}\,dx = o(1).$$

Therefore,

$$c_\epsilon = \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \left(\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}\right)\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s_2}}}{|z|^{s_2}}\,dx + o(1).$$

Since the quantity $\left(\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}\right)$ is positive, we get

$$c_\epsilon \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + o(1).$$

Consequently, the sequence $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

Remark 2.3. Define

$$c_\epsilon := \inf_{u\in\mathcal{N}_\epsilon} \Psi_\epsilon(u) \quad \text{and} \quad \delta_\epsilon := \inf_{u\in\mathcal{N}_\epsilon}\int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

By Lemmas 2.1 and 2.2, we have:

1) for any $\epsilon \in (0, s_1)$, it holds

$$c_\epsilon \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right) \delta_\epsilon > \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right) \delta_0 > 0;$$

2) any $(PS)_c$ sequence for $\Psi_\epsilon |_{\mathcal{N}_\epsilon}$ is a $(PS)_c$ sequence for Ψ_ϵ.

Lemma 2.4. *The sequence $(c_\epsilon)_{0\leq\epsilon<s_1}$ is strictly increasing.*

Proof. Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ be fixed. Then for all $\epsilon \in (0, s_1)$, there exist $t_{\epsilon,u} > 0$ implicitly given by

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \lambda t_{\epsilon,u}^{2^*_{s_1}-2} \int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}} \, dx - t_{\epsilon,u}^{2^*_{s_2}-2} \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s_2}}}{|z|^{s_2}} \, dx = 0. \tag{2.6}$$

Thanks to the Implicit Function Theorem, the map $\epsilon \longmapsto t(\epsilon) := t_\epsilon$ is of class $\mathcal{C}^1$. Moreover, by Lemma 2.1, we have

$$\frac{d}{d\epsilon}(\Psi_\epsilon(t_\epsilon u)) > 0.$$

Differentiating $\Psi_\epsilon(t_\epsilon u)$ with respect to ϵ and using (2.6), we obtain

$$\frac{d}{d\epsilon}(\Psi_\epsilon(t_\epsilon u)) = -\frac{\lambda}{2^*_{s_1}} t_\epsilon^{2^*_{s_1}} \frac{d}{d\epsilon} \int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}} \, dx > 0.$$

Hence the sequence $(c_\epsilon)_\epsilon$ is strictly increasing. □

We are now in position to prove the following.

Proposition 2.5. *Let $N \geq 3$ and $0 \leq s_2 < s_1 < 2$. Then for any $\epsilon \in (0, s_1)$, problem (2.2) admits a ground state solution having the least energy*

$$c_\epsilon < \left(\frac{1}{2} - \frac{1}{2^*_{s_2}}\right) S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}}, \tag{2.7}$$

where S_{N,s_2} is the cylindrical Hardy–Sobolev best constant given by

$$S_{N,s_2} := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |z|^{-s_2}|u|^{2^*_{s_2}}\right)^{2/2^*_{s_2}}}.$$

Proof. Let U be the least energy solution of

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^N} |z|^{-s_2}|u|^{2^*_{s_2}} \, dx.$$

We have

$$c_\epsilon \leq \max_{t>0} \Psi_\epsilon(tU)$$

$$= \max_{t>0}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla U|^2\,dx - \lambda\frac{t^{2^*_{s_1}}}{2^*_{s_1}}\int_{\mathbb{R}^N} a_\epsilon(z)|U|^{2^*_{s_1}}\,dx - \frac{t^{2^*_{s_2}}}{2^*_{s_2}}\int_{\mathbb{R}^N}|z|^{-s_2}|U|^{2^*_{s_2}}\,dx\right)$$
$$< \max_{t>0}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla U|^2\,dx - \frac{t^{2^*_{s_2}}}{2^*_{s_2}}\int_{\mathbb{R}^N}|z|^{-s_2}|U|^{2^*_{s_2}}\,dx\right) = \left(\frac{1}{2}-\frac{1}{2^*_{s_2}}\right)S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}}.$$

Therefore,

$$c_\epsilon < \left(\frac{1}{2}-\frac{1}{2^*_{s_2}}\right)S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}}. \tag{2.8}$$

Next, we let $(u_n)_n$ be a Palais–Smale sequence of Ψ_ϵ with

$$c_\epsilon < \left(\frac{1}{2}-\frac{1}{2^*_{s_2}}\right)S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}}.$$

Up to a subsequence, we may assume that

$$u_n \rightharpoonup u_0 \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{and} \quad u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N.$$

Assuming by contradiction that $u_n \nrightarrow u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we have

$$c_\epsilon = \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx - \lambda\frac{t^{2^*_{s_1}}}{2^*_{s_1}}\int_{\mathbb{R}^N} a_\epsilon(z)|u_n|^{2^*_{s_1}}\,dx - \frac{t^{2^*_{s_2}}}{2^*_{s_2}}\int_{\mathbb{R}^N}|z|^{-s_2}|u_n|^{2^*_{s_2}}\,dx + o(1)$$

and, for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}\nabla u_n\cdot\nabla\varphi\,dx - \lambda\int_{\mathbb{R}^N} a_\epsilon(z)|u_n|^{2^*_{s_1}-2}u_n\varphi\,dx - \int_{\mathbb{R}^N}|z|^{-s_2}|u_n|^{2^*_{s_2}-2}u_n\varphi\,dx = o(1).$$

Thanks to the weak convergence of the sequence $(u_n)_n$ into u_0, we have

$$\int_{\mathbb{R}^N}|\nabla(u_n-u_0)|^2\,dx = \int_{\mathbb{R}^N}|\nabla u_n|^2\,dx - \int_{\mathbb{R}^N}|\nabla u_0|^2\,dx + o(1). \tag{2.9}$$

Further, by the Brezis–Lieb lemma, we have

$$\int_{\mathbb{R}^N}|z|^{-s_2}|u_n-u_0|^{2^*_{s_2}}\,dx = \int_{\mathbb{R}^N}|z|^{-s_2}|u_n|^{2^*_{s_2}}\,dx - \int_{\mathbb{R}^N}|z|^{-s_2}|u_0|^{2^*_{s_2}}\,dx + o(1), \tag{2.10}$$

as $n \to \infty$. The embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*_{s_1}}(\mathbb{R}^N, a_\epsilon(z)\,dx)$ is compact (see, for instance, [32, Lemma 14]). Then we have

$$\int_{\mathbb{R}^N} a_\epsilon(z)|u_n|^{2^*_{s_1}}\,dx = \int_{\mathbb{R}^N} a_\epsilon(z)|u_0|^{2^*_{s_1}}\,dx + o(1). \tag{2.11}$$

Therefore, combining (2.9)–(2.11), we obtain

$$c_\epsilon = \Psi_\epsilon(u_0) + \Phi(u_n-u_0) + o(1) \quad \text{as } n \to \infty.$$

Since $\Psi'_\epsilon(u_0) = 0$, we have $\Psi_\epsilon(u_0) \geq 0$. Then it immediately follows that

$$c_\epsilon \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_2}}\right) S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}},$$

which contradicts (2.8). Thus

$$u_n \to u_0 \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N),$$

which ends the proof. □

Next, we let $\epsilon \in (0, s_1)$ and define the mountain pass value

$$\tilde{c}_\epsilon := \inf_{\gamma \in \Gamma_\epsilon} \max_{0 \leq t \leq 1} \Psi_\epsilon(\gamma(t)),$$

where

$$\Gamma_\epsilon := \left\{\gamma : [0,1] \to \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ continuous} : \gamma(0) = 0 \text{ and } \Psi_\epsilon(\gamma(1)) < 0\right\}.$$

It is well known that

$$c_\epsilon = \tilde{c}_\epsilon. \tag{2.12}$$

As an immediate consequence of Lemma 2.1, Proposition 2.5, and (2.12), it follows that

$$\lim_{\epsilon \to 0} \tilde{c}_\epsilon = \tilde{c}_0 = c_0.$$

Proposition 2.6. *Let $N \geq 3$, $0 \leq s_2 < s_1 < 2$, and $\epsilon \in (0, s_1)$. Then any solution of (2.2) satisfies*

$$\int_{B(0,1)} \frac{|u(x)|^{2^*_{s_1}}}{|z|^{s_1-\epsilon}}\, dx = \int_{\mathbb{R}^N \setminus B(0,1)} \frac{|u(x)|^{2^*_{s_1}}}{|z|^{s_1+\epsilon}}\, dx.$$

Proof. We define $P : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$P(x,u) = \frac{a_\epsilon(z)|u|^{2^*_{s_1}}}{2^*_{s_1}} + \frac{1}{2^*_{s_2}} \frac{|u|^{2^*_{s_2}}}{|z|^{s_2}}.$$

We have

$$2N \int_{\mathbb{R}^N} P(x,u)\, dx + 2\sum_{i=1}^{N} \int_{\mathbb{R}^N} x_i \frac{\partial P}{\partial x_i}(x,u)\, dx = (N-2)\int_{\mathbb{R}^N} |\nabla u|^2\, dx,$$

see [21, Proposition 2.1]. In other words,

$$\frac{2N}{2^*_{s_1}} \int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}}\, dx + \frac{2N}{2^*_{s_2}} \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s_2}}}{|z|^{s_2}}\, dx + \frac{2}{2^*_{s_1}} \sum_{i=1}^{N} \int_{\mathbb{R}^N} x_i \frac{\partial}{\partial x_i}\left[a_\epsilon(z)\right] |u|^{2^*_{s_1}}\, dx$$
$$+ \frac{2}{2^*_{s_2}} \sum_{i=1}^{N} \int_{\mathbb{R}^N} x_i \frac{\partial}{\partial x_i}\left[\frac{1}{|z|^{s_2}}\right] |u|^{2^*_{s_2}}\, dx = (N-2)\int_{\mathbb{R}^N} |\nabla u|^2\, dx. \tag{2.13}$$

Direct computations lead to

$$\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}\left[a_\epsilon(z)\right] = \epsilon \frac{1_{B(0,1)}}{|z|^{s_1-\epsilon}} - \epsilon \frac{1_{\mathbb{R}^N\setminus B(0,1)}}{|z|^{s_1+\epsilon}} - s_1 a_\epsilon(x),$$
$$\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}\left[\frac{1}{|z|^{s_2}}\right] = -\frac{s_2}{|z|^{2^*_{s_2}}}. \tag{2.14}$$

Replacing (2.14) into (2.13), we get

$$(N-2)\int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}}\,dx + (N-2)\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s_2}}}{|z|^{s_2}}dx + \frac{2\epsilon}{2^*_{s_1}}\int_{B(0,1)}\frac{|u|^{2^*_{s_1}}}{|z|^{s_1-\epsilon}}\,dx$$
$$-\frac{2\epsilon}{2^*_{s_1}}\int_{\mathbb{R}^N\setminus B(0,1)}\frac{|u|^{2^*_{s_1}}}{|z|^{s_1+\epsilon}}\,dx = (N-2)\int_{\mathbb{R}^N}|\nabla u|^2\,dx. \tag{2.15}$$

Next, we multiply (2.2) by u and integrate by parts to get

$$(N-2)\int_{\mathbb{R}^N} a_\epsilon(z)|u|^{2^*_{s_1}}\,dx + (N-2)\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s_2}}}{|z|^{s_2}}\,dx = (N-2)\int_{\mathbb{R}^N}|\nabla u|^2\,dx. \tag{2.16}$$

Hence, by (2.15) and (2.16), we obtain the desired result. □

2.2. Existence of a global (i.e., defined on the whole $\mathbb{R}^N$) solution in $\mathbb{R}^N$. We let $\epsilon > 0$. There exists $u_\epsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}$, a positive solution of

$$-\Delta u_\epsilon = \lambda a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}-1} + \frac{|u_\epsilon|^{2^*_{s_2}-1}}{|z|^{s_2}} \quad \text{in } \mathbb{R}^N.$$

Then u_ϵ satisfies

$$c_\epsilon = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_\epsilon|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N}\frac{|u_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx,$$

and

$$\int_{\mathbb{R}^N}|\nabla u_\epsilon|^2\,dx = \lambda\int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}\,dx + \int_{\mathbb{R}^N}\frac{|u_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx.$$

Therefore,

$$c_\epsilon = \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right)\int_{\mathbb{R}^N}|\nabla u_\epsilon|^2\,dx + \left(\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}\right)\int_{\mathbb{R}^N}\frac{|u_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx.$$

Since the quantity $\left(\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}\right)$ is positive, we have

$$c_0 \geq c_\epsilon \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right)\int_{\mathbb{R}^N}|\nabla u_\epsilon|^2\,dx.$$

Consequently, $(u_\epsilon)_\epsilon$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Thus, there exists a subsequence (still denoted by u_ϵ) and $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$u_\epsilon \rightharpoonup u_0 \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N) \qquad \text{and} \qquad u_\epsilon \to u_0 \quad \text{a.e. in } \mathbb{R}^N. \tag{2.17}$$

Lemma 2.7. *We have*

$$\Psi_0'(u_0) = 0.$$

Proof. Let $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $\varphi \geq 0$. By the Fatou lemma, we have

$$\int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s_1}-2}}{|z|^{s_1}} u_0 \varphi \, dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi \, dx. \tag{2.18}$$

By (2.3), we have

$$a_\epsilon(z) \leq \frac{1}{|z|^{s_1}} \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

such that for all $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ with $\varphi \geq 0$, we get

$$\int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi \, dx \leq \int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2^*_{s_1}-2}}{|z|^{s_1}} u_\epsilon \varphi \, dx. \tag{2.19}$$

By (2.17), we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi}{|z|^{s_1}} \, dx = \int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s_1}-2} u_0 \varphi}{|z|^{s_1}} \, dx. \tag{2.20}$$

By (2.19) and (2.20), we obtain

$$\limsup_{\epsilon \to 0} \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi \, dx \leq \int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s_1}-2} u_0 \varphi}{|z|^{s_1}} \, dx. \tag{2.21}$$

Hence, by (2.18) and (2.21), we get

$$\int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s_1}-2}}{|z|^{s_1}} u_0 \varphi \, dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi \, dx, \qquad \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ with } \varphi \geq 0. \tag{2.22}$$

Next, for any test function $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, we write

$$\varphi = \varphi^+ - \varphi^-,$$

with $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$. Since both φ^+ and φ^- are nonnegative, they satisfy (2.22). Then, by summing the corresponding identities, we obtain

$$\int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s_1}-2}}{|z|^{s_1}} u_0 \varphi \, dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi \, dx, \quad \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

This then ends the proof. □

Lemma 2.8. *We have* $0 \leq \Psi_0(u_0) \leq c_0$. *Moreover, if* $u_0 \neq 0$, *then we have*

$$\Psi_0(u_0) = c_0 > 0.$$

Proof. By Lemma 2.7, we have

$$\Psi_0'(u_0) = 0. \tag{2.23}$$

If $u_0 = 0$, it is obvious that

$$\Psi_0(u_0) = 0. \tag{2.24}$$

Next, we assume that $u_0 \neq 0$. By (2.23), we have

$$\langle \Psi_0'(u_0), u_0 \rangle = 0.$$

Therefore

$$\Psi_0(u_0) \geq c_0 > 0. \tag{2.25}$$

Recall that, up to a sequence, we have

$$c_\epsilon = \Psi_\epsilon(u_\epsilon) = \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right) \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 \, dx + \left(\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}\right) \int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}} \, dx.$$

Using the Fatou lemma, we get

$$\Psi_0(u_0) \leq \lim_{\epsilon \to 0^+} \inf c_\epsilon = c_0. \tag{2.26}$$

Then the result follows directly from (2.24)–(2.26). $\square$

In the sequel, we will prove that $u_0 \neq 0$ so that the following existence result is immediate.

Proposition 2.9. *Let $N \geq 3$, $0 \leq s_2 < s_1 < 2$. Then the problem*

$$\begin{cases} -\Delta u = \lambda \dfrac{u^{2^*_{s_1}-1}}{|z|^{s_1}} + \dfrac{u^{2^*_{s_2}-1}}{|z|^{s_2}} & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and } u > 0 & \text{in } \mathbb{R}^N \end{cases}$$

has a least-energy solution.

The proof is mainly based on several lemmas.

Lemma 2.10. *We have*

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}} \, dx > 0.$$

Proof. We assume (by contradiction) that

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}} \, dx = 0. \tag{2.27}$$

Let $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. By the Hölder inequality, we have

$$\left| \int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}-2} u_\epsilon \varphi \, dx \right| \leq \left(\int_{\mathbb{R}^N} a_\epsilon(z) |u_\epsilon|^{2^*_{s_1}} dx \right)^{\frac{2^*_{s_1}-1}{2^*_{s_1}}} \left(\int_{\mathbb{R}^N} a_\epsilon(z) |\varphi|^{2^*_{s_1}} dx \right)^{\frac{1}{2^*_{s_1}}}.$$

Further, the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*_{s_1}}(\mathbb{R}^N, a_\epsilon(x)dx)$ is compact (see, for instance, [32, Lemma 14]). Then there exists $C \geq 0$ such that

$$\left(\int_{\mathbb{R}^N} a_\epsilon(z)|\varphi|^{2^*_{s_1}}\,dx\right)^{\frac{2}{2^*_{s_1}}} \leq C\left(\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx\right), \quad \varphi \in \mathcal{D}^{1,2}\big(\mathbb{R}^N\big).$$

Therefore,

$$\left|\int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}-2}u_\epsilon\varphi\,dx\right| \leq \tilde{C}\left(\int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}\,dx\right)^{\frac{1}{2^*_{s_1}}}, \tag{2.28}$$

for some positive constant $\tilde{C}$ independent of ϵ. Consequently, by (2.27) and (2.28), we obtain

$$\int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}-2}u_\epsilon\varphi dx = o(1) \quad \text{as } \epsilon \to 0. \tag{2.29}$$

Hence, from (2.27), (2.29), and the fact that $(u_\epsilon)_\epsilon$ is a $(PS)_{c_\epsilon}$ sequence for Φ, we deduce that

$$\Phi(u_\epsilon) = c_\epsilon + o(1) \quad \text{and} \quad \Phi'(u_\epsilon) = o(1) \quad \text{in } \big(\mathcal{D}^{1,2}\big(\mathbb{R}^N\big)\big)'.$$

Since $\lim_{\epsilon\to 0^+}\inf c_\epsilon > 0$, the sequence $(u_\epsilon)_\epsilon$ does not converge to 0 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Therefore,

$$\left(\frac{1}{2} - \frac{1}{2^*_{s_2}}\right) S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}} \leq \lim_{\epsilon\to 0^+} \Phi(u_\epsilon) = \lim_{\epsilon\to 0^+}\left(\Psi_\epsilon(u_\epsilon) - \int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon(x)|^{2^*_{s_1}}\,dx\right) = c_0,$$

which contradicts (2.7). Hence

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}dx > 0. \qquad \square$$

Lemma 2.11. *For $\epsilon > 0$, let $(\varphi_\epsilon)_\epsilon \subset \mathcal{D}^{1,2}\big(\mathbb{R}^N\big)$ be a bounded sequence such that, as $\epsilon \longmapsto 0$:*

$$L_\epsilon(\varphi_\epsilon) := \int_{\mathbb{R}^N} |\nabla\varphi_\epsilon|^2\,dx - \lambda\int_{\mathbb{R}^N} a_\epsilon(z)|\varphi_\epsilon|^{2^*_{s_1}}\,dx - \int_{\mathbb{R}^N}\frac{|\varphi_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx = o(1).$$

Let also

$$\varphi_\epsilon \nrightarrow 0 \text{ in } L^{2^*_{s_2}}(\mathbb{R}^N, |z|^{-s_2}dx) \quad \text{and} \quad \liminf_{\epsilon\to 0^+}\int_{\mathbb{R}^N} a_\epsilon(z)|\varphi_\epsilon|^{2^*_{s_1}}dx > 0.$$

Then, up to a subsequence, we have

$$\lim_{\epsilon\to 0}\Psi_\epsilon(\varphi_\epsilon) \geq \lim_{\epsilon\to 0^+} c_\epsilon = c_0 > 0.$$

Proof. By the cylindrical Hardy–Sobolev inequality, there exists a positive constant C_{N,s_2} independent of ϵ such that

$$C_{N,s_2}\left(\int_{\mathbb{R}^N}\frac{|\varphi_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx\right)^{\frac{2}{2^*_{s_2}}}\le\int_{\mathbb{R}^N}|\nabla\varphi_\epsilon|^2\,dx, \tag{2.30}$$

see, for instance, Fabbri, Mancini, and Sandeep [15]. Since the sequence $(\varphi_\epsilon)_\epsilon$ does not converge to 0 in $L^{2^*_{s_2}}(\mathbb{R}^N,|z|^{-s_2}\,dx)$, we have

$$\varphi_\epsilon\nrightarrow 0\text{ in }\mathcal{D}^{1,2}\big(\mathbb{R}^N\big).$$

Further, $(\varphi_\epsilon)_\epsilon$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then there exist two constants $0<d_1<d_2$ such that

$$d_1\le\int_{\mathbb{R}^N}|\nabla\varphi_\epsilon|^2\,dx\le d_2. \tag{2.31}$$

Using the fact that $\varphi_\epsilon\nrightarrow 0$ in $L^{2^*_{s_2}}(\mathbb{R}^N,|z|^{-s_2}dx)$ and by (2.30), there exist two constants $0<d_3<d_4$ such that

$$d_3\le\int_{\mathbb{R}^N}\frac{|\varphi_\epsilon|^{2^*_{s_2}}}{|z|^{s2}}\,dx\le d_4. \tag{2.32}$$

Therefore, up to a subsequence, we can assume that

$$\lim_{\epsilon\to0^+}\int_{\mathbb{R}^N}|\nabla\varphi_\epsilon|^2\,dx=a^*>0\quad\text{and}\quad\lim_{\epsilon\to0^+}\int_{\mathbb{R}^N}\frac{|\varphi_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx=b^*>0.$$

Set

$$\eta:=\liminf_{\epsilon\to0}\int_{\mathbb{R}^N}a_\epsilon(z)|\varphi_\epsilon|^{2^*_{s_1}}\,dx>0. \tag{2.33}$$

We have

$$a^*-\eta-b^*=0. \tag{2.34}$$

By Lemma 2.1, there exists $t_\epsilon>0$ unique such that $t_\epsilon\varphi_\epsilon\in\mathcal{N}_\epsilon$. Therefore,

$$\int_{\mathbb{R}^N}|\nabla\varphi_\epsilon|^2\,dx-ct_\epsilon^{2^*_{s_1}-2}\int_{\mathbb{R}^N}a_\epsilon(z)|\varphi_\epsilon|^{2^*_{s_1}}\,dx-t_\epsilon^{2^*_{s_2}-2}\int_{\mathbb{R}^N}\frac{|\varphi_\epsilon|^{2^*_{s_2}}}{|z|^{s_2}}\,dx=0. \tag{2.35}$$

By (2.31)–(2.33) and (2.35), we can easily see that $(t_\epsilon)_\epsilon$ is bounded and, up to a subsequence, we obtain

$$\lim_{\epsilon\to0}t_\epsilon=t_0>0.$$

By the identity $L_\epsilon(t_\epsilon\varphi_\epsilon)\equiv0$ and the fact that $(\varphi_\epsilon)_\epsilon$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we deduce that

$$\lim_{\epsilon\to0^+}L_\epsilon(t_0\varphi_\epsilon)=0.$$

Therefore,

$$a^*-\eta t_0^{2^*_{s_1}-2}-b^*t_0^{2^*_{s_2}-2}=0.$$

Next, we define

$$g(t) := a^* - \eta t^{2^*_{s_1}-2} - b^* t^{2^*_{s_2}-2}.$$

By the classical Intermediate Value Theorem, there exists $\bar{t} > 0$ unique such that

$$g(\bar{t}) = 0.$$

Then, by this and (2.34), we deduce that

$$t_0 = 1.$$

By the boundedness of $(\varphi_\epsilon)_\epsilon$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we easily obtain

$$\lim_{\epsilon\to 0^+} \Psi_\epsilon(\varphi_\epsilon) = \lim_{\epsilon\to 0^+} \Psi_\epsilon(t_\epsilon \varphi_\epsilon).$$

By the definition of t_ϵ, we have $t_\epsilon\varphi_\epsilon \in \mathcal{N}_\epsilon$. Then

$$\Psi_\epsilon(t_\epsilon\varphi_\epsilon) \geq c_\epsilon.$$

Therefore, we conclude that

$$\lim_{\epsilon\to 0^+} \Psi_\epsilon(\varphi_\epsilon) \geq \lim_{\epsilon\to 0^+} c_\epsilon = c_0 > 0. \qquad \square$$

Proof of Proposition 2.9. Let $(u_\epsilon)_\epsilon$ and u_0 be given by (2.17). By Lemma 2.8, we only need to prove that $u_0 \neq 0$. We assume by contradiction that $u_0 = 0$. By Lemma 2.10, we have

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}\,dx =: \tau > 0.$$

By Proposition 2.6, we have

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_1} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}\,dx = \lim_{\epsilon\to 0^+} \int_{B_1} a_\epsilon(z)|u_\epsilon|^{2^*_{s_1}}\,dx = \frac{\tau}{2}. \tag{2.36}$$

Next, we let $\eta_1, \eta_2 \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ such that

$$\eta_1 \equiv 1 \text{ in } B_{\frac{1}{2}}, \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^N\setminus B_1, \quad \eta_2 \equiv 1 \text{ in } B_1, \quad \text{and} \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^N \setminus B_2.$$

Define

$$Z_\epsilon := \eta_1 u_\epsilon + (1-\eta_2)u_\epsilon \quad \text{and} \quad v_\epsilon := u_\epsilon - Z_\epsilon = (\eta_2 - \eta_1)u_\epsilon.$$

Then $\operatorname{supp}(v_\epsilon) \subset \Omega$ with $\Omega := B_2\setminus B_{1/2}$. By the Rellich–Kondrachov compactness theorem, we have

$$v_\epsilon \to 0 \quad \text{in } L^{2^*_{s_1}}(\Omega, a_\epsilon(z)\,dx).$$

Then $(v_\epsilon)_\epsilon$ is a Palais–Smale sequence for Φ. By the Brezis–Lieb lemma, we have

$$\Psi_\epsilon(u_\epsilon) = \Psi_\epsilon(Z_\epsilon) + \Phi(v_\epsilon) + o(1) \quad \text{as } \epsilon \to 0.$$

Moreover, $\Psi'_\epsilon(u_\epsilon) = 0$. Then we deduce that

$$\Phi'(Z_\epsilon) = o(1) \quad \text{as } \epsilon \to 0. \tag{2.37}$$

Therefore,

$$\Phi(Z_\epsilon) \geq 0. \tag{2.38}$$

Now, if $v_\epsilon \nrightarrow 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, then we get

$$\lim_{\epsilon \to 0^+} \Phi(v_\epsilon) \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_2}}\right) S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}}. \tag{2.39}$$

By (2.37)–(2.39), we obtain

$$\lim_{\epsilon \to 0^+} c_\epsilon \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_2}}\right) S_{N,s_2}^{\frac{2^*_{s_2}}{2^*_{s_2}-2}},$$

which contradicts the result in Proposition 2.5. Hence $v_\epsilon \to 0$ in $\mathcal{D}^{1,2}\big(\mathbb{R}^N\big)$. By the Brezis–Lieb lemma, we get

$$\Psi_\epsilon(u_\epsilon) = \Psi_\epsilon(Z_\epsilon) + o(1) = \Psi_\epsilon(\eta_1 u_\epsilon) + \Psi_\epsilon((1-\eta_2)u_\epsilon) + o(1) \qquad \text{as } \epsilon \to 0.$$

Since $\Psi'_\epsilon(u_\epsilon) = 0$, we have

$$\langle \Psi'_\epsilon(u_\epsilon), \eta_1 u_\epsilon \rangle = 0.$$

Then $v_\epsilon \to 0$ in $\mathcal{D}^{1,2}\big(\mathbb{R}^N\big)$ so that

$$\langle \Psi'_\epsilon(\eta_1 u_\epsilon), \eta_1 u_\epsilon \rangle = o(1) \quad \text{as } \epsilon \to 0. \tag{2.40}$$

Next we have

$$\int_{\mathbb{R}^N} a_\epsilon(z) |\eta_1 u_\epsilon|^{2^*_{s_1}}\, dx = \int_{B_1} a_\epsilon(z) |\eta_1 u_\epsilon|^{2^*_{s_1}}\, dx.$$

Therefore, by this and (2.36), we obtain

$$\lim_{\epsilon \to 0^+} \inf \int_{\mathbb{R}^N} a_\epsilon(z) |\eta_1 u_\epsilon|^{2^*_{s_1}}\, dx = \frac{\tau}{2}. \tag{2.41}$$

Hence, by (2.40), (2.41), and Lemma 2.11, we get

$$\liminf_{\epsilon \to 0} \Phi_\epsilon(\eta_1 u_\epsilon) \leq \lim_{\epsilon \to 0^+} c_\epsilon = c_0 > 0.$$

Using the same arguments as before, we obtain

$$\liminf_{\epsilon \to 0} \Phi_\epsilon((1-\eta_2) u_\epsilon) \leq \lim_{\epsilon \to 0^+} c_\epsilon = c_0 > 0.$$

Hence

$$c_0 \geq 2c_0,$$

which is false since $c_0 > 0$. Consequently, we obtain that

$$u_0 \neq 0,$$

which ends the proof. □

Next we will establish symmetry and decay estimates properties of positive solutions $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ of the following Euler–Lagrange equations:

$$- \Delta u = \lambda \frac{u^{2^*_{s_1}-1}}{|z|^{s_1}} + \frac{u^{2^*_{s_2}-1}}{|z|^{s_2}} \quad \text{in } \mathbb{R}^N, \tag{2.42}$$

where for $N \geq 3$, we have $x := (y,z) \in \mathbb{R} \times \mathbb{R}^{N-1}$, $0 \leq s_2 < s_1 < 2$ and $2^*_{s_i} := \frac{2(N-s_i)}{N-2}$, $i = 1,2$, are two Hardy–Sobolev critical exponents. Next, rewrite equation (2.42) as follows:

$$-\Delta u = \frac{f(x)}{|z|^{s_1}} u + \frac{g(x)}{|z|^{s_1}},$$

where $f, g \in L^P_{loc}(\mathbb{R}^N)$ for some $p > \frac{N}{2-s_1}$. Then the following result follows from [15, Lemma 3.2 and Lemma 3.3].

Proposition 2.12. *Let u be a solution of (2.42). Let also*

$$\begin{cases} s_1 < 1 + \dfrac{1}{N} & \text{if } N \geq 4, \\ s_1 < \dfrac{3}{2} & \text{if } N = 3. \end{cases}$$

Then $u \in \mathcal{C}^\infty$ in the z variable while in the y variable, it is $\mathcal{C}^{1,\alpha}$ for all $\alpha < 1 - s_1$ if $s_1 < 1$ and $\mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ for all $\alpha < 2 - s_1$ if $1 \leq s_1 < 2$.

This then allows us to prove the following symmetry and decay estimates result.

Proposition 2.13. *Let u be a solution of the Euler–Lagrange equation (2.42). Then*

(i) *the function u depends only on $|y|$ and $|z|$,*
(ii) *there exist two constants $0 < c_1 < c_2$ such that*

$$\frac{c_1}{1 + |x|^{N-2}} \leq u(x) \leq \frac{c_2}{1 + |x|^{N-2}}, \quad x \in \mathbb{R}^N.$$

Proof. The proof of the symmetry is based on the moving plane method (see, for instance, [6, 27]). We let $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. For $\mu > 0$, we define

$$\Omega_\mu = \left\{ x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : y_1 > \mu \right\}$$

and for all $x \in \Omega_\mu$, we set

$$x_\mu = (2\mu - y_1, y_2, \ldots, y_k, z) \in \Omega_\mu$$

and define

$$w_\mu(x) := u_\mu(x) - u(x) = u(x_\mu) - u(x) \quad \text{in } \Omega_\mu.$$

Then $w_\mu \in H^1_0(\Omega_\mu)$.

Step 1: We first prove that

$$w_\mu \geq 0 \quad \text{in } \Omega_\mu \tag{2.43}$$

for μ large enough. Thanks to (2.42), we have

$$-\Delta w_\mu(x) = \lambda \left(\frac{u^{2^*_{s_1}-1}(x)}{|z|^{s_1}} - \frac{u_\mu^{2^*_{s_1}-1}(x)}{|z|^{s_1}} \right) + \left(\frac{u^{2^*_{s_2}-1}(x)}{|z|^{s_2}} - \frac{u_\mu^{2^*_{s_2}-1}(x)}{|z|^{s_2}} \right) \quad \text{in } \Omega_\mu. \tag{2.44}$$

We multiply (2.44) by $w_\mu^- := \min\{w_\mu, 0\}$ and integrate by parts to get

$$\begin{aligned} \int_{\Omega_\mu} |\nabla w_\mu^-|^2 dx =& \lambda \int_{\Omega_\mu} w_\mu^-(x) \left(\frac{u^{2^*_{s_1}-1}(x)}{|z|^{s_1}} - \frac{u_\mu^{2^*_{s_1}-1}(x)}{|z|^{s_1}} \right) dx \\ &+ \int_{\Omega_\mu} w_\mu^-(x) \left(\frac{u^{2^*_{s_2}-1}(x)}{|z|^{s_2}} - \frac{u_\mu^{2^*_{s_2}-1}(x)}{|z|^{s_2}} \right) dx \\ \leq& \lambda \int_{\Omega_\mu} \frac{w_\mu^-(x)}{|z|^{s_1}} \left(u^{2^*_{s_1}-1}(x) - u_\mu^{2^*_{s_1}-1}(x) \right) dx \\ &+ \int_{\Omega_\mu} \frac{w_\mu^-(x)}{|z|^{s_2}} \left(u^{2^*_{s_2}-1}(x) - u_\mu^{2^*_{s_2}-1}(x) \right) dx. \end{aligned}$$

On the support of w_μ^-, we have $u_\mu(x) \leq u(x)$. Then, using the convexity of the function $t \mapsto t^{2^*_{s_i}}$ $(i = 1, 2)$ on $(0, \infty)$, we get

$$\begin{aligned} u_\mu^{2^*_{s_i}-1}(x) - u^{2^*_{s_i}-1}(x) &\leq (2^*_{s_i} - 1) u^{2^*_{s_i}-2}(x) \left(u(x) - u_\mu(x) \right) \\ &= (1 - 2^*_{s_i}) u^{2^*_{s_i}-2}(x) w_\mu^-(x) \quad \text{in } \Omega_\mu. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Omega_\mu} |\nabla w_\mu^-|^2 \, dx \leq& \lambda (2^*_{s_1} - 1) \int_{\Omega_\mu} \frac{|w_\mu^-(x)|^2}{|z|^{s_1}} |u(x)|^{2^*_{s_1}-2} \, dx \\ &+ (2^*_{s_2} - 1) \int_{\Omega_\mu} \frac{|w_\mu^-(x)|^2}{|z|^{s_2}} |u(x)|^{2^*_{s_2}-2} dx. \end{aligned}$$

Next, by the Hölder inequality, we have

$$\begin{aligned} &\int_{\Omega_\mu} \frac{|w_\mu^-(x)|^2}{|z|^{s_i}} |u(x)|^{2^*_{s_i}-2} \, dx \\ &\qquad \leq \left(\int_{\Omega_\mu} \frac{|w_\mu^-(x)|^{2^*_{s_i}}}{|z|^{s_i}} \, dx \right)^{2/2^*_{s_i}} \left(\int_{M_\mu \cap \Omega_\mu} \frac{|u(x)|^{2^*_{s_i}}}{|z|^{s_i}} \, dx \right)^{\frac{2^*_{s_i}-2}{2^*_{s_i}}}, \end{aligned}$$

where $i = 1, 2$ and $M_\mu := \{x \in \Omega_\mu : \ u(x) > u_\mu(x)\}$. Since

$$\lim_{\mu \to \infty} \int_{M_\mu \cap \Omega_\mu} \frac{|u(x)|^{2^*_{s_1}}}{|z|^{s_1}} \, dx = \lim_{\mu \to \infty} \int_{M_\mu \cap \Omega_\mu} \frac{|u(x)|^{2^*_{s_2}}}{|z|^{s_2}} \, dx = 0,$$

we deduce that

$$\int_{\Omega_\mu} |\nabla w_\mu^-|^2\,dx < S_{N,0}\left(\int_{\Omega_\mu} |w_\mu^-|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{2N}},$$

where $S_{N,0}$ is the Sobolev best constant. As a consequence, for μ large enough, we have $w_\mu^- = 0$, and hence $u_\mu(x) \geq u(x)$, which proves (2.43).

Now we let

$$\mu^* := \inf\{\mu > 0 :\ u(x) \leq u_{\mu'}(x) \quad \text{for all } x \in \Omega_{\mu'} \text{ and all } \mu' > \mu\} < \infty.$$

Step 2: Then we will prove that $\mu^* = 0$. By contradiction, we assume that $\mu^* > 0$. Then

$$-\Delta w_{\mu^*}(x) = c_{\mu^*}(x) w_{\mu^*}(x) \quad \text{in } \Omega_{\mu^*},$$

where

$$c_{\mu^*} := \begin{cases} \left(\dfrac{u_{\mu^*}^{2^*_{s_1}-1}(x) - u^{2^*_{s_1}-1}(x)}{|z|^{s_1}} - \dfrac{u_{\mu^*}^{2^*_{s_2}-1}(x) - u^{2^*_{s_2}-1}(x)}{|z|^{s_2}}\right) w_{\mu^*}^{-1} & \text{if } w_{\mu^*} \neq 0,\\ 0 & \text{if } w_{\mu^*} = 0. \end{cases}$$

Clearly, $c_{\mu^*} \in L^\infty(\Omega_{\mu^*})$. Moreover, $w_{\mu^*} \geq 0$ in $\partial\Omega_{\mu^*}$. Then, applying the maximum principle, we have

$$w_{\mu^*} > 0 \quad \text{in } \Omega_{\mu^*}.$$

Let D be any smooth compact set in Ω_{μ^*} such that $|\Omega_\mu \setminus D|$ is sufficiently small for any μ near μ^*. Since

$$w_{\mu^*}(x) \geq \delta > 0 \quad \text{in } D,$$

by continuity, we have

$$w_\mu(x) \geq 0 \quad \text{in D}$$

for all μ near μ^*. In particular,

$$w_\mu(x) \geq 0 \quad \text{on } \partial(\Omega_\mu \setminus D).$$

Using again the maximum principle, we have

$$w_\mu(x) \geq 0 \quad \text{in } \Omega_\mu \setminus D,$$

and thus $w_\mu(x) \geq 0$ in Ω_μ, contrary to the definition of μ^*. We therefore conclude that $\mu^* = 0$. Consequently,

$$u(-y_1, y_2, \ldots, y_k, z) \geq u(y_1, \cdots, y_k, z), \quad y_1 > 0.$$

Applying the same arguments as before to the function

$$v(y,z) = u(-y_1, y_2, \ldots, y_k, z) \quad \text{in } \mathbb{R}^N$$

leads to

$$u(-y_1, y_2, \ldots, y_k, z) \le u(y_1, y_2, \ldots, y_k, z).$$

Therefore,

$$u(-y_1, y_2, \ldots, y_k, z) = u(y_1, y_2, \ldots, y_k, z) \quad \text{in } \mathbb{R}^N.$$

Hence the solution u of (2.42) is symmetric with respect to y_1.

Step 3: Repeating the same arguments as before to the functions

$$x \longmapsto w(\mathcal{R}_k y, z) \quad \text{and} \quad x \longmapsto w(y, \mathcal{R}_{N-k} z),$$

where $\mathcal{R}_k \in O(k)$ and $\mathcal{R}_{N-k} \in O(N-k)$ are respectively k-dimensional and $(N-k)$-dimensional rotations, allows us to conclude that w depends only on $|y|$ and $|z|$ and w is strictly decreasing in $|y|$. This then ends the proof of (i).

For the decay estimate, we write the Euler-Lagrange equation (2.42) as follows:

$$-\Delta w(x) = A(x) w \quad \text{in } \mathbb{R}^N,$$

where

$$A(x) = \lambda \frac{w^{2^*_{s_1}-2}(x)}{|z|^{s_1}} + \frac{w^{2^*_{s_2}-2}(x)}{|z|^{s_2}}.$$

For $x \neq 0$, let $v(x) = |x|^{2-N} w\big(x|x|^{-2}\big)$ be the Kelvin transformation of w. It also satisfies (2.42). Therefore, using the fact that the solution of (2.42) is bounded, we can find two constants $0 < c_1 < c_2$ such that

$$\frac{c_1}{1+|x|^{N-2}} \le w(x) \le \frac{c_2}{1+|x|^{N-2}}, \quad x \in \mathbb{R}^N.$$

This then ends the proof. □

To close this section, we assume that $k = 1$ and we will prove the following decay properties of w involving its higher derivatives.

Proposition 2.14. *Let $N \ge 3$ and $k = 1$. Let also w be a positive solution of (2.42). Then there exists a positive constant C, depending only on N and s_1 and s_2, such that*

(i) *For $|x| = |(y,z)| \le 1$,*

$$|\nabla w(x)| + |x||D^2 w(x)| \le C|z|^{1-s_1}.$$

(ii) *For $|x| = |(y,z)| \ge 1$,*

$$|\nabla w(x)| + |x||D^2 w(x)| \le C \max(1, |z|^{-s_1})|x|^{1-N}.$$

Proof. Let $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that

$$w(x) = w(y,z) = \theta(|y|, |z|).$$

Using polar coordinates, the function $\theta = \theta(y, \rho)$ satisfies

$$\rho^{2-N}(\rho^{N-2}\theta_2)_2 + \theta_{11} = \lambda \rho^{-s_1}\theta^{2^*_{s_1}-1} + \rho^{-s_2}\theta^{2^*_{s_2}-1} \quad \text{for } y, \rho \in \mathbb{R}_+, \tag{2.45}$$

where θ_1 and θ_2 are the derivatives of θ with respect to the first and the second variables. Then, integrating this identity in the ρ variable, we therefore get that for every $\rho > 0$,

$$\theta_2(y,\rho) = -\frac{1}{\rho^{N-2}}\int_0^\rho r^{N-2}\theta_{11}(y,r)dr + \frac{\lambda}{\rho^{N-2}}\int_0^\rho r^{N-2}r^{-s_1}\theta^{2^*_{s_1}-1}(y,r)\,dr + \frac{1}{\rho^{N-2}}\int_0^\rho r^{N-2}r^{-s_2}\theta^{2^*_{s_2}-1}(y,r)\,dr.$$

Next, differentiating with respect to the first variable, we get

$$\begin{aligned}\theta_{12}(y,\rho) = &\frac{-1}{\rho^{N-2}}\int_0^\rho r^{N-2}\theta_{111}(y,r)dr\\ &+\frac{\lambda}{\rho^{N-2}}\int_0^\rho r^{N-2}r^{-s_1}\theta_1(y,r)\theta^{2^*_{s_1}-2}(y,r)\,dr\\ &+\frac{1}{\rho^{N-2}}\int_0^\rho r^{N-2}r^{-s_2}\theta_1(y,r)\theta^{2^*_{s_2}-2}(y,r)\,dr.\end{aligned}$$

By Proposition 2.12 and the fact that $2^*_{s_2} > 2^*_{s_1} \geq 2$, we obtain

$$|\theta_2(y,\rho)| + |\theta_{12}(y,\rho)| \leq C\left(\rho + \rho^{1-s_1} + \rho^{1-s_2}\right) \leq C\rho^{1-s_1} \quad \text{for } |(y,\rho)| \leq 1. \tag{2.46}$$

Now, using this in (2.45), we get

$$|\theta_{22}| \leq C\rho^{s_1} \quad \text{for } |(y,\rho)| \leq 1. \tag{2.47}$$

By (2.46) and (2.47), we obtain

$$|\theta_2(y,\rho)| + |\theta_{12}(y,\rho)| + \rho|\theta_{22}| \leq C\rho^{1-s_1}.$$

Therefore, it easily follows that

$$|\nabla w(x)| + |x||D^2w(x)| \leq C|z|^{1-s_1} \quad \text{for all } |x| = |(y,z)| \leq 1$$

and

$$|\nabla w(x)| + |x||D^2w(x)| \leq C\max(1,|z|^{-s_1})|x|^{1-N} \quad \text{for all } |x| \geq 1.$$

This then completes the proof. □

3. Local parametrization and metric

Let $\Gamma \subset \mathbb{R}^N$ be a smooth closed curve. Let $(E_1;\dots;E_N)$ be an orthonormal basis of $\mathbb{R}^N$. For $y_0 \in \Gamma$ and $r > 0$ small, we consider the curve $\gamma : (-r,r) \to \Gamma$, parameterized by the arc length such that $\gamma(0) = y_0$. Up to a translation and a rotation, we may assume that $\gamma'(0) = E_1$. We choose a smooth orthonormal frame field $(E_2(y);\dots;E_N(y))$ on the normal bundle of Γ such that $(\gamma'(y);E_2(y);\dots;E_N(y))$ is an oriented basis of $\mathbb{R}^N$ for every $y \in (-r,r)$, with $E_i(0) = E_i$. We set

$$Q_r := (-r,r) \times B_{\mathbb{R}^{N-1}}(0,r),$$

where $B_{\mathbb{R}^{N-1}}(0, r)$ denotes the ball in $\mathbb{R}^{N-1}$ with radius r centered at the origin. Provided $r > 0$ small, the map $F_{y_0} : Q_r \to \Omega$, given by

$$(y, z) \mapsto F_{y_0}(y, z) := \gamma(y) + \sum_{i=2}^{N} z_i E_i(y),$$

is smooth and it parametrizes a neighborhood of $y_0 = F_{y_0}(0, 0)$. We consider $\rho_\Gamma : \Gamma \to \mathbb{R}$, the distance function to the curve given by

$$\rho_\Gamma(y) = \min_{\overline{y} \in \mathbb{R}^N} |y - \overline{y}|.$$

In the above coordinates, we have

$$\rho_\Gamma \left(F_{y_0}(x) \right) = |z| \quad \text{ for every } x = (y, z) \in Q_r. \tag{3.1}$$

Clearly, for every $t \in (-r, r)$ and $i = 2, \dots, N$, there are real numbers $\kappa_i(y)$ and $\tau_i^j(y)$ such that

$$E_i'(y) = \kappa_i(y)\gamma'(y) + \sum_{j=2}^{N} \tau_i^j(y) E_j(y).$$

The quantity $\kappa_i(y)$ is the curvature in the $E_i(y)$-direction while $\tau_i^j(y)$ is the torsion from the osculating plane spanned by $\{\gamma'(y); E_j(y)\}$ in the direction E_i. We note that provided $r > 0$ small, κ_i and τ_i^j are smooth functions on $(-r, r)$. Moreover, it is easy to see that

$$\tau_i^j(y) = -\tau_j^i(y) \quad \text{ for } i, j = 2, \dots, N.$$

The curvature vector $\kappa : \Gamma \to \mathbb{R}^N$ is defined as

$$\kappa(\gamma(y)) := \sum_{i=2}^{N} \kappa_i(y) E_i(y)$$

with the norm

$$|\kappa(\gamma(y))| := \sqrt{\sum_{i=2}^{N} \kappa_i^2(y)}.$$

In the sequel, we set

$$\beta_{ij} = \sum_{l=2}^{N} \tau_i^l \tau_j^l.$$

Now we derive the expansion of the metric induced by the parameterization F_{y_0} defined above. For $x = (y, z) \in Q_r$, we define

$$\begin{aligned} g_{11}(x) &= \partial_y F_{y_0}(x) \cdot \partial_y F_{y_0}(x), \\ g_{1i}(x) &= \partial_y F_{y_0}(x) \cdot \partial_{z_i} F_{y_0}(x), \\ g_{ij}(x) &= \partial_{z_j} F_{y_0}(x) \cdot \partial_{z_i} F_{y_0}(x). \end{aligned}$$

Then we have

Lemma 3.1. *There exits $r > 0$, depending only on Γ and N, such that for every $x = (y, z) \in Q_r$,*

$$\begin{cases} g_{11}(x) = 1 + 2\sum_{i=2}^{N} z_i\kappa_i(0) + 2y\sum_{i=2}^{N} z_i\kappa_i'(0) + \sum_{ij=2}^{N} z_iz_j\kappa_i(0)\kappa_j(0) \\ \qquad\qquad + \sum_{ij=2}^{N} z_iz_j\beta_{ij}(0) + O\left(|x|^3\right), \\ g_{1i}(x) = \sum_{j=2}^{N} z_j\tau_j^i(0) + y\sum_{j=2}^{N} z_j\left(\tau_j^i\right)'(0) + O\left(|x|^3\right), \\ g_{ij}(x) = \delta_{ij}. \end{cases}$$

As a consequence, we have the following result.

Lemma 3.2. *There exists $r > 0$, depending only on Γ and N, such that for every $x \in Q_r$, we have*

$$\sqrt{|g|}(x) = 1 + \sum_{i=2}^{N} z_i\kappa_i(0) + y\sum_{i=2}^{N} z_i\kappa_i'(0) + \frac{1}{2}\sum_{ij=2}^{N} z_iz_j\kappa_i(0)\kappa_j(0) + O\left(|x|^3\right). \quad (3.2)$$

Here and in what follows, $|g|$ stands for the determinant of g. Moreover, the components of the inverse metric g^{-1} are given by

$$\begin{cases} g^{11}(x) = 1 - 2\sum_{i=2}^{N} z_i\kappa_i(0) - 2y\sum_{i=2}^{N} z_i\kappa_i'(0) + 3\sum_{ij=2}^{N} z_iz_j\kappa_i(0)\kappa_j(0) + O\left(|x|^3\right), \\ g^{i1}(x) = -\sum_{j=2}^{N} z_j\tau_j^i(0) - y\sum_{j=2}^{N} z_j\left(\tau_j^i\right)'(0) + 2\sum_{j=2}^{N} z_lz_j\kappa_l(0)\tau_j^i(0) + O\left(|x|^3\right), \\ g^{ij}(x) = \delta_{ij} + \sum_{l,m=2}^{N} z_lz_m\tau_l^j(0)\tau_m^i(0) + O\left(|x|^3\right). \end{cases}$$

We will also need the following estimates result.

Lemma 3.3. *Let $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $N \geq 3$, satisfy $v(y, z) = \overline{\theta}(|y|, |z|)$ for some function $\overline{\theta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. Then for $0 < r < R$, we have*

$$\begin{aligned} \int_{Q_R\setminus Q_r} |\nabla v|_g^2\sqrt{|g|}\,dx &= \int_{Q_R\setminus Q_r} |\nabla v|^2\,dx \\ &\quad + \frac{|\kappa(x_0)|^2}{N-1}\int_{Q_R\setminus Q_r} |z|^2\left|\partial_y v\right|^2\,dx \\ &\quad + \frac{|\kappa(x_0)|^2}{2(N-1)}\int_{Q_R\setminus Q_r} |z|^2|\nabla v|^2\,dx + O\left(\int_{Q_R\setminus Q_r} |x|^3|\nabla v|^2\,dx\right). \end{aligned}$$

For the proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3, we refer to the paper of Fall and Thiam [16] and Ijaodoro and Thiam [19].

4. The existence result in domains

We recall that

$$\beta^* = \inf_{\substack{u\in\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)\\ u>0}} \max_{t\geq 0} \Pi(tu) \quad \text{and} \quad c^* := \inf_{\substack{u\in H^1_0(\Omega)\\ u>0}} \max_{t\geq 0} J(tu) \tag{4.1}$$

with

$$\Pi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^{-s_1}|u|^{2^*_{s_1}}\, dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N} |z|^{-s_2}|u|^{2^*_{s_2}}\, dx$$

and

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2\, dx + \frac{1}{2}\int_{\Omega} h(x)u^2\, dx - \frac{\lambda}{2^*_{s_1}}\int_{\Omega} \frac{|u|^{2^*_{s_1}}}{\rho_\Gamma^{s_1}(x)}\, dx - \frac{1}{2^*_{s_2}}\int_{\Omega} \frac{|u|^{2^*_{s_2}}}{\rho_\Gamma^{s_2}(x)}\, dx.$$

The aim of this section is to prove the following compactness result.

Proposition 4.1. *Let $N \geq 4$, $0 \leq s_2 < s_1 < 2$, $\lambda > 0$ and let Ω be a bounded domain of $\mathbb{R}^N$. Consider a smooth closed curve Γ contained in Ω. Let h be a continuous function such that the linear operator $-\Delta + h$ is coercive. Let also*

$$c^* < \beta^*.$$

Then there exists a positive function $u \in H^1_0(\Omega)$ which is a solution of

$$-\Delta u + hu = \lambda\rho_\Gamma^{-s_1}u^{2^*_{s_1}-1} + \rho_\Gamma^{-s_2}u^{2^*_{s_2}-1} \quad \textit{in } \Omega. \tag{4.2}$$

The proof of Proposition 4.1 is divided into various preliminary results. We start by the following.

Lemma 4.2. *Let $\alpha < \beta^*$ and let $(u_n)_n \subset H^1_0(\Omega)$ be a Palais–Smale sequence for J at level α. Then, up to a subsequence, there exists $u \in H^1_0(\Omega)$ such that*

$$\begin{cases} u_n \to u & \textit{in } H^1_0(\Omega),\\ J(u) = \alpha, \\ J'(u) = 0. \end{cases}$$

Proof. Let $(u_n)_n \subset H^1_0(\Omega)$ be a Palais–Smale sequence for J at level α:

$$J(u_n) = \frac{1}{2}\int_{\Omega} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\Omega} hu_n^2\, dx - \frac{\lambda}{2^*_{s_1}}\int_{\Omega} \rho_\Gamma^{-s_1}|u_n|^{2^*_{s_1}}\, dx - \frac{1}{2^*_{s_2}}\int_{\Omega} \rho_\Gamma^{-s_2}|u_n|^{2^*_{s_2}}\, dx = \alpha + o_n(1) \tag{4.3}$$

and, for all $\varphi \in H^1_0(\Omega)$ as $n \to \infty$, we have

$$\langle J'(u_n), \varphi\rangle = \int_{\Omega} \nabla u_n \nabla\varphi\, dx + \int_{\Omega} hu_n\varphi\, dx - \lambda\int_{\Omega} \rho_\Gamma^{-s_1}|u_n|^{2^*_{s_1}-2}u_n\varphi\, dx$$

$$- \int_\Omega \rho_\Gamma^{-s_2} |u_n|^{2^*_{s_2}-2} u_n \varphi \, dx = o_n(1). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\alpha = \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right) \int_\Omega |\nabla u_n|^2 \, dx + \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right) \int_\Omega h u_n^2 \, dx + \left(\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}\right) \int_\Omega \rho_\Gamma^{-s_2} |u_n|^{2^*_{s_2}} \, dx + o_n(1). \tag{4.5}$$

Since both $\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}}$ and $\frac{1}{2} - \frac{1}{2^*_{s_1}}$ are positive and the operator $-\Delta + h$ is coercive, we obtain

$$\frac{\alpha}{\left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right)} + o(1) \geq \int_\Omega |\nabla u_n|^2 \, dx + \int_\Omega h u_n^2 \, dx \geq c \|u_n\|_{H^1_0(\Omega)}$$

for some positive constant c independent of n. Consequently, up to a subsequence, there exists $u \in H^1_0(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H^1_0(\Omega), \\ u_n &\to u \quad \text{a.e. in } \Omega, \\ u_n &\to u \quad \text{strongly in } L^2(\Omega). \end{aligned} \tag{4.6}$$

Set the defect $v_n := u_n - u$. By the Brezis–Lieb lemma and the weak convergence above, we have

$$\int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v_n|^2 \, dx + o(1)$$

and for $i = 1, 2$,

$$\int_\Omega \rho_\Gamma^{-s_i} |u_n|^{2^*_{s_i}} \, dx = \int_\Omega \rho_\Gamma^{-s_i} |u|^{2^*_{s_i}} \, dx + \int_\Omega \rho_\Gamma^{-s_i} |v_n|^{2^*_{s_i}} \, dx + o_n(1).$$

Therefore,

$$J(u_n) = J(u) + J_{\text{mod}}(v_n) + o_n(1)$$

with

$$J_{\text{mod}}(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{\lambda}{2^*_{s_1}} \int_\Omega \rho_\Gamma^{-s_1} |v|^{2^*_{s_1}} \, dx - \frac{1}{2^*_{s_2}} \int_\Omega \rho_\Gamma^{-s_2} |v|^{2^*_{s_2}} \, dx.$$

Moreover,

$$J'(u) = 0 \quad \text{and} \quad J'_{\text{mod}}(v_n) = o(1) \quad \text{in } (H^1_0(\Omega))'. \tag{4.7}$$

For small $r > 0$, define the concentration functions

$$Q_{n,i}(x, r) := \int_{F_x(Q_r) \cap \Omega} \rho_\Gamma^{-s_i} |v_n|^{2^*_{s_i}} \, dx, \quad i = 1, 2.$$

We have three alternatives.

Alternative 1: *Vanishing.* Let $r > 0$. We have

$$\lim_{n\to+\infty} \sup_{x\in\Omega} Q_{n,i}(x,r) = 0, \quad i = 1,2.$$

Since $\overline{\Omega}$ is compact, we can cover it by a finite number of open sets $\left(F_{x_j}(Q_r)\right)_{1\le j\le m}$. Consequently,

$$\Omega \subset \sqcup_{j=1}^{m} F_{x_j}(Q_r).$$

Therefore,

$$\int_\Omega \rho_\Gamma^{-s_i} |v_n|^{2^*_{s_i}}\, dx \le \sum_{j=1}^{m} \int_{F_{x_j}(Q_r)\cap\Omega} \rho_\Gamma^{-s_i} |v_n|^{2^*_{s_i}}\, dx \le m \sup_{x\in\Omega} Q_{n,i}(x,r) = o_n(1).$$

Thus

$$\lim_{n\to\infty} \int_\Omega \rho_\Gamma^{-s_i} |v_n|^{2^*_{s_i}}\, dx = 0, \quad i = 1,2.$$

By this, (4.6) and the second identity in (4.7), we obtain

$$\int_\Omega (|\nabla v_n|^2 + h v_n^2)\, dx = \lambda \int_\Omega \rho_\Gamma^{-s_1} |v_n|^{2^*_{s_1}}\, dx + \int_\Omega \rho_\Gamma^{-s_2} |v_n|^{2^*_{s_2}}\, dx = o_n(1).$$

Since the operator $-\Delta + h$ is coercive, we immediately obtain that

$$u_n \to u \text{ strongly in } H_0^1(\Omega).$$

Alternative 2: *Compactness.* There exist $x_0 \in \Omega$ and $r_0 > 0$:

$$v_n \longrightarrow v \not\equiv 0 \quad \text{in } L^{2^*_{s_i}}(F\left(Q_{r_0}(x_0)\right), \rho_\Gamma^{-s_i}\, dx).$$

Hence $v_n \longrightarrow v \not\equiv 0$ in $H_0^1(\Omega)$ thanks to local compactness and the Euler–Lagrange residual in (4.7). Thus we are already done.

Alternative 3: *Concentration.* There exist $\varepsilon_0 > 0$, sequence points $(y_n)_n \subset \overline{\Omega}$, and $r > 0$ such that

$$\liminf_{n\to\infty} Q_{n,1}(y_n, r) \ge \varepsilon_0 \quad \text{or} \quad \liminf_{n\to\infty} Q_{n,2}(y_n, r) \ge \varepsilon_0. \tag{4.8}$$

Since $\overline{\Omega}$ is compact, then (up to a subsequence) $y_n \to y_0$ in $\overline{\Omega}$. It is obvious that $y_0 \in \Gamma$. Indeed, otherwise $\rho_\Gamma(y_0) \ge r_0 > 0$ for some $r_0 > 0$. Then $\rho_\Gamma^{-s_i}$ is bounded on $B_{r_0/2}(y_0)$ and the usual Rellich compactness yields strong $L^{2^*_{s_i}}\left(B_{r_0/2}(y_0), \rho_\Gamma^{-s_i} dx\right)$ convergence of v_n. Thus $Q_{n,i}(y_n, r) \to 0$, contradicts (4.8). Hence $y_0 \in \Gamma$.

In the sequel, we will assume that concentration occurs. Without loss of generality, we may assume that

$$\liminf_{n\to\infty} Q_{n,1}(y_n, r) \ge \varepsilon_0.$$

The argument is the same if we assume that $\liminf_{n\to\infty} Q_{n,2}(y_n, r) \geq \varepsilon_0$. Let $R \geq 1$ and let $\eta \in (0, \varepsilon_0/2)$ be fixed. Since the map $r \longmapsto Q_{n,1}(y_n, r)$ is continuous, there exists $r_n > 0$ such that

$$\int_{F_{y_n}(Q_{Rr_n})} \rho_\Gamma^{-s_1} |v_n|^{2^*_{s_1}}\, dx = \eta. \tag{4.9}$$

Define the rescaled functions

$$w_n(\xi) = r_n^{\frac{N-2}{2}}\, v_n\left(F_{y_n}(r_n\xi)\right) \chi\left(\tfrac{\xi}{R}\right),$$

where $\chi \in C_c^\infty(Q_2)$ is a cut-off function with $\chi \equiv 1$ on Q_1. In Fermi coordinates along Γ (near y_n), we have

$$\rho_\Gamma\left(F_{y_n}(r_n\xi)\right) = r_n|z| \quad \text{and} \quad dx = r_n^N(1 + o_n(1))\, d\xi \tag{4.10}$$

uniformly for $|\xi| \leq 2R$ with $\xi = (y, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$, see Section 3. Combining (4.9) and (4.10), we obtain

$$\int_{Q_R} |z|^{-s_1} |w_n|^{2^*_{s_1}}\, dx = \int_{F_{y_n}(Q_{Rr_n})} \rho_\Gamma^{-s_1} |v_n|^{2^*_{s_1}}\, dx + o(1) = \eta + o_n(1).$$

In particular, the sequence $(w_n)_n$ is bounded in $\mathcal{D}^{1,2}(Q_R)$. Consequently,

$$w_n \rightharpoonup w \text{ in } \mathcal{D}^{1,2}(Q_R), \quad w_n \to w \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \leq q < 2^*_0 = \frac{2N}{N-2}$$

and

$$w_n \to w \quad \text{a.e. in } Q_R.$$

Moreover, the sequence $\left(|z|^{-s_1}|w_n|^{2^*_{s_1}}\right)_n$ is uniformly integrable on Q_R. We use the Vitali theorem to obtain

$$0 < \eta = \lim_{n\to\infty} \int_{Q_R} |z|^{-s_1} |w_n|^{2^*_{s_1}}\, dx = \int_{Q_R} |z|^{-s_1} |w|^{2^*_{s_1}}\, dx.$$

Therefore $w \not\equiv 0$. Taking the limit as $R \to \infty$, we obtain $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Next, we let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and define the rescaled test functions

$$\varphi_n(x) = r_n^{\frac{2-N}{2}}\, \varphi\left(\frac{F_{y_n}^{-1}(x)}{r_n}\right) \chi\left(\frac{F_{y_n}^{-1}(x)}{R}\right).$$

By the change of variables formula $x = F_{y_n}(r_n\xi)$, (4.10), and letting $n \to \infty$ then $R \to \infty$, the metric errors vanish by the Dominated Convergence Theorem and the smallness of the cut-off gradient region, we get

$$\int_\Omega \nabla v_n \cdot \nabla \varphi_n\, dx = \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi\, dx + o_n(1)$$

and

$$\int_\Omega \rho_\Gamma^{-s_i}|v_n|^{2^*_{s_i}-2}v_n\varphi_n\,dx = \int_{\mathbb{R}^N}|z|^{-s_i}|w|^{2^*_{s_i}-2}w\varphi\,dx + o_n(1), \quad i=1,2.$$

Combining these two identities and taking φ_n as a test function in the second identity of (4.7), we obtain

$$\int_{\mathbb{R}^N}\nabla w\cdot\nabla\varphi\,dx = \lambda\int_{\mathbb{R}^N}|z|^{-s_1}|w|^{2^*_{s_1}-2}w\varphi\,dx + \int_{\mathbb{R}^N}|z|^{-s_2}|w|^{2^*_{s_2}-2}w\varphi\,dx. \quad (4.11)$$

We recall that the functional $\Pi : \mathcal{D}^{1,2}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ is given by

$$w \mapsto \Pi(w) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N}|z|^{-s_1}|w|^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N}|z|^{-s_2}|w|^{2^*_{s_2}}\,dx,$$

whose mountain-pass level is β^*.

Hence, thanks to (4.11), w is a nontrivial weak solution of the Euler–Lagrange equation for the functional Π.

Next, we claim that

$$\liminf_{n\to\infty} J_{mod}(v_n) \geq \beta^*$$

with β^* as in (4.1). To prove the claim, we fix $\varepsilon > 0$. For $R > 1$, we set $\chi_R(x) = \chi\left(\frac{x}{R}\right)$ and define a localized bubble as

$$\tilde{v}_{n,R}(x) = r_n^{\frac{2-N}{2}}\chi_R\left(\frac{F_{y_n}^{-1}(x)}{r_n}\right)w\left(\frac{F_{y_n}^{-1}(x)}{r_n}\right),$$

where the cut-off function χ is defined as above. By the change of variables formula, we have

$$\int_\Omega|\nabla\tilde{v}_{n,R}|^2\,dx = \int_{Q_{2R}}g^{ij}(\xi)\partial_i\left(\tilde{v}_{n,R}\circ F_{y_n}\right)(\xi)\partial_j\left(\tilde{v}_{n,R}\circ F_{y_n}\right)(\xi)\sqrt{|g|}(\xi)\,d\xi.$$

Using

$$(\Phi_R w)(\xi) = r_n^{\frac{N-2}{2}}\tilde{v}_{n,R}\left(F_{y_n}(\xi)\right),$$

we have

$$\nabla_x(\Phi_R w)(x) = r_n^{\frac{N-2}{2}}DF_{y_n}(x)^{\intercal}\nabla_x\tilde{v}_{n,R}(F_{y_n}(\xi)), DF_{y_n}(\xi) = r_n\left(I_n + O\left(r_n|\xi|\right)\right).$$

Therefore,

$$\int_\Omega|\nabla\tilde{v}_{n,R}|^2\,dx = \int_{Q_{2R}}|\nabla\Phi_R w|^2\,d\xi + o_n(1)\int_{Q_{2R}}\left(|\nabla(\Phi_R w)|^2 + R^{-2}|\Phi_R w|^2\right)\,d\xi.$$

Now, $\Phi_R \equiv 1$ on Q_R, $\nabla\Phi_R$ is supported in $A_R := Q_{2R}\setminus Q_R$ and $|\nabla\Phi_R| \leq \frac{C}{R}$ for some positive constant C. By the Hardy inequality on A_R, we have

$$\int_{A_R}w^2\,d\xi \leq R^2\int_{A_R}|\nabla w|^2\,d\xi.$$

Thus

$$\left| \int_\Omega |\nabla \tilde{v}_{n,R}|^2 \, dx - \int_{\mathbb{R}^N} |\nabla \left(\Phi_R w\right)|^2 \, d\xi \right| \le o_n(1) + C \int_{\mathbb{R}^N \setminus Q_R} |\nabla w|^2 \, d\xi. \tag{4.12}$$

By the change of variables formula, (4.10), we have

$$\int_\Omega \rho^{-s_1} |\tilde{v}_{n,R}|^{2^*_{s_1}} \, dx = \int_{Q_{2R}} |z|^{-s_1} |\Phi_R w|^{2^*_{s_1}} \, d\xi + o_n(1) \int_{Q_{2R}} |z|^{-s_1} |\Phi_R w|^{2^*_{s_1}} \, d\xi.$$

Therefore,

$$\left| \int_\Omega \rho^{-s_1} |\tilde{v}_{n,R}|^{2^*_{s_1}} \, dx - \int_{Q_{2R}} |z|^{-s_1} |\Phi_R w|^{2^*_{s_1}} \, d\xi \right| \le o_n(1) + \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_1} |w|^{2^*_{s_1}} \, d\xi. \tag{4.13}$$

The same computation gives

$$\left| \int_\Omega \rho^{-s_2} |\tilde{v}_{n,R}|^{2^*_{s_2}} \, dx - \int_{Q_{2R}} |z|^{-s_2} |\Phi_R w|^{2^*_{s_2}} \, d\xi \right| \le o_n(1) + \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_2} |w|^{2^*_{s_2}} \, d\xi. \tag{4.14}$$

From (4.12)–(4.14) and the definition of J_{mod} and Π, we obtain

$$\begin{aligned} |J_{mod}(\tilde{v}_{n,R}) - \Pi(\Phi_R w)| \le o_n(1) &+ C \int_{\mathbb{R}^N \setminus Q_R} |\nabla w|^2 \, d\xi \\ &- \frac{\lambda}{2^*_{s_1}} \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_1} |w|^{2^*_{s_1}} \, d\xi - \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_2} |w|^{2^*_{s_2}} \, d\xi. \end{aligned}$$

Since $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $|z|^{-s_i}|w|^{2^*_{s_i}} \in L^1(\mathbb{R}^N)$ thanks to the limiting equation and Hardy–Sobolev, the three tail integrals go to zero as R tends to ∞. Hence

$$J_{mod}(\tilde{v}_{n,R}) - \Pi(\Phi_R w) = o_R(1) + o_n(1).$$

Now, for each R, define the Nehari scaling $t_{n,R} > 0$ by

$$E_{n,R}(t) := \langle J'_{\text{mod}}(t\tilde{v}_{n,R}), t\tilde{v}_{n,R} \rangle = 0 \quad \text{at } t = t_{n,R}.$$

Since the map $t \longmapsto E_{n,R}(t)$ is continuous, $E_{n,R}(0) = 0$, and $E'_{n,R}(0) > 0$, while $E_{n,R}(t) \to -\infty$ as $t \to \infty$, such a $t_{n,R}$ exists and it is unique. By the convergence of coefficients under blow-up, we have

$$t_{n,R} \longrightarrow t_R > 0,$$

where t_R is the Nehari multiplier for Π and the truncated profile $\Psi_R w$. Furthermore, the maximal value satisfies

$$\max_{t>0} J_{\text{mod}}(t\tilde{v}_{n,R}) = J_{\text{mod}}(t_{n,R}\tilde{v}_{n,R}) \ge \Pi(t_R \Psi_R w) - \varepsilon \quad \text{for all large } n.$$

After choosing R so large and then n large, we get $o_R(1) < \varepsilon/2$ and $o_n(1) < \varepsilon/2$. We obtain the desired estimate.

Since $\Psi_R w \longrightarrow w$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and in the weighted $L^{2^*_{s_i}}\left(\mathbb{R}^N, |z|^{-s_i}\,dx\right)$ spaces as $R \to \infty$, the Nehari multiplier $t_R \to t_*$, where t_* maximizes $\Pi(tw)$, and

$$J_{\mathbb{R}^N}(t_R\Psi_R w) \longrightarrow \max_{t>0} J_{\mathbb{R}^N}(tw) \geq \beta^*.$$

Combining the previous displays and letting first $n \to \infty$, then $R \to \infty$, and finally $\varepsilon \to 0$, yield the localized lower bound

$$\liminf_{n\to\infty} \max_{t>0} J_{\mathrm{mod}}(t\tilde{v}_{n,R}) \geq \beta^*.$$

It remains to relate $J_{\mathrm{mod}}(v_n)$ to $\max_t J_{\mathrm{mod}}(t\tilde{v}_{n,R})$. For R fixed, choose n-dependent cut-off $\chi_{n,R} \in C_c^\infty(\Omega)$ such that

$$\chi_{n,R} \equiv 1 \quad \text{in } F_{y_n}\left(Q_{Rr_n}\right) \quad \text{and} \quad \chi_{n,R} \equiv 0 \quad \text{outside } F_{y_n}\left(Q_{2Rr_n}\right).$$

Consider the split

$$v_n = \chi_{n,R} v_n + (1-\chi_{n,R})v_n = Z_{n,R} + W_{n,R},$$

where we have set

$$Z_{n,R} := \chi_{n,R} v_n \quad \text{and} \quad W_{n,R} := (1-\chi_{n,R})v_n.$$

By construction, for any $\delta > 0$, there exist $n_0 \in \mathbb{N}$ and R_0 such that for all $R \geq R_0$,

$$\left|J_{\mathrm{mod}}(Z_{n,R}) - J_{\mathrm{mod}}(v_n)\right| \leq \delta \quad \text{for all } n \geq n_0,$$

and the interaction terms between $Z_{n,R}$ and $W_{n,R}$ are $o_n(1)$ because the supports are separated and the weights are locally integrable.

Comparing Z_{nR} with $\tilde{v}_{n,R}$, we get

$$\liminf_{n\to\infty} J_{\mathrm{mod}}(v_n) \geq \liminf_{n\to+\infty} J_{\mathrm{mod}}(Z_{n,R}) - \delta \geq \liminf_{n\to+\infty} \max_{t>0} J_{\mathrm{mod}}(t\tilde{v}_{n,R}) - \delta \geq \beta^* - \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\liminf_{n\to+\infty} J_{\mathrm{mod}}(v_n) \geq \beta^*,$$

which proves the claim. Recall that

$$\alpha = \lim_n J(u_n) = J(u) + \lim_n J_{\mathrm{mod}}(v_n).$$

If a nontrivial bubble occurs, then $\lim_n J_{\mathrm{mod}}(v_n) \geq \beta^* + o(1)$, so that $\alpha \geq J(u) + \beta^* \geq \beta^*$. This contradicts the hypothesis $\alpha < \beta^*$. Thus, no nontrivial bubble can appear, and necessarily $v_n \to 0$ (so $u_n \to u$) in $H_0^1(\Omega)$. □

Next, we will need the Mountain Pass Lemma of Ambrosetti–Robinowitz, see [1].

Lemma 4.3 (Mountain Pass Lemma)**.** *Assume that* $(X, \|\cdot\|_X)$ *is a Banach space and* $\Psi : X \to \mathbb{R}$ *is a functional of class* $\mathcal{C}^1$. *Assume also that*

1) $\Psi(0) = 0$;
2) *there exist* $\lambda_0, r > 0$ *such that* $\Psi(u) \geq \lambda_0$ *for all* $u \in X$ *satisfying* $\|u\|_X = r$;
3) *there exists* $u_0 \in X$ *such that*

$$\limsup_{t\to+\infty} \Psi(tu_0) < 0.$$

Consider $t_0 > 0$ *sufficiently large such that* $\|t_0 u_0\|_X > r$ *and* $\Psi(t_0 u_0) < 0$. *Define* $\beta \leq \lambda_0$ *as*

$$\beta := \inf_{\gamma\in\mathcal{P}} \sup_{t\in[0,1]} \Psi(\gamma(t)),$$

where

$$\mathcal{P} = \left\{\gamma \in \mathcal{C}^0([0,1]; X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = t_0 u_0\right\}.$$

Then there exists a sequence $(u_n)_n \subset X$ *such that* $\Psi(u_n) \to \beta$ *and* $\Psi'(u_n) \to 0$ *strongly in* X'. *Moreover, we have*

$$\beta \leq \sup_{t\geq 0} \Psi(tu_0).$$

Lemma 4.4. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$, Γ *be a closed curve included in* Ω, *and let* h *be a continuous function such that the linear operator* $-\Delta + h$ *is coercive. Let* $u_0 \in H^1_0(\Omega)\backslash\{0\}$. *Then there exists a positive constant* c_0 *depending on* u_0 *and* $(u_n)_n \subset H^1_0(\Omega)$, *a Palais–Smale sequence for* J *at level* c_0. *Moreover,*

$$c_0 \leq \sup_{t\geq 0} J(tu_0).$$

Proof. We let $t \in \mathbb{R}$. Recall that for all $u \in H^1_0(\Omega)$, we have

$$J(tu) := \frac{t^2}{2}\int_\Omega (|\nabla u|^2 + hu^2)\,dx - \lambda\frac{|t|^{2^*_{s_1}}}{2^*_{s_1}}\int_\Omega \rho_\Gamma^{-s_1}|u|^{2^*_{s_1}}\,dx - \frac{|t|^{2^*_{s_2}}}{2^*_{s_2}}\int_\Omega \rho_\Gamma^{-s_2}|u|^{2^*_{s_2}}\,dx.$$

Then $J \in \mathcal{C}^1(H^1_0(\Omega), \mathbb{R})$. Since $0 \leq s_2 < s_1 < 2$ and taking into consideration the fact that the function

$$s \longmapsto 2^*_s := \frac{2(N-s)}{N-2}$$

is decreasing, we have

$$\lim_{t\to\infty} J(tu) = -\infty.$$

Moreover, using the fact that $2^*_{s_1}, 2^*_{s_2} > 2$, then there exist positive numbers λ_0, r such that

$$\inf_{\|u\|_{H^1_0(\Omega)}=r} J(u) \geq \lambda_0.$$

Therefore, by the Mountain Pass Lemma 4.3, we get the desired result. □

Proof of Proposition 4.1. Let $u_0 \in H^1_0(\Omega)$ be a nonnegative, nonvanishing function such that

$$\sup_{t\geq 0} J(tu_0) < \beta^*.$$

Then, by Lemma 4.4, there exists $c_0 > 0$ depending on u_0 and a Palais-Smale sequence $(u_n)_n \subset H^1_0(\Omega)$ for J at level c_0 such that

$$c_0 \leq \sup_{t\geq 0} J(tu_0) < \beta^*.$$

By Lemma 4.2, there exists $u \in H^1_0(\Omega) \setminus \{0\}$ such that, up to a subsequence,

$$u_n \longmapsto u \quad \text{strongly in } H^1_0(\Omega) \text{ as } n \to \infty \text{ and } J'(u) = 0.$$

The last equality corresponds exactly to the Euler–Lagrange equation (4.2). This then ends the proof. □

5. Proof of Theorem 1.3

Let $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfy

$$-\Delta w = \lambda |z|^{-s_1} w^{2^*_{s_1}-1} + |z|^{-s_2} w^{2^*_{s_2}-1} \quad \text{in } \mathbb{R}^N. \tag{5.1}$$

We recall that

$$\beta^* = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2\, dx - \frac{\lambda}{2^*_{s_1}} \int_{\mathbb{R}^N} |z|^{-s_1} |w|^{2^*_{s_1}}\, dx - \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^N} |z|^{-s_2} |w|^{2^*_{s_2}}\, dx.$$

In what follows, $A^N_{s_1,s_2}$ is defined such that for all $N \geq 5$, we have

$$\begin{aligned} 2(N-1)A^N_{s_1,s_2} \int_{\mathbb{R}^N} w^2\, dx &:= \int_{\mathbb{R}^N} |z|^2 |\partial_y w|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |z|^2 |\nabla w|^2\, dx \\ &\quad - \frac{\lambda}{2^*_{s_1}} \int_{\mathbb{R}^N} |z|^{2-s_1} |w|^{2^*_{s_1}}\, dx - \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^N} |z|^{2-s_2} |w|^{2^*_{s_2}}\, dx \end{aligned} \tag{5.2}$$

and

$$A^4_{s_1,s_2} := 3/2. \tag{5.3}$$

Lemma 5.1. *Let $N \geq 4$. Then we have*

$$A^N_{s_1,s_2} > 0.$$

Proof. The function w satisfies

$$-\Delta w = \lambda \frac{w^{2^*_{s_1}}}{|z|^{s_1}} + \frac{w^{2^*_{s_2}-1}}{|z|^{s_2}} \quad \text{in } \mathbb{R}^N.$$

We multiply this equation by $|z|^2 w$ and integrate by parts to get

$$0 = -\int_{\mathbb{R}^N} \left(|z|^2 w\right) \Delta w\, dx - \lambda \int_{\mathbb{R}^N} |z|^{2-s_1} w^{2^*_{s_1}}\, dx - \int_{\mathbb{R}^N} |z|^{2-s_2} w^{2^*_{s_2}}\, dx$$

$$= \int_{\mathbb{R}^N} |z|^2|\nabla w|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} \Delta(|z|^2)w^2\,dx - \lambda\int_{\mathbb{R}^N} |z|^{2-s_1}w^{2^*_{s_1}}\,dx - \int_{\mathbb{R}^N} |z|^{2-s_2}w^{2^*_{s_2}}\,dx$$

$$= \int_{\mathbb{R}^N} |z|^2|\nabla w|^2\,dx - (N-1)\int_{\mathbb{R}^N} w^2\,dx - \lambda\int_{\mathbb{R}^N} |z|^{2-s_1}w^{2^*_{s_1}}\,dx - \int_{\mathbb{R}^N} |z|^{2-s_2}w^{2^*_{s_2}}\,dx.$$

Therefore,

$$\int_{\mathbb{R}^N} |z|^2|\nabla w|^2\,dx - \lambda\int_{\mathbb{R}^N} |z|^{2-s_1}w^{2^*_{s_1}}\,dx - \int_{\mathbb{R}^N} |z|^{2-s_2}w^{2^*_{s_2}}\,dx = (N-1)\int_{\mathbb{R}^N} w^2\,dx > 0.$$

It immediately follows that

$$\frac{1}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^2|\nabla w|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^{2-s_1}w^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^{2-s_2}w^{2^*_{s_2}}\,dx > 0.$$

The function $s \mapsto 2^*_s = \frac{2(N-s)}{N-2}$ is decreasing on $[0,2]$, and $0 < s_2 < s_1 < 2$. Consequently, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N} |z|^2|\nabla w|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^{2-s_1}w^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N} |z|^{2-s_2}w^{2^*_{s_2}}\,dx > 0.$$

We finish by adding a positive term which yields

$$\int_{\mathbb{R}^N} |z|^2|\partial_y w|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |z|^2|\nabla w|^2\,dx - \frac{\lambda}{2^*_{s_1}}\int_{\mathbb{R}^N} |z|^{2-s_1}w^{2^*_{s_1}}\,dx - \frac{1}{2^*_{s_2}}\int_{\mathbb{R}^N} |z|^{2-s_2}w^{2^*_{s_2}}\,dx > 0.$$

The result follows directly from this, (5.2) and (5.3). □

Then we have the following result.

Proposition 5.2. *For $N \ge 4$, let Ω be a bounded domain of $\mathbb{R}^N$. Let also*

$$A^N_{s_1,s_2}|\kappa(y_0)|^2 + h(y_0) < 0 \tag{5.4}$$

at some point $y_0 \in \Gamma$. Then there exists $u \in H^1_0(\Omega)\setminus\{0\}$ such that

$$c^* := \max_{t\ge 0} J(tu) < \beta^*.$$

Let Ω be a bounded domain of $\mathbb{R}^N$ and $\Gamma \subset \Omega$ be a smooth closed curve. We let $\eta \in \mathcal{C}^\infty_c(F_{y_0}(Q_{2r}))$ be such that

$$0 \le \eta \le 1 \quad \text{and} \quad \eta \equiv 1 \quad \text{in } Q_r.$$

For $\epsilon > 0$, we consider the test function $u_\epsilon : \Omega \to \mathbb{R}$ given by

$$u_\epsilon(y) := \epsilon^{\frac{2-N}{2}} \eta(F_{y_0}^{-1}(y)) w\left(\epsilon^{-1} F_{y_0}^{-1}(y)\right). \tag{5.5}$$

In particular, for every $x = (y, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$, we have

$$u_\epsilon\left(F_{y_0}(x)\right) := \epsilon^{\frac{2-N}{2}} \eta\left(x\right) \theta\left(\frac{|y|}{\epsilon}, \frac{|z|}{\epsilon}\right). \tag{5.6}$$

It is clear that $u_\epsilon \in H_0^1(\Omega)$. Moreover, for $t \geq 0$, we have

$$J(tu_\epsilon) = \frac{t^2}{2} \int_\Omega \left(|\nabla u_\epsilon|^2 + h(x) u_\epsilon^2\right) dx - \lambda \frac{t^{2^*_{s_1}}}{2^*_{s_1}} \int_\Omega \rho_\Gamma^{-s_1} |u_\epsilon|^{2^*_{s_1}} \, dx \\ - \frac{t^{2^*_{s_2}}}{2^*_{s_2}} \int_\Omega \rho_\Gamma^{-s_2} |u_\epsilon|^{2^*_{s_2}} \, dx.$$

To simplify the notations, further we will simply write F instead of F_{y_0}. Recalling (5.5), we write

$$u_\epsilon(y) = \epsilon^{\frac{2-N}{2}} \eta(F^{-1}(y)) W_\epsilon(y), \tag{5.7}$$

where $W_\epsilon(y) = w\left(\frac{F^{-1}(y)}{\epsilon}\right)$.

Lemma 5.3. *As $\epsilon \to 0$, for $N \geq 5$, we have*

$$\int_\Omega |\nabla u_\epsilon|^2 \, dx + \int_\Omega h(x) u_\epsilon^2(x) \, dx = \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \epsilon^2 \frac{|\kappa(y_0)|^2}{N-1} \int_{\mathbb{R}^N} |z|^2 \left|\partial_y w\right|^2 \, dx \\ + \epsilon^2 \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 \, dx + \epsilon^2 h(y_0) \int_{\mathbb{R}^N} w^2(x) \, dx + O\left(\epsilon^{N-2}\right).$$

If $N = 4$, then there exists $C > 0$ such that

$$\int_\Omega |\nabla u_\epsilon|^2 \, dx + \int_\Omega h(x) u_\epsilon^2(x) \, dx \\ \leq \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + C\epsilon^2 \left(\frac{3}{2}|\kappa(y_0)|^2 + h(y_0)\right) |\ln(\epsilon)| + O(\epsilon^2).$$

Proof. By (5.7), we have

$$|\nabla u_\epsilon|^2 = \epsilon^{2-N} \left(\eta^2 |\nabla W_\epsilon|^2 + \eta^2 |\nabla W_\epsilon|^2 + \frac{1}{2} \nabla W_\epsilon^2 \cdot \nabla \eta^2\right).$$

Integrating by parts, we get

$$\int_\Omega |\nabla u_\epsilon|^2 \, dx = \epsilon^{2-N} \int_{F(Q_{2r})} \eta^2 |\nabla W_\epsilon|^2 \, dx \\ + \epsilon^{2-N} \int_{F(Q_{2r}) \setminus F(Q_r)} W_\epsilon^2 \left(|\nabla \eta|^2 - \frac{1}{2} \Delta \eta^2\right) dx$$

$$= \epsilon^{2-N} \int_{F(Q_{2r})} \eta^2 |\nabla W_\epsilon|^2 \, dx - \epsilon^{2-N} \int_{F(Q_{2r}) \setminus F(Q_r)} W_\epsilon^2 \eta \Delta \eta \, dx$$
$$= \epsilon^{2-N} \int_{F(Q_{2r})} \eta^2 |\nabla W_\epsilon|^2 \, dx + O\left(\epsilon^{2-N} \int_{F(Q_{2r}) \setminus F(Q_r)} W_\epsilon^2 \, dx \right).$$

By the change of variables formula $\epsilon y = F(x)$, Lemma 3.3 and (5.6), we get

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon|^2 \, dx &= \int_{Q_{r/\epsilon}} |\nabla w|^2_{g_\epsilon} \sqrt{|g_\epsilon|} \, dx \\ &\quad + O\left(\epsilon^2 \int_{Q_{2r/\epsilon} \setminus Q_{r/\epsilon}} w^2 \, dx \int_{Q_{2r/\epsilon} \setminus Q_{r/\epsilon}} |\nabla w|^2 \, dx \right) \\ &= \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \epsilon^2 \frac{|\kappa(y_0)|^2}{N-1} \int_{Q_{r/\epsilon}} |z|^2 \left| \partial_y w \right|^2 \, dx \\ &\quad + \epsilon^2 \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{Q_{r/\epsilon}} |z|^2 |\nabla w|^2 \, dx + O\left(\rho(\epsilon)\right), \end{aligned}$$

where

$$\begin{aligned} \rho(\epsilon) = \epsilon^3 & \int_{Q_{r/\epsilon}} |x|^3 |\nabla w|^2 \, dx + \epsilon^2 \int_{Q_{2r/\epsilon} \setminus Q_{r/\epsilon}} |w|^2 \, dx \\ & + \int_{\mathbb{R}^N \setminus Q_{r/\epsilon}} |\nabla w|^2 \, dx + \epsilon^2 \int_{Q_{2r/\epsilon} \setminus Q_{r/\epsilon}} |z|^2 |\nabla w|^2 \, dx. \end{aligned}$$

By Proposition 2.14, for $N \geq 4$, we have

$$\rho(\epsilon) = O(\epsilon^{N-2}).$$

For $N \geq 5$, we use polar coordinates to obtain

$$\int_{\mathbb{R}^N \setminus Q_{r/\epsilon}} w^2 \, dx + \int_{\mathbb{R}^N \setminus Q_{r/\epsilon}} |z|^2 \left| \partial_y w \right|^2 \, dx + \int_{\mathbb{R}^N \setminus Q_{r/\epsilon}} |z|^2 |\nabla w|^2 \, dx = O(\epsilon^{N-4}).$$

Therefore,

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon|^2 \, dx = & \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \epsilon^2 \frac{|\kappa(y_0)|^2}{N-1} \int_{\mathbb{R}^N} |z|^2 \left| \partial_y w \right|^2 dx \\ & + \epsilon^2 \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 \, dx + O\left(\epsilon^{N-2}\right), \quad N \geq 5. \end{aligned} \tag{5.8}$$

If $N = 4$, then we have

$$\int_\Omega |\nabla u_\epsilon|^2 \, dx \leq \int_{\mathbb{R}^4} |\nabla w|^2 \, dx + \epsilon^2 \frac{|\kappa(y_0)|^2}{2} \int_{Q_{r/\epsilon}} |z|^2 \left| \nabla w \right|^2 \, dx + O\left(\epsilon^2\right). \tag{5.9}$$

Next, by the change of variables formula $\epsilon y = F(x)$, (5.6) and the continuity of the function h, we have

$$\int_\Omega h(x) u_\epsilon^2(x) \, dx = \epsilon^2 h(y_0) \int_{Q_{r/\epsilon}} w^2(x) \, dx + \epsilon^2 \int_{Q_{2r/\epsilon} \setminus Q_{r/\epsilon}} w^2(x) \, dx.$$

By the estimate in Proposition 2.13, we can easily prove that

$$\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} w^2(x)\,dx = O\left(\epsilon^{N-2}\right).$$

Moreover, for $N \geq 5$, we have

$$\int_{\mathbb{R}^N\setminus Q_{r/\epsilon}} w^2(x)\,dx = O\left(\epsilon^{N-2}\right).$$

Therefore,

$$\int_\Omega u_\epsilon^2(x)\,dx = \epsilon^2 h(y_0)\int_{\mathbb{R}^N} w^2(x)\,dx + o\left(\epsilon^2\right). \tag{5.10}$$

If $N = 4$, we have

$$\int_\Omega u_\epsilon^2(x)\,dx = \epsilon^2 h(y_0)\int_{Q_{r/\epsilon}} w^2(x)\,dx + O\left(\epsilon^2\right). \tag{5.11}$$

Next, we assume that $N = 4$ and we let $\eta_\epsilon(x) = \eta(\epsilon x)$. We multiply (5.1) by $|z|^2\eta_\epsilon w$ and integrate by parts to get

$$\begin{aligned}
\lambda\int_{Q_{2r/\epsilon}} \eta_\epsilon|z|^{2-s_1}w^{2^*_{s_1}}\,dx + \int_{Q_{2r/\epsilon}} \eta_\epsilon|z|^{2-s_2}w^{2^*_{s_2}}\,dx &= \int_{Q_{2r/\epsilon}} \nabla w\cdot\nabla\left(\eta_\epsilon|z|^2 w\right)dx\\
&= \int_{Q_{2r/\epsilon}} \eta_\epsilon|z|^2|\nabla w|^2\,dx + \frac{1}{2}\int_{Q_{2r/\epsilon}} \nabla w^2\cdot\nabla\left(|z|^2\eta_\epsilon\right)dx\\
&= \int_{Q_{2r/\epsilon}} \eta_\epsilon|z|^2|\nabla w|^2\,dx - \frac{1}{2}\int_{Q_{2r/\epsilon}} w^2\Delta\left(|z|^2\eta_\epsilon\right)dx\\
&= \int_{Q_{2r/\epsilon}} \eta_\epsilon|z|^2|\nabla w|^2\,dx - 3\int_{Q_{2r/\epsilon}} w^2\eta_\epsilon\,dx\\
&= -\frac{1}{2}\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} w^2(|z|^2\Delta\eta_\epsilon + 4\nabla\eta_\epsilon\cdot z)\,dx.
\end{aligned}$$

We then deduce that

$$\begin{aligned}
&\lambda\int_{Q_{2r/\epsilon}} |z|^{2-s_1}w^{2^*_{s_1}}\,dx + \int_{Q_{2r/\epsilon}} |z|^{2-s_2}w^{2^*_{s_2}}\,dx\\
&= \int_{Q_{r/\epsilon}} |z|^2|\nabla w|^2 dx - (N-1)\int_{Q_{r/\epsilon}} w^2\,dx\\
&+ O\left(\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^{2-s}w^{2^*_s}\,dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^2|\nabla w|^2\,dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} w^2\,dx\right)\\
&+ O\left(\epsilon\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z||\nabla w|\,dx + \epsilon^2\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^2 w^2\,dx\right).
\end{aligned}$$

By Proposition 2.13, we have

$$\lambda\int_{Q_{2r/\epsilon}} |z|^{2-s_1}w^{2^*_{s_1}}\,dx + \int_{Q_{2r/\epsilon}} |z|^{2-s_2}w^{2^*_{s_2}}\,dx = O(1)$$

and

$$\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^{2-s} w^{2_s^*}\,dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^2|\nabla w|^2\,dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} w^2\,dx + \epsilon \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z||\nabla w|\,dx + \epsilon^2 \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^2 w^2\,dx = O(\epsilon^2).$$

Therefore,

$$\int_{Q_{r/\epsilon}} |z|^2|\nabla w|^2\,dx = 3\int_{Q_{r/\epsilon}} w^2\,dx + O(1). \tag{5.12}$$

To end the proof, we use Proposition 2.13 and polar coordinates to get

$$\int_{Q_{r/\epsilon}} w^2\,dx \le C\int_{Q_{r/\epsilon}} \frac{dx}{1+|x|^4} = C|S^3|\int_0^{r\epsilon} \frac{t^3 dt}{1+t^4} \le C(1+|\ln(\epsilon)|) \tag{5.13}$$

for some positive constant C. Thus the result follows immediately from (5.8)–(5.13). □

Lemma 5.4. *Let $N \ge 3$ and $s \in (0,2)$. Then we have*

$$\int_\Omega \rho_\Gamma^{-s}|u_\epsilon|^{2_s^*}\,dx = \int_{\mathbb{R}^N} |z|^{-s} w^{2_s^*}\,dx + \epsilon^2 \frac{|\kappa(y_0)|^2}{2(N-1)}\int_{\mathbb{R}^N} |z|^{2-s} w^{2_s^*}\,dx + O\left(\epsilon^{N-s}\right).$$

Proof. By the change of variable $y = \frac{F(x)}{\epsilon}$, (3.1) and (3.2), we have

$$\begin{aligned}
\int_\Omega \rho_\Gamma^{-s}|u_\epsilon|^{2_s^*}\,dx &= \int_{Q_{r/\epsilon}} |z|^{-s} w^{2_s^*}\sqrt{|g_\epsilon|}\,dx + O\left(\int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^{-s}(\eta(\epsilon x)w)^{2_s^*}\,dx\right)\\
&= \int_{Q_{r/\epsilon}} |z|^{-s} w^{2_s^*}\,dx + \epsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\epsilon}} |z|^{2-s} w^{2_s^*}\,dx\\
&\quad + O\left(\epsilon^3\int_{Q_{r/\epsilon}} |x|^3|z|^{-s} w^{2_s^*}\,dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^{-s} w^{2_s^*}\,dx\right)\\
&= \int_{\mathbb{R}^N} |z|^{-s} w^{2_s^*}\,dx + \epsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\epsilon}} |z|^{2-s} w^{2_s^*}\,dx\\
&\quad + O\left(\epsilon^3\int_{Q_{r/\epsilon}} |x|^3|z|^{-s} w^{2_s^*}\,dx\right.\\
&\quad \left. + \int_{\mathbb{R}^N\setminus Q_{r/\epsilon}} |z|^{-s} w^{2_s^*} dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^{-s} w^{2_s^*}\,dx\right).
\end{aligned}$$

By Proposition 2.13, we have

$$\epsilon^3\int_{Q_{r/\epsilon}} |x|^3|z|^{-s} w^{2_s^*}\,dx + \int_{\mathbb{R}^N\setminus Q_{r/\epsilon}} |z|^{-s} w^{2_s^*}\,dx + \int_{Q_{2r/\epsilon}\setminus Q_{r/\epsilon}} |z|^{-s} w^{2_s^*}\,dx = O\left(\epsilon^{N-s}\right)$$

and

$$\int_{\mathbb{R}^N \setminus Q_{r/\epsilon}} |z|^{2-s} w^{2_s^*} dx = O\left(\epsilon^{N-2-s}\right), \quad N \geq 4. \tag{5.14}$$

Thus,

$$\int_\Omega \rho_\Gamma^{-s} |u_\epsilon|^{2_s^*} \, dx = \int_{\mathbb{R}^N} |z|^{-s} w^{2_s^*} \, dx + \epsilon^2 \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{\mathbb{R}^N} |z|^{2-s} w^{2_s^*} \, dx + O\left(\epsilon^{N-s}\right) \tag{5.15}$$

as $\epsilon \to 0$, which ends the proof. □

Proof of Proposition 5.2. We let $t \geq 0$ and $u \in H_0^1(\Omega)$. We have

$$\begin{aligned} J(tu) := & \frac{t^2}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{t^2}{2} \int_\Omega h(x) u^2 \, dx \\ & - t^{2_{s_1}^*} \frac{\lambda}{2_{s_1}^*} \int_\Omega \frac{|u|^{2_{s_1}^*}}{\rho_\Gamma^{s_1}(x)} \, dx - t^{2_{s_2}^*} \frac{1}{2_{s_2}^*} \int_\Omega \frac{|u|^{2_{s_2}^*}}{\rho_\Gamma^{s_2}(x)} \, dx. \end{aligned}$$

By (2.4), Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned} J\left(tu_\epsilon\right) = \Pi(tw) &+ \epsilon^2 t^2 \frac{|\kappa(y_0)|^2}{2(N-1)} \left(\int_{Q_{r/\epsilon}} |z|^2 \left|\partial_y w\right|^2 \, dx + \frac{1}{2} \int_{Q_{r/\epsilon}} |z|^2 |\nabla w|^2 dx \right) \\ &+ \epsilon^2 t^2 h(y_0) \int_{Q_{r/\epsilon}} w^2 \, dx - \epsilon^2 \lambda \frac{t^{2_{s_1}^*}}{2_{s_1}^*} \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{Q_{r/\epsilon}} |z|^{2-s_1} w^{2_{s_1}^*} \, dx \\ &- \epsilon^2 \frac{t^{2_{s_2}^*}}{2_{s_2}^*} \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{Q_{r/\epsilon}} |z|^{2-s_2} w^{2_{s_2}^*} \, dx + O\left(\epsilon^{N-2}\right) \quad \text{for } N \geq 5. \end{aligned}$$

For $N = 4$, we have

$$J(tu_\epsilon) \leq \Pi(tw) + C\epsilon^2 t^2 \left(\frac{3}{2} |\kappa(y_0)|^2 + h(y_0) \right) |\ln(\epsilon)| + O(\epsilon^2).$$

Since $2_{s_2}^* > 2_{s_1}^*$, $J(tu_\epsilon)$ has a unique maximum, we have

$$\max_{t \geq 0} \Pi(tw) = \Pi(w) = \beta^*.$$

Therefore, the maximum of $J(tu_\epsilon)$ occurs at $t_\epsilon := 1 + o_\epsilon(1)$. Next, we set

$$\begin{aligned} \mathcal{G}_\epsilon(tw) : = & \epsilon^2 t^2 \frac{|\kappa(y_0)|^2}{2(N-1)} \left(\int_{\mathbb{R}^N} |z|^2 \left|\partial_t w\right|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 \, dx \right) \\ & + \epsilon^2 t^2 h(y_0) \int_{\mathbb{R}^N} w^2 \, dx - \epsilon^2 \lambda \frac{t^{2_{s_1}^*}}{2_{s_1}^*} \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{\mathbb{R}^N} |z|^{2-s_1} w^{2_{s_1}^*} \, dx \\ & - \epsilon^2 \frac{t^{2_{s_2}^*}}{2_{s_2}^*} \frac{|\kappa(y_0)|^2}{2(N-1)} \int_{\mathbb{R}^N} |z|^{2-s_2} w^{2_{s_2}^*} \, dx + o(\epsilon^2) \quad \text{for } N \geq 5, \end{aligned}$$

and

$$\mathcal{G}_\epsilon(tw) = C\epsilon^2 |\ln(\epsilon)| t^2 \left(\frac{3}{2} |\kappa(y_0)|^2 + h(y_0) \right) + O(\epsilon^2) \quad \text{for } N = 4.$$

Due to assumption (5.4), we have

$$\mathcal{G}_\epsilon(w) < 0.$$

Therefore,

$$\max_{t\geq 0} J(tu_\epsilon) := J(t_\epsilon u_\epsilon) \leq \Pi(t_\epsilon w) + \epsilon^2 \mathcal{G}(t_\epsilon w) < \Pi(t_\epsilon w) \leq \Pi(w) = \beta^*.$$

We thus get the desired result. □

The proof of Theorem 1.3 is a direct consequence of Proposition 4.1 and Proposition 5.2.

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References

[1] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), No. 4, 349–381.

[2] T. Bartsch, S. Peng, and Z. Zhang, *Existence and non-existence of solutions to elliptic equations related to the Caffarelli–Kohn–Nirenberg inequalities*, Calc. Var. Partial Differential Equations **30** (2007), No. 1, 113–136.

[3] D. Cao and P. Han, *Solutions to critical equation with multi-singular inverse square potentials*, J. Differential Equations **224** (2006), No. 2, 332–372.

[4] F. Catrina and Z.Q. Wang, *On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions*, Comm. Pure Appl. Math **54** (2001), No. 2, 229–258.

[5] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), No. 3, 332–372.

[6] J-L. Chern and C-S Lin, *Minimizers of Caffarelli–Kohn–Nirenberg Inequalities with the singularity on the boundary*, Arch. Rational Mech. Anal. **197** (2010), 401–432.

[7] K.S. Chou and C.W. Chu, *On the best constant for a weighted Sobolev–Hardy inequality*, J. London Math. Soc. (2) **48** (1993), No. 1, 137–151.

[8] A.V. Demyanov and A.I. Nazarov, *On solvability of Dirichlet problem to semilinear Schrodinger equation with singular potential*, Zapiski Nauchnykh Seminarov POMI **336** (2006), 25–45.

[9] J. Dolbeault, M.J. Esteban, and G. Tarantello, *The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli–Kohn–Nirenberg inequalities in two space dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), No. 2, 313–341.

[10] M.M. Fall, I.A. Minlend, and E.H.A. Thiam, *The role of the mean curvature in a Hardy–Sobolev trace inequality*, NoDEA Nonlinear Differential Equations Appl. **22** (2015), No. 5, 1047–1066.

[11] N. Ghoussoub and X. S. Kang, *Hardy–Sobolev critical elliptic equations with boundary singularities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), No. 6, 767–793.

[12] N. Ghoussoub and F. Robert, *The effect of curvature on the best constant in the Hardy–Sobolev inequalities*, Geom. Funct. Anal. **16** (2006), No. 6, 1201–1245.

[13] N. Ghoussoub and F. Robert, *On the Hardy–Schrödinger operator with a boundary singularity*, Anal. PDE **10** (2017), 1017–1079.

[14] N. Ghoussoub and F. Robert, *Concentration estimates for Emden–Fowler equations with boundary singularities and critical growth*, Int. Math. Res. Pap. IMRP **2006** (2006), Paper No. 21867, 85 pp.

[15] I. Fabbri, G. Mancini, and K. Sandeep, *Classification of solutions of a critical Hardy–Sobolev operator*, J Differential Equations **224** (2006), 258–276.

[16] M.M. Fall and E.H.A. Thiam, *Hardy–Sobolev inequality with singularity a curve*, Topol. Methods Nonlinear Anal. **51** (2018), No. 1, 151–181.

[17] Y.Y. Li, *Prescribing scalar curvature on S^n and related problems, I*, J. Differential Equations **120** (1995), No. 2, 319–410.

[18] C.H. Hsia, C.S. Lin, and H. Wadade, *Revisiting an idea of Brézis and Nirenberg*, J. Funct. Anal. **259** (2010), 1816–1849.

[19] I.E. Ijaodoro and E.H.A. Thiam, *Influence of an L^p-perturbation on Hardy–Sobolev inequality with singularity a curve*, Opuscula Math. **41** (2021), No. 2, 187–204.

[20] H. Jaber, *Hardy–Sobolev equations on compact Riemannian manifolds*, Nonlinear Anal. **103** (2014), 39–54.

[21] R. Lehrer and L.A. Maia, *Positive solutions of asymptotically linear equations via Pohozaev manifold*, J. Funct. Anal. **266** (2014), 213–246.

[22] E.H. Lieb, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. Math. **118** (1983), No. 2, 349–374.

[23] Y. Li and C Lin, *A nonlinear elliptic PDE with two Sobolev–Hardy critical exponents*, Arch. Ration. Mech. Anal. **203** (2012), 943–968.

[24] C. S. Lin, *Interpolation inequalities with weights*, Comm. Partial Differential Equations **11** (1986), No. 14, 1515–1538.

[25] C.S. Lin and Z.Q. Wang, *Symmetry of extremal functions for the Caffarelli–Kohn–Nirenberg inequalities*, Proc. Amer. Math. Soc. **132** (2004), No. 6, 1685–1691.

[26] R. Musina, *Ground state solutions of a critical problem involving cylindrical weights*, Nonlinear Anal. **68** (2008), No. 12, 3972–3986.

[27] J. Serrin, *A symmetry theorem in potential theory,* Arch. Ration. Mech. Anal. **43** (1971), 304–318.

[28] E.H.A. Thiam, *Hardy and Hardy–Sobolev Inequalities on Riemannian manifolds*, Imhotep Mathematical Journal **2** (2017), No. 1, 14–35.

[29] E.H.A. Thiam, *Hardy–Sobolev inequality with higher dimensional singularities*, Analysis (Berlin) **39** (2019), No. 3, 79–96.

[30] E.H.A. Thiam, *Mass effect on an elliptic PDE involving two Hardy–Sobolev critical exponents*, Differ. Equ. Appl. **16** (2024), No. 3, 183–198.

[31] E.H.A. Thiam, M. Ciss, and A. Diatta, *A Nonlinear elliptic PDE with curve singularity on the boundary*, Moroccan J. Pure Appl. Anal. **11** (2025), 181–202.

[32] X. Zhong and W. Zou, *A nonlinear elliptic PDE with multiple Hardy–Sobolev critical exponents in R^N*, J. Differential Equations **292** (2021), 354–387.

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Abdourahmane Diatta,
Université Assane Seck de Ziguinchor, BP 525 Ziguinchor, Senegal,
E-mail: a.diatta20160578@zig.univ.sn

El Hadji Abdoulaye Thiam,
Université Iba Der Thiam, UFR des Sciences et Technologies, Département de Mathématiques, BP A967, Thiès, Senegal,
E-mail: elhadjiabdoulaye.thiam@univ-thies.sn

Нелінійне рівняння з частинними похідними із двома критичними показниками Гарді–Соболєва та одновимірною особливістю

Abdourahmane Diatta and El Hadji Abdoulaye Thiam

Для $N \geq 4$ нехай Ω є обмеженою областю в $\mathbb{R}^N$, а Γ є замкненою кривою, що міститься в Ω. Ми досліджуємо існування додатних розв'язків $u \in H_0^1(\Omega)$ рівняння

$$-\Delta u + hu = \lambda \rho^{-s_1} \Gamma u^{2^* s_1 - 1} + \rho^{-s_2} \Gamma u^{2^* s_2 - 1} \quad \text{в } \Omega, \tag{1}$$

де $h : \Omega \longrightarrow \mathbb{R}$ є неперервною функцією, λ є додатним дійсним параметром, $0 \leq s_2 < s_1 < 2$, а ρ_Γ є функцією відстані до Γ. У цій роботі ми доводимо існування розв'язків типу гірського переходу (mountain pass solutions) для рівняння Ейлера–Лагранжа (1) залежно від локальної геометрії кривої та потенціалу h. Ми також вивчаємо існування, симетрію та оцінки спадання глобальних додатних розв'язків (1) при $\Omega = \mathbb{R}^N$, де Γ є прямою.

Ключові слова: два показники Гарді–Соболєва, кривина, розв'язок типу гірського переходу, особливість на кривій