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Controllability of Hilfer Fractional Noninstantaneous Impulsive Differential Inclusions in Banach Spaces

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In this paper, we study the existence and controllability of Hilfer fractional differential inclusions with noninstantaneous impulses in Banach spaces. The analysis is conducted using various mathematical tools and the set-valued version of Mönch's fixed point theorem, which relies on several properties of the Kuratowski measure of noncompactness. To demonstrate the applicability of our results, we conclude the study with a detailed example.

Key words: Measure of noncompactness, fixed point theory, fractional calculus

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1. Introduction

Fractional calculus is a branch of mathematical analysis that generalizes the principles of classical integer-order calculus to encompass derivatives and integrals of arbitrary real or complex orders. Its distinguishing feature lies in fractional operators, which inherently capture memory and hereditary properties. These attributes make fractional calculus an invaluable tool for modeling complex real-world phenomena with greater accuracy. This field has garnered significant attention of researchers in mathematics, physics and various applied sciences, serving as a foundation for studying diverse phenomena through advanced mathematical models [4–8, 16, 30, 38, 48]. For more details on the applications of fractional calculus, readers are referred to the works of Baleanu et al. [14], Kilbas et al. [33], Samko et al. [42], and Zhou [50]. Additionally, Abbas et al. [1, 2] studied several problems involving advanced fractional differential and integral equations presenting various applications. Benchohra et al. [17, 18] demonstrated the existence, stability and uniqueness of solutions for diverse problems using various fractional derivatives and different types of conditions.

Impulsive fractional differential equations and inclusions were studied and developed by several researchers, see Bainov and Simenov [12], Benchohra *et al.* [19], Samoilenko and Perestyuk [43] and Lakshmikantham *et al.* [34] and references therein.

Recently, a number of works were published about differential equations and inclusions with noninstantaneous impulses [10, 20, 36] as well as those that study fractional differential equations involving Hilfer derivatives [25, 27, 30, 32].

In order to propose new results on the field of controllability of fractional differential inclusions, Antonov and Debbouche [21] introduced a new concept called impulsive control inclusion condition of the following system:

$$\begin{cases} D^{\rho_1,\rho_2}_{\delta_\imath^+}\xi(\delta) \in \mathfrak{Z}\xi(\delta) + \aleph(\delta,\xi(\delta)) + \overline{\mathfrak{Z}}(\varkappa(\delta)), \quad \delta \in \Xi'' := \Xi' - \{\delta_1,\delta_2,\dots,\delta_\gamma\}, \\ I^{(1-\rho_1)(1-\rho_2)}_{0+}[\xi(\delta)]|_{\delta=0} = \xi_0, \\ I^{(1-\rho_1)(1-\rho_2)}_{\delta_\imath^+}\xi(\delta_\imath^+) \in \xi(\delta_\imath^-) + \Psi_\imath(\xi(\delta_\imath^-)) + Dv(\delta_\imath^-), \quad \imath = 1,2,\dots,\gamma, \end{cases}$$

where D^{ρ_1,ρ_2}_{0+} denotes the Hilfer fractional derivative of order ρ_2 and type ρ_1 such that $0 \le \rho_1 \le 1$, $\frac{1}{2} < \rho_2 < 1$, the state $\xi(\cdot)$ takes its values in a Banach space ∇ with the norm $\|\cdot\|$ and $\xi_0 \in \nabla$, $\mathfrak{Z}$ is the infinitesimal generator of a strongly continuous semigroup $S(\delta)$, $\delta \geqslant 0$. The control functions $\varkappa$ and v are given in $L^2(\Xi, U)$, with U as a Banach space, $\overline{\mathfrak{Z}}$ and D are bounded linear operators from U into ∇.

In [24], Du *et al.* investigated the approximate controllability of the following impulsive fractional differential inclusions involving the Hilfer fractional derivative:

$$\begin{cases} D^{\rho_1,\rho_2}_{0+}\xi(\delta) \in \mathfrak{Z}\xi(\delta) + \aleph(\delta,\xi(\delta)) + \overline{\mathfrak{Z}}(\varkappa(\delta)), \quad \delta \in (0,b], \quad \delta \ne \delta_\jmath, \\ \Delta I^{1-\rho_3}_{0+}\xi(\delta)|_{\delta=\delta_\jmath} = \Psi_\jmath(\delta,\xi(\delta_\jmath^-), \quad \jmath = 1,2,\dots,\gamma, \\ I^{1-\rho_3}_{0+}\xi(\delta)|_{\delta=0} = \xi_0 \in \nabla, \end{cases}$$

where D^{ρ_1,ρ_2}_{0+} denotes the Hilfer fractional derivative of order ρ_1 and type ρ_2, $0 \le \rho_2 \le 1$, $\frac{1}{2} < \rho_1 < 1$, and $\rho_3 = \rho_2 + \rho_1 - \rho_1\rho_2$, $\xi(\cdot)$ takes its values in a Banach space ∇ with the norm $\|\cdot\|$, $\mathfrak{Z} : D(\mathfrak{Z}) \subseteq \nabla \to \nabla$ is the infinitesimal generator of a C_0-semigroup $\{S(\delta), \delta \geqslant 0\}$ on ∇.

In addition, there is another problem studied by J. Wang *et al.* [49] about controllability of

$$\begin{cases} D^{\rho_1,\rho_2}_{\zeta_\imath^+}\xi(\delta) \in \mathfrak{Z}\xi(\delta) + \aleph(\delta,\xi(\delta)) + \overline{\mathfrak{Z}}(\varkappa(\delta)) \text{ for a.a. } \delta \in (\zeta_\imath, \delta_{\imath+1}], \imath = 0,1,\dots,\gamma, \\ \xi(\delta_\imath^+) = \wp_\imath(\delta_\imath, \xi(\delta_\imath^-), \ \xi(\delta) = \wp_\imath(\delta, \xi(\delta_\imath^-), \quad \delta \in (\delta_\imath, \zeta_\imath], \ \imath = 1,2,\dots,\gamma, \\ I^{1-\rho_3}_{0+}\xi(0) = \xi_0 + g(\xi), \ I^{1-\rho_3}_{\zeta_\imath^+}\xi(\zeta_\imath^+) = \wp_\imath(\zeta_\imath, \xi(\delta_\imath^-), \quad \imath = 1,2,\dots,\gamma, \end{cases}$$

where $0 < \rho_1 < 1$, $0 \le \rho_2 \le 1$, $\rho_3 = \rho_1 + \rho_2 - \rho_1\rho_2$. Let $\Xi = [0,b], b > 0$. In addition, $0 = \zeta_0 < \delta_1 < \zeta_1 < \delta_2 < \cdots < \delta_\gamma < \zeta_\gamma < \delta_{\gamma+1} = b$, $\xi(\delta_\imath^+)$, $\xi(\delta_\imath^-)$ are the right and left limits of ξ at the point $\delta_\imath$, respectively, $I^{1-\rho_3}_{\zeta_\imath^+}$ is the left-sided Riemann–Liouville integral of order $1-\rho_3$ with lower limit at $\zeta_\imath$, and $I^{1-\rho_3}_{\zeta_\imath^+}\xi(\zeta_\imath^+) = \lim_{\delta\to\zeta_\imath^+} I^{1-\rho_3}_{\zeta_\imath^+}(\delta)$.

In this paper, we present some new results about the controllability of Hilfer fractional differential inclusions with noninstantaneous impulses in Banach spaces:

$$\begin{cases} {}^{H}D_{0^+}^{\rho_1,\rho_2}\xi(\delta) \in \mathfrak{Z}\xi(\delta) + \aleph(\delta,\xi(\delta)) + \overline{\mathfrak{Z}}(\varkappa(\delta)), \quad \delta \in \bigcup_{\jmath=0}^{\gamma}(\zeta_\jmath,\delta_{\jmath+1}], \\ \xi(\delta) = \Psi_\jmath\left(\delta,\xi(\delta)\right), \quad \delta \in \bigcup_{\jmath=1}^{\gamma}(\delta_\jmath,\zeta_\jmath], \\ I_{0^+}^{1-\rho_3}\xi(0) = \xi_0, \end{cases} \tag{1.1}$$

where ${}^{H}D_{0^+}^{\rho_1,\rho_2}$ is the Hilfer fractional derivative of order ρ_1,ρ_2 such that $0 < \rho_1 \le 1$, $0 \le \rho_2 \le 1$, and $0 \le \rho_3 \le 1$ with $\rho_3 = \rho_1 + \rho_2(1-\rho_1)$, $\mathfrak{Z} : D(\mathfrak{Z}) \subset \nabla \to \nabla$ is a linear operator, and the infinitesimal generator of a strongly continuous semigroup $(T(\delta))_{\delta\geqslant 0}$ in ∇ (∇ is a Banach space with the norm $\|\cdot\|$), $\aleph : \Xi\times\nabla \to P(\nabla)$ is a multivalued map satisfying some assumptions that will be specified later, ($P(\nabla)$ is the family of all nonempty subsets of the separable Banach space $(\nabla, \|\cdot\|)$), $\Psi_\jmath : (\delta_\jmath,\zeta_\jmath]\times\nabla \to \nabla$, $\jmath = 1,2,\ldots,\gamma$ are given functions. The control function $\varkappa$ is given in $L^{\flat}(\Xi,E)$, $\flat > \frac{1}{\rho_1}$, a Banach space of admissible control functions, with E being a real Banach space, $\overline{\mathfrak{Z}}$ is a bounded linear operator from E into E, and $\xi_0 \in \nabla$, $\Xi = [0,\Im]$, $\Im > 0$, $0 = \zeta_0 < \delta_1 < \zeta_1 < \cdots < \delta_\gamma < \zeta_\gamma < \delta_{\gamma+1} := \Im$, $I_{0^+}^{1-\rho_3}(\cdot)$ is the Riemann–Liouville fractional integral of order $1-\rho_3 > 0$.

The novelty of this paper lies in the generalized nature of the problem incorporating the ψ-Hilfer fractional derivative, thereby extending all previously referenced works. In particular, we advance the results of [21, 24] by introducing a new framework for impulses and controllability specifically focusing on non-instantaneous impulses. Furthermore, we build upon [49] by modifying several conditions and exploring the existence of solutions using the set-valued version of Mönch's fixed point theorem, along with the key properties of the Kuratowski measure of noncompactness. This comprehensive approach underscores the significant contributions of this study to the field.

The following is the structure of our paper. In Section 2, we present some notations and important preliminary definitions concerning multifunctions and fractional calculus. In Section 3, we study the controllability of (1.1). Then, an example is given in Section 4 to interpret our main results.

2. Preliminaries

Let $C(\Xi,\nabla)$ denote the Banach space of all $\nabla-$valued continuous functions from Ξ into ∇ with the norm

$$\|\xi\|_{C(\Xi,\nabla)} = \sup\{\|\xi(\delta)\| : \delta \in \Xi\}.$$

Let $L(\nabla)$ be the Banach space of all linear and bounded operators on ∇, with the norm $\|T\|$ of a bounded linear operator $T \in L(\nabla)$ defined as the least upper bound of the set $\{\|T\xi\| : \|\xi\| \le 1\}$. In other words,

$$\|T\| = \sup_{\|\xi\|\le 1}\|T\xi\|.$$

Let $L^{\infty}(\Xi, \nabla)$ be the Banach space of measurable functions on Ξ which are essentially bounded with the norm

$$\|\eta\|_{\infty} = \sup\{c > 0 : \|\eta(\delta)\| \leq c \quad \text{for a.a. } \delta \in \Xi\}.$$

Let $L^{\flat}(\Xi, \nabla)$ be the Banach space of measurable functions which are Bochner integrable with the norm

$$\|v\|_{L^{\flat}(\Xi,\nabla)} = \left(\int_{\Xi} \|v(\delta)\|^{\flat} d\delta\right)^{\frac{1}{\flat}}, \quad \flat \in [1, \infty).$$

On the other hand, the weighted space of functions ξ on $\Xi^{'} := (0, \Im]$ is defined by

$$C_{1-\rho_3}(\Xi, \nabla) = \{\xi \in C(\Xi^{'}, \nabla) : \delta^{1-\rho_3}\xi(\delta) \in C(\Xi, \nabla)\},$$

where $0 \leq \rho_3 \leq 1$, $C_{1-\rho_3}(\Xi, \nabla)$ is a Banach space with the norm

$$\|\xi\|_{C_{1-\rho_3}} = \sup_{\delta \in \Xi} \|\delta^{1-\rho_3}\xi(\delta)\|.$$

Let $\Xi_{\jmath} := (\zeta_{\jmath}, \delta_{\jmath+1}]$, $\Xi_{\jmath}^{'} := (\delta_{\jmath}, \zeta_{\jmath}]$, and $\Xi^{*} = [0, \Im] \setminus \bigcup_{\jmath=1}^{\gamma}(\delta_{\jmath}, \zeta_{\jmath}]$. Now we consider the Banach space

$$\begin{aligned} PC_{1-\rho_3}(\Xi, \nabla) = \Big\{ & \xi \in C(\Xi_{\jmath}^{'}, \nabla) : (\delta - \zeta_{\jmath})^{1-\rho_3}\xi \in C(\Xi_{\jmath}, \nabla), \\ & \exists \lim_{\delta \to \zeta_{\jmath}^{+}} (\delta - \zeta_{\jmath})^{1-\rho_3}\xi(\delta), \quad \jmath = 0, 1, \ldots, \gamma, \quad \text{and} \\ & \exists \lim_{\delta \to \delta_{\jmath}^{+}} \xi(\delta), \quad \jmath = 1, 2, \ldots, \gamma \Big\}, \end{aligned}$$

with

$$\|\xi\|_{PC_{1-\rho_3}(\Xi,\nabla)} = \max\left\{ \max_{\jmath=0,1,\ldots,\gamma} \sup_{\delta \in \Xi_{\jmath}} (\delta - \zeta_{\jmath})^{1-\rho_3}\|\xi(\delta)\|, \max_{\jmath=1,2,\ldots,\gamma} \sup_{\delta \in \Xi_{\jmath}^{'}} \|\xi(\delta)\| \right\}.$$

Consider the following set:

$$\lambda_r = \{\xi \in PC_{1-\rho_3}(\Xi, \nabla) : \|\xi\|_{C_{1-\rho_3}} \leq r\}.$$

Notice that for each constant $r > 0$, λ_r is a bounded closed convex set in $PC_{1-\rho_3}(\Xi, \nabla)$.

Let $AC(\Xi, \nabla)$ be the space of absolutely continuous functions.

Remark 2.1. If $\xi \in PC_{1-\rho_3}(\Xi, \nabla)$, then for any $\jmath = 0, 1, \ldots, \gamma$, the following hold:

(i) ξ is not necessarily defined at $\zeta_{\jmath}$, but $\lim_{\delta \to \zeta_{\jmath}^{+}}(\delta - \zeta_{\jmath})\xi(\delta)$ and $\xi(\zeta_{\jmath+1}^{-})$ exist.

(ii) $\xi(\delta_{\jmath+1}) = \xi(\delta_{\jmath+1}^{-})$ and $\xi(\delta_{\jmath+1}^{+})$ exists. Moreover,

$$(\delta_{\jmath+1} - \zeta_{\jmath})^{1-\rho_3}\|\xi(\delta_{\jmath+1}^{-})\| \leq \|\xi\|_{PC_{1-\rho_3}(\Xi,\nabla)}.$$

(iii) If $\xi_\tau \to \xi$ in $PC_{1-\rho_3}(\Xi,\nabla)$, then $\xi_\tau(\delta) \to \xi(\delta)$, $\delta \in (\delta_\jmath, \zeta_\jmath], \jmath = 1,2,\dots,\gamma$, and

$$(\delta-\zeta_\jmath)^{1-\rho_3}\xi_\tau(\delta) \to (\delta-\zeta_\jmath)^{1-\rho_3}\xi(\delta), \delta \in (\zeta_\jmath, \delta_{\jmath+1}].$$

Consequently, $\xi_\tau(\delta) \to \xi(\delta), \delta \in (\zeta_\jmath, \delta_{\jmath+1}]$, and hence

$$\xi_\tau(\delta_{\jmath+1}) = \xi_\tau(\delta_{\jmath+1}^-) \to \xi(\delta_{\jmath+1}) = \xi(\delta_{\jmath+1}^-), \jmath = 0,1,\dots,\gamma.$$

Then $\xi_\tau(\delta) \to \xi(\delta)$ for a.a. $\delta \in \Xi$.

After that, the function $\eta_{PC_{1-\rho_3}(\Xi,\nabla)} : P_b(PC_{1-\rho_3}(\Xi,\nabla)) \to [0,\infty)$, defined by

$$\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y) = \max\left\{\max_{\jmath=0,1,\dots,\gamma}\eta_{C(\overline{\Xi_\jmath},\nabla)}(Y_{|\overline{\Xi_\jmath}}), \max_{\jmath=1,2,\dots,\gamma}\eta_{C(\overline{\Xi'_\jmath},\nabla)}(Y_{|\overline{\Xi'_\jmath}})\right\},$$

is the measure of noncompactness on $PC_{1-\rho_3}(\Xi,\nabla)$, where

$$\begin{aligned} Y_{|\overline{\Xi_\jmath}} = \Big\{ & \overline{\xi}^* \in C(\overline{\Xi_\jmath},\nabla) : \overline{\xi}^*(\delta) = (\delta-\zeta_\jmath)^{1-\rho_3}\overline{\xi}(\delta), \\ & \delta \in \Xi_\jmath,\ \overline{\xi}^*(\zeta_\jmath) = \lim_{\delta\to\zeta_\jmath^+}(\delta-\zeta_\jmath)^{1-\rho_3}\overline{\xi}(\delta), \overline{\xi} \in Y\Big\} \end{aligned}$$

and

$$Y_{|\overline{\Xi'_\jmath}} = \{\overline{\xi}^* \in C(\overline{\Xi'_\jmath},\nabla) : \overline{\xi}^*(\delta) = \overline{\xi}(\delta),\ \delta \in \Xi'_\jmath,\ \overline{\xi}^*(\delta_\jmath) = \overline{\xi}(\delta_\jmath^+),\ \overline{\xi} \in Y\}.$$

Let us define the following subsets of $P(\nabla)$:

- $P_b(\nabla) = \{\mathbb{B} \subseteq P(\nabla) : \mathbb{B}$ is nonempty and bounded$\}$,
- $P_{cl}(\nabla) = \{\mathbb{B} \subseteq P(\nabla) : \mathbb{B}$ is nonempty convex and closed$\}$,
- $P_{ck}(\nabla) = \{\mathbb{B} \subseteq P(\nabla) : \mathbb{B}$ is nonempty convex and compact$\}$,
- $P_{cl,b}(\nabla) = \{\mathbb{B} \subseteq P(\nabla) : \mathbb{B}$ is nonempty closed and bounded$\}$.

Let $(\operatorname{conv}\mathbb{B})$ (respectively, $\overline{\operatorname{conv}}(\mathbb{B})$) be the convex hull (respectively, the convex closed hull in ∇) of a subset $\mathbb{B}$.

Definition 2.2 ([23, 26, 31])**.** A multivalued map $H : \nabla \to P(\nabla)$ is said to be convex (closed) valued if $H(\xi)$ is convex (closed) for all $\xi \in \nabla$. A multivalued map H is bounded on bounded sets if $H(\overline{\mathfrak{Z}}) = \cup_{\xi\in\overline{\mathfrak{Z}}}H(\xi)$ is bounded in ∇ for all $\overline{\mathfrak{Z}} \in P_b(\nabla)$, i.e.,

$$\exists \sup_{\xi\in\overline{\mathfrak{Z}}}\big\{\sup\{|y| : y \in H(\xi)\}\big\}.$$

Definition 2.3 ([23, 26, 31])**.** A multivalued map $H : \nabla \to P(\nabla)$ is called upper semi-continuous (u.s.c.) on ∇ if $H(\xi_0) \in P_{cl}(\nabla)$; for each $\xi_0 \in \nabla$, and for each open set $D \subset \nabla$ with $H(\xi_0) \in D$, there exists an open neighborhood D_0 of ξ_0 such that $H(D_0) \subset D$. G is said to be completely continuous if $H(\overline{\mathfrak{Z}})$ is relatively compact for every $\overline{\mathfrak{Z}} \in P_b(\nabla)$. An element $\xi \in \nabla$ is a fixed point of H if $\xi \in H(\xi)$.

Lemma 2.4 ([31]). *Let $H : \nabla \to P(\nabla)$ be completely continuous with nonempty compact values. Then H is u.s.c. if and only if H has a closed graph, that is,*

$$\xi_\tau \to \xi_*, y_\tau \to y_*, \ y_\tau \in H(\xi_\tau) \Rightarrow y_* \in H(\xi_*).$$

Definition 2.5. A multivalued map $H : \Xi \to P_{cl}(\nabla)$ is said to be measurable if for every $y \in \nabla$, the function

$$\delta \to d(y, H(\delta)) = \inf\{|y - \bar{\xi}| : \bar{\xi} \in H(\delta)\}$$

is measurable.

Definition 2.6. A multivalued map $\aleph : \Xi \times \nabla \to P(\nabla)$ is said to be Carathéodory if

(i) $\aleph(\cdot, \xi)$ is measurable for each $\xi \in \nabla$,
(ii) $\aleph(\delta, \cdot)$ is upper semicontinuous for almost all $\delta \in \Xi$.

$\aleph$ is said to be L^1-Carathéodory if $\aleph$ satisfies the following condition:

(iii) For each $r > 0$, there exists $\varphi_r \in L^1(\Xi, \mathbb{R}_+)$ such that

$$\|\aleph(\delta, \xi)\|_P = \sup\{|\mathfrak{H}| : \mathfrak{H} \in \aleph(\delta, \xi)\} \le \varphi_r \text{ for all } \|\xi\| \le r \text{ and for a.a. } \delta \in \Xi.$$

For each $\xi \in C(\Xi, \nabla)$, define the set of selections of $\aleph$ by

$$S_{\aleph \circ \xi} = \{\mathfrak{H} \in L^1(\Xi, \nabla) : \mathfrak{H}(\delta) \in \aleph(\delta, \xi(\delta)) \quad \text{for a.a. } \delta \in \Xi\}.$$

Let (∇, d) be a metric space induced from the normed space $(\nabla, |\cdot|)$. The function $H_d : P(\nabla) \times P(\nabla) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(\mathfrak{W}_1, \mathfrak{W}_2) = \max\left\{\sup_{\mathfrak{w}_1 \in \mathfrak{W}_1} d(\mathfrak{w}_1, \mathfrak{W}_2), \sup_{\mathfrak{w}_2 \in \mathfrak{W}_2} d(\mathfrak{W}_1, \mathfrak{w}_2)\right\}$$

is known as the Hausdorff–Pompeiu metric. For more details on multivalued maps see the book of Hu and Papageorgiou [31].

Let M_∇ be the class of all bounded subsets of a metric space ∇.

Definition 2.7 ([15, 47]). Let ∇ be a Banach space and denote by M_∇ the family of bounded subsets of ∇. The map $\eta : \mathsf{M}_\nabla \to [0, \infty)$ defined by

$$\eta(\mathsf{M}_\nabla) = \inf\left\{\varepsilon > 0 : M \subset \bigcup_{j=1}^{\gamma} \Bbbk_j, \ \operatorname{diam}(\Bbbk_j) \le \varepsilon\right\}, \quad M \in \mathsf{M}_\nabla,$$

is called the Kuratowski measure of noncompactness.

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [15]):

- $\eta(\mathfrak{W}_2) = 0 \Leftrightarrow \overline{\mathfrak{W}_2}$ is compact ($\mathfrak{W}_2$ is relatively compact);
- $\eta(\mathfrak{W}_2) = \eta(\overline{\mathfrak{W}_2})$;

- $\mathfrak{W}_1 \subset \mathfrak{W}_2 \Rightarrow \eta(\mathfrak{W}_1) \leq \eta(\mathfrak{W}_2)$;
- $\eta(\mathfrak{W}_1 + \mathfrak{W}_2) \leq \eta(\mathfrak{W}_1) + \eta(\mathfrak{W}_2)$;
- $\eta(cB) = |c|\eta(\mathfrak{W}_2)$, $c \in \mathbb{R}$;
- $\eta(\operatorname{conv} B) = \eta(\mathfrak{W}_2)$.

Theorem 2.8 ([29]). *Let ∇ be a Banach space. Let $C \subset L^1(\Xi, \nabla)$ be a countable set with $\|\xi(\delta)\| \leq l(\delta)$ for a.a. $\delta \in \Xi$ and every $\xi \in C$, where $l \in L^1(\Xi, \mathbb{R}_+)$.*

Then $\psi(\delta) = \eta(C(\delta)) \in L^1(\Xi, \mathbb{R}_+)$ and verifies

$$\eta\left(\left\{\int_0^{\Im} \xi(s)\, ds : \xi \in C\right\}\right) \leq 2\int_0^{\Im} \eta(C(s))\, ds,$$

where η is the Kuratowski measure of noncompactness on the Banach space ∇.

Lemma 2.9 ([35]). *Let $\aleph$ be a Carathéodory multivalued map and $\Upsilon : L^1(\Xi, \nabla) \to C(\Xi, \nabla)$ be a linear continuous map. Then the operator*

$$\Upsilon \circ S_{\aleph \circ \xi} : C(\Xi, \nabla) \to P_{cv,cp}(C(\Xi, \nabla)),$$
$$\xi \mapsto (\Upsilon \circ S_{\aleph \circ \xi})(\xi) = \Upsilon(S_{\aleph \circ \xi})$$

is a closed graph operator in $C(\Xi, \nabla) \times C(\Xi, \nabla)$.

Definition 2.10. Let ∇ be a Banach space. A multivalued mapping $\mathcal{T} : \nabla \to P_{cl,b}(\nabla)$ is called $\jmath$-set-Lipshitz if there exists a constant $\jmath > 0$ such that

$$\eta(\mathcal{T}(W)) \leq \jmath\eta(W) \quad \text{for all } W \in P_{cl,b}(\nabla).$$

If $\jmath < 1$, then $\mathcal{T}$ is called a $\jmath$-set-contraction on ∇.

For completeness, we recall some fundamental definitions of the theory of fractional calculus.

Definition 2.11 ([22]). Let $\rho_1 > 0$ and $\mathfrak{H} \in L^1(\Xi, \nabla)$. The Riemann–Liouville integral is defined by

$$I_\delta^{\rho_1} \mathfrak{H}(\delta) = \frac{1}{\Gamma(\rho_1)} \int_0^\delta \frac{\mathfrak{H}(s)}{(\delta - s)^{1-\rho_1}}\, ds.$$

Definition 2.12 ([42, 45, 46]). The Hilfer fractional derivative $D_{a+}^{\rho_1,\rho_2}$ of function $\mathfrak{H} \in C^\tau(a, b)$ of order $\tau - 1 < \rho_1 < \tau$ and type $0 \leq \rho_2 \leq 1$, is defined by

$$D_{a+}^{\rho_1,\rho_2} \mathfrak{H}(\xi) = I_{a+}^{\rho_3-\rho_1} \left(\frac{d}{dx}\right)^\tau I_{a+}^{(1-\rho_2)(\tau-\rho_1)} \mathfrak{H}(\xi),$$

where $I_{a+}^{\rho_1}$ is the Riemann–Liouville fractional integral.

In order to define the mild solution of the problem (1.1) we recall the following definition.

Definition 2.13 ([28]). A closed and linear operator $\mathfrak{Z}$ is said to be sectorial if there are constants $\omega \in \mathbb{R}, \mu \in [\frac{\pi}{2}, \pi], \widehat{\Phi} > 0$, such that the following two conditions are satisfied:

1) $\sum\limits_{(\mu,\omega)} := \{\lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \mu\} \subset \rho(\mathfrak{Z})$, where ($\rho(\mathfrak{Z})$ is the resolvent set of $\mathfrak{Z}$;

2) $\|R(\lambda, \mathfrak{Z})\|_{L(\nabla)} \leq \dfrac{\widehat{\Phi}}{|\lambda - \omega|}$, $\lambda \in \sum_{(\mu,\omega)}$.

Definition 2.14 ([9]). Let $\mathfrak{Z}$ be a closed linear operator with domain $D(\mathfrak{Z})$ defined on a Banach space ∇ and $\rho_1 > 0$. We say that $\mathfrak{Z}$ is the generator of a ρ_1-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_{\rho_1} : \mathbb{R}_+ \to L(\nabla)$ such that $\{\lambda^{\rho_1} : Re(\lambda) > \omega\} \subset \rho(\mathfrak{Z})$ and

$$(\lambda^{\rho_1} I - \mathfrak{Z})^{-1}\xi = \int_0^\infty e^{-\lambda\delta} S_{\rho_1}(\delta)\xi \, d\delta, \quad \operatorname{Re}\lambda > \omega, \ \xi \in \nabla.$$

In this case, $S_{\rho_1}(\delta)$ is called the ρ_1-resolvent family generated by $\mathfrak{Z}$.

Definition 2.15 ([3]). Let $\mathfrak{Z}$ be a closed linear operator with domain $D(\mathfrak{Z})$ defined on a Banach space ∇ and $\rho_1 > 0$, then we say that $\mathfrak{Z}$ is the generator of a solution operator if there exists $\omega \geq 0$ and a strongly continuous function $S_{\rho_1} : \mathbb{R}_+ \to L(\nabla)$ such that $\{\lambda^{\rho_1} : Re(\rho_1) > \omega\} \subset \rho(\mathfrak{Z})$ and

$$\lambda^{\rho_1 - 1}(\lambda^{\rho_1} I - \mathfrak{Z})^{-1}\xi = \int_0^\infty e^{-\lambda\delta} S_{\rho_1}(\delta)\xi \, d\delta, \quad \operatorname{Re}\lambda > \omega, \ \xi \in \nabla.$$

In this case, $S_{\rho_1}(\delta)$ is called the solution operator generated by $\mathfrak{Z}$. For more details see [37, 41].

Definition 2.16 ([27]). Let $\mathfrak{H} : \Xi \times \nabla \to \nabla$. By a mild solution of

$$\begin{aligned} &D_{0^+}^{\rho_1,\rho_2}\xi(\delta) = \mathfrak{Z}\xi(\delta) + \mathfrak{H}(\delta, \xi(\delta)), \quad \delta \in (0, \Im], \\ &I_{0^+}^{1-\rho_3}\xi(0) = \xi_0, \end{aligned}$$

we mean a function $\xi \in C((0, \Im], \nabla)$, which verifies

$$\xi(\delta) = Q_{\rho_1,\rho_2}(\delta)\xi_0 + \int_0^\delta R_{\rho_1}(\delta - s)\mathfrak{H}(s, \xi(s)) \, ds, \quad \delta \in (0, \Im], \tag{2.1}$$

where, $R_{\rho_1}(\delta) = \delta^{\rho_1 - 1} P_{\rho_1}(\delta)$, $P_{\rho_1}(\delta) = \int_0^\infty \rho_1 \mu \Bbbk_{\rho_1}(\mu) T(\delta^{\rho_1}\mu) \, d\mu$, $\delta \geqslant 0$,

$$\Bbbk_\varpi(\mu) = \sum_{\tau=1}^{\infty} (-\mu)^{\tau-1}/((\tau - 1)\Gamma(1 - \varpi\tau)), \varpi \in (0, 1), \quad \mu \in \mathbb{C},$$

and

$$Q_{\rho_1,\rho_2}(\delta) = I_{0_+}^{\rho_2(1-\rho_1)} R_{\rho_1}(\delta).$$

Note that $\Bbbk_\varpi(\mu)$ verifies

$$\int_0^\infty \mu^{\Im} \Bbbk_\varpi(\mu) d\mu = \Gamma(1 + \Im)/\Gamma(1 + \Im\varpi) \quad \text{for } \mu \geqslant 0.$$

Remark 2.17. From [27], we have:

(i) $D_{0+}^{\rho_2(1-\rho_1)}Q_{\rho_1,\rho_2}(\delta) = R_{\rho_1}(\delta), \delta \in (0, \Im]$.

(ii) When $\rho_2 = 0$, (2.1) reduces to the equation in ([51], Definition 3.1), we have

$$Q_{\rho_1,0}(\delta) = R_{\rho_1}(\delta) = \delta^{\rho_1-1}P_{\rho_1}(\delta).$$

(iii) When $\rho_2 = 1$, (2.1) reduces to the equation in ([51], Definition 3.1). Note $Q_{\rho_1,1} = Q_{\rho_1}(\delta)$, where $Q_{\rho_1}(\delta)$ is defined in [51].

Lemma 2.18 ([27])**.** *Suppose the semigroup* $(T(\delta))_{\delta\geqslant 0}$, *satisfies the condition*

(H_T) $T(\delta)$ *is continuous for the uniform operator topology for* $\delta > 0$, *and there is* $\widehat{\Phi} > 1$ *such that* $\sup\limits_{\delta\geqslant 0}\|T(\delta)\| \leq \widehat{\Phi}$.

Then we have

(i) $P_{\rho_1}(\delta)$ *is continuous for the uniform operator topology for* $\delta > 0$.

(ii) *For any fixed* $\delta > 0, Q_{\rho_1,\rho_2}(\delta)$ *and* $R_{\rho_1}(\delta)$ *are linear bounded operators, and for any fixed* $\xi \in \nabla$,

$$\|Q_{\rho_1,\rho_2}(\delta)\xi\| \leq (\widehat{\Phi}\delta^{\rho_3-1}/\Gamma(\rho_3))\|\xi\|, \rho_3 = \rho_1 + \rho_2 - \rho_1\rho_2,$$

and

$$\|R_{\rho_1}(\delta)\xi\| \leq (\widehat{\Phi}\delta^{\rho_1-1}/\Gamma(\rho_1))\|\xi\|.$$

(iii) $\{R_{\rho_1}(\delta), \delta > 0\}$ *and* $\{Q_{\rho_1,\rho_2}(\delta), \delta > 0\}$ *are strongly continuous, which means that for any* $\xi \in \nabla$ *and* $0 < \delta_1 < \delta_2 \leq \Im$ *we have*

$$\|R_{\rho_1}(\delta_1)\xi - R_{\rho_1}(\delta_2)\xi\| \to 0 \text{ and } \|Q_{\rho_1,\rho_2}(\delta_1)\xi - Q_{\rho_1,\rho_2}(\delta_2)\xi\| \to 0 \text{ as } \delta_1 \to \delta_2.$$

Theorem 2.19 ([13, 44])**.** *If* $\rho_1 \in (0,1)$ *and* $\mathfrak{Z} \in \mathcal{A}^{\rho_1}(\mu_0, \omega_0)$, *then for any* $\xi \in \nabla$ *and* $\delta > 0$, *we have*

$$\|Q_{\rho_1,\rho_2}(\delta)\|_{L(\nabla)} \leq \widehat{\Phi}e^{\omega\delta} \text{and} \|R_{\rho_1}(\delta)\|_{L(\nabla)} \leq Ce^{\omega\delta}(1 + \delta^{\rho_1-1}), \delta > 0, \omega > \omega_0.$$

Let

$$\overset{\frown}{M}_Q = \sup_{0\leq\delta\leq\Im} \|Q_{\rho_1,\rho_2}(\delta)\|_{L(\nabla)}, \quad \overset{\frown}{M}_R = \sup_{0\leq\delta\leq\Im} Ce^{\omega\delta}(1 + \delta^{\rho_1+1}).$$

So we have

$$\|Q_{\rho_1,\rho_2}(\delta)\|_{L(\nabla)} \leq \overset{\frown}{M}_Q, \quad \|R_{\rho_1}(\delta)\|_{L(\nabla)} \leq \delta^{\rho_1-1}\overset{\frown}{M}_R.$$

Denote $\mathcal{A}^{\rho_1}(\mu_0, \omega_0) := \{\mathcal{A} \in \mathcal{O} : \mathcal{A}$ generates analytic solution operators $S_{\rho_1}(\delta)$(the ρ_1-resolvent family generated by $\mathfrak{Z}$) of type $(\mu_0, \omega_0)\}$, where $\mathcal{O}^{\rho_1}(\omega) := \cup\left\{\mathcal{O}^{\rho_1}(\widehat{\Phi}, \omega) : \widehat{\Phi} \geq 1\right\}$ and $\mathcal{O}^{\rho_1} := \cup\{\mathcal{O}^{\rho_1}(\omega) : \omega \geq 0\}$.

Theorem 2.20 ([39])**.** *Let* D *be a closed convex subset of a Banach space* $\mathcal{E}$ *and* $L : D \to P_c(D)$. *Assume the graph of* L *is closed,* L *maps compact sets into relatively compact sets and that for some* $\xi_0 \in U$, *one has*

$$\begin{aligned} &Y \subseteq D, \quad Y \subset \operatorname{conv}(\{0\} \cup L(Y)), \quad \overline{Y} = \overline{C} \text{ with } C \subseteq Y \text{ countable} \\ &\qquad\qquad\qquad\qquad \Rightarrow Y \text{ relatively compact.} \end{aligned} \tag{2.2}$$

Then L *has a fixed point.*

3. Main results

Firstly, based on (2.1), we present the concept of mild solution of the problem (1.1).

Definition 3.1. A function $\xi \in PC_{1-\rho_3}(\Xi, \nabla)$ is called a mild solution of the problem (1.1) if there is $\mathfrak{H} \in S^1_{\aleph(\cdot,\xi(\cdot))}$ such that

$$\xi(\delta) = \begin{cases} Q_{\rho_1,\rho_2}(\delta)\xi_0 + \displaystyle\int_0^\delta R_{\rho_1}(\delta - s)\left(\mathfrak{H}(s,\xi(s)) + \mathfrak{Z}\varkappa(s)\right) ds, \quad \delta \in [0, \delta_1], \\ \Psi_\jmath(\delta, \xi(\delta_\jmath^-)), \quad \delta \in \Xi_\jmath^{'}, \jmath = 1, 2, \dots, \gamma, \\ Q_{\rho_1,\rho_2}(\delta)\Psi_\jmath(\zeta_\jmath, \xi(\delta_\jmath^-)) + \displaystyle\int_{\zeta_\jmath}^\delta R_{\rho_1}(\delta - s)\left(\mathfrak{H}(s,\xi(s)) + \mathfrak{Z}\varkappa(s)\right) ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad \delta \in \Xi_\jmath, \jmath = 1, 2, \dots, \gamma, \end{cases}$$

Definition 3.2. The system (1.1) is said to be controllable on Ξ if for every $\xi_0, \xi_1 \in \nabla$, there exists a control function $\varkappa \in L^\flat(\Xi, E)$ such that the mild solution of (1.1) satisfies $\xi(\Im) = \xi_1$.

The hypotheses:

($\mathcal{F}_1$) For every $\xi \in PC_{1-\rho_3}(\Xi, \nabla)$, the multifunction $\delta \to \aleph(\delta, \xi(\delta))$ has a strong measurable selection, and for almost every $\delta \in \Xi, \overline{\xi} \to \aleph(\delta, \overline{\xi})$ is upper semi-continuous.

($\mathcal{F}_2$) $\aleph : \Xi \times \nabla \to P_{ck}(\nabla)$ is a multifunction, and there exists a function $\hbar \in L^\flat(\Xi, \mathbb{R}^+)$ and a continuous nondecreasing function $\Omega : [0, \infty) \to (0, \infty)$ such that

$$\|\aleph(\delta, \xi)\| \leq \hbar(\delta)\Omega(\|\xi\|) \quad \text{for } \delta \in \Xi \text{ and for all } \xi \in \nabla.$$

($\mathcal{F}_3$) There exists $\kappa \in L^\flat(\Xi, \mathbb{R}^+)$ such that for any $D \subseteq \nabla$ and any $\jmath = 0, 1, \dots, \gamma$,

$$\eta(\aleph(\delta, D)) \leq (\delta - \zeta_\jmath)^{1-\rho_3}\kappa(\delta)\eta(D) \quad \text{for a.a. } \delta \in \Xi,$$

and

$$\Im^{1-\rho_3}\widehat{\Phi}\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}\left(\frac{2\zeta}{\Gamma(\rho_1)} + \frac{2\zeta\Phi^2}{\Gamma(\rho_1)^2}\right) < 1, \tag{3.1}$$

where

$$\zeta = \Im^{\rho_1 - 1/\flat}((\flat - 1)/(\flat\rho_1 - 1))^{(\flat-1)/\flat},$$

where $\flat$ is a real number such that $\flat > \frac{1}{\rho_1}$ and η is the Kuratowski measure of noncompactness on ∇.

($\mathcal{H}_{\Psi_\jmath}$) For every $\jmath = 1, 2, \dots, \gamma, \Psi_\jmath : [\delta_\jmath, \zeta_\jmath] \times \nabla \to \nabla$ is uniformly continuous on bounded sets, and for any $\delta \in \Xi, \Psi_\jmath(\delta, \cdot)$ is compact, and there exists a positive constant $\wp$ such that

$$\|\Psi_\jmath(\delta, \xi)\| \leq \wp(\delta_\jmath - \zeta_{\jmath-1})^{1-\rho_3}\|\xi\| \quad \text{for all } \xi \in \nabla, \ \delta \in [\delta_\jmath, \zeta_\jmath].$$

$(\mathcal{H}_Z)$ The linear bounded operator $Z : L^\flat(\Xi, E) \to \nabla$, defined by

$$Z(\varkappa) = \int_{\zeta_\gamma}^{\Im} R_{\rho_1}(\Im - s)\overline{\mathfrak{Z}}(\varkappa(s))\, ds,$$

has an invertible operator $Z^{-1} : \nabla \to L^\flat(\Xi, E)/\ker(Z)$, and there exists a positive constant Φ such that $\|Z^{-1}\| \le \Phi$ and $\|\overline{\mathfrak{Z}}\| \le \Phi$.

Theorem 3.3. *Let the assumptions* $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$, $(\mathcal{H}_{\Psi_\jmath})$, *and* $(\mathcal{H}_Z)$ *hold. Then problem* (1.1) *is controllable on* Ξ, *provided that*

$$\frac{\widehat{\Phi}\zeta v\Im^{1-\rho_3}}{\Gamma(\rho_1)}\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)} + \frac{\widehat{\Phi}\Im^{1-\rho_3}\Phi^2}{\Gamma(\rho_1)}\zeta\left[\frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_1)}\wp + \frac{v\widehat{\Phi}}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right] + \wp + \frac{\wp\widehat{\Phi}}{\Gamma(\rho_1)} < 1. \tag{3.2}$$

Proof. By assumption, Z is well defined. Since $\flat > 1/\rho_1$, $s \to (\Im - s)^{\rho_1-1}$ are in $L^{\flat/(\flat-1)}([0,\Im],\mathbb{R}^+)$, then, by the Hölder inequality, for any $\varkappa \in L^\flat(\Xi, E)$, we have

$$\begin{aligned}\|Z(\varkappa)\| &\le \frac{\widehat{\Phi}\Phi}{\Gamma(\rho_1)}\int_{\zeta_\gamma}^{\Im}(\Im - s)^{\rho_1-1}\|\varkappa(s)\|ds\\ &\le \frac{\widehat{\Phi}\Phi}{\Gamma(\rho_1)}\|\varkappa\|_{L^\flat(\Xi,E)}\left(\frac{\flat-1}{\rho_1\flat-1}\right)^{(\flat-1)/\flat}\Im^{\rho_1-1/\flat}.\end{aligned} \tag{3.3}$$

By $(\mathcal{F}_1)$, for every $\xi \in PC_{1-\rho_3}(\Xi,\nabla)$, the multifunction $\delta \to F(\delta,\xi(\delta))$ has a measurable selection $\mathfrak{H}$, and by $(\mathcal{F}_2)$,

$$\|\mathfrak{H}(\delta)\| \le \hbar(\delta)\Omega(\|\xi\|_{PC_{1-\rho_3}(\Xi,\nabla)}).$$

Hence,

$$\mathfrak{H} \in S^\flat_{\aleph(\cdot,\xi(\cdot))} = \left\{w \in L^\flat(\Xi,\nabla) : w(\delta) \in \aleph(\delta,\xi(\delta)) \quad \text{for a.a. } \delta \in \bigcup_{\jmath=0}^{\jmath=\gamma}(\zeta_\jmath,\delta_{\jmath+1}]\right\}.$$

Next, we have

$$\|Q_{\rho_1,\rho_2}(\Im-\zeta_\gamma)\Psi_\gamma(\zeta_\gamma,\xi(\delta_\gamma^-))\| \le (\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}/\Gamma(\rho_3))\wp(\delta_\gamma-\zeta_{\gamma-1})^{1-\rho_3}\times\|\xi(\delta_\gamma^-)\|.$$

Thus, for all $\xi \in PC_{1-\rho_3}(\Xi,\nabla)$ and any $\mathfrak{H} \in S^\flat_{\aleph(\cdot,\xi(\cdot))}$, by using $(\mathcal{H}_Z)$, we can define the control function $\varkappa_{\xi,\mathfrak{H}} \in L^\flat(\Xi,E)$ by

$$\varkappa_{\xi,\mathfrak{H}} = Z^{-1}\left[\xi_1 - Q_{\rho_1,\rho_2}(\Im-\zeta_\gamma)\Psi_\gamma(\zeta_\gamma,\xi(\delta_\gamma^-)) - \int_{\zeta_\gamma}^{\Im}R_{\rho_1}(\Im-s)\mathfrak{H}(s,\xi(s))\,ds\right]. \tag{3.4}$$

Therefore, we can define a multifunction $S : PC_{1-\rho_3}(\Xi,\nabla) \to PC_{1-\rho_3}(\Xi,\nabla)$ as follows. For all $\xi \in PC_{1-\rho_3}(\Xi,\nabla)$, a function $\overline{\xi} \in S(\xi)$ if and only if

$$\overline{\xi}(\delta) = \begin{cases} Q_{\rho_1,\rho_2}(\delta)\xi_0 + \displaystyle\int_0^{\delta} R_{\rho_1}(\delta - s)\left(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds, \quad \delta \in [0,\delta_1], \\ \Psi_\jmath(\delta,\xi(\delta_\jmath^-)), \quad \delta \in \Xi_\jmath^{'}, \jmath = 1,2,\ldots,\gamma, \\ Q_{\rho_1,\rho_2}(\delta)\Psi_\jmath(\zeta_\jmath,\xi(\delta_\jmath^-)) \\ \quad + \displaystyle\int_{\zeta_\jmath}^{\delta} R_{\rho_1}(\delta - s)\left(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds, \quad \delta \in \Xi_\jmath, \jmath = 1,2,\ldots,\gamma, \end{cases}$$

where $\mathfrak{H} \in S^{\flat}_{\aleph(\cdot,\xi(\cdot))}$.

Now, using (3.4), we will prove that for any fixed point of S is a mild solution for (1.1) and verifies $\xi(0) = \xi_0$ and $\xi(\Im) = \xi_1$.

So, if ξ is a fixed point for S, then from (3.4) we have

$$\begin{aligned} \xi(\Im) &= Q_{\rho_1,\rho_2}(\delta)(\Im - \zeta_\jmath)\Psi_\gamma(\zeta_\gamma,\xi(\delta_\gamma^-)) \\ &\quad + \int_{\zeta_\gamma}^{\Im} R_{\rho_1}(\Im - s)\mathfrak{H}(s,\xi(s))\, ds + \int_{\zeta_\gamma}^{\Im} R_{\rho_1}(\Im - s)\overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\, ds \\ &= Q_{\rho_1,\rho_2}(\delta)(\Im - \zeta_\jmath)\Psi_\gamma(\zeta_\gamma,\xi(\delta_\gamma^-)) + \int_{\zeta_\gamma}^{\Im} R_{\rho_1}(\Im - s)\mathfrak{H}(s,\xi(s))ds + Z(\varkappa_{\xi,\mathfrak{H}}) \\ &= Q_{\rho_1,\rho_2}(\delta)(\Im - \zeta_\jmath)\Psi_\gamma(\zeta_\gamma,\xi(\delta_\gamma^-)) + \int_{\zeta_\gamma}^{\Im} R_{\rho_1}(\Im - s)\mathfrak{H}(s,\xi(s))ds \\ &\quad + \xi_1 - Q_{\rho_1,\rho_2}(\delta)(\Im - \zeta_\jmath)\Psi_\gamma(\zeta_\gamma,\xi(\delta_\gamma^-)) - \int_{\zeta_\gamma}^{\Im} R_{\rho_1}(\Im - s)\mathfrak{H}(s,\xi(s))\, ds = \xi_1. \end{aligned}$$

Using (2.20), we will prove that S has a fixed point. The proof will be given in several steps. It is not easy to see that the values of S are convex.

Step 1: Here, let K where $S(\Pi_\jmath) \subseteq \Pi_K$, where

$$\Pi_K = \left\{\xi \in PC_{1-\rho_3}(\Xi,\nabla) : \|\xi\|_{PC_{1-\rho_3}(\Xi,\nabla)} \leq K\right\}.$$

Suppose the contrary. Then for any $K \in \mathbb{R}_+$, there are $\xi_K, y_K \in PC_{1-\rho_3}(\Xi,\nabla)$ with $y_K \in S(\xi_K)$, $\|\xi_K\|_{PC_{1-\rho_3}(\Xi,\nabla)} \leq K$ and $\|y_K\|_{PC_{1-\rho_3}(\Xi,\nabla)} > K$. Then there is $\mathfrak{H}_K \in S^{\flat}_{\aleph(\cdot,\xi(\cdot))}$, $K \geqslant 1$, such that

$$\begin{aligned} &\overline{\xi}_K(\delta) \\ &= \begin{cases} Q_{\rho_1,\rho_2}(\delta)\xi_0 \\ \quad + \displaystyle\int_0^{\delta} R_{\rho_1}(\delta - s)\left(\mathfrak{H}_K(s,\xi_K(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s))\right) ds, \quad \delta \in (0,\delta_1], \\ \Psi_\jmath(\delta,\xi_K(\delta_\jmath^-)), \quad \delta \in \Xi_\jmath^{'}, \jmath = 1,2,\ldots,\gamma, \\ Q_{\rho_1,\rho_2}(\delta - \zeta_\jmath)\Psi_\jmath(\zeta_\jmath,\xi_K(\delta_\jmath^-)) + \displaystyle\int_{\zeta_\jmath}^{\delta} R_{\rho_1}(\delta - s)\big(\mathfrak{H}_K(s,\xi_K(s)) \\ \qquad\qquad + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s))\big)\, ds, \quad \delta \in \Xi_\jmath, \jmath = 1,2,\ldots,\gamma. \end{cases} \end{aligned} \tag{3.5}$$

Then, if $\delta \in [0, \delta_1]$, using the Hölder inequality, we have

$$\begin{aligned}\sup_{\delta\in[0,\delta_1]} \delta^{1-\rho_3}\|\overline{\xi}_K(\delta)\| &\le \sup_{\delta\in\delta_1} \delta^{1-\rho_3}\|Q_{\rho_1,\rho_2}(\delta)\xi_0\| \\ &\quad + \sup_{\delta\in[0,\delta_1]} \frac{\widehat{\Phi}\delta^{1-\rho_3}\Omega(\|\xi_K\|_{PC_{1-\rho_3}(\Xi,\nabla)})}{\Gamma(\rho_1)} \int_0^\delta (\delta-s)^{\rho_1-1}\hbar(s)\,ds \\ &\quad + \sup_{\delta\in[0,\delta_1]} \frac{\widehat{\Phi}\Phi\delta^{1-\rho_3}}{\Gamma(\rho_1)} \int_0^\delta (\delta-s)^{\rho_1-1}\|\varkappa_{\xi_K,\mathfrak{H}_K}(s)\|\,ds. \\ &\le \frac{\widehat{\Phi}}{\Gamma(\rho_3)}\|\xi_0\| + \frac{\widehat{\Phi}\delta^{1-\rho_3}}{\Gamma(\rho_1)}\Omega(K)\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\zeta \\ &\quad + \frac{\widehat{\Phi}\Phi\delta^{1-\rho_3}}{\Gamma(\rho_1)}\|\varkappa_{\xi_K,\mathfrak{H}_K}\|_{L^\flat(\Xi,\mathbb{R}^+)}\zeta. \end{aligned} \tag{3.6}$$

From (3.6), we get

$$\begin{aligned} &\|\varkappa_{\xi_K,\mathfrak{H}_K}\|_{L^\flat(\Xi,\mathbb{R}^+)} \\ &\le \|Z^{-1}\|\left[\|\xi_1\| + \|Q_{\rho_1,\rho_2}(\Im-\zeta_\gamma)\Psi_\gamma(\zeta_\gamma - \xi(\delta_\gamma^-))\| + \int_{\zeta_\gamma}^{\Im}\|R_{\rho_1}(\Im-s)\mathfrak{H}(s,\xi_s)\|\,ds\right] \\ &\le \Phi\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp(\delta_\gamma-\zeta_{\gamma-1})^{1-\rho_3}\|\xi(\delta_\gamma^-))\|\right. \\ &\quad \left. + \frac{\widehat{\Phi}\Omega(\|\xi_K\|_{PC_{1-\rho_3}(\Xi,\nabla)})}{\Gamma(\rho_1)}\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\zeta\right] \\ &= \Phi\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp\|\xi_K\|_{PC_{1-\rho_3}(\Xi,\nabla)} + \frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right] \\ &\le \Phi\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp K + \frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right]. \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} \sup_{\delta\in[0,\delta_1]} \delta^{1-\rho_3}\|\overline{\xi}_K(\delta)\| &\le \frac{\widehat{\Phi}}{\Gamma(\rho_3)}\|\xi_0\| + \zeta\frac{\widehat{\Phi}\Im^{1-\rho_3}}{\Gamma(\rho_1)}\Omega(K)\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)} \\ &\quad + \frac{\widehat{\Phi}\Im^{1-\rho_3}\Phi^2}{\Gamma(\rho_1)}\zeta\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp K + \frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right]. \end{aligned} \tag{3.8}$$

If $\delta \in (\delta_\jmath, \zeta_\jmath]$, $\jmath = 1, 2, \dots, \gamma$, then

$$\sup_{\delta\in[\delta_\jmath,\zeta_\jmath]} \|\overline{\xi}_K(\delta)\| \le \wp(\delta_\jmath - \zeta_{\jmath-1})^{1-\rho_3}\|\xi_K(\delta_\jmath^-)\| \le \wp\|\xi_K\|_{PC_{1-\rho_3}(\Xi,\nabla)} \le \wp K. \tag{3.9}$$

In a similar way, for $\delta \in (\zeta_\jmath, \delta_{\jmath+1}], \jmath = 1, 2, \dots, \gamma$, we get

$$\sup_{\delta\in[\zeta_\jmath,\delta_{\jmath+1}]} (\delta-\zeta_\jmath)^{1-\rho_3}\|\overline{\xi}_K(\delta)\|$$

$$
\begin{aligned}
&\leq \sup_{\delta\in[\zeta_j,\delta_{j+1}]} \frac{\widehat{\Phi}\Psi_j(\zeta_j,\xi_K(\delta_j^-))}{\Gamma(\rho_3)} + \frac{\widehat{\Phi}\Im^{1-\rho_3}}{\Gamma(\rho_1)}\Omega(K)\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\zeta \\
&\quad + \frac{\widehat{\Phi}\Im^{1-\rho_3}\Phi^2}{\Gamma(\rho_1)}\zeta\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp K + \frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right] \\
&\leq \frac{\widehat{\Phi}\wp K}{\Gamma(\rho_3)} + \frac{\widehat{\Phi}\Im^{1-\rho_3}}{\Gamma(\rho_1)}\Omega(K)\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\zeta \\
&\quad + \frac{\widehat{\Phi}\Im^{1-\rho_3}\Phi^2}{\Gamma(\rho_1)}\zeta\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp K + \frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\zeta\right]. \qquad (3.10)
\end{aligned}
$$

From (3.8),(3.9) and (3.10), we have

$$
\begin{aligned}
K < \|\overline{\xi}_K(\delta)\|_{PC_{1-\rho_3}(\Xi,\nabla)} &\leq \frac{\widehat{\Phi}}{\Gamma(\rho_3)}\|\xi_0\| + \zeta\frac{\widehat{\Phi}\Im^{1-\rho_3}}{\Gamma(\rho_1)}\Omega(K)\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)} \\
&\quad + \frac{\widehat{\Phi}\Im^{1-\rho_3}\Phi^2}{\Gamma(\rho_1)}\zeta\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp K + \frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right] \\
&\quad + \wp K + \frac{\widehat{\Phi}\wp K}{\Gamma(\rho_3)}.
\end{aligned}
$$

We divide the both sides by K and pass to the limit as $K\to\infty$ to obtain

$$
\begin{aligned}
1 &\leq \frac{\widehat{\Phi}\zeta\upsilon\Im^{1-\rho_3}}{\Gamma(\rho_1)}\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)} \\
&\quad + \frac{\widehat{\Phi}\Im^{1-\rho_3}\Phi^2}{\Gamma(\rho_1)}\zeta\left[\frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp + \frac{\upsilon\widehat{\Phi}}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^\flat(\Xi,\mathbb{R}^+)}\right] + \wp + \frac{\wp\widehat{\Phi}}{\Gamma(\rho_3)},
\end{aligned}
$$

which contradicts (3.2).

Then we conclude that there is a number K_0 such that $S(\Pi_{K_0})\subseteq\Pi_{K_0}$.

Step 2: Let $K=\{q\in PC_{1-\rho_3}(\Xi,\nabla), q\in S(\Pi_{K_0})\}$. We claim that the subsets $K_{|\overline{\Xi_j}}(j=0,1,\ldots,\gamma)$ and $K_{|\overline{\Xi_{j'}}}(j=1,2,\ldots,\gamma)$ are equicontinuous, where

$$
\begin{aligned}
K_{|\overline{\Xi_j}} &= \{q:\overline{\Xi_j}\to\nabla,\ q(\delta)=(\delta-\zeta_j)^{1-\rho_3}\overline{\xi}(\delta),\ \delta\in\Xi_j, \\
&\qquad q(\zeta_j)=\lim_{\delta\to\zeta_j}(\delta-\zeta_j)^{1-\rho_3}q(\delta),\ \overline{\xi}\in S(\xi),\ \xi\in\Pi_{n_0}\} \\
K_{|\overline{\Xi_{j'}}} &= \{\overline{\xi}^*\in C(\overline{\Xi_{j'}},\nabla):\overline{\xi}^*(\delta)=\overline{\xi}(\delta),\ \delta\in[\delta_j,\zeta_j], \\
&\qquad \overline{\xi}^*(\delta_j)=\overline{\xi}(\delta_j^+),\ \overline{\xi}\in S(\xi),\ \xi\in\Pi_{n_0}\}.
\end{aligned}
$$

► **Case 1 :** Let $q\in K_{|\overline{\Xi_0}}$. Then there is $\xi\in\Pi_{K_0}$ and $\mathfrak{H}\in S^\flat_{\aleph(\cdot,\xi(\cdot))}$ such that for $\delta\in(0,\delta_1]$,

$$
q(\delta)=\delta^{1-\rho_3}\left[Q_{\rho_1,\rho_2}(\delta)\xi_0+\int_0^\delta R_{\rho_1}(\delta-s)\left(\mathfrak{H}(s,\xi(s))+\overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right)ds\right],
$$

and $q(0) = \lim_{\delta \to 0+} \delta^{1-\rho_3}\overline{\xi}(\delta)$. It follows for $\delta = 0$, $\sigma \in (0, \delta_1]$ that

$$\lim_{\sigma \to 0+} q(\sigma) = \lim_{\sigma \to 0+} \sigma^{1-\rho_3}\overline{\xi}(\sigma) = \lim_{\delta \to 0+} \delta^{1-\rho_3}\overline{\xi}(\delta) = q(0).$$

Let $\delta, \delta + \sigma \in (0, \delta_1]$. Then

$$\begin{aligned}\|q(\delta+\sigma) - q(\delta)\| \leq{} & \|(\delta+\sigma)^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta+\sigma)\xi_0 - \delta^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta)\xi_0\| \\ & + \|(\delta+\sigma)^{1-\rho_3} \int_0^{\delta+\sigma} R_{\rho_1}(\delta+\sigma-s)\left(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds \\ & - \delta^{1-\rho_3} \int_0^{\delta} R_{\rho_1}(\delta-s)\left(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds\| \leq \sum_{j=1}^{j=8} I_j,\end{aligned}$$

where

$$\begin{aligned} I_1 &= (\delta+\sigma)^{1-\rho_3}\|Q_{\rho_1,\rho_2}(\delta+\sigma)\xi_0 - Q_{\rho_1,\rho_2}(\delta)\xi_0\|, \\ I_2 &= |(\delta+\sigma)^{1-\rho_3} - \delta^{1-\rho_3}|\|Q_{\rho_1,\rho_2}(\delta)\xi_0\|, \\ I_3 &= \left\|(\delta+\sigma)^{1-\rho_3}\int_\delta^{\delta+\sigma} R_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))ds\right\|, \\ I_4 &= \left\|\int_0^\delta [(\delta+\sigma)^{1-\rho_3} R_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s)) \right. \\ & \qquad \left. - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))]ds\right\|, \\ I_5 &= \left\|\int_0^\delta \left[\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s) - \delta^{1-\rho_3}R_{\rho_1}(\delta-s)\right]\mathfrak{H}(s,\xi(s))\, ds\right\|, \\ I_6 &= \left\|(\delta+\sigma)^{1-\rho_3}\int_\delta^{\delta+\sigma} R_{\rho_1}(\delta+\sigma-s)\overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) ds\right\|, \\ I_7 &= \left\|\int_0^\delta \left[(\delta+\sigma)^{1-\rho_3}R_{\rho_1}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\right] \right. \\ & \qquad \left. \times \overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) ds\right\|, \\ I_8 &= \left\|\int_0^\delta \left[\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s) - \delta^{1-\rho_3}R_{\rho_1}(\delta-s)\right]\overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) ds\right\|. \end{aligned}$$

By Lemma 2.18, it follows that

$$\lim_{\sigma\to 0} I_1 = \lim_{\sigma\to 0} (\delta+\sigma)^{1-\rho_3}\|Q_{\rho_1,\rho_2}(\delta+\sigma)\xi_0 - Q_{\rho_1,\rho_2}(\delta)\xi_0\| = 0,$$

and

$$\begin{aligned}\lim_{\sigma\to 0} I_2 ={} & \lim_{\sigma\to 0} |(\delta+\sigma)^{1-\rho_3} - \delta^{1-\rho_3}|\|Q_{\rho_1,\rho_2}(\delta)\xi_0\| \\ \leq{} & \frac{\widehat{\Phi}\delta^{\rho_3-1}}{\Gamma(\rho_3)}\|\xi_0\| \lim_{\sigma\to 0}|(\delta+\sigma)^{1-\rho_3} - \delta^{1-\rho_3}| = 0.\end{aligned}$$

From Lemma 2.18 and $(\mathcal{F}_2)$, we get

$$\begin{aligned}\lim_{\sigma\to 0} I_3 &= \lim_{\sigma\to 0}\left\|(\delta+\sigma)^{1-\rho_3}\int_{\delta}^{\delta+\sigma} R_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))ds\right\|\\ &\le \frac{\widehat{\Phi}\Omega(K_0)}{\Gamma(\rho_1)}\lim_{\sigma\to 0}(\delta+\sigma)^{1-\rho_3}\int_{\delta}^{\delta+\sigma}(\delta+\sigma-s)^{\rho_1-1}\hbar(s)\,ds = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{\sigma\to 0} I_4 \le \lim_{\sigma\to 0}&\bigg\|\int_0^{\delta}\Big[(\delta+\sigma)^{1-\rho_3}R_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))\\ &\quad - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))\Big]ds\bigg\|\\ = \lim_{\sigma\to 0}&\bigg\|\int_0^{\delta}\Big[(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))\\ &\quad - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))\Big]ds\bigg\|\\ \le \frac{\widehat{\Phi}\Omega(n_0)}{\Gamma(\rho_1)}&\lim_{\sigma\to 0}\int_0^{\delta}|(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s)^{\rho_1-1}-\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}|\hbar(s)\,ds.\end{aligned}$$

Since $\hbar\in L^{\flat}(\Xi,\mathbb{R}^+)$ and

$$\int_0^{\delta}\left[(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s)^{\rho_1-1}-\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\right]\hbar(s)\,ds$$

exists, we have $\lim_{\sigma\to 0} I_4 = 0$.

For I_5, note that

$$\begin{aligned}\lim_{\sigma\to 0} I_5 &= \lim_{\sigma\to 0}\bigg\|\int_0^{\delta}\big[\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\\ &\qquad\qquad - \delta^{1-\rho_3}R_{\rho_1}(\delta-s)\big]\mathfrak{H}(s,\xi(s))\,ds\bigg\|\\ &= \lim_{\sigma\to 0}\bigg\|\int_0^{\delta}[\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta+\sigma-s)\\ &\qquad\qquad - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}P_{\rho_1}(\delta-s)]\mathfrak{H}(s,\xi(s))\,ds\bigg\|.\end{aligned}$$

To find this limit, suppose $\varepsilon>0$ to be enough small. We have

$$\begin{aligned}&\lim_{\sigma\to 0} I_5\\ &\le \Omega(K_0)\delta^{1-\rho_3}\lim_{\sigma\to 0}\int_0^{\delta-\varepsilon}(\delta-s)^{\rho_1-1}\hbar(s)\sup_{s\in[0,\delta-\varepsilon]}\|P_{\rho_1}(\delta+\sigma-s)-P_{\rho_1}(\delta-s)\|\,ds\\ &\quad + \lim_{\sigma\to 0}\int_{\delta-\varepsilon}^{\delta}(\delta-s)^{\rho_1-1}\|P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s))-P_{\rho_1}(\delta-s)\mathfrak{H}(s,\xi(s))\,ds\|\end{aligned}$$

$$
\begin{aligned}
&\leq \Omega(K_0)\delta^{1-\rho_3}\lim_{\sigma\to 0}\int_0^{\delta-\varepsilon}(\delta-s)^{\rho_1-1}\hbar(s)\sup_{s\in[0,\delta-\varepsilon]}\|P_{\rho_1}(\delta+\sigma-s)-P_{\rho_1}(\delta-s)\|\,ds\\
&\quad+\frac{2M\Omega(K_0)}{\Gamma(\rho_1)}\lim_{\sigma\to 0}\int_{\delta-\varepsilon}^{\delta}\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\hbar(s)\,ds\\
&\leq \Omega(K_0)\delta^{1-\rho_3}\lim_{\sigma\to 0}\int_0^{\delta-\varepsilon}(\delta-s)^{\rho_1-1}\hbar(s)\sup_{s\in[0,\delta-\varepsilon]}\|P_{\rho_1}(\delta+\sigma-s)-P_{\rho_1}(\delta-s)\|\,ds\\
&\quad+\frac{2M\Omega(K_0)}{\Gamma(\rho_1)}\lim_{\sigma\to 0}\biggl[\int_0^{\delta}\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\hbar(s)\,ds\\
&\qquad\qquad\qquad\qquad-\int_0^{\delta-\varepsilon}(\delta-\varepsilon)^{1-\rho_3}(\delta-\varepsilon-s)^{\rho_1-1}\hbar(s)\,ds\biggr]\\
&\quad+\frac{2M\Omega(K_0)}{\Gamma(\rho_1)}\lim_{\sigma\to 0}\biggl[\int_0^{\delta-\varepsilon}(\delta-\varepsilon)^{1-\rho_3}(\delta-\varepsilon-s)^{\rho_1-1}\hbar(s)\,ds\\
&\qquad\qquad\qquad\qquad-\int_0^{\delta-\varepsilon}\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\hbar(s)\,ds\biggr].
\end{aligned}
$$

From Lemma 2.18, $\lim_{\sigma\to 0}\sup_{s\in[0,\delta-\varepsilon]}\|P_{\rho_1}(\delta+\sigma-s)-P_{\rho_1}(\delta-s)\|=0$, and since $\hbar\in L^{\flat}(\Xi,\mathbb{R}^+)$, we have $I_5\to 0$ as $\sigma\to 0$ and $\varepsilon\to 0$.

Next, from (3.7), we have

$$
\begin{aligned}
\lim_{\sigma\to 0}I_6&=\lim_{\sigma\to 0}(\delta+\sigma)^{1-\rho_3}\left\|\int_\delta^{\delta+\sigma}R_{\rho_1}(\delta+\sigma-s)\overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right)ds\right\|\\
&\leq\lim_{\sigma\to 0}\frac{\widehat{\Phi}(\delta+\sigma)^{1-\rho_3}\Phi}{\Gamma(\rho_1)}\int_\delta^{\delta+\sigma}(\delta+\sigma-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\|\,ds\\
&\leq\lim_{\sigma\to 0}\frac{\widehat{\Phi}(\delta+\sigma)^{1-\rho_3}\Phi}{\Gamma(\rho_1)}\|\varkappa_{\xi,\mathfrak{H}}\|_{L^{\flat}(\Xi,E)}\left(\int_\delta^{\delta+\sigma}(\delta+\sigma-s)^{(\rho_1-1)\flat/(\flat-1)}\,ds\right)^{(\flat-1)/\flat}\\
&\leq\Phi\left[\|\xi_1\|+\frac{\widehat{\Phi}(\Im-\zeta_\gamma)^{\rho_3-1}}{\Gamma(\rho_3)}\wp K+\frac{\widehat{\Phi}\Omega(K)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^{\flat}(\Xi,\mathbb{R}^+)}\right]\\
&\quad\times\lim_{\sigma\to 0}\frac{\widehat{\Phi}(\delta+\sigma)^{1-\rho_3}\Phi}{\Gamma(\rho_1)}\left(\int_0^{\delta-\varepsilon}(\delta+\sigma-s)^{(\rho_1-1)\flat/(\flat-1)}\,ds\right)^{(\flat-1)/\flat}=0.
\end{aligned}
$$

For I_7, it should be noticed that

$$
\begin{aligned}
&\|[(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s)^{\rho_1-1}-\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}]\mathfrak{H}(s,\xi(s))\|\\
&\quad\leq\Omega(K_0)[(\delta+\sigma)^{1-\rho_3}(\delta-s)^{\rho_1-1}+\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}]\hbar(s)\quad\text{for a.a. } s\in[0,\delta].
\end{aligned}
$$

Since $\hbar\in L^{\flat}(\Xi,\mathbb{R}^+)$ and

$$
\int_0^{\delta}[(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s)^{\rho_1-1}-\delta^{1-\rho_3}(\delta-s)^{\rho_1-1}]\hbar(s)ds
$$

exists, then from the Lebesgue dominated convergence theorem, we see that

$$\begin{aligned}
\lim_{\sigma\to 0} I_7 &\le \lim_{\sigma\to 0}\bigg\| \int_0^{\delta} \left[(\delta+\sigma)^{1-\rho_3} R_{\rho_1}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1} P_{\rho_1}(\delta+\sigma-s)\right] \\
&\quad \times \overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) ds \bigg\| \\
&= \lim_{\sigma\to 0}\int_0^{\delta} |(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}| \\
&\quad \times \|P_{\rho_1}(\delta+\sigma-s)\overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right)\| \, ds \\
&\le \frac{\Phi\widehat{\Phi}\Omega(K_0)}{\Gamma(\rho_1)} \lim_{\sigma\to 0}\int_0^{\delta} |(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}| \times \|\varkappa_{\xi,\mathfrak{H}}(s)\| \, ds \\
&\le \frac{\Phi\widehat{\Phi}\Omega(K_0)}{\Gamma(\rho_1)} \|\varkappa_{\xi,\mathfrak{H}}\|_{L^{\flat}(\Xi,E)} \\
&\quad \times \lim_{\sigma\to 0}\left(\int_0^{\delta} |(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}|^{\flat/(\flat-1)} ds\right)^{(\flat-1)/\flat} \\
&\le \frac{\Phi\widehat{\Phi}\Omega(K_0)}{\Gamma(\rho_1)}\left[\|\xi_1\| + \frac{\widehat{\Phi}(\Im-\zeta_{\gamma})^{\rho_3-1}}{\Gamma(\rho_3)}\wp K_0 + \frac{\widehat{\Phi}\Omega(K_0)}{\Gamma(\rho_1)}\zeta\|\hbar\|_{L^{\flat}(\Xi,\mathbb{R}^+)}\right] \\
&\quad \times \lim_{\sigma\to 0}\left(\int_0^{\delta} |(\delta+\sigma)^{1-\rho_3}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}|^{\flat/(\flat-1)} \, ds\right)^{(\flat-1)/\flat} = 0.
\end{aligned}$$

Next,

$$\begin{aligned}
\lim_{\sigma\to 0} I_8 &= \lim_{\sigma\to 0}\bigg\| \int_0^{\delta} \left[\delta^{1-\rho_3}(\delta-s)^{\rho_1-1} P_{\rho_1}(\delta+\sigma-s) - \delta^{1-\rho_3} R_{\rho_1}(\delta-s)\right] \\
&\qquad \times \overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) ds \bigg\| \\
&= \lim_{\sigma\to 0}\bigg\| \int_0^{\delta} \left[\delta^{1-\rho_3}(\delta-s)^{\rho_1-1} P_{\rho_1}(\delta+\sigma-s) - \delta^{1-\rho_3}(\delta-s)^{\rho_1-1} P_{\rho_1}(\delta-s)\right] \\
&\qquad \times \overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) ds \bigg\|.
\end{aligned}$$

To find this limit, let $\varepsilon > 0$ be enough small. We have

$$\begin{aligned}
\lim_{\sigma\to 0} I_8 &\le \Phi\Omega(K_0)\delta^{1-\rho_3}\lim_{\sigma\to 0}\int_0^{\delta-\varepsilon} (\delta-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\| \\
&\qquad \times \sup_{s\in[0,\delta-\varepsilon]} \|P_{\rho_1}(\delta+\sigma-s) - P_{\rho_1}(\delta-s)\| \, ds \\
&\quad + \lim_{\sigma\to 0}\int_{\delta-\varepsilon}^{\delta} \delta^{1-\rho_3}(\delta-s)^{\rho_1-1} \\
&\qquad \times \|P_{\rho_1}(\delta+\sigma-s)\overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right) - P_{\rho_1}(\delta-s)\overline{\mathfrak{Z}}\left(\varkappa_{\xi,\mathfrak{H}}(s)\right)\| \, ds \\
&\le \Phi\Omega(K_0)\delta^{1-\rho_3}\lim_{\sigma\to 0}\int_0^{\delta-\varepsilon} (\delta-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\|
\end{aligned}$$

$$\begin{aligned}
&\times \sup_{s\in[0,\delta-\varepsilon]} \|P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s)) - P_{\rho_1}(\delta-s)\|\, ds \\
&+ \frac{2M\Phi\Omega(K_0)}{\Gamma(\rho_1)} \lim_{\sigma\to 0}\int_{\delta-\varepsilon}^{\delta} \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\|\, ds \\
\leq\ & \Phi\Omega(K_0)\delta^{1-\rho_3}\lim_{\sigma\to 0}\int_0^{\delta-\varepsilon} (\delta-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\| \\
&\times \sup_{s\in[0,\delta-\varepsilon]} \|P_{\rho_1}(\delta+\sigma-s)\mathfrak{H}(s,\xi(s)) - P_{\rho_1}(\delta-s)\| ds \\
&+ \frac{2\Phi\widehat{\Phi}\Omega(K_0)}{\Gamma(\rho_1)} \lim_{\sigma\to 0}\bigg[\int_0^{\delta} \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\| ds \\
&- \int_0^{\delta-\varepsilon} (\delta-\varepsilon)^{1-\rho_3}(\delta-\varepsilon-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\|\, ds\bigg] \\
&+ \frac{2M\Phi\Omega(K_0)}{\Gamma(\rho_1)} \lim_{\sigma\to 0}\bigg[\int_0^{\delta-\varepsilon} (\delta-\varepsilon)^{1-\rho_3}(\delta-\varepsilon-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\|\, ds \\
&- \int_0^{\delta-\varepsilon} \delta^{1-\rho_3}(\delta-s)^{\rho_1-1}\|\varkappa_{\xi,\mathfrak{H}}(s)\|\, ds\bigg].
\end{aligned}$$

From Lemma 2.18, $\lim_{\sigma\to 0}\sup_{s\in[0,\delta-\varepsilon]}\|P_{\rho_1}(\delta+\sigma-s)-P_{\rho_1}(\delta-s)\| = 0$, and since $\varkappa_{\xi,\mathfrak{H}} \in L^{\flat}(\Xi, E)$, then $I_8 \to 0$ as $\sigma\to 0$ and $\varepsilon \to 0$.

▶ **Case 2:** Let $\overline{\xi} \in K_{|\Xi'_\jmath}$, $\jmath = 1,2,\ldots,\gamma$. Then $\overline{\xi}(\delta) = \Psi_\jmath(\delta,\xi(\delta_\jmath^-))$, $\delta \in (\delta_\jmath,\zeta_\jmath]$, $\jmath = 1,2,\ldots,\gamma$. Let $\jmath \in \{1,2,\ldots,\gamma\}$ be fixed and $\delta, \delta+\sigma \in (\delta_\jmath,\zeta_\jmath]$. Since $\|\xi\|_{PC_{1-\rho_3}(\Xi,\nabla)} \leq K_0$, it follows from the uniform continuity of $\Psi_\jmath$ on bounded sets that

$$\lim_{\sigma\to 0}\|\overline{\xi}(\delta+\sigma)-\overline{\xi}(\delta)\| = \lim_{\sigma\to 0}\|\Psi_\jmath(\delta+\sigma,\xi(\delta_\jmath^-)) - \Psi_\jmath(\delta,\xi(\delta_\jmath^-))\| = 0,$$

independent of ξ.

For $\delta = \delta_\jmath$, $\jmath = 1,2,\ldots,\gamma$, let $\sigma > 0$ be such that $\delta_\jmath+\sigma \in (\delta_\jmath,\zeta_\jmath]$ and $\lambda > 0$ such that $\delta_\jmath < \lambda < \delta_\jmath+\sigma \leq \zeta_\jmath$. Then we have

$$\|\overline{\xi}^*(\delta_\jmath+\sigma)-\overline{\xi}^*(\delta_\jmath)\| = \lim_{\lambda\to\delta_\jmath^+}\|\overline{\xi}(\delta_\jmath+\sigma)-\overline{\xi}(\lambda)\| = 0.$$

▶ **Case 3:** Let $q \in K_{|\Xi_\jmath}$, $\jmath = 1,2,\ldots,\gamma$. Then there is $\xi \in \Pi_{K_0}$ and $\mathfrak{H} \in S^{\flat}_{\aleph(\cdot,(\cdot))}$ such that for $\delta \in (\zeta_\jmath,\delta_{\jmath+1}]$,

$$\begin{aligned}
q(\delta) = (\delta-\zeta_\jmath)^{1-\rho_3}\bigg[&Q_{\rho_1,\rho_2}(\delta-\zeta_\jmath)\Psi_\jmath(\zeta_\jmath,\xi(\delta_\jmath^-)) \\
&+ \int_{\zeta_\jmath}^{\delta} R_{\rho_1}(\delta-s)\left(\mathfrak{H}_K(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds\bigg].
\end{aligned}$$

Let $\jmath \in \{1,2,\ldots,\gamma\}$ be fixed. If $\delta = \zeta_\jmath$ and $\sigma > 0$, then

$$\lim_{\sigma\to 0+} q(\zeta_\jmath+\sigma) = \lim_{\sigma\to 0+}(\zeta_\jmath+\sigma-\zeta_\jmath)^{1-\rho_3}\overline{\xi}(\zeta_\jmath+\sigma) = \lim_{\delta\to\zeta_\jmath+}(\delta-\zeta_\jmath)^{1-\rho_3}\overline{\xi}(\delta) = q(\zeta_\jmath).$$

Next, let $\delta, \delta+\sigma \in (\zeta_\jmath, \delta_{\jmath+1}], \sigma > 0$. Then we have

$$\begin{aligned}\|q(\delta+\sigma) - q(\delta)\| = {} & \|(\delta+\sigma-\zeta_\jmath)^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta+\sigma-\zeta_\jmath)\Psi_\jmath(\zeta_\jmath, \xi(\delta_\jmath^-)) \\ & - (\delta-\zeta_\jmath)^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta-\zeta_\jmath)\Psi_\jmath(\zeta_\jmath, \xi(\delta_\jmath^-))\| \\ & + \|(\delta+\sigma-\zeta_\jmath)^{1-\rho_3} \int_{\zeta_\jmath}^{\delta} R_{\rho_1}(\delta+\sigma-s)(\mathfrak{H}_K(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s)))\, ds \\ & - (\delta-\zeta_\jmath)^{1-\rho_3} \int_{\zeta_\jmath}^{\delta+\sigma} R_{\rho_1}(\delta-s)(\mathfrak{H}_K(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s)))ds\|.\end{aligned}$$

According to the deduction as in case 1, we conclude that $\lim_{\sigma\to 0}\|q(\delta+\sigma) - q(\delta)\| = 0$.

Step 3: The graph of the multivalued function $S_{|\Pi_{K_0}} : \Pi_{K_0} \to 2^{\Pi_{K_0}}$ is closed. Consider $\{\xi_K\}_{K\geqslant 1}$ in Π_{K_0} with $\xi_K \to \xi$ in Π_{K_0} and let $\overline{\xi}_K \in S(\xi_K)$ with $\overline{\xi}_K \to \overline{\xi}$ in $PC_{1-\rho_3}(\Xi, \nabla)$. We need to show $\overline{\xi} \in S(\xi)$. Recalling the definition of S, for any $K \geqslant 1$, there is a $\mathfrak{H}_K \in S^{\flat}_{\aleph(\cdot,\xi(\cdot))}$ such that (3.5) holds.

It is seen that $\|\mathfrak{H}_K(\delta)\| \leq \hbar(\delta)\Omega(K_0)$ for every $K \geqslant 1$ and for a.a. $\delta \in \Xi$. Then $\{\mathfrak{H}_K, K \geqslant 1\}$ is bounded in $L^\flat(\Xi, \nabla)$. Because $\flat > 1, L^\flat(\Xi,\nabla)$ is reflexive, and hence, without loss of generality, we can assume that $\{\mathfrak{H}_K\}$ converges weakly to a function $\mathfrak{H} \in L^\flat(\Xi, \nabla)$. From Mazur's lemma, for every natural number $\imath$, there is a natural number $k_0(\imath) > \imath$ and a sequence of nonnegative real numbers $\lambda_{\imath,\jmath}$, $\jmath = k_0(\imath), \ldots, \imath$, such that $\sum_{\jmath=\imath}^{k_0} \lambda_{\imath,\jmath} = 1$, and the sequence of convex combinations $q_\imath = \sum_{\jmath=\imath}^{k_0} \lambda_{\imath,\jmath}\mathfrak{H}_\jmath, \imath \geqslant 1$, converges strongly to $\mathfrak{H} \in L^1(\Xi, \nabla)$ as $\imath \to \infty$.

Take $\overline{\overline{\xi}}_K(\delta) = \sum_{\jmath=\tau}^{k_0(K)} \lambda_{K,\jmath}\overline{\xi}_\jmath(\delta)$. Then

$$\overline{\overline{\xi}}_K(\delta) = \begin{cases} Q_{\rho_1,\rho_2}(\delta)\xi_0 + \displaystyle\int_0^\delta R_{\rho_1}(\delta-s)\left(q_K(s) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,q_K}(s))\right) ds, & \delta \in (0, \delta_1], \\ \Psi_\jmath(\delta, \xi(\delta_\jmath^-)), \qquad \delta \in \Xi_\jmath', \jmath = 1, 2, \ldots, \gamma, \\ Q_{\rho_1,\rho_2}(\delta-\zeta_\jmath)\Psi_\jmath(\zeta_\jmath, \xi_K(\delta_\jmath^-)) \\ \quad + \displaystyle\int_{\zeta_\jmath}^\delta R_{\rho_1}(\delta-s)\left(q_K(s) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,q_K}(s))\right) ds, & \delta \in \Xi_\jmath,\ \jmath = 1, 2, \ldots, \gamma, \end{cases}$$

Thus, $\lim_{K\to\infty}\varkappa_{\xi_K,q_K}(\delta) = \varkappa_{\xi,\mathfrak{H}}(\delta)$ for a.a. $\delta \in \Xi$.

By the continuity of $\overline{\mathfrak{Z}}$ and by the uniform continuity of $\Psi_\jmath$ on bounded sets, it follows from the Lebesgue dominated convergence theorem that $\overline{\overline{\xi}}_K(\delta) \to \upsilon(\delta)$, where

$$\upsilon(\delta) = \begin{cases} Q_{\rho_1,\rho_2}(\delta)\xi_0 + \displaystyle\int_0^\delta R_{\rho_1}(\delta-s)\left(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds, & \delta \in (0, \delta_1] \\ \Psi_\jmath(\delta, \xi(\delta_\jmath^-)), \qquad \delta \in \Xi_\jmath', \jmath = 1, 2, \ldots, \gamma, \\ Q_{\rho_1,\rho_2}(\delta-\zeta_\jmath)\Psi_\jmath(\zeta_\jmath, \xi(\delta_\jmath^-)) \\ \quad + \displaystyle\int_{\zeta_\jmath}^\delta R_{\rho_1}(\delta-s)\left(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\right) ds, & \delta \in \Xi_\jmath,\ \jmath = 1, 2, \ldots, \gamma, \end{cases}$$

Since $\overline{\xi}_K \to \overline{\xi}$, then $\overline{\xi} = \upsilon$. For almost everywhere $\delta, \aleph(\delta, \cdot)$ is upper semicontinuous with closed convex values, so from [11], it follows that $\mathfrak{H}(\delta) \in \aleph(\delta, \xi(\delta))$ for a.a. $\delta \in \Xi$, and hence S is closed.

Step 4: We show that (2.2) holds. Let $Y \subseteq \Pi_{K_0}, Y = \mathrm{conv}(\{0\} \cup S(Y)), \overline{Y} = \overline{C}$ with $C \subseteq Y$ countable. We claim that Y is relatively compact in $PC_{1-\rho_3}(\Xi, \nabla)$. Since C is countable and $C \subseteq Y = \mathrm{conv}(\{0\} \cup S(Y))$, we can find a countable set $\mathcal{F} = \{\overline{\xi}_K, \tau \geqslant 1\} \subseteq S(Y)$ with $C \subseteq \mathrm{conv}(\{0\} \cup \mathcal{F})$.

Now, for any $K \geqslant 1$, there exists $\xi_K \in Y \subseteq \Pi_{K_0}$ with $\overline{\xi}_K \in S(\xi_K)$. Thus there is a $\mathfrak{H}_K \in S^{\flat}_{\aleph(\cdot, \xi_K(\cdot))}$ such that (3.5) holds. According to the definition of $\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)$, we obtain

$$\begin{aligned}
\eta PC_{1-\rho_3}(\Xi, \nabla)(Y) &= \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(\overline{Y}) = \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(\overline{C}) = \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(C) \\
&\leq \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(\mathrm{conv}(\{\xi_0\} \cup \mathcal{F})) = \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(\mathcal{F}) \\
&= \max\left\{ \max_{\jmath=0,1,\ldots,\gamma} \eta_{C(\overline{\Xi_\jmath},\nabla)}\left(\mathcal{F}_{|\overline{\Xi_\jmath}}\right), \max_{\jmath=1,2,\ldots,\gamma} \eta_{C(\overline{\Xi'_\jmath},\nabla)}\left(\mathcal{F}_{|\overline{\Xi'_\jmath}}\right) \right\}.
\end{aligned}$$

Since $Y_{|\overline{\Xi_\jmath}}$ and $Y_{|\overline{\Xi'_\jmath}}$ are equicontinuous, then the last inequality becomes

$$\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y) \leq \max\left\{ \max_{\substack{\jmath=0,1,\ldots,\gamma \\ \delta\in\overline{\Xi_\jmath}}} \eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\}, \max_{\substack{\jmath=1,2,\ldots,\gamma \\ \delta\in\overline{\Xi'_\jmath}}} \eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \right\}, \tag{3.11}$$

where

$$\overline{\xi}^*_K(\delta) = \begin{cases}
\delta^{1-\rho_3}\overline{\xi}(\delta), & \delta \in (0, \delta_1], \\
\lim\limits_{\delta\to 0} \delta^{1-\rho_3}\overline{\xi}(\delta), & \delta = 0, \\
\Psi_\jmath(\delta, \xi_K(\delta_\jmath^-)), & \delta \in (\delta_\jmath, \zeta_\jmath], \jmath = 1, 2, \ldots, \gamma, \\
\overline{\xi}_\tau(\delta_\jmath^+), & \delta = \delta_\jmath, \\
(\delta - \zeta_\jmath)^{1-\rho_3}\overline{\xi}(\delta), & \delta \in (\zeta_\jmath, \delta_{\jmath+1}], \jmath = 1, 2, \ldots, \gamma, \\
\lim\limits_{\delta\to\zeta_\jmath} (\delta - \zeta_\jmath)^{\rho_3-1}\overline{\xi}(\delta), & \delta = \zeta_\jmath, \jmath = 1, 2, \ldots, \gamma.
\end{cases}$$

That is,

$$\begin{aligned}
\overline{\xi}^*_K(\delta) &= \delta^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta)\xi_0 \\
&\qquad + \delta^{1-\rho_3}\int_0^\delta R_{\rho_1}(\delta - s)\left(\mathfrak{H}_K(s, \xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s))\right) ds, \quad \delta \in (0, \delta_1], \\
\overline{\xi}^*_K(\delta) &= \lim_{\delta\to 0} \delta^{1-\rho_3}\overline{\xi}(\delta), \quad \delta = 0, \\
\overline{\xi}^*_K(\delta) &= \Psi_\jmath(\delta, \xi_K(\delta_\jmath^-)), \quad \delta \in (\delta_\jmath, \zeta_\jmath],\ \jmath = 1, 2, \ldots, \gamma, \\
\overline{\xi}^*_K(\delta) &= \Psi_\jmath(\delta, \xi_K(\delta_\jmath^-)), \quad \delta = \delta_\jmath, \\
\overline{\xi}^*_K(\delta) &= (\delta - \zeta_\jmath)^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta - \zeta_\jmath)\Psi_\jmath(\delta, \xi_K(\delta_\jmath^-))
\end{aligned}$$

$$+ (\delta - \zeta_\jmath)^{1-\rho_3} \left[\int_{\zeta_\jmath}^{\delta} R_{\rho_1}(\delta - s) \big(\mathfrak{H}_K(s, \xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s))\big)\, ds \right],$$

$$\delta \in (\zeta_\jmath,\ \delta_{\jmath+1}],\ \jmath = 1, 2, \dots, \gamma,$$

$$\overline{\xi}^*_K(\delta) = \lim_{\delta \to \zeta_\jmath^+} (\delta - \zeta_\jmath)^{\rho_3 - 1} \overline{\xi}(\delta), \quad \delta = \zeta_\jmath,\ \jmath = 1, 2, \dots, \gamma.$$

Then, using the properties of the measure of noncompactness, we have

$$\begin{aligned}
\eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \le{}& \eta\{\delta^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta)\xi_0, \tau \geqslant 1\} \\
&+ \delta^{1-\rho_3} \eta\Big\{ \int_0^{\delta} (\delta - s)^{\rho_1 - 1} P_{\rho_1}(\delta - s) \mathfrak{H}_K(s, \xi(s))\, ds, \tau \geqslant 1 \Big\} \\
&+ \delta^{1-\rho_3} \eta\Big\{ \int_0^{\delta} (\delta - s)^{\rho_1 - 1} P_{\rho_1}(\delta - s) \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s))\, ds, K \geqslant 1 \Big\}, \\
&\hspace{20em} \delta \in (0, \delta_1],
\end{aligned}$$

$$\eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \le \eta\Big\{ \lim_{\delta \to 0+} \delta^{1-\rho_3} \overline{\xi}_K(\delta), K \geqslant 1 \Big\}, \quad \delta = 0,$$

$$\eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \le \eta\big\{ \Psi_\jmath(\delta, \xi_K(\delta_\jmath^-)), K \geqslant 1 \big\}, \quad \delta \in (\delta_\jmath, \zeta_\jmath],\ \jmath = 1, 2, \dots, \gamma,$$

$$\eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \le \eta\big\{ \Psi_\jmath(\delta, \xi_K(\delta_\jmath^-)), K \geqslant 1 \big\}, \quad \delta = \delta_\jmath,\ \jmath = 1, 2, \dots, \gamma,$$

$$\begin{aligned}
\eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \le{}& \eta\big\{ (\delta - \zeta_\jmath)^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta - \zeta_\jmath) \Psi_\jmath(\delta, \xi_K(\delta_\jmath^-)), K \geqslant 1 \big\} \\
&+ (\delta - \zeta_\jmath)^{1-\rho_3} \eta\Big\{ \int_{\zeta_\gamma}^{\delta} (\delta - s)^{\rho_1 - 1} \mathfrak{H}_K(s, \xi(s)) ds, K \geqslant 1 \Big\} \\
&+ (\delta - \zeta_\jmath)^{1-\rho_3} \eta\Big\{ \int_{\zeta_\gamma}^{\delta} (\delta - s)^{\rho_1 - 1} \overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s))\, ds, K \geqslant 1 \Big\}, \\
&\hspace{12em} \delta \in (\zeta_\jmath, \delta_{\jmath+1}],\ \jmath = 1, 2, \dots, \gamma,
\end{aligned}$$

$$\eta\{\overline{\xi}^*_K(\delta), K \geqslant 1\} \le \eta\Big\{ \lim_{\delta \to \zeta_\jmath} (\delta - \zeta_\jmath)^{1-\rho_3} \overline{\xi}_K(\delta), K \geqslant 1 \Big\}, \quad \delta = \zeta_\jmath.$$

From the continuity of Q_{ρ_1,ρ_2}, it follows that $\eta\{\delta^{1-\rho_3} Q_{\rho_1,\rho_2}(\delta) \times \xi_0, K \geqslant 1\} = 0$.

Then, if $\delta \in \Xi_0$, using (2.8), we get that

$$\begin{aligned}
\eta\{\overline{\xi}^*_K, K \geqslant 1\} \le{}& \frac{2\delta^{1-\rho_3}}{\Gamma(\rho_1)} \int_0^{\delta} (\delta - s)^{\rho_1 - 1} \eta\{\mathfrak{H}_K(s, \xi(s)), K \geqslant 1\}\, ds \\
&+ \frac{2\delta^{1-\rho_3}}{\Gamma(\rho_1)} \int_0^{\delta} (\delta - s)^{\rho_1 - 1} \eta\{\overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s)), K \geqslant 1\}\, ds.
\end{aligned}$$

Observe that from $(\mathcal{F}_3)$ we obtain

$$\eta\{\mathfrak{H}_K(s, \xi(s)), K \geqslant 1\} \le \kappa(s) s^{1-\rho_3} \eta\{\xi_\jmath(s), \jmath \geqslant 1\} \le \kappa(s)_{\eta_{PC_{1-\rho_3}(\Xi,\nabla)}}(Y)$$

for a.a. $s \in \Xi_0$. We have

$$\int_0^{\delta} (\delta - s)^{\rho_1 - 1} \eta\{\mathfrak{H}_K(s, \xi(s)), K \geqslant 1\}\, ds$$

$$\leq \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)\int_0^\delta (\delta-s)^{\rho_1-1}\kappa(s)\,ds \leq \eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)\zeta\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}. \quad (3.12)$$

To estimate

$$\eta\Big\{\int_0^\delta (\delta-s)^{\rho_1-1}\overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s)), K \geqslant 1\Big\},$$

we consider $\Theta : L^\flat(\overline{\Xi_0}, E) \to C(\Xi,\nabla)$:

$$\Theta(g)(\delta) = \int_0^\delta (\delta-s)^{\rho_1-1}\overline{\mathfrak{Z}}(g(s))ds,$$

where $g \in L^\flat(\overline{\Xi_0}, E)$. Now Θ is linear, and for any $\wp_1, \wp_2 \in L^\flat(\overline{\Xi_0}, E)$ and any $\delta \in \Xi$, we have

$$\begin{aligned}
\|\Theta(\wp_1)(\delta) - \Theta(\wp_2)(\delta)\| &\leq \int_0^\delta (\delta-s)^{\rho_1-1}\|\overline{\mathfrak{Z}}(\wp_1(s)) - \overline{\mathfrak{Z}}(\wp_2(s))\|\,ds \\
&\leq \|\overline{\mathfrak{Z}}\|\int_0^\delta (\delta-s)^{\rho_1-1}\|\wp_1(s) - \wp_2(s)\|\,ds \\
&\leq \Phi\zeta\|\wp_1 - \wp_2\|_{L^\flat(\Xi,E)}.
\end{aligned}$$

Then Θ is linear and continuous (bounded). Moreover, from the linearity and boundedness of Z^{-1}, the compactness of $\Psi_\jmath$, the continuity of $Q_{\rho_1,\rho_2}(\Im - \zeta_\gamma)$, (3.4) and (3.12), we have

$$\begin{aligned}
\eta_{L^\flat(\Xi,E)}\{\varkappa_{\xi_K,\mathfrak{H}_K}, K \geqslant 1\} &\leq \eta_{L^\flat(\Xi,E)}\Big(Z^{-1}\Big\{\xi_1 - Q_{\rho_1,\rho_2}(\Im-\zeta_\gamma)\Psi_\jmath(\zeta_\jmath, \xi(\delta_\jmath^-)) \\
&\qquad - \frac{1}{\Gamma(\rho_1)}\int_{\zeta_\jmath}^{\Im} R_{\rho_1}(\Im-s)^{\rho_1-1}\mathfrak{H}_K(s,\xi(s))\,ds, K \geqslant 1\Big\}\Big) \\
&\leq \Phi\Big[\eta\Big\{\xi_1 - Q_{\rho_1,\rho_2}(\Im-\zeta_\gamma)\Psi_\jmath(\zeta_\jmath, \xi(\delta_\jmath^-)) \\
&\qquad - \frac{1}{\Gamma(\rho_1)}\int_{\zeta_\jmath}^{\Im} R_{\rho_1}(\Im-s)^{\rho_1-1}\mathfrak{H}_K(s,\xi(s))\,ds, K \geqslant 1\Big\}\Big] \\
&= \frac{\Phi}{\Gamma(\rho_1)}\eta\Big\{\int_{\zeta_\jmath}^{\Im} R_{\rho_1}(\Im-s)^{\rho_1-1}\mathfrak{H}_K(s,\xi(s))ds, K \geqslant 1\Big\} \\
&\leq \frac{2M\Phi}{\Gamma(\rho_1)}\int_0^\delta (\delta-s)^{\rho_1-1}\eta\{\mathfrak{H}_K(s,\xi(s)), K \geqslant 1\}\,ds \\
&\leq \frac{2\zeta\widehat{\Phi}\Phi}{\Gamma(\rho_1)}\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\eta\Big\{\int_0^\delta (\delta-s)^{\rho_1-1}\overline{\mathfrak{Z}}(\varkappa_{\xi_K,\mathfrak{H}_K}(s)), K \geqslant 1\Big\} &= \eta\{\Theta(\varkappa_{\xi_K,\mathfrak{H}_\tau}), K \geqslant 1\} \\
\leq \|\Theta\|\eta_{L^\flat(\Xi,E)}\{\varkappa_{\xi_K,\mathfrak{H}_K}, K \geqslant 1\} &\leq \frac{2\zeta\widehat{\Phi}\Phi^2}{\Gamma(\rho_1)}\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}.
\end{aligned}$$

This inequality with (3.12) gives us

$$\begin{aligned}\max_{\delta\in(0,\delta]}&\eta\{\overline{\xi}^*_K(\delta),K\geqslant 1\}\\&\le\eta_{PC_{1-\rho_3}(\Xi,\nabla)}\Im^{1-\rho_3}\widehat{\Phi}\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}\left[\frac{2\zeta}{\Gamma(\rho_1)}+\frac{2\zeta\Phi^2}{\Gamma(\rho_1)^2}\right].\end{aligned}\tag{3.13}$$

Next, we obtain

$$\begin{aligned}\eta\{\overline{\xi}^*_K(0),K\geqslant 1\}&=\eta\left\{\lim_{\delta\to 0^+}\delta^{1-\rho_3}\overline{\xi}_\tau(\delta),K\geqslant 1\right\}\\&=\eta\left\{\frac{1}{\Gamma(\rho_3)}(\xi_0),K\geqslant 1\right\}=0.\end{aligned}\tag{3.14}$$

Moreover, since $\xi_K(\delta_\jmath^-)\to\xi(\delta_\jmath^-)$, the set $\{\xi_K(\delta_\jmath^-),K\geqslant 1\}$ is bounded for every $\jmath=1,2,\dots,\gamma$. Then from the compactness of $\Psi_\jmath$, for $\jmath=1,2,\dots,\gamma$, we get

$$\eta\{\Psi_\jmath(\delta,\xi_K(\delta_\jmath^-)),K\geqslant 1\}=0,\quad \delta\in(\delta_\jmath,\zeta_\jmath]\tag{3.15}$$

and $\eta\{\Psi_\jmath(\delta,\xi_K(\delta_\jmath^-)),K\geqslant 1\}=0$. Then for $\jmath=1,2,\dots,\gamma$, we obtain

$$\begin{aligned}\eta\{\overline{\xi}^*_K(\zeta_\jmath),K\geqslant 1\}&=\eta\left\{\lim_{\delta\to\zeta_\jmath^+}(\delta-\zeta_\jmath)^{\rho_3-1}\overline{\xi}(\delta),K\geqslant 1\right\}\\&=\eta\left\{\frac{\widehat{\Phi}}{\Gamma(\rho_3)}\Psi_\jmath(\zeta_\jmath,\xi_K(\delta_\jmath^-),K\geqslant 1\right\}=0.\end{aligned}\tag{3.16}$$

As above, $\eta\{(\delta-\zeta_\jmath)^{1-\rho_3}Q_{\rho_1,\rho_2}(\delta-\zeta_\jmath)\Psi_\jmath(\zeta_\jmath,\xi_K(\delta_\jmath^-),K\geqslant 1\}=0,\jmath=1,2,\dots,\gamma$.

Then, arguing as above, we see that for any $\imath=1,2,\dots,\gamma$, we have

$$\max_{\delta\in\Xi_\imath]}\eta\{\overline{\xi}^*_K(\delta),K\geqslant 1\}\le\eta_{PC_{1-\rho_3}(\Xi,\nabla)}\Im^{1-\rho_3}\widehat{\Phi}\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}\left[\frac{2\zeta}{\Gamma(\rho_1)}+\frac{2\zeta\Phi^2}{\Gamma(\rho_1)^2}\right].\tag{3.17}$$

From (3.1), (3.11), (3.13)–(3.17), we get

$$\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)\le\eta_{PC_{1-\rho_3}(\Xi,\nabla)}\Im^{1-\rho_3}\widehat{\Phi}\|\kappa\|_{L^\flat(\Xi,\mathbb{R}^+)}\left[\frac{2\zeta}{\Gamma(\rho_1)}+\frac{2\zeta\Phi^2}{\Gamma(\rho_1)^2}\right]<\eta_{PC}(Y).$$

Thus $\eta_{PC_{1-\rho_3}(\Xi,\nabla)}(Y)=0$, and hence Y is relatively compact.

Step 5: The mapping S ensures that compact sets are mapped into relatively compact sets.

Let $\mathfrak{Z}$ be a compact subset of Π_{K_0}. Consider a sequence $(\overline{\xi}_K),\tau\geqslant 1$, in $S(\mathfrak{Z})$. Then there exists a sequence $(\xi_K),K\geqslant 1$, in $\mathfrak{Z}$ such that $\overline{\xi}_K\in S(\xi_K)$. Consequently, there exists $\mathfrak{H}_K\in S^\flat_{\aleph(\cdot,\xi_K(\cdot))}$ for $\delta\in\Xi$, satisfying (3.5). We demonstrate that the set $Y=\{\overline{\xi}_K,K\geqslant 1\}$ is relatively compact in $PC_{1-\rho_3}(\Xi,\nabla)$. Since $\mathfrak{Z}$ is compact in $PC_{1-\rho_3}(\Xi,\nabla)$, we can assume, without loss of generality, that $\xi_K\to\xi$ in $\mathfrak{Z}$. As in Step 3, a subsequence of $(\overline{\xi}_K)$ converges to a function $\upsilon\in S(\mathfrak{Z})$.

Thus, the set $\{\overline{\xi}_K, K \geqslant 1\}$ is relatively compact in $PC_{1-\rho_3}(\Xi,\nabla)$. Consequently, $S(\overline{\mathfrak{Z}})$ is relatively compact.

Now, by applying Lemma 2.20, there is a $\xi \in PC_{1-\rho_3}(\Xi,\nabla)$ and $\mathfrak{H} \in S^{\flat}_{\aleph(\cdot,\xi_K(\cdot))}$ such that

$$\xi(\delta) = \begin{cases} Q_{\rho_1,\rho_2}(\delta)\xi_0 + \displaystyle\int_0^\delta R_{\rho_1}(\delta - s)\big(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}(\varkappa_{\xi,\mathfrak{H}}(s))\big)\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad \delta \in [0,\delta_1], \\ \Psi_\jmath(\delta,\xi(\delta_\jmath^-)), \delta \in (\delta_\jmath,\zeta_\jmath],\ \jmath = 1,2,\dots,\gamma, \\ Q_{\rho_1,\rho_2}(\delta)\Psi_\jmath(\zeta_\jmath,\xi(\delta_\jmath^-)) + \displaystyle\int_{\zeta_\jmath}^\delta R_{\rho_1}(\delta - s)\big(\mathfrak{H}(s,\xi(s)) + \overline{\mathfrak{Z}}\varkappa(s)\big)\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad \delta \in (\zeta_\jmath,\delta_{\jmath+1}],\ \jmath = 1,2,\dots,\gamma. \end{cases}$$

This completes the proof. □

4. The example

In this section, we give an example illustrating our results. We consider the following problem:

$$\begin{cases} D^{1/2,\rho_2}_{\zeta_\jmath^+}\overline{\xi}(\delta,\Im) \in \dfrac{\partial^2\overline{\xi}}{\partial\Im^2}(\delta,\Im) + P(\delta,\overline{\xi}(\delta,\Im)) + \overline{\mathfrak{Z}}(\varkappa(\delta)), \\ \qquad\qquad\qquad\qquad\qquad \delta \in (\zeta_\jmath,\delta_{\jmath+1}],\ \jmath = 0,1, \\ \overline{\xi}(\delta,\Im) = \Psi_1(\delta,\overline{\xi}(\delta_1^-),\Im)), \quad \delta \in (\delta_1,\zeta_1],\ \Im \in \Xi, \\ I^{1-\rho_3}_{0^+}\overline{\xi}(0,\Im) = \overline{\xi}_0, \end{cases} \tag{4.1}$$

where $\varkappa \in L^2(\Xi, L^2(\Xi))$, $\rho_1 = 1/2$, $0 \le \rho_2 \le 1$, $\rho_3 = \rho_1 + \rho_2 - \rho_1\rho_2$, $\Xi = [0,T]$, and $\nabla = E = L^2[0,T]$; ∇ is a separable Hilbert space. Set $\zeta_0 = 0$, $\delta_1 = 1/4$, $\zeta_1 = 1/2$, and $\delta_2 = \Im = 1$. For any function $\overline{\xi} : \Xi \to L^2(\Xi)$ and any $\delta \in [0,T]$, we let $\xi(\delta)(\Im) := \overline{\xi}(\delta,\Im), y \in \Xi$.

Let $\aleph : \Xi \times \nabla \to P_{ck}(\nabla)$ be such that $y \in \aleph(\delta,\overline{\xi}) \iff y(\Im) \in P(\delta,\overline{\xi}(\delta,\Im))$, where $P : \Xi \times \mathbb{R} \to P_{ck}(\mathbb{R})$ is chosen such that $(\mathcal{F}_1)$, $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$ are satisfied. Define $\Psi_1 : [\delta_1,\zeta_1] \times \nabla \to \nabla$ as $\Psi_1(\delta,\xi) = \delta^{1-\rho_3}L(\xi)$, where $L : D(L) = \nabla \to \nabla$ is a compact linear bounded operator. Thus, we can easily show that $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$, $(\mathcal{H}_{\Psi_\jmath})$ and $(\mathcal{H}_{\Psi_1})$ hold. Let $\overline{\mathfrak{Z}} : \nabla \to \nabla, \overline{\mathfrak{Z}} = \rho_3 I_d$, where I_d is the identity operator and $\rho_3 > 0$.

Define the operator $\mathfrak{Z} : D(\mathfrak{Z}) \subseteq L^2[0,1] \to L^2[0,1]$ by

$$\mathfrak{Z}\overline{\xi} = \frac{\partial^2\overline{\xi}}{\partial\xi^2},$$

where the domain $\mathfrak{Z}$ is given by

$$D(\mathfrak{Z}) = \{\overline{\xi} \in L^2[0,1] : \overline{\xi}, \overline{\xi}_\xi \text{ are a.c. } \overline{\xi}_{xx} \in L^2[0,1],\ \ \overline{\xi}(\delta,0) = \overline{\xi}(\delta,1) = 0\}.$$

Then $\mathfrak{Z}$ can be written as

$$\mathfrak{Z}\xi = \sum_{\tau=1}^{\infty} \tau^2 \left\langle \overline{\xi}, \overline{\xi}_\tau \right\rangle, \quad \overline{\xi}_\tau, \overline{\xi} \in D(\mathfrak{Z}),$$

where $\overline{\xi}_\tau(y) = \sqrt{2}\sin ny, \tau = 1, 2, \ldots$, is the orthonormal basis of ∇.

For any $\overline{\xi} \in L^2[0,1]$, we have

$$T(\delta)(\overline{\xi}) = \sum_{\tau=1}^{\infty} e^{-\tau^2\delta} \left\langle \overline{\xi}, \overline{\xi}_\tau \right\rangle \overline{\xi}_\tau.$$

Here, $\mathfrak{Z}$ is the infinitesimal generator of the strongly continuous semigroup $\{T(\delta), \delta \geqslant 0\}$.

Moreover, the operator $P_{1/2}(\cdot)$ can be written as

$$P_{1/2}(\delta) = (1/2)\int_0^\infty \mu\varepsilon_{3/4}(\mu) T(\delta^{1/2}\mu)\, d\mu.$$

We define $Z : L^2(\Xi, L^2(\Xi)) \to L^2(\Xi)$ by

$$Z(\varkappa) := \int_{1/2}^1 (1-s)^{-1/2} T(1-s)\varkappa(s)\, ds.$$

Z is linear and bounded. Now we show that Z is surjective.

Let $\overline{\xi} \in L^2(\Xi)$. Consider the Mittag–Leffler function as follows:

$$E_{1/2}(-\tau^2/\sqrt{2}) = \int_0^\infty \mathbb{k}_{1/2}(\mu) e^{(-\tau^2/\sqrt{2})\mu} d\mu, \quad \tau \in \mathbb{N}.$$

Notice that for any natural number τ and any $\mu > 0$, we have $\mu/\sqrt{2} < \mu\tau^2/\sqrt{2}$, and hence, $e^{(-\tau^2/\sqrt{2})\mu} \le e^{-\mu/\sqrt{2}} < 1$. Thus,

$$E_{1/2}\left(-\frac{\tau^2}{\sqrt{2}}\right) \le E_{1/2}\left(-\frac{1}{\sqrt{2}}\right) < \int_0^\infty \mathbb{k}_{1/2}(\mu) d\mu = 1.$$

Then

$$0 < 1 - E_{1/2}\left(-\frac{1}{\sqrt{2}}\right) \le 1 - E_{1/2}\left(-\frac{\tau^2}{\sqrt{2}}\right) < 1.$$

Define a function $\varkappa : \Xi \to L^2(\Xi)$ by

$$\varkappa(\delta) = \sum_{\tau=1}^{\infty} \frac{\tau^2}{\rho_3} \frac{\left\langle \overline{\xi}, \overline{\xi}_\tau \right\rangle \overline{\xi}_\tau}{1 - E_{1/2}\left(-\tau^2/\sqrt{2}\right)}, \quad \delta \in \Xi. \tag{4.2}$$

Then

$$Z(\varkappa) = \rho_3 \int_1^{1/2} (1-s)^{-1/2} P_{1/2}(1-s)\varkappa(s)\, ds$$

$$
\begin{aligned}
&= \int_1^{1/2} (1-s)^{-1/2} P_{1/2}(1-s) \sum_{\tau=1}^{\infty} \tau^2 \frac{\langle \bar{\xi}, \bar{\xi}_\tau \rangle \bar{\xi}_\tau}{1 - E_{1/2}\left(-\tau^2/\sqrt{2}\right)}\, ds \\
&= \int_1^{1/2} (1-s)^{-1/2} \Bigg(\frac{1}{2} \int_0^{\infty} \mu \Bbbk_{1/2}(\mu) T\left((1-s)^{1/2}\mu\right) \\
&\quad \times \sum_{\tau=1}^{\infty} \tau^2 \frac{\langle \bar{\xi}, \bar{\xi}_\tau \rangle \bar{\xi}_\tau}{1 - E_{1/2}\left(-\tau^2/\sqrt{2}\right)}\, d\mu \Bigg) ds \\
&= \int_1^{1/2} (1-s)^{-1/2} \Bigg(\frac{1}{2} \int_0^{\infty} \mu \Bbbk_{1/2}(\mu) \\
&\quad \times \sum_{\gamma=1}^{\infty} e^{-\mu\gamma^2(1-s)^{1/2}} \left\langle \sum_{\tau=1}^{\infty} \tau^2 \frac{\langle \bar{\xi}, \bar{\xi}_\tau \rangle \bar{\xi}_\tau}{1 - E_{1/2}\left(-\tau^2/\sqrt{2}\right)}, \bar{\xi}_\gamma \right\rangle \bar{\xi}_\gamma \, d\mu \Bigg) ds \\
&= \int_0^{\infty} \Bbbk_{1/2}(\mu) \sum_{\gamma=1}^{\infty} \frac{1}{1 - E_{1/2}\left(-\gamma^2/\sqrt{2}\right)} \\
&\quad \times \left(\int_{1/2}^{1} \frac{\gamma^2 \mu}{2} e^{-\mu\gamma^2(1-s)^{1/2}} (1-s)^{1/2} ds \right) \langle \bar{\xi}, \bar{\xi}_\gamma \rangle \bar{\xi}_\gamma \, d\mu \\
&= \int_0^{\infty} \Bbbk_{1/2}(\mu) \sum_{\gamma=1}^{\infty} \frac{1}{1 - E_{1/2}\left(-\gamma^2/\sqrt{2}\right)} \left[1 - e^{-\gamma^2\mu/\sqrt{2}}\right] d\mu \, \langle \bar{\xi}, \bar{\xi}_\gamma \rangle \bar{\xi}_\gamma \\
&= \sum_{\gamma=1}^{\infty} \frac{1}{1 - E_{1/2}\left(-\gamma^2/\sqrt{2}\right)} \int_0^{\infty} \Bbbk_{1/2}(\mu) \left[1 - e^{-\gamma^2\mu/\sqrt{2}}\right] d\mu \, \langle \bar{\xi}, \bar{\xi}_\gamma \rangle \bar{\xi}_\gamma \\
&= \sum_{\gamma=1}^{\infty} \frac{1}{1 - E_{1/2}\left(-\gamma^2/\sqrt{2}\right)} \\
&\quad \times \left[\int_0^{\infty} \Bbbk_{1/2}(\mu) d\mu - \int_0^{\infty} \Bbbk_{1/2}(\mu) e^{-\gamma^2\mu/\sqrt{2}}\, d\mu \right] \langle \bar{\xi}, \bar{\xi}_\gamma \rangle \bar{\xi}_\gamma \\
&= \sum_{\gamma=1}^{\infty} \frac{1}{1 - E_{1/2}\left(-\gamma^2/\sqrt{2}\right)} \left[1 - \int_0^{\infty} \Bbbk_{1/2}(\mu) e^{-\gamma^2\mu/\sqrt{2}}\, d\mu \right] \langle \bar{\xi}, \bar{\xi}_\gamma \rangle \bar{\xi}_\gamma \\
&= \sum_{\gamma=1}^{\infty} \langle \bar{\xi}, \bar{\xi}_\gamma \rangle \bar{\xi}_\gamma = \xi.
\end{aligned}
$$

From the above computations Z is surjective, where $Z^{-1}\bar{\xi} = \varkappa$ and $\varkappa$ is given by (4.2). Note that Z^{-1} is linear, and for $\bar{\xi} \in D(\mathfrak{Z})$,

$$
\|\bar{\xi}\| = \|\mathfrak{Z}(\bar{\xi})\| := \sqrt{\sum_{\tau=1}^{\infty} \tau^4 \langle \bar{\xi}, \bar{\xi}_\tau \rangle^2}.
$$

Then

$$
\|Z^{-1}(\bar{\xi})(\delta)\| = \sqrt{\sum_{\tau=1}^{\infty} \frac{\tau^4}{{\rho_3}^2} \frac{\langle \bar{\xi}, \bar{\xi}_\tau \rangle^2}{[1 - E_{1/2}\left(-\tau^2/\sqrt{2}\right)]^2}}
$$

$$\leq \frac{1}{\rho_3\left[1 - E_{1/2}\left(-1/\sqrt{2}\right)\right]}\sqrt{\sum_{\tau=1}^{\infty}\tau^4\left\langle\bar{\xi},\bar{\xi}_\tau\right\rangle^2}$$
$$= \frac{\|\bar{\xi}\|}{\rho_3\left[1 - E_{1/2}\left(-1/\sqrt{2}\right)\right]}.$$

We observe that $Z^{-1}(\xi)$ is independent of $\delta \in [0,1]$.

Consequently, we obtain

$$\|Z^{-1}\| \leq \frac{1}{\rho_3\left[1 - E_{1/2}\left(-1/\sqrt{2}\right)\right]}.$$

Then the system (4.1) is controllable.

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Керованість дробових немиттєвих імпульсних диференціальних включень Хілфера в банахових просторах

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У цій роботі ми досліджуємо існування та керованість дробових диференціальних включень Хілфера з немиттєвими імпульсами в банахових просторах. Аналіз проведено з використанням різноманітних математичних інструментів та версії теореми Мьонча про нерухому точку для багатозначних функцій, яка спирається на кілька властивостей міри некомпактності Куратовського. Для демонстрації застосовності наших результатів ми завершуємо дослідження детальним прикладом.

Ключові слова: міра некомпактності, теорія нерухомої точки, дробове числення