

# Cheng–Yau Logarithmic Gradient Estimates for a Nonlinear Elliptic Equation on Smooth Metric Measure Spaces

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In this paper, we consider the nonlinear elliptic equation

$$\Delta_f v^\tau + \lambda v = 0$$

on a complete smooth metric measure space with the  $m$ -Bakry–Émery Ricci curvature bounded from below, where  $\tau > 0$  and  $\lambda$  are constants. We obtain some new local and global universal log-gradient estimates for positive solutions to the equation using the Nash–Moser iteration technique. As applications of these estimates, we obtain a Liouville type theorem, a Harnack inequality and the global gradient estimates for such solutions. Our results generalize and improve the estimates established by Wang (J. Differential Equations **260** (2016), 567–585) and Zhao (Arch. Math. (Basel) **114** (2020), 457–469).

*Key words:* Smooth metric measure space, gradient estimate, Liouville theorem, Harnack inequality

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## 1. Introduction

In this paper, we are concerned with the nonlinear elliptic equation

$$\Delta_f v^\tau + \lambda v = 0 \tag{1.1}$$

on a smooth metric measure space  $(M^n, g, e^{-f} d\nu)$  whose  $m$ -Bakry–Émery Ricci tensor

$$\text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{1}{m-n} df \otimes df, \quad m > n, \tag{1.2}$$

satisfies  $\text{Ric}_f^m \geq -(m-1)Kg$  for some constant  $K \geq 0$ , where  $\tau > 0$  and  $\lambda$  are constant, and

$$\Delta_f(\cdot) = \Delta(\cdot) - \langle \nabla f, \nabla(\cdot) \rangle$$

is the weighted Laplacian for some smooth potential function  $f$ . Here,  $d\nu$ ,  $\text{Ric}$ ,  $\Delta$ ,  $\nabla^2$  and  $\nabla$  are the volume form, Ricci tensor, the usual Laplacian, Hessian and gradient operators associated with the metric  $g$ , respectively.

The goal of the present paper is to establish the Cheng–Yau type log-gradient estimates for positive solutions to the above equation, more precisely, some universal log-gradient estimates which do not depend on the weight functions and  $L^\infty$ -bounds of positive solutions.

Obviously, when  $f$  is a constant,  $\rho = v^\tau$  and  $p = \frac{1}{\tau}$ , equation (1.1) becomes

$$\Delta\rho + \lambda\rho^p = 0. \tag{1.3}$$

Equation (1.1) appears also in differential geometry. Indeed, for any  $n$ -dimensional ( $n \geq 3$ ) complete manifold  $(M, g)$ , consider a pointwise conformal metric  $\tilde{g} = \rho^{\frac{4}{n-2}}g$  for some smooth positive function  $\rho$ . Then the scalar curvature  $\tilde{R}$  of metric  $\tilde{g}$  related to the scalar curvature  $R$  of metric  $g$  is given by

$$\Delta\rho - \frac{n-2}{4(n-1)}R\rho + \frac{n-2}{4(n-1)}\tilde{R}\rho^{\frac{n+2}{n-2}} = 0. \tag{1.4}$$

When  $R = 0$  and  $\tilde{R}$  is a constant, (1.4) reduces to (1.3) with  $\lambda = \frac{n-2}{4(n-1)}\tilde{R}$  and  $\tau = \frac{n-2}{n+2}$ . If  $M$  is compact and  $\tilde{R}$  is constant, the existence of a positive solution  $\rho$  is the well-known Yamabe problem. For more details, we refer to [1].

Another important reason of studying equation(1.1) is that its parabolic counterpart is related to the porous medium equation and the fast diffusion equation. It is a nonlinear heat equation in the form

$$\partial_t v = \Delta v^\tau. \tag{1.5}$$

The above equation (1.5) with  $\tau > 1$  is called the porous medium equation, which describes the flow of an isentropic gas through a porous medium [2]; and it with  $0 < \tau < 1$  is called the fast diffusion equation, which appears in various geometric flows such as the Ricci flow on surfaces [3] and the Yamabe flow [4]. There is a lot of literature on the study of gradient estimates of the porous medium equation and the fast diffusion equation. We only cite [5–13] here and one can find more references therein.

On the other hand, inspired by the works of Yau [14] and Li and Wang [15], many gradient estimates for equation (1.1) on Riemannian manifolds were obtained. In [16], Wang established gradient estimates for positive solutions to equation (1.1) on any smooth metric measure space whose  $m$ -Bakry–Émery Ricci tensor is bounded from below by  $-(m-1)K$  with  $K \geq 0$ . We state his main result as below.

**Theorem 1.1** ([16]). *Let  $(M^n, g, e^{-f}dv)$  be a smooth metric measure space with  $\text{Ric}_f^n \geq -(m-1)Kg$  for some  $K \geq 0$  and  $m > 2$ . Suppose that  $v$  is a smooth positive solution to equation (1.1) and  $u = \frac{\tau}{\tau-1}v^{\tau-1}$ .*

(i) *If  $\tau > 1$ , then*

$$\lambda \leq C_1 \left( m, \tau, K, \sup_M u \right), \quad \frac{|\nabla u|^2}{u} \leq C_2 \left( m, \tau, \lambda, K, \sup_M u \right), \tag{1.6}$$

*where  $C_1 > 0$  and  $C_2 \geq 0$  are explicit.*

(ii) *If  $1 - \frac{2}{m} < \tau < 1$ , then*

$$\lambda \leq \bar{C}_1 \left( m, \tau, K, \sup_M u \right), \quad \frac{|\nabla u|^2}{u} \leq -\bar{C}_2 \left( m, \tau, \lambda, K, \sup_M u \right), \tag{1.7}$$

*where  $\bar{C}_1 > 0$  and  $\bar{C}_2 \geq 0$  are explicit and  $u < 0$ .*

As a corollary, the author of [16] also derived a uniform bound for positive solutions to equation (1.1).

Recently, Zhao [17] studied gradient estimates for positive solutions of the nonlinear elliptic equation

$$\Delta_V v^\tau + \lambda v = 0 \tag{1.8}$$

on a Riemannian manifold  $(M, g)$  with the  $k$ -Bakry–Émery Ricci curvature bounded from below, where  $\Delta_V$  denotes a weighted Laplacian associated to a certain vector field  $V$ . We state his main result below.

**Theorem 1.2** ([17]). *Let  $(M^n, g)$  be a complete Riemannian manifold. Suppose  $\text{Ric}_V^k \geq -Kg$  for some  $K \geq 0$  and  $k > 2$ . Let  $u$  be a positive solution to equation (1.8). For any  $\sigma > \frac{1}{(k-1)^2}$ , if  $1 \leq \tau < 1 + \frac{1}{(1+\sigma)(k-1)}$ , then*

$$\sup_{B_{R/2}(o)} \frac{|\nabla u|}{u} \leq C_3 \left( \sigma, k, \tau, K, \lambda, R, \left( \inf_{B_{R/2}(o)} u \right)^{1-\tau} \right). \tag{1.9}$$

The method employed in [16, 17] is the maximum principle. The gradient estimates in Theorem 1.1 and Theorem 1.2 depend on the bound of the solution. Moreover, the gradient estimates are not of Cheng–Yau type.

In [18], Cheng and Yau proved the well-known Cheng–Yau type gradient estimate for positive harmonic functions. We state their main result below.

**Theorem 1.3** ([18]). *Let  $(M^n, g)$  be a complete Riemannian manifold. Suppose  $\text{Ric} \geq -Kg$  for some  $K \geq 0$ . Then for any positive harmonic function  $v$  in a geodesic ball  $B_{R/2}(o)$ , there is a constant  $C_4(n)$  depending only on  $n$  such that*

$$\sup_{B_{R/2}(o)} \frac{|\nabla v|}{v} \leq C_4(n) \frac{1 + \sqrt{KR}}{R}. \tag{1.10}$$

We note that Cheng–Yau type gradient estimates depend only on  $n$ ,  $K$  and  $R$ , and they do not depend on any other geometric quantities. Moreover, Cheng–Yau type gradient estimates are more significant and useful in geometric analysis since they can derive a strong Liouville theorem and Harnack inequalities. There have been extensive studies on Cheng–Yau type gradient estimates over the recent decades; see, for example, [19–25] and references therein.

Inspired by these works, we employ the Nash–Moser iteration technique to prove a Cheng–Yau type gradient estimate for positive solutions to equation(1.1) on any smooth metric measure space with the  $m$ -Bakry–Émery Ricci tensor bounded from below. Our first main result is stated as follows.

**Theorem 1.4.** *Let  $(M^n, g, e^{-f} dv)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$  for some  $K \geq 0$  and  $m > 2$ . Suppose that  $\tau > 0$  and  $v$  is a smooth positive solution to equation (1.1) on a geodesic ball  $B_R(o) \subset M$ . If  $m$ ,  $\lambda$  and  $\tau$  satisfy*

$$\lambda \left( \frac{m+1}{m-1} - \frac{1}{\tau} \right) \geq 0, \tag{1.11}$$

or

$$1 - \frac{2 + \sqrt{2}}{m + 1 + \sqrt{2}} \leq \tau \leq 1 - \frac{2 - \sqrt{2}}{m + 1 - \sqrt{2}}, \quad (1.12)$$

then there holds

$$\sup_{B_{R/2}(o)} \frac{|\nabla v|}{v} \leq c(m, \tau, \lambda) \frac{1 + \sqrt{KR}}{R}, \quad (1.13)$$

where the constant  $c(m, \tau, \lambda)$  depends only on  $m$ ,  $\tau$  and  $\lambda$ .

After careful analysis of the conditions (1.11) and (1.12) in Theorem 1.4, it is seen that the following results hold.

**Corollary 1.5.** *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$  for some  $K \geq 0$ . Suppose that  $\tau > 0$  and  $v$  is a smooth positive solution to equation (1.1) on a geodesic ball  $B_R(o) \subset M$ . If  $m$ ,  $\lambda$  and  $\tau$  satisfy*

$$\lambda \geq 0, \quad \tau \geq 1 - \frac{2}{m+1}, \quad (1.14)$$

or

$$\lambda \leq 0, \quad 0 < \tau \leq 1 - \frac{2 - \sqrt{2}}{m + 1 - \sqrt{2}}, \quad (1.15)$$

then we have

$$\sup_{B_{R/2}(o)} \frac{|\nabla v|}{v} \leq c(m, \lambda, \tau) \frac{1 + \sqrt{KR}}{R}.$$

**Remark 1.6.** Here we want to give several remarks of the above Cheng-Yau type gradient estimates.

- (i) Compared with the previous work (cf. Theorem 1.1 and Theorem 1.2), Theorem 1.4 and Corollary 1.5 extend  $\tau$  to a larger range.
- (ii) In Theorem 1.4 and Corollary 1.5, the gradient estimates on a positive solution  $v$  do not involve the bound of  $v$ . Meanwhile, the gradient estimates obtained in Theorem 1.1 and Theorem 1.2 concern the bound of  $v$ . Hence we improve the previous results in Theorem 1.1 and Theorem 1.2.
- (iii) Theorem 1.4 and Corollary 1.5 provide a unified expression of previous gradient estimates in Theorem 1.1 for various ranges of  $\tau$ .

As an application of Theorem 1.4, we get the following Liouville type theorem.

**Corollary 1.7.** *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq 0$ . Suppose that  $\tau > 0$  and  $v$  is a smooth positive solution to equation (1.1) on a geodesic ball  $B_R(o) \subset M$ . If  $m$ ,  $\lambda$  and  $\tau$  satisfy*

$$\lambda \geq 0, \quad \tau \geq 1 - \frac{2}{m+1},$$

or

$$\lambda \leq 0, \quad 0 < \tau \leq 1 - \frac{2 - \sqrt{2}}{m + 1 - \sqrt{2}},$$

then  $v$  is a positive constant on  $M$ .

**Remark 1.8.**

- (i) We extend the range of the corresponding power  $\tau$  for the same problem as in [16]. In addition, comparing with the Liouville type result derived in [16], we do not need the bound of  $v$ .
- (ii) The gradient estimate in [17] is not of Cheng–Yau type, which cannot derive the Liouville property.

Another consequence of Theorem 1.4 is the following Harnack inequality.

**Corollary 1.9.** *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m - 1)Kg$  for some  $K \geq 0$ . Suppose that  $\tau > 0$  and  $v$  is a smooth positive solution to equation (1.1), defined on a geodesic ball  $B_R(o) \subset M$ , with constants  $m, \lambda, \tau$  satisfying (1.14) or (1.15). Then, for any  $x, y \in B_{R/2}(o)$ , there holds*

$$v(x) \leq e^{c(\tau, m, \lambda)(1 + \sqrt{KR})} v(y).$$

Moreover, using these local gradient estimates, we can obtain the global gradient estimates for positive solutions to equation (1.1).

**Theorem 1.10.** *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m - 1)Kg$ , where  $K$  is a nonnegative constant. Let  $v$  be a global positive solution of (1.1) in  $M$ .*

- (i) *If  $m, \lambda$  and  $\tau$  satisfy (1.11), then*

$$\frac{|\nabla v|}{v} \leq \frac{(m - 1)\sqrt{K}}{\tau}.$$

- (ii) *If  $m, \lambda$  and  $\tau$  satisfy (1.12), then*

$$\frac{|\nabla v|}{v} \leq \frac{\sqrt{K}}{\tau \sqrt{\frac{1}{2} \left( \frac{\sqrt{2-m-1}}{m-1} + \frac{1}{\tau} \right) \left( \frac{\sqrt{2+m+1}}{m-1} - \frac{1}{\tau} \right)}}.$$

The paper is organized as follows. In Section 2, we give a meticulous estimate of  $\Delta_f(|\nabla u|^{2\alpha})$  at  $u = -\ln v$ . We also recall the L. Saloff-Coste’s Sobolev embedding theorem in Section 2. In Section 3, we establish a universal integral estimate on  $|\nabla \ln v|^{2\alpha}$  and then use delicately the Nash–Moser iteration to prove main results of this paper. The proofs of Corollaries 1.7 and 1.9 are given in Section 4. The last section is devoted to the proof of Theorem 1.10.

**2. Preliminaries**

In this section we need some preliminary results to prove the gradient estimate in (1.13) of Theorem 1.4.

We observe that equation (1.1) is rewritten as follows:

$$\Delta_f v + (\tau - 1)v^{-1}|\nabla v|^2 + \tau^{-1}\lambda v^{2-\tau} = 0. \tag{2.1}$$

Now, let  $u = -\ln v$ . Then, with respect to (2.1),  $u$  satisfies the following nonlinear elliptic equation:

$$\Delta_f u - \tau |\nabla u|^2 - \tau^{-1} \lambda e^{(\tau-1)u} = 0. \tag{2.2}$$

Set  $h = |\nabla u|^2$ . Then we have

$$\Delta_f u - \tau h - \tau^{-1} \lambda e^{(\tau-1)u} = 0. \tag{2.3}$$

Firstly, we need to establish the following lemma.

**Lemma 2.1.** *Let  $(M^n, g, e^{-f} dv)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$ , where  $K$  is a nonnegative constant and  $m > 2$ . Assume that  $v$  is a smooth positive solution to equation (1.1). If  $m, \lambda, \tau \in W_1 \cup W_2$ , where*

$$W_1 = \left\{ (m, \lambda, \tau) : \lambda \left( \frac{m+1}{m-1} - \frac{1}{\tau} \right) \geq 0 \right\};$$

$$W_2 = \left\{ (m, \lambda, \tau) : 1 - \frac{2 + \sqrt{2}}{m+1 + \sqrt{2}} \leq \tau \leq 1 - \frac{2 - \sqrt{2}}{m+1 - \sqrt{2}} \right\},$$

then there exists a constant  $\alpha_0 > 1$  and some  $\beta_0 > 0$  such that the inequality

$$\Delta_f h^{\alpha_0} \geq 2\alpha_0 \beta_0 h^{\alpha_0+1} - 2\alpha_0(m-1)Kh^{\alpha_0} - \alpha_0 a_1 |\nabla h| h^{\alpha_0 - \frac{1}{2}} \tag{2.4}$$

holds point-wisely in  $\{x : h(x) > 0\}$ , where  $a_1 = (2 - \frac{2}{m-1})\tau$  and  $\alpha_0, \beta_0$  depend only on  $m, \lambda$  and  $\tau$ . The  $\alpha_0$  and  $\beta_0$  can be defined differently in different  $W_1$  and  $W_2$ .

*Proof.* For any  $\alpha > 1$ , a direct computation gives

$$\Delta_f h^\alpha = \alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + \alpha h^{\alpha-1} \Delta_f h.$$

Using the Bochner formula

$$\frac{1}{2} \Delta_f h = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f(\nabla u, \nabla u),$$

we obtain

$$\Delta_f h^\alpha = \alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + 2\alpha h^{\alpha-1} (|\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f(\nabla u, \nabla u)). \tag{2.5}$$

Let  $\{e_1, e_2, \dots, e_n\}$  be an orthonormal frame of  $TM$  on a domain with  $h \neq 0$  such that  $e_1 = \frac{\nabla u}{|\nabla u|}$ . If we denote  $\nabla u = \sum_{i=1}^n u_i e_i$ , it is easy to see that  $u_1 = |\nabla u| = h^{1/2}$  and  $u_i = 0$  for any  $2 \leq i \leq n$ . Then the following identities hold (see (3.1) and (3.3) in [19]):

$$u_{11} = \frac{1}{2} h^{-1} \langle \nabla u, \nabla h \rangle, \tag{2.6}$$

$$\sum_{i=1}^n u_{1i}^2 = \frac{1}{4} h^{-1} |\nabla h|^2, \tag{2.7}$$

$$\sum_{i=2}^n u_{ii} = \tau h + \tau^{-1} \lambda e^{(\tau-1)u} - u_{11} + \langle \nabla f, \nabla u \rangle. \tag{2.8}$$

Using the Cauchy inequality, we get

$$|\nabla^2 u|^2 \geq \sum_{i=1}^n u_{1i}^2 + \sum_{i=2}^n u_{ii}^2 \geq u_{11}^2 + \frac{1}{n-1} \left( \sum_{i=2}^n u_{ii} \right)^2. \tag{2.9}$$

Then, using the inequality (see (5) in [26]):

$$(a + b)^2 \geq \frac{a^2}{1 + \delta} - \frac{b^2}{\delta}, \quad \delta = \frac{m - n}{n - 1},$$

we can derive from (2.8) that

$$\begin{aligned} \frac{1}{n-1} \left( \sum_{i=2}^n u_{ii} \right)^2 &= \frac{1}{n-1} \left( \tau h + \tau^{-1} \lambda e^{(\tau-1)u} - u_{11} + \langle \nabla f, \nabla u \rangle \right)^2 \\ &\geq \frac{1}{n-1} \left( \frac{n-1}{m-1} \left( \tau h + \tau^{-1} \lambda e^{(\tau-1)u} - u_{11} \right)^2 \right. \\ &\quad \left. - \frac{n-1}{m-n} \langle \nabla f, \nabla u \rangle^2 \right) \\ &= \frac{1}{m-1} \left( \tau h + \tau^{-1} \lambda e^{(\tau-1)u} - u_{11} \right)^2 - \frac{1}{m-n} \langle \nabla f, \nabla u \rangle^2 \\ &= \frac{1}{m-1} \left( \tau^2 h^2 + \tau^{-2} \lambda^2 e^{2(\tau-1)u} + u_{11}^2 + 2\lambda h e^{(\tau-1)u} \right. \\ &\quad \left. - 2\tau h u_{11} - 2\tau^{-1} \lambda e^{(\tau-1)u} u_{11} \right) - \frac{1}{m-n} \langle \nabla f, \nabla u \rangle^2. \end{aligned} \tag{2.10}$$

Then we obtain

$$\begin{aligned} |\nabla^2 u|^2 &\geq u_{11}^2 + \frac{1}{m-1} \left( \tau^2 h^2 + \tau^{-2} \lambda^2 e^{2(\tau-1)u} + u_{11}^2 + 2\lambda h e^{(\tau-1)u} \right. \\ &\quad \left. - 2\tau h u_{11} - 2\tau^{-1} \lambda e^{(\tau-1)u} u_{11} \right) - \frac{1}{m-n} \langle \nabla f, \nabla u \rangle^2. \end{aligned} \tag{2.11}$$

We can derive from (2.3) that

$$\begin{aligned} \langle \nabla u, \nabla \Delta_f u \rangle &= \langle \nabla u, \nabla (\tau h + \tau^{-1} \lambda e^{(\tau-1)u}) \rangle \\ &= \tau \langle \nabla u, \nabla h \rangle + \tau^{-1} (\tau - 1) \lambda e^{(\tau-1)u} h. \end{aligned} \tag{2.12}$$

Substituting (2.6), (2.7), (2.11), and (2.12) into (2.5), we obtain

$$\begin{aligned} \frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha &\geq \left( 2\alpha - 1 + \frac{1}{m-1} \right) u_{11}^2 + \left( 2 - \frac{2}{m-1} \right) \tau h u_{11} \\ &\quad + \text{Ric}_f^m(\nabla u, \nabla u) + \frac{1}{m-1} \tau^2 h^2 + \frac{1}{m-1} \tau^{-2} \lambda^2 e^{2(\tau-1)u} \\ &\quad + \lambda \left( \frac{m+1}{m-1} - \frac{1}{\tau} \right) h e^{(\tau-1)u} - \frac{2}{m-1} \tau^{-1} \lambda e^{(\tau-1)u} u_{11}. \end{aligned} \tag{2.13}$$

By applying the inequality  $a^2 + 2ab \geq -b^2$ , we have

$$\begin{aligned} \left(2\alpha - 1 + \frac{1}{m-1}\right) u_{11}^2 - \frac{2}{m-1} \tau^{-1} \lambda e^{(\tau-1)u} u_{11} \\ \geq -\frac{1}{((2\alpha - 1)(m-1) + 1)(m-1)} \tau^{-2} \lambda^2 e^{2(\tau-1)u}. \end{aligned} \tag{2.14}$$

By (2.14) and the assumption that  $\text{Ric}_f^m \geq -(m-1)Kg$ , we obtain

$$\begin{aligned} \frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq & \left(2 - \frac{2}{m-1}\right) \tau h u_{11} - (m-1)Kh + \frac{1}{m-1} \tau^2 h^2 \\ & + \frac{1}{m-1} \tau^{-2} \lambda^2 e^{2(\tau-1)u} + \lambda \left(\frac{m+1}{m-1} - \frac{1}{\tau}\right) h e^{(\tau-1)u} \\ & - \frac{1}{((2\alpha - 1)(m-1) + 1)(m-1)} \tau^{-2} \lambda^2 e^{2(\tau-1)u}. \end{aligned} \tag{2.15}$$

To estimate  $\Delta_f h^\alpha$ , we divide the arguments into two cases:

Case 1. If  $(m, \lambda, \tau) \in W_1$ , we have

$$\lambda \left(\frac{m+1}{m-1} - \frac{1}{\tau}\right) h e^{(\tau-1)u} \geq 0.$$

Furthermore, we observe that

$$\left(\frac{1}{m-1} - \frac{1}{((2\alpha - 1)(m-1) + 1)(m-1)}\right) \tau^{-2} \lambda^2 e^{2(\tau-1)u} \geq 0.$$

Thus we can always find  $\alpha = \alpha(m, \tau, \lambda)$  large enough such that

$$\frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq \frac{1}{m-1} \tau^2 h^2 + \left(2 - \frac{2}{m-1}\right) \tau h u_{11} - (m-1)Kh.$$

Case 2. If  $(m, \lambda, \tau) \in W_2$ , we have

$$\frac{1}{m-1} - \frac{m-1}{2} \left(\frac{m+1}{m-1} - \frac{1}{\tau}\right)^2 \geq 0. \tag{2.16}$$

Estimate (2.15) can be written as follows:

$$\begin{aligned} \frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq & \left(2 - \frac{2}{m-1}\right) \tau h u_{11} - (m-1)Kh + \frac{1}{m-1} \tau^2 h^2 \\ & + \left(\frac{1}{2(m-1)} \tau^{-2} \lambda^2 e^{2(\tau-1)u} + \lambda \left(\frac{m+1}{m-1} - \frac{1}{\tau}\right) h e^{(\tau-1)u}\right) \\ & + \left(\frac{1}{2(m-1)} - \frac{1}{((2\alpha - 1)(m-1) + 1)(m-1)}\right) \tau^{-2} \lambda^2 e^{2(\tau-1)u}. \end{aligned} \tag{2.17}$$

According to  $a^2 + 2ab \geq -b^2$ , we get

$$\frac{1}{2(m-1)} \tau^{-2} \lambda^2 e^{2(\tau-1)u} + \lambda \left(\frac{m+1}{m-1} - \frac{1}{\tau}\right) h e^{(\tau-1)u}$$

$$\geq -\frac{m-1}{2} \left( \frac{m+1}{m-1} - \frac{1}{\tau} \right)^2 \tau^2 h^2. \tag{2.18}$$

Meanwhile, we note that

$$\left( \frac{1}{2(m-1)} - \frac{1}{((2\alpha-1)(m-1)+1)(m-1)} \right) \tau^{-2} \lambda^2 e^{2(\tau-1)u} \geq 0. \tag{2.19}$$

Let

$$B_{m,\tau} = \left( \frac{1}{m-1} - \frac{m-1}{2} \left( \frac{m+1}{m-1} - \frac{1}{\tau} \right)^2 \right) \tau^2.$$

Using (2.16)–(2.19), we can always find  $\alpha = \alpha(m, \tau)$  large enough such that

$$\frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq B_{m,\tau} h^2 + \left( 2 - \frac{2}{m-1} \right) \tau h u_{11} - (m-1)Kh.$$

We denote

$$\frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq B_{m,\tau} h^2 + \left( 2 - \frac{2}{m-1} \right) \tau h u_{11} - (m-1)Kh, \tag{2.20}$$

and

$$\beta_0 := \beta_0(m, \tau) = \begin{cases} \frac{\tau^2}{m-1}, & (m, \lambda, \tau) \text{ lies in } W_1, \\ B_{m,\tau}, & (m, \lambda, \tau) \text{ lies in } W_2 \setminus W_1. \end{cases}$$

Thus we finish the proof. □

To prove Theorem 1.4, we also need L. Saloff-Coste’s Sobolev embedding theorem.

**Lemma 2.2** ([27–29]). *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$ , where  $K$  is a nonnegative constant. For  $m > 2$ , there exists some positive constant  $c(m)$  depending only on  $m$ , such that for all  $B \subset M$  of radius  $R$ , we have for  $\phi \in C_0^\infty(B)$ :*

$$\left( \int_B |\phi|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \leq e^{c(m)(1+\sqrt{KR})} V_f^{-\frac{2}{m}} R^2 \int_B (|\nabla \phi|^2 + R^{-2} \phi^2) e^{-f} d\nu,$$

where  $V_f = \int_B e^{-f} d\nu$ .

### 3. Proof of Theorem 1.4

In this section, we finish the proof of Theorem 1.4. We proceed with the following integral inequality on the solutions to equation (1.1).

**Lemma 3.1.** *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$ , where  $K$  is a nonnegative constant, and  $\Omega = B_R(o) \subset M$  be a geodesic ball. Suppose that  $v$  is a positive solution*

to equation (1.1) with the constants  $m > 2$ ,  $\lambda$  and  $\tau$  which satisfy the condition in Theorem 1.4 (or Corollary 1.5),  $u = -\ln v$  and  $h = |\nabla u|^2$ . Then there exist constants  $a_2$ ,  $a_3$  and  $a_4$  depending only on  $m$ ,  $\lambda$  and  $\tau$  such that for any  $t \geq t_0$ , where  $t_0$  is defined in (3.14), there holds

$$\begin{aligned} & \beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \frac{a_2}{t} e^{-t_0} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ & \leq a_4 t_0^2 R^{-2} \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{a_3}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu, \end{aligned} \quad (3.1)$$

where  $\eta \in C_0^\infty(\Omega, \mathbb{R})$ .

*Proof.* We set  $U = \{x \in \Omega | h(x) = 0\}$ . We note that  $v$  is smooth on  $\Omega \setminus U$ . For a nonnegative function  $\psi$  with compact support in  $\Omega \setminus U$ , we multiply both sides of equation (2.4) by the function  $\psi$  and then integrate by parts to obtain

$$\begin{aligned} \int_{\Omega} (\Delta_f h^{\alpha_0}) \psi e^{-f} d\nu &= - \int_{\Omega} \alpha_0 h^{\alpha_0-1} \langle \nabla h, \nabla \psi \rangle e^{-f} d\nu \\ &\geq 2\beta_0 \alpha_0 \int_{\Omega} h^{\alpha_0+1} \psi e^{-f} d\nu - 2(m-1)\alpha_0 K \int_{\Omega} h^{\alpha_0} \psi e^{-f} d\nu \\ &\quad - a_1 \alpha_0 \int_{\Omega} h^{\alpha_0-\frac{1}{2}} |\nabla h| \psi e^{-f} d\nu. \end{aligned} \quad (3.2)$$

For constants  $\epsilon > 0$  and  $t > 1$ , we choose

$$\psi = h_\epsilon^t \eta^2,$$

where  $h_\epsilon = (h - \epsilon)^+$ ,  $\eta \in C_0^\infty(\Omega, \mathbb{R})$  is nonnegative and less than or equal to 1, and  $t$  is to be determined later. Then a direct calculation shows that

$$\nabla \psi = t h_\epsilon^{t-1} \eta^2 \nabla h + 2 h_\epsilon^t \eta \nabla \eta.$$

Inserting this identity into (3.2), we get

$$\begin{aligned} & 2\beta_0 \int_{\Omega} h^{\alpha_0+1} h_\epsilon^t \eta^2 e^{-f} d\nu + \int_{\Omega} t h^{\alpha_0-1} h_\epsilon^{t-1} |\nabla h|^2 \eta^2 e^{-f} d\nu \\ & \leq 2(m-1)K \int_{\Omega} h^{\alpha_0} h_\epsilon^t \eta^2 e^{-f} d\nu - \int_{\Omega} 2h^{\alpha_0-1} h_\epsilon^t \langle \nabla h, \nabla \eta \rangle \eta e^{-f} d\nu \\ & \quad + a_1 \int_{\Omega} h^{\alpha_0-\frac{1}{2}} h_\epsilon^t |\nabla h| \eta^2 e^{-f} d\nu. \end{aligned} \quad (3.3)$$

Since

$$h_\epsilon^t \langle \nabla h, \nabla \eta \rangle \geq -h_\epsilon^t |\nabla h| |\nabla \eta|, \quad (3.4)$$

inserting (3.4) into (3.3) and letting  $\epsilon \rightarrow 0$ , we have

$$2\beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \int_{\Omega} t h^{\alpha_0+t-2} |\nabla h|^2 \eta^2 e^{-f} d\nu$$

$$\begin{aligned} &\leq 2(m-1)K \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + a_1 \int_{\Omega} h^{\alpha_0+t-\frac{1}{2}} |\nabla h| \eta^2 e^{-f} d\nu \\ &\quad + 2 \int_{\Omega} h^{\alpha_0+t-1} |\nabla h| |\nabla \eta| \eta e^{-f} d\nu. \end{aligned} \tag{3.5}$$

Applying the Young inequality, we get

$$\begin{aligned} a_1 h^{\alpha_0+t-\frac{1}{2}} |\nabla h| \eta^2 &\leq \frac{t}{4} h^{\alpha_0+t-2} |\nabla h|^2 \eta^2 + \frac{a_1^2}{t} h^{\alpha_0+t+1} \eta^2, \\ 2h^{\alpha_0+t-1} |\nabla h| |\nabla \eta| \eta &\leq \frac{t}{4} h^{\alpha_0+t-2} |\nabla h|^2 \eta^2 + \frac{4}{t} h^{\alpha_0+t} |\nabla \eta|^2. \end{aligned} \tag{3.6}$$

Now we choose  $t$  large enough such that

$$\frac{a_1^2}{t} \leq \beta_0. \tag{3.7}$$

It can be concluded from (3.5)–(3.7) that

$$\begin{aligned} \beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \frac{t}{2} \int_{\Omega} h^{\alpha_0+t-2} |\nabla h|^2 \eta^2 e^{-f} d\nu \\ \leq 2(m-1)K \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{4}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu. \end{aligned} \tag{3.8}$$

Using

$$\begin{aligned} \left| \nabla \left( h^{\frac{\alpha_0+t}{2}} \eta \right) \right|^2 &\leq 2 \left| \nabla h^{\frac{\alpha_0+t}{2}} \right|^2 \eta^2 + 2h^{\alpha_0+t} |\nabla \eta|^2 \\ &= \frac{(\alpha_0+t)^2}{2} h^{\alpha_0+t-2} |\nabla h|^2 \eta^2 + 2h^{\alpha_0+t} |\nabla \eta|^2, \end{aligned} \tag{3.9}$$

we have by (3.8)

$$\begin{aligned} \beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \frac{4t}{(2\alpha_0+2t)^2} \int_{\Omega} \left| \nabla \left( h^{\frac{\alpha_0+t}{2}} \eta \right) \right|^2 e^{-f} d\nu \\ \leq 2(m-1)K \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{4}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu \\ + \frac{8t}{(2\alpha_0+2t)^2} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu. \end{aligned} \tag{3.10}$$

In what follows,  $a_i (i = 1, 2, 3, \dots)$  stand for positive constants depending on  $m, \tau$  and  $\lambda$ , and  $t$  is chosen in such a way that

$$\frac{a_2}{t} \leq \frac{4t}{(2\alpha_0+2t)^2} \quad \text{and} \quad \frac{8t}{(2\alpha_0+2t)^2} + \frac{4}{t} \leq \frac{a_3}{t}. \tag{3.11}$$

It can be concluded from (3.10) and (3.11) that

$$\beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \frac{a_2}{t} \int_{\Omega} \left| \nabla \left( h^{\frac{\alpha_0+t}{2}} \eta \right) \right|^2 e^{-f} d\nu$$

$$\leq 2(m-1)K \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{a_3}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu. \quad (3.12)$$

Now letting  $\phi = h^{\frac{\alpha_0+t}{2}} \eta$  in Lemma 2.2, we have

$$\begin{aligned} e^{-c(m)(1+\sqrt{KR})} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ \leq \int_{\Omega} \left| \nabla \left( h^{\frac{\alpha_0+t}{2}} \eta \right) \right|^2 e^{-f} d\nu + R^{-2} \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu. \end{aligned}$$

Combining the above inequality with (3.12), we obtain

$$\begin{aligned} \beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \frac{a_2}{t} e^{-c(m)(1+\sqrt{KR})} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ \leq 2(m-1)K \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{a_3}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu \\ + \frac{a_2}{t} \int_{\Omega} R^{-2} h^{\alpha_0+t} \eta^2 e^{-f} d\nu. \end{aligned} \quad (3.13)$$

Now we let

$$t_0 = c_{m,\lambda,\tau} \left( 1 + \sqrt{KR} \right) \quad \text{and} \quad c_{m,\lambda,\tau} = \max \left\{ c(m) + 1, \frac{a_1^2}{\beta_0} \right\}. \quad (3.14)$$

Thus, we can choose  $t$  large enough such that for any  $t \geq t_0$ , the following estimates hold true:

$$2(m-1)KR^2 \leq \frac{2(m-1)}{c_{m,\lambda,\tau}^2} t_0^2 \quad \text{and} \quad \frac{a_2}{t} \leq \frac{a_2}{c_{m,\lambda,\tau}}.$$

Furthermore, there exists  $a_4 = a_4(m, \lambda, \tau) > 0$  such that

$$2(m-1)KR^2 + \frac{a_2}{t} \leq a_4 t_0^2 = a_4 c_{m,\lambda,\tau}^2 \left( 1 + \sqrt{KR} \right)^2. \quad (3.15)$$

Hence, it follows from (3.13) and (3.15) that

$$\begin{aligned} \beta_0 \int_{\Omega} h^{\alpha_0+t+1} \eta^2 e^{-f} d\nu + \frac{a_2}{t} e^{-t_0} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ \leq a_4 t_0^2 R^{-2} \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{a_3}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu. \end{aligned}$$

This is the required inequality and we finish the proof of this lemma.  $\square$

Using inequality (3.1) in Lemma 3.1, we will get a local estimate of  $h$  stated in the lemma below.

**Lemma 3.2.** *Let  $(M^n, g, e^{-f}d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$ , where  $K$  is a nonnegative constant, and  $\Omega = B_R(o) \subset M$  be a geodesic ball. Suppose that  $h$  is the same as in Lemma 3.1. If the constants  $m > 2$ ,  $\lambda$  and  $\tau$  satisfy the condition in Theorem 1.4 (or Corollary 1.5), then, for  $\beta = (\alpha_0 + t_0) \frac{m}{m-2}$ , there exists a nonnegative constant  $a_7 = a_7(m, \lambda, \tau) > 0$  such that*

$$\left( \int_{B_{3R/4}(o)} h^\beta e^{-f} d\nu \right)^{\frac{1}{\beta}} \leq a_7 V_f^{\frac{1}{\beta}} \frac{t_0^2}{R^2}. \tag{3.16}$$

*Proof.* Letting  $t = t_0$  in (3.1), we have

$$\begin{aligned} \beta_0 \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu + \frac{a_2}{t_0} e^{-t_0} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t_0}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ \leq a_4 t_0^2 R^{-2} \int_{\Omega} h^{\alpha_0+t_0} \eta^2 e^{-f} d\nu + \frac{a_3}{t_0} \int_{\Omega} h^{\alpha_0+t_0} |\nabla \eta|^2 e^{-f} d\nu. \end{aligned} \tag{3.17}$$

Now we let  $D = \left\{ x \in \Omega | h(x) \geq \frac{2a_4 t_0^2}{\beta_0 R^2} \right\}$ . Hence, we have

$$\begin{aligned} a_4 t_0^2 R^{-2} \int_{\Omega} h^{\alpha_0+t_0} \eta^2 e^{-f} d\nu \\ = a_4 t_0^2 R^{-2} \int_D h^{\alpha_0+t_0} \eta^2 e^{-f} d\nu + a_4 t_0^2 R^{-2} \int_{\Omega \setminus D} h^{\alpha_0+t_0} \eta^2 e^{-f} d\nu \\ \leq \frac{\beta_0}{2} \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu + \frac{a_4 t_0^2}{R^2} \left( \frac{2a_4 t_0^2}{\beta_0 R^2} \right)^{\alpha_0+t_0} V_f. \end{aligned} \tag{3.18}$$

Applying (3.18) to (3.17), we obtain

$$\begin{aligned} \frac{\beta_0}{2} \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu + \frac{a_2}{t_0} e^{-t_0} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t_0}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ \leq \frac{a_4 t_0^2}{R^2} \left( \frac{2a_4 t_0^2}{\beta_0 R^2} \right)^{\alpha_0+t_0} V_f + \frac{a_3}{t_0} \int_{\Omega} h^{\alpha_0+t_0} |\nabla \eta|^2 e^{-f} d\nu. \end{aligned} \tag{3.19}$$

We choose a function  $\zeta \in C_0^\infty(B_R(o))$  that satisfies

$$\begin{cases} 0 \leq \zeta(x) \leq 1, & |\nabla \zeta(x)| \leq \frac{C}{R}, & x \in B_R(o); \\ \zeta(x) \equiv 1, & & x \in B_{3R/4}(o), \end{cases}$$

and choose  $\eta = \zeta^{\alpha_0+t_0+1}$ . Then we get

$$a_3 R^2 |\nabla \eta|^2 \leq a_3 C^2 (\alpha_0 + t_0 + 1)^2 \eta^{\frac{2\alpha_0+2t_0}{\alpha_0+t_0+1}} \leq a_5 t_0^2 \eta^{\frac{2\alpha_0+2t_0}{\alpha_0+t_0+1}}. \tag{3.20}$$

According to (3.20) and the Hölder inequality, we obtain

$$\frac{a_3}{t_0} \int_{\Omega} h^{\alpha_0+t_0} |\nabla \eta|^2 e^{-f} d\nu$$

$$\begin{aligned}
&\leq \frac{a_5 t_0}{R^2} \int_{\Omega} \left( h^{\alpha_0+t_0} \eta^{\frac{2\alpha_0+2t_0}{\alpha_0+t_0+1}} (e^{-f})^{\frac{\alpha_0+t_0}{\alpha_0+t_0+1}} \right) \left( (e^{-f})^{\frac{1}{\alpha_0+t_0+1}} \right) d\nu \\
&\leq \frac{a_5 t_0}{R^2} \left( \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu \right)^{\frac{\alpha_0+t_0}{\alpha_0+t_0+1}} \left( \int_{\Omega} e^{-f} d\nu \right)^{\frac{1}{\alpha_0+t_0+1}} \\
&= \frac{a_5 t_0}{R^2} \left( \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu \right)^{\frac{\alpha_0+t_0}{\alpha_0+t_0+1}} V_f^{\frac{1}{\alpha_0+t_0+1}}. \tag{3.21}
\end{aligned}$$

Then we use the Young inequality, which is described as follows: for any  $\varepsilon > 0$ ,

$$\Theta \Lambda \leq \varepsilon \frac{\Theta^p}{p} + \varepsilon^{-\frac{q}{p}} \frac{\Lambda^q}{q} \quad \text{with } q = \frac{p}{p-1} \text{ relating to } p.$$

We choose

$$\varepsilon = \frac{\beta_0 R^2}{2a_5 t_0}, \quad p = \frac{\alpha_0 + t_0 + 1}{\alpha_0 + t_0}, \quad q = \alpha_0 + t_0 + 1, \quad \Lambda = V_f^{\frac{1}{\alpha_0+t_0+1}}$$

and

$$\Theta = \left( \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu \right)^{\frac{\alpha_0+t_0}{\alpha_0+t_0+1}}.$$

Then we can get

$$\begin{aligned}
&\frac{a_5 t_0}{R^2} \left( \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu \right)^{\frac{\alpha_0+t_0}{\alpha_0+t_0+1}} V_f^{\frac{1}{\alpha_0+t_0+1}} \\
&\leq \frac{a_5 t_0}{R^2} \left[ \frac{\beta_0 R^2}{2a_5 t_0} \frac{\alpha_0 + t_0}{\alpha_0 + t_0 + 1} \left( \left( \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu \right)^{\frac{\alpha_0+t_0}{\alpha_0+t_0+1}} \right)^{\frac{\alpha_0+t_0+1}{\alpha_0+t_0}} \right. \\
&\quad \left. + \left( \frac{\beta_0 R^2}{2a_5 t_0} \right)^{-(\alpha_0+t_0)} \frac{1}{\alpha_0 + t_0 + 1} \left( V_f^{\frac{1}{\alpha_0+t_0+1}} \right)^{\alpha_0+t_0+1} \right] \\
&\leq \frac{\beta_0}{2} \left[ \int_{\Omega} h^{\alpha_0+t_0+1} \eta^2 e^{-f} d\nu + \left( \frac{2a_5 t_0}{\beta_0 R^2} \right)^{\alpha_0+t_0+1} V_f \right]. \tag{3.22}
\end{aligned}$$

By (3.19) and (3.22), we have

$$\begin{aligned}
&\left( \int_{\Omega} h^{\frac{m(\alpha_0+t_0)}{m-2}} \eta^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\
&\leq \frac{t_0}{a_2} e^{t_0} V_f^{1-\frac{2}{m}} R^2 \left[ \frac{2a_4 t_0^2}{R^2} \left( \frac{2a_4 t_0^2}{\beta_0 R^2} \right)^{\alpha_0+t_0} + \frac{a_5 t_0^2}{R^2} \left( \frac{2a_5 t_0}{\beta_0 R^2} \right)^{\alpha_0+t_0} \right] \\
&\leq a_6^{t_0} V_f^{1-\frac{2}{m}} t_0^3 \left( \frac{t_0^2}{R^2} \right)^{\alpha_0+t_0}, \tag{3.23}
\end{aligned}$$

where the constant  $a_6$  depends only on  $m$ ,  $\lambda$ ,  $\tau$  and satisfies

$$a_6^{t_0} \geq \frac{2a_4}{a_2} e^{t_0} \left( \frac{4a_4}{\beta_0} \right)^{\alpha_0+t_0} + \frac{a_5}{a_2} e^{t_0} \left( \frac{4a_5}{\beta_0 t_0} \right)^{\alpha_0+t_0}.$$

Here we have used the fact that  $t_0 \geq 1$ . Taking  $\frac{1}{\alpha_0+t_0}$  power of the both sides of (3.23) gives

$$\left( \int_{\Omega} (h\eta^{\frac{2}{\alpha_0+t_0}})^{\beta} e^{-f} d\nu \right)^{\frac{1}{\beta}} \leq a_6^{\frac{t_0}{\alpha_0+t_0}} V_f^{\frac{1}{\beta}} t_0^{\frac{3}{\alpha_0+t_0}} \frac{t_0^2}{R^2} \leq a_7 V_f^{\frac{1}{\beta}} \frac{t_0^2}{R^2}, \tag{3.24}$$

where the constant  $a_7$  depends only on  $m, \lambda, \tau$  and satisfies

$$a_7 \geq a_6^{\frac{t_0}{\alpha_0+t_0}} t_0^{\frac{3}{\alpha_0+t_0}}.$$

Since  $\eta \equiv 1$  in  $B_{3R/4}(o)$ , we have

$$\left( \int_{B_{3R/4}(o)} h^{\beta} e^{-f} d\nu \right)^{\frac{1}{\beta}} \leq a_7 V_f^{\frac{1}{\beta}} \frac{t_0^2}{R^2}.$$

Thus we obtain the desired estimate. □

Now we are in the position to give the proof of Theorem 1.4 by applying the Nash–Moser iteration method.

*Proof.* Now we let

$$\left\| h^{\frac{\alpha_0+t}{2}} \eta \right\|_{L_f^{\frac{2m}{m-2}}(\Omega)}^2 := \left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}}.$$

Suppose that  $v$  is a positive solution to equation (1.1) on a complete  $n$ -dimensional smooth metric measure space  $(M^n, g, e^{-f} d\nu)$  with the constants  $m > 2, \lambda$  and  $\tau$  which satisfy the condition in Theorem 1.4 (or Corollary 1.5),  $u = -\ln v$  and  $h = |\nabla u|^2$ .

We get rid of the first term in (3.1) and thereby obtain

$$\begin{aligned} & \frac{a_2}{t} e^{-t_0} V_f^{\frac{2}{m}} R^{-2} \left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ & \leq \frac{a_4 t_0^2}{R^2} \int_{\Omega} h^{\alpha_0+t} \eta^2 e^{-f} d\nu + \frac{a_3}{t} \int_{\Omega} h^{\alpha_0+t} |\nabla \eta|^2 e^{-f} d\nu \\ & \leq \int_{\Omega} \left( \frac{a_4 t_0^2}{R^2} \eta^2 + \frac{a_3}{t} |\nabla \eta|^2 \right) h^{\alpha_0+t} e^{-f} d\nu, \end{aligned}$$

which is equivalent to

$$\left( \int_{\Omega} |h^{\frac{\alpha_0+t}{2}} \eta|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \leq e^{t_0} V_f^{-\frac{2}{m}} \int_{\Omega} \left( \frac{a_4}{a_2} t_0^2 t \eta^2 + \frac{a_3}{a_2} R^2 |\nabla \eta|^2 \right) h^{\alpha_0+t} e^{-f} d\nu. \tag{3.25}$$

Then we take an increasing sequence  $\{\beta_k\}_{k=1}^{\infty}$  such that

$$\beta_1 = \beta \quad \text{and} \quad \beta_{k+1} = \frac{m\beta_k}{m-2}, \quad k = 1, 2, \dots,$$

and a decreasing sequence  $\{r_k\}_{k=1}^\infty$  such that

$$r_k = \frac{R}{2} + \frac{R}{4^k}, \quad k = 1, 2, \dots$$

Let  $\Omega_k = B_{r_k}(o)$ . Then we can choose  $\{\eta_k\}_{k=1}^\infty \subset C_0^\infty(B_R(o))$  such that

$$\eta_k \in C_0^\infty(B_{r_k}(o)), \quad \eta_k = 1 \text{ in } B_{r_{k+1}}(o), \quad \text{and} \quad |\nabla \eta_k| \leq \frac{C4^k}{R}.$$

By letting  $t = t_k$ ,  $\eta = \eta_k$  and  $t_k + \alpha_0 = \beta_k$  in (3.25), we derive that

$$\begin{aligned} & \left( \int_{\Omega_k} |h^{\frac{\alpha_0+t_k}{2}} \eta_k|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \\ & \leq e^{t_0} V_f^{-\frac{2}{m}} \int_{\Omega_k} \left( \frac{a_4}{a_2} t_0^2 t_k \eta_k^2 + \frac{a_3}{a_2} R^2 |\nabla \eta_k|^2 \right) h^{\alpha_0+t_k} e^{-f} d\nu \\ & \leq e^{t_0} V_f^{-\frac{2}{m}} \int_{\Omega_k} \left( \frac{a_4}{a_2} t_0^2 t_k \eta_k^2 + \frac{a_3}{a_2} C^2 16^k \right) h^{\alpha_0+t_k} e^{-f} d\nu \\ & \leq e^{t_0} V_f^{-\frac{2}{m}} \left( \frac{a_4}{a_2} t_0^2 t_k + \frac{a_3}{a_2} C^2 16^k \right) \int_{\Omega_k} h^{\alpha_0+t_k} e^{-f} d\nu \\ & \leq e^{t_0} V_f^{-\frac{2}{m}} \left( \frac{a_4}{a_2} t_0^2 (t_0 + \alpha_0) \left( \frac{m}{m-2} \right)^k + \frac{a_3}{a_2} C^2 16^k \right) \int_{\Omega_k} h^{\alpha_0+t_k} e^{-f} d\nu \\ & \leq e^{t_0} V_f^{-\frac{2}{m}} \left( \frac{a_4}{a_2} t_0^2 (t_0 + \alpha_0) 16^k + \frac{a_3}{a_2} C^2 16^k \right) \int_{\Omega_k} h^{\alpha_0+t_k} e^{-f} d\nu. \end{aligned} \quad (3.26)$$

We can find some constant  $a_8$  such that

$$\left( \int_{\Omega_k} |h^{\frac{\alpha_0+t_k}{2}} \eta_k|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m}} \leq a_8 t_0^3 16^k e^{t_0} V_f^{-\frac{2}{m}} \int_{\Omega_k} h^{\beta_k} e^{-f} d\nu. \quad (3.27)$$

Then, taking power of  $1/\beta_k$  of the both sides of (3.27), we have

$$\left( \int_{\Omega_k} |h^{\frac{\alpha_0+t_k}{2}} \eta_k|^{\frac{2m}{m-2}} e^{-f} d\nu \right)^{\frac{m-2}{m\beta_k}} \leq (a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}})^{\frac{1}{\beta_k}} 16^{\frac{k}{\beta_k}} \left( \int_{\Omega_k} h^{\beta_k} e^{-f} d\nu \right)^{\frac{1}{\beta_k}}. \quad (3.28)$$

Thus,

$$\left( \int_{\Omega_{k+1}} h^{\beta_{k+1}} e^{-f} d\nu \right)^{\frac{1}{\beta_{k+1}}} \leq (a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}})^{\frac{1}{\beta_k}} 16^{\frac{k}{\beta_k}} \left( \int_{\Omega_k} h^{\beta_k} e^{-f} d\nu \right)^{\frac{1}{\beta_k}},$$

which implies that

$$\|h\|_{L_f^{\beta_{k+1}}(\Omega_{k+1})} \leq \left( a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}} \right)^{\frac{1}{\beta_k}} 16^{\frac{k}{\beta_k}} \|h\|_{L_f^{\beta_k}(\Omega_k)}. \quad (3.29)$$

By iteration, we have

$$\|h\|_{L_f^{\beta_{k+1}}(\Omega_{k+1})} \leq \left( a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}} \right)^{\sum_{k=1}^{\infty} \frac{1}{\beta_k}} 16^{\sum_{k=1}^{\infty} \frac{k}{\beta_k}} \|h\|_{L_f^{\beta_1}(B_{3R/4}(o))}. \tag{3.30}$$

We note that

$$\sum_{k=1}^{\infty} \frac{1}{\beta_k} = \frac{m}{2\beta_1} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{k}{\beta_k} = \frac{m^2}{4\beta_1}.$$

Then, by letting  $k \rightarrow \infty$  in (3.30), we obtain the following inequality:

$$\|h\|_{L_f^{\beta_{k+1}}(\Omega_{k+1})} \leq (a_8 t_0^3 e^{t_0})^{\frac{m}{2\beta_1}} 16^{\frac{m^2}{4\beta_1}} V_f^{-\frac{1}{\beta_1}} \|h\|_{L_f^{\beta_1}(B_{3R/4}(o))}. \tag{3.31}$$

There exists  $a_9 > 0$  such that

$$a_9 \geq (a_8 t_0^3 e^{t_0})^{\frac{m}{2\beta_1}} 16^{\frac{m^2}{4\beta_1}}.$$

Thus we have

$$\|h\|_{L_f^\infty(B_{R/2}(o))} \leq a_9 V_f^{-\frac{1}{\beta_1}} \|h\|_{L_f^{\beta_1}(B_{3R/4}(o))}. \tag{3.32}$$

According to (3.16), we have

$$\|h\|_{L_f^\infty(B_{R/2}(o))} \leq a_{10} \frac{(1 + \sqrt{KR})^2}{R^2}, \tag{3.33}$$

where  $a_{10} = a_9 a_7 c_{m,\lambda,\tau}^2$ . Recalling  $h = |\nabla u|^2$  and  $u = -\ln v$ , we derive the required estimate. Consequently, we conclude the proof of Theorem 1.4.  $\square$

### 4. Proofs of Corollaries 1.7 and 1.9

*Proof of Corollary 1.7.* Under the conditions in Corollary 1.7, then by Theorem 1.4, we have that for any  $x \in B_{R/2}(o) \subset M$ ,

$$\frac{|\nabla v(x)|}{v(x)} \leq \sup_{B_{R/2}(o)} \frac{|\nabla v|}{v} \leq \frac{c(m, \tau, \lambda)}{R}. \tag{4.1}$$

Letting  $R \rightarrow \infty$  in (4.1), we get

$$\nabla v(x) = 0, \quad \forall x \in M.$$

Therefore,  $v$  is a positive constant on  $M$ .  $\square$

*Proof of Corollary 1.9.* Under the same conditions as in Theorem 1.4, let  $x, y \in B_{R/2}(o)$  be any two points with minimal geodesic  $\gamma$  connecting them. By using the gradient estimate (1.13) and the fact that  $\text{length}(\gamma) \leq R$ , we have

$$\begin{aligned} \ln v(x) - \ln v(y) &= \int_\gamma |\nabla \ln v| dt \\ &\leq \int_\gamma c(\tau, m, \lambda) \frac{1 + \sqrt{KR}}{R} dt \leq c(\tau, m, \lambda)(1 + \sqrt{KR}). \end{aligned}$$

Thus, for any  $x, y \in B_{R/2}(o)$ , we obtain

$$v(x) \leq e^{c(\tau, m, \lambda)(1 + \sqrt{KR})} v(y). \tag{4.2} \quad \square$$

### 5. Global gradient estimate

By the local gradient estimate of positive solution  $v$  to equation (1.1), we have

$$\sup_{B_{R/2}(o)} \frac{|\nabla v|}{v} \leq c(m, \tau, \lambda) \frac{1 + \sqrt{KR}}{R}.$$

If  $v$  is a global solution, then, letting  $R \rightarrow \infty$  in the above local gradient estimate, we obtain

$$\frac{|\nabla v|}{v} \leq c(m, \tau, \lambda) \sqrt{K}, \quad x \in M.$$

In this section, we give an explicit expression of the above  $c(m, \tau, \lambda)$ . By almost the same idea as that adopted in [20], we can establish the following key lemmas.

**Lemma 5.1.** *Let  $(M^n, g, e^{-f} dv)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^n \geq -(m-1)Kg$ , where  $K$  is a nonnegative constant. Let  $v$  be a global positive solution of (1.1) in  $M$ .*

(1) *If  $m, \tau$  and  $\lambda$  satisfy (1.11), we denote*

$$y_1 = \frac{(m-1)^2}{\tau^2} K.$$

*Then, for any  $\delta > 0$ ,  $\omega = (h - y_1 - \delta)^+$  and  $\alpha$  large enough, there are some positive constants  $l_1, l_2$  depending on  $m, \tau, \lambda, K$  and  $\delta$  such that*

$$\Delta_f \omega^\alpha \geq 2\alpha \omega^{\alpha-1} (l_1 \omega - l_2 |\nabla \omega|).$$

(2) *If  $m, \tau$  and  $\lambda$  satisfy (1.12), we denote*

$$y_2 = \frac{K}{\frac{\tau^2}{2} \left( \frac{\sqrt{2-m-1}}{m-1} + \frac{1}{\tau} \right) \left( \frac{\sqrt{2+m+1}}{m-1} - \frac{1}{\tau} \right)}.$$

*Then, for any  $\delta > 0$ ,  $\omega = (h - y_2 - \delta)^+$  and  $\alpha$  large enough, there are some positive constants  $l_3, l_4$  depending on  $m, \tau, K$  and  $\delta$  such that*

$$\Delta_f \omega^\alpha \geq 2\alpha \omega^{\alpha-1} (l_3 \omega - l_4 |\nabla \omega|).$$

*Proof.* (1) By Lemma 2.1, if  $m, \tau$  and  $\lambda$  satisfy (1.11), then

$$\frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq \frac{1}{m-1} \tau^2 h^2 - (m-1)Kh - \frac{\alpha_1}{2} h^{\frac{1}{2}} |\nabla h| \tag{5.1}$$

holds.

We note that  $\nabla w = \nabla h$  in  $\{h \geq y_1 + \delta\}$ , but  $\nabla w$  causes a distribution on  $\{h = y_1 + \delta\}$ . If we assume  $\alpha > 1$ , then the distribution caused by  $\nabla w$  on  $\{h = y_1 + \delta\}$  is eliminated by  $w$  since  $w = 0$  on  $\{h = y_1 + \delta\}$ .

Since  $h \geq w$ , it follows that

$$\Delta_f w^\alpha = -\alpha w^{\alpha-1} \langle \nabla w, \nabla f \rangle + \text{div}(\alpha w^{\alpha-1} \nabla w)$$

$$\begin{aligned}
 &= -\alpha w^{\alpha-1} \langle \nabla h, \nabla f \rangle + \operatorname{div}(\alpha w^{\alpha-1} \nabla h) \\
 &= \alpha w^{\alpha-1} \Delta_f h + \alpha(\alpha - 1) w^{\alpha-2} |\nabla h|^2 \\
 &= \frac{w^{\alpha-1}}{h^{\alpha-1}} (\alpha h^{\alpha-1} \Delta_f h + \alpha(\alpha - 1) w^{-1} h^{\alpha-1} |\nabla h|^2) \\
 &\geq \frac{w^{\alpha-1}}{h^{\alpha-1}} (\alpha h^{\alpha-1} \Delta_f h + \alpha(\alpha - 1) h^{\alpha-2} |\nabla h|^2) = \frac{w^{\alpha-1}}{h^{\alpha-1}} \Delta_f h^\alpha. \tag{5.2}
 \end{aligned}$$

Substituting (5.1) into (5.2), we get

$$\Delta_f w^\alpha \geq 2\alpha w^{\alpha-1} \left( \frac{1}{m-1} \tau^2 h^2 - (m-1)Kh - \frac{a_1}{2} h^{\frac{1}{2}} |\nabla h| \right).$$

Since

$$h \geq y_1 + \delta = \frac{(m-1)^2}{\tau^2} K + \delta,$$

the above inequality can be written as

$$\begin{aligned}
 \Delta_f w^\alpha &\geq 2\alpha w^{\alpha-1} h^{\frac{1}{2}} \left( h^{\frac{1}{2}} \left( \frac{1}{m-1} \tau^2 h - (m-1)K \right) - \frac{a_1}{2} |\nabla w| \right) \\
 &\geq 2\alpha w^{\alpha-1} h^{\frac{1}{2}} \left( h^{\frac{1}{2}} \frac{\delta \tau^2}{m-1} - \frac{a_1}{2} |\nabla w| \right).
 \end{aligned}$$

There holds

$$y_1 + \delta \leq h \leq c(m, \tau, \lambda) \sqrt{K}$$

on  $\{h \geq y_1 + \delta\}$ , and thus we have

$$\Delta_f w^\alpha \geq 2\alpha w^{\alpha-1} (l_1 w - l_2 |\nabla w|),$$

where  $l_1$  and  $l_2$  are two positive constants depending on  $m, K, \tau, \lambda$  and  $\delta$ .

(2) If  $m, \tau$  and  $\lambda$  satisfy (1.12), it follows from (2.20) that

$$\frac{h^{1-\alpha}}{2\alpha} \Delta_f h^\alpha \geq B_{m,\tau} h^2 - (m-1)Kh - \frac{a_1}{2} h^{\frac{1}{2}} |\nabla h|, \tag{5.3}$$

where

$$B_{m,\tau} = \left( \frac{1}{m-1} - \frac{m-1}{2} \left( \frac{m+1}{m-1} - \frac{1}{\tau} \right)^2 \right) \tau^2.$$

Then, on  $\{h \geq y_2 + \delta\}$ ,

$$h \geq y_2 + \delta = \frac{(m-1)K}{B_{m,\tau}} + \delta.$$

So (5.3) can be written as

$$\begin{aligned}
 \Delta_f w^\alpha &\geq 2\alpha w^{\alpha-1} h^{\frac{1}{2}} \left( h^{\frac{1}{2}} (B_{m,\tau} h - (m-1)K) - \frac{a_1}{2} |\nabla w| \right) \\
 &\geq 2\alpha w^{\alpha-1} h^{\frac{1}{2}} \left( h^{\frac{1}{2}} \delta B_{m,\tau} - \frac{a_1}{2} |\nabla w| \right).
 \end{aligned}$$

There holds

$$y_2 + \delta \leq h \leq c(m, \tau, \lambda)\sqrt{K}$$

on  $\{h \geq y_2 + \delta\}$ , and thus we obtain

$$\Delta_f w^\alpha \geq 2\alpha w^{\alpha-1}(l_3 w - l_4 |\nabla w|),$$

where  $l_3$  and  $l_4$  are two positive constants depending on  $m, K, \tau$  and  $\delta$ . □

Next, we use an iteration scheme similar to that in [20, 27, 30] to derive an explicit bound of the gradient.

**Lemma 5.2.** *Let  $(M^n, g, e^{-f} d\nu)$  be a complete  $n$ -dimensional smooth metric measure space with  $\text{Ric}_f^m \geq -(m-1)Kg$ , where  $K$  is a nonnegative constant. Let  $v$  be a global positive solution of (1.1) in  $M$ . For some  $y > 0$ , we let  $w = (h - y)^+$ . If  $w$  satisfies the following inequality:*

$$\Delta_f w^\alpha \geq 2\alpha w^{\alpha-1}(l_3 w - l_4 |\nabla w|), \tag{5.4}$$

where  $l_3$  and  $l_4$  and  $\alpha$  are some positive constants, then  $w = 0$ , i.e.,  $h \leq y$ .

*Proof.* Let  $\eta \in C_0^\infty(M, \mathbb{R})$  be a cut-off function to be determined later. For a constant  $\gamma > 1$ , we choose  $w^\gamma \eta^2$  as a test function. Now we multiply the both sides of (5.4) by  $w^\gamma \eta^2$  and integrate the obtained inequality over  $M$  to get

$$\int_M (w^\gamma \eta^2) \Delta_f w^\alpha e^{-f} d\nu \geq \int_M 2\alpha w^{\alpha+\gamma-1} (l_3 w - l_4 |\nabla w|) \eta^2 e^{-f} d\nu.$$

Using the integration by parts, we have

$$\begin{aligned} - \int_M \alpha \gamma w^{\alpha+\gamma-2} |\nabla w|^2 \eta^2 e^{-f} d\nu - \int_M 2\alpha w^{\alpha+\gamma-1} \langle \nabla w, \nabla \eta \rangle \eta e^{-f} d\nu \\ \geq \int_M 2\alpha w^{\alpha+\gamma-1} (l_3 w - l_4 |\nabla w|) \eta^2 e^{-f} d\nu. \end{aligned}$$

Since

$$\langle \nabla w, \nabla \eta \rangle \geq -|\nabla w| |\nabla \eta|,$$

we conclude that

$$\begin{aligned} \int_M 2l_3 w^{\alpha+\gamma} \eta^2 e^{-f} d\nu + \int_M \gamma w^{\alpha+\gamma-2} |\nabla w|^2 \eta^2 e^{-f} d\nu \\ \leq \int_M 2l_4 w^{\alpha+\gamma-1} |\nabla w| \eta^2 e^{-f} d\nu + \int_M 2w^{\alpha+\gamma-1} |\nabla w| |\nabla \eta| \eta e^{-f} d\nu. \end{aligned}$$

Applying the Young inequality, we get

$$\begin{aligned} 2l_4 w^{\alpha+\gamma-1} |\nabla w| \eta^2 &\leq l_4 w^{\alpha+\gamma-2} \frac{|\nabla w|^2}{\epsilon} \eta^2 + \epsilon l_4 w^{\alpha+\gamma} \eta^2; \\ 2w^{\alpha+\gamma-1} |\nabla w| |\nabla \eta| \eta &\leq w^{\alpha+\gamma-2} \frac{|\nabla w|^2}{\epsilon} \eta^2 + \epsilon w^{\alpha+\gamma} |\nabla \eta|^2. \end{aligned}$$

Now we choose  $\epsilon$  such that

$$\frac{l_4 + 1}{\epsilon} = \gamma,$$

then we obtain

$$\int_M 2l_3 w^{\alpha+\gamma} \eta^2 e^{-f} d\nu \leq \int_M \epsilon w^{\alpha+\gamma} |\nabla \eta|^2 e^{-f} d\nu + \int_M \epsilon l_4 w^{\alpha+\gamma} \eta^2 e^{-f} d\nu.$$

Choosing  $\gamma$  large enough such that  $\epsilon l_4 < l_3$ , we have

$$\int_M l_3 w^{\alpha+\gamma} \eta^2 e^{-f} d\nu \leq \int_M \epsilon w^{\alpha+\gamma} |\nabla \eta|^2 e^{-f} d\nu.$$

If  $w \neq 0$ , without loss of generality, we can suppose that  $w|_{B_1} \neq 0$  for some geodesic ball  $B_1$  with radius 1, and we observe that

$$\int_{B_1} e^{-f} d\nu > 0$$

always holds true.

We use  $B_k$  to represent the geodesic ball with radius  $k$ . Then we choose  $\eta \in C_0^\infty(B_{k+1})$  such that

$$\eta \equiv 1 \text{ in } B_k \quad \text{and} \quad |\nabla \eta| \leq 4 \text{ in } B_{k+1}.$$

Then we derive that

$$\int_{B_{k+1}} 16\epsilon w^{\alpha+\gamma} e^{-f} d\nu \geq \int_{B_{k+1}} l_3 w^{\alpha+\gamma} e^{-f} d\nu \geq \int_{B_k} l_3 w^{\alpha+\gamma} e^{-f} d\nu.$$

By iteration on  $k$ , we get

$$\int_{B_k} w^{\alpha+\gamma} e^{-f} d\nu \geq \left(\frac{\hat{C}_1}{\epsilon}\right)^k \int_{B_1} w^{\alpha+\gamma} e^{-f} d\nu, \tag{5.5}$$

where  $\hat{C}_1$  is a positive constant. Since  $w$  is uniformly bounded, we have

$$\hat{C}_2^{\alpha+\gamma} V_f(B_k) \geq \int_{B_k} w^{\alpha+\gamma} e^{-f} d\nu \tag{5.6}$$

and

$$\int_{B_1} w^{\alpha+\gamma} e^{-f} d\nu \geq \hat{C}_3^{\alpha+\gamma} V_f(B_1), \tag{5.7}$$

where  $\hat{C}_2$  and  $\hat{C}_3$  are positive constants, and  $V_f(B_k) = \int_{B_k} e^{-f} d\nu$ . By the volume comparison theorem, we have

$$\frac{V_f(B_k)}{V_f(B_1)} \leq e^{(m-1)k\sqrt{K}}. \tag{5.8}$$

Let  $\hat{C}_4 = \frac{\hat{C}_2}{\hat{C}_3}$ . Then, plugging (5.6)–(5.8) into (5.5), we infer that

$$\hat{C}_4^{\alpha+\gamma} e^{(m-1)k\sqrt{K}} \geq e^{k \ln \frac{\hat{C}_1}{\epsilon}}. \quad (5.9)$$

We choose  $\gamma$  such that

$$\ln \frac{\hat{C}_1}{\epsilon} > 2(m-1)\sqrt{K} + 2.$$

If  $k$  is chosen sufficiently large, (5.9) can not hold. Hence,  $w \equiv 0$ , and thus we finish the proof of this lemma.  $\square$

Now we are in the position to give the proof of Theorem 1.10.

*Proof.* By Lemmas 5.1 and 5.2, we can prove Theorem 1.10. If  $m, \lambda, \tau$  satisfy (1.11) or (1.12), then, by Lemma 5.1, for any  $\delta > 0$  and  $w = (h - y_i - \delta)^+$ ,  $i = 1, 2$ , we obtain

$$\Delta_f w^\alpha \geq 2\alpha w^{\alpha-1} (l_3 w - l_4 |\nabla w|).$$

According to Lemma 5.2, we obtain  $w \equiv 0$ , i.e.,  $h \leq y_i + \delta$ . Since  $\delta$  can be arbitrarily small, it is clear that  $h \leq y_i$ . Since

$$h = |\nabla u|^2 = |(\nabla v)/v|^2,$$

we get

$$\frac{|\nabla v|}{v} \leq \sqrt{y_i},$$

and thus we finish the proof of Theorem 1.10.  $\square$

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## Логарифмічні градієнтні оцінки Ченя–Яу для нелінійного еліптичного рівняння на гладких метричних просторах із мірою

Cheng Jin, Youde Wang, and Fanqi Zeng

У цій роботі ми розглядаємо нелінійне еліптичне рівняння

$$\Delta_f v^\tau + \lambda v = 0$$

на повному гладкому метричному просторі з мірою, де  $m$ -кривина Річчі Бакрі–Емері обмежена знизу, а  $\tau > 0$  та  $\lambda \in \mathbb{R}$  є константами. Ми одержуємо

нові локальні та глобальні універсальні log-градієнтні оцінки для додатних розв'язків цього рівняння, використовуючи метод ітерацій Неша–Мозера. Як застосування цих оцінок, ми одержуємо теорему типу Ліувілля, нерівність Гарнака та глобальні градієнтні оцінки для таких розв'язків. Наші результати узагальнюють та покращують оцінки, встановлені Вангом (*J. Differential Equations* 260 (2016), 567–585) та Чжао (*Arch. Math. (Basel)* 114 (2020), 457–469).

*Ключові слова:* Гладкий метричний простір з мірою, градієнтна оцінка, теорема Ліувілля, нерівність Гарнака