

Bounded Solutions to Strongly Nonlinear Elliptic Problems without Sign Restrictions in Weighted Sobolev Spaces

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In this paper, we study the existence and L^∞ -regularity of solutions for a general class of strongly nonlinear elliptic problems associated with the differential inclusion

$$\gamma(u) + A(u) + \Phi(x, u, \nabla u) \ni f,$$

where A is a Leray–Lions operator from $W_0^{1,p}(\Omega, \omega)$ into its dual $W^{-1,p'}(\Omega, \omega^*)$, γ is a maximal monotone mapping with $0 \in \gamma(0)$, Φ is a nonlinear term satisfying only a growth condition, without any sign condition on s , and the right-hand side f is assumed to belong to $L^\infty(\Omega)$.

Key words: inclusion problems, L^∞ -estimate, p -growth, weak solution, weighted Sobolev spaces

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1. Introduction

Nonlinear elliptic problems with natural growth conditions have been extensively studied over the past decades due to their rich mathematical structure and wide range of applications in physics, engineering and material sciences. A fundamental framework for such problems is provided by Leray–Lions type operators, which generalize the p -Laplacian and admit heterogeneous and anisotropic behaviors. In the classical (unweighted) setting, existence and regularity results have been established under various assumptions on the data and nonlinearities. In particular, the pioneering work of Brézis and Strauss [16] addressed semilinear elliptic equations with L^1 -data, while Boccardo and Gallouët [12] developed fundamental techniques for dealing with nonlinear problems involving low-regularity data. These ideas were further extended by Boccardo, Murat and Puel [14], who obtained L^∞ -estimates for nonlinear elliptic equations. A systematic L^1 -theory for nonlinear elliptic problems was later established by Bénéilan et al. [10], which has become a cornerstone in the study of weak, renormalized and entropy solutions. Further developments in this direction were achieved by Dal Maso et al. [17], who extended the notion of renormalized solutions to problems with general measure data.

More recently, increasing attention has been paid to extensions of these results in more general functional frameworks including weighted Sobolev spaces

and variable exponent spaces. Weighted Sobolev spaces provide a natural setting for modeling nonhomogeneous media, where spatial heterogeneities are described through weight functions (see, for instance, [19, 22, 24]). However, the introduction of weights significantly complicates the analysis due to the lack of uniform ellipticity, their possible degeneracy, and the delicate nature of compact embeddings making it more difficult to derive a priori estimates and to apply standard compactness arguments.

In this paper, we study the existence and boundedness of solutions to a class of strongly nonlinear elliptic inclusion problems of the form

$$\begin{cases} \gamma(u) + A(u) + \Phi(x, u, \nabla u) \ni f & \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega, \omega), \quad \Phi(x, u, \nabla u) \in L^1(\Omega), \end{cases} \quad (1.1)$$

within the weighted Sobolev space $W_0^{1,p}(\Omega, \omega)$. Here, A is a Leray–Lions type operator, γ is a maximal monotone graph, and Φ is a nonlinear lower-order term exhibiting natural growth with respect to the gradient. A distinctive feature of our framework is the absence of any sign condition on Φ , which introduces additional analytical difficulties, particularly in the derivation of L^∞ -estimates.

The study of such problems in the unweighted setting is now well established. In particular, existence and regularity results for nonlinear elliptic equations with natural growth terms and L^1 or bounded data have been obtained using truncation methods and refined compactness techniques (see, for example, [10, 13, 29]). Extensions to inclusion problems involving maximal monotone operators have also been considered in previous works. In particular, the authors in [7] established existence and boundedness results for strongly nonlinear elliptic problems under a sign condition on the nonlinear term (see also [8, 23] for related results in the variable exponent framework). Later, in the case of L^1 data, they obtained existence results under additional coercivity assumptions on the perturbation term [6]. More recently, Ouboufettal and Akdim [28] removed the sign condition in the unweighted setting by combining truncation methods and compactness arguments (see also [31] for the case $\Phi = 0$).

The main contribution of the present paper is twofold. First, we extend these results to the framework of weighted Sobolev spaces, where the presence of weights requires nontrivial adaptations of classical methods. In particular, the derivation of a priori estimates relies on weighted Hardy-type inequalities and compact embeddings whose validity depends on delicate conditions on the weights. Second, we remove both the usual sign condition and the coercivity assumption on the nonlinear term Φ of the form

$$|\Phi(x, s, \xi)| \geq \alpha, \omega(x)|\xi|^p \quad \text{for } |s| \geq \sigma,$$

with constants $\sigma > 0$ and $\alpha > 0$, which ensures additional regularity of the solution (see [6]). This prevents the direct application of standard energy estimates. To overcome this difficulty, we employ a combination of truncation techniques and exponential test functions allowing us to control the nonlinear terms without

relying on sign and/or coercivity assumptions. (See, for instance, [1–3, 11, 20, 21] for related works in weighted settings).

Our approach is based on the approximation of the maximal monotone graph via Yosida regularization [15, 31] combined with the theory of pseudo-monotone operators developed by Lions [25] and Leray–Lions [26]. The passage to the limit requires careful analysis including strong convergence results obtained through monotonicity arguments and the use of Vitali’s theorem. The weighted setting introduces additional challenges in each of these steps, particularly in establishing compactness and equi-integrability properties.

We emphasize that the assumptions on the weights, including the Hardy-type inequality and the compact embedding, play a crucial role in our analysis. These conditions are satisfied for a wide class of weights, including those depending on the distance to the boundary and weights related to the classical theory of weighted Sobolev spaces (see [24]). We provide an illustrative example in the final section to demonstrate the applicability of our framework.

The paper is organized as follows. Section 2 introduces the notation, the functional framework and some auxiliary results. Section 3 presents the assumptions and the main theorem. Section 4 collects key technical lemmas. Section 5 is devoted to the proofs of the main results. Finally, Section 6 provides an illustrative example that highlights the applicability of our assumptions.

2. Preliminary results and notations

In this subsection, we recall several standard definitions and properties that will be used throughout the paper. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N \geq 2$. We assume that $1 < p < \infty$ and the weight vector $\omega(x)$ satisfies, for any $i = 0, \dots, N$,

$$\omega_i \in L^1_{\text{loc}}(\Omega), \quad (2.1)$$

$$\omega_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega). \quad (2.2)$$

We now define the weighted Lebesgue space $L^p(\Omega, \omega_0)$ with weight ω_0 as

$$L^p(\Omega, \omega_0) = \left\{ u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } \int_{\Omega} |u(x)|^p \omega_0(x) dx < \infty \right\},$$

where the norm is

$$\|u\|_{p, \omega_0} = \left(\int_{\Omega} |u(x)|^p \omega_0(x) dx \right)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega, \omega)$ the space of real-valued functions $u \in L^p(\Omega, \omega_0)$ such that $\partial_i u \in L^p(\Omega, \omega_i)$ for $i = 1, \dots, N$, which is a Banach space under the norm

$$\|u\|_{1,p,\omega} = \left(\int_{\Omega} |u(x)|^p \omega_0(x) dx + \sum_{i=1}^N \int_{\Omega} |\partial_i u|^p \omega_i(x) dx \right)^{\frac{1}{p}}. \quad (2.3)$$

By condition (2.1), we can define the subspace

$$X = W_0^{1,p}(\Omega, \omega)$$

of $W^{1,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, if (2.2) holds, then X and $W^{1,p}(\Omega, \omega)$ are separable and reflexive Banach spaces. We recall that the dual of X is equivalent to $W^{-1,p'}(\Omega, \omega^*)$, where $\omega^*(x) = (\omega_i^*(x) = \omega_i^{1-p'}(x))_{\{1 \leq i \leq N\}}$ and $p' = \frac{p}{p-1}$ is the conjugate of p (see [19, 22] for more details).

In what follows, we make use of the following notation. For any $u : \Omega \rightarrow \mathbb{R}$ and $k \geq 0$, we write $\{|u| \leq (<, \geq, >, =)k\}$ for the set $\{x \in \Omega : |u(x)| \leq (<, \geq, >, =)k\}$.

For a fixed $k > 0$, let the truncation functions T_k and G_k of the real variable s be defined as follows:

$$T_k(s) = \max(-k, \min(k, s)) \quad \text{and} \quad G_k(s) = s - T_k(s).$$

Moreover, we denote by $\tau^+ = \max\{\tau, 0\}$ and $\tau^- = -\min\{\tau, 0\}$ the positive and the negative parts of τ .

Finally, we note that the constants C_i for $i = 1, 2, \dots$ used throughout this paper depend only on the data of our problem and never on the indices of the sequences that we frequently introduce.

3. Statement of the problem and main result

The analysis presented in this paper is conducted under the following standing assumptions:

(H1) Ω is a bounded open domain in $\mathbb{R}^N$ ($N \geq 2$), $1 < p < \infty$, and $\omega(x) = (\omega_i(x))_{\{0 \leq i \leq N\}}$ is a vector of weight functions on Ω such that (2.1) and (2.2) hold. In addition, the expression

$$\|u\|_X = \left(\sum_{i=1}^N \int_{\Omega} |\partial_i u|^p \omega_i(x) dx \right)^{\frac{1}{p}} \quad (3.1)$$

is a norm defined on X which is equivalent to the norm (2.3).

There exists a weight function σ on Ω and parameters $1 < q < \infty$ and $p < r$ such that

$$\sigma^{1-q'} \in L^1(\Omega) \quad (3.2)$$

and the Hardy-type inequality

$$\left(\int_{\Omega} |u(x)|^q \sigma(x) dx \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^N \int_{\Omega} |\partial_i u|^p \omega_i(x) dx \right)^{\frac{1}{p}} \quad (3.3)$$

holds for every $u \in X$, with a constant C independent of u . Moreover, the embedding

$$X \hookrightarrow L^q(\Omega, \sigma) \quad (3.4)$$

is compact, and the embedding

$$X \hookrightarrow L^r(\Omega) \quad (3.5)$$

is continuous.

- (H2)** The function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying the assumptions:

$$a(x, s, \xi) \xi \geq \alpha \sum_{i=1}^N \omega_i(x) |\xi_i|^p, \quad (3.6)$$

for some constant $\alpha > 0$, for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$; for any $i = 1, \dots, N$,

$$|a_i(x, s, \xi)| \leq \omega_i(x)^{\frac{1}{p}} \left(k(x) + \sigma(x)^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + \sum_{j=1}^N \omega_j(x)^{\frac{1}{p'}} |\xi_j|^{p-1} \right), \quad (3.7)$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$, where σ and q are given by **(H1)** and k is a nonnegative function in $L^{p'}(\Omega)$;

$$\left(a(x, s, \xi) - a(x, s, \xi') \right) (\xi - \xi') > 0, \quad (3.8)$$

for a.a. $x \in \Omega$, $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$ such that $\xi \neq \xi'$.

- (H3)** The function $\Phi : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function such that for almost every $x \in \Omega$, for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, we have

$$|\Phi(x, s, \xi)| \leq \Phi_0(x) + \Phi_1(s) \sum_{i=1}^N \omega_i(x) |\xi_i|^p, \quad (3.9)$$

where $\Phi_1 : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function belonging to $L^1(\mathbb{R})$, and $\Phi_0 \in L^\infty(\Omega)$.

- (H4)** The nonlinearity $\gamma : \mathbb{R} \longrightarrow 2^\mathbb{R}$ is a set valued, maximal monotone mapping such that $0 \in \gamma(0)$.
- (H5)** The function f belongs to $L^\infty(\Omega)$.

We now provide the definitions of solutions to (1.1), followed by the statement of our main result.

Definition 3.1. Let $f \in L^\infty(\Omega)$. We say that a couple of functions $(u, v) \in X \times L^1(\Omega)$ is a weak solution to problem (1.1) if

$$u(x) \in D(\gamma) \text{ and } v \in \gamma(u(x)) \text{ a.e. in } \Omega, \quad (3.10)$$

$$a(x, u, \nabla u) \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}), \quad (3.11)$$

$$\Phi(x, u, \nabla u) \in L^1(\Omega), \quad (3.12)$$

and

$$\int_{\Omega} v \varphi \, dx + \int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} \Phi(x, u, \nabla u) \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad (3.13)$$

for any $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

Remark 3.2.

1. In view of Definition 3.1 of a weak solution, all the terms appearing in the weak formulation (3.13) are well defined. Indeed, for any $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, the duality between $L^p(\Omega, \omega_i)$ and $L^{p'}(\Omega, \omega_i^*)$, for all $i = 1, \dots, N$, ensures that the integral $\int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi \, dx$ is finite. Moreover, since $\varphi \in L^\infty(\Omega)$ and $v, f, \Phi(x, u, \nabla u) \in L^1(\Omega)$, the remaining terms in the weak formulation are also well defined.
2. We also emphasize the role of the auxiliary function $v \in \gamma(u)$, which naturally arises from the multivalued maximal monotone operator framework. This formulation allows us to treat the inclusion $\gamma(u)$ in a variational setting by introducing v as a measurable selection of the graph γ . Such an approach is a standard in the theory of nonlinear elliptic inclusions and is closely related to the notion of weak solutions.

Our main result is the following theorem:

Theorem 3.3. *Assume that (H1)–(H5) hold. Then there exists at least one weak solution $(u, v) \in X \times L^\infty(\Omega)$ of problem (1.1).*

Remark 3.4.

1. First, we notice that the expression (3.1) does not, in general, define a norm on the space X . However, this property holds under additional assumptions on the weight functions. More precisely, if we assume that $\omega_0(x) \equiv 1$ and that there exists $\nu \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right)$ such that $\omega_i^{-\nu} \in L^1(\Omega)$ for all $i = 1, \dots, N$, then the expression (3.1) defines a norm on X which is equivalent to (2.3) (see [19]). Moreover, under these assumptions, the embedding (3.5) holds for all $1 \leq r < p_1^*$ if $p\nu < N(\nu + 1)$, and for all $r \geq 1$ if $p\nu \geq N(\nu + 1)$, where

$$p_1 = \frac{p\nu}{\nu + 1}, \quad p_1^* = \frac{Np_1}{N - p_1} = \frac{Np\nu}{N(\nu + 1) - p\nu}$$

denotes the Sobolev conjugate of p_1 (see [18, 19]). In particular, hypothesis (H1) is satisfied in the case $\sigma \equiv 1$.

2. Since we are dealing with a Dirichlet problem, it is natural to assume a Hardy-type inequality, which can be viewed as a weighted generalization of the classical Poincaré inequality. This is a standard assumption in the study of weighted Sobolev spaces and it holds for a large class of weights, for instance, Muckenhoupt weights (see [27]). Furthermore, the compact embedding (3.4) is essential in our analysis as it ensures the compactness properties required for the approximation and convergence arguments.

3. Finally, hypothesis (3.5) in **(H1)** can be replaced by a more explicit integrability condition. More precisely, it suffices to assume that there exists a parameter r such that $1 < p < r < q$ and

$$\sigma^{-\frac{r}{q-r}} \in L^1(\Omega). \quad (3.14)$$

This condition guarantees the validity of the embedding (3.5).

4. Useful lemmas

The following lemmas play an important role in the remainder of this work.

Lemma 4.1 ([4]). *Assume that **(H1)** holds. Let $u \in X$, then u^+ and u^- lie in X . Moreover, we have*

$$\begin{aligned} \partial_i(u^+) &= \begin{cases} \partial_i u & \text{if } u > 0, \\ 0 & \text{if } u \leq 0, \end{cases} \\ \partial_i(u^-) &= \begin{cases} 0 & \text{if } u \geq 0, \\ -\partial_i u & \text{if } u < 0. \end{cases} \end{aligned}$$

Lemma 4.2 ([4]). *Assume that **(H1)** holds. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in X such that $u_\varepsilon \rightharpoonup u$ weakly in X . Then*

$$\begin{aligned} u_\varepsilon^+ &\rightharpoonup u^+ \text{ weakly in } X, \\ u_\varepsilon^- &\rightharpoonup u^- \text{ weakly in } X. \end{aligned}$$

Lemma 4.3 ([4, Lemma 3.2]). *Assume that **(H1)** and **(H2)** are satisfied, and let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in X such that $u_\varepsilon \rightharpoonup u$ weakly in X and*

$$\int_{\Omega} [a(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, u_\varepsilon, \nabla u)] \cdot \nabla(u_\varepsilon - u) \, dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (4.1)$$

Then $u_\varepsilon \rightarrow u$ strongly in X .

5. Proof of our result

The proof is divided into four steps.

Step 1: Approximating problems. For $0 < \varepsilon \leq 1$, let $\gamma_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the Yosida approximation of γ (see [15] for more details). We consider the following approximate problem:

$$\begin{cases} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \operatorname{div}(a(x, u_\varepsilon, \nabla u_\varepsilon)) + \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where

$$\Phi_\varepsilon(x, s, \xi) = \frac{\Phi(x, s, \xi)}{1 + \varepsilon |\Phi(x, s, \xi)|}, \quad (5.2)$$

which effectively satisfies, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|\Phi_\varepsilon(x, s, \xi)| \leq |\Phi(x, s, \xi)| \text{ and } |\Phi_\varepsilon(x, s, \xi)| \leq \frac{1}{\varepsilon}. \quad (5.3)$$

Lemma 5.1. *For each $0 < \varepsilon \leq 1$, there exists a weak solution u_ε to problem (5.1) such that $u_\varepsilon \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.*

Proof. Let $0 < \varepsilon \leq 1$. We define the operator $G_\varepsilon : W_0^{1,p}(\Omega, \omega) \rightarrow W^{-1,p'}(\Omega, \omega^{1-p'})$ by

$$G_\varepsilon = B_\varepsilon + L_\varepsilon,$$

where, for all $u, \varpi \in W_0^{1,p}(\Omega, \omega)$,

$$\langle B_\varepsilon u, \varpi \rangle = \int_{\Omega} \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) - \operatorname{div}(a(x, u, \nabla u)) \right) \varpi \, dx$$

and

$$\langle L_\varepsilon u, \varpi \rangle = \int_{\Omega} \Phi_\varepsilon(x, u, \nabla u) \varpi \, dx.$$

Further, we will show that the operator G_ε is bounded, coercive and pseudo-monotone.

Let $u, \varpi \in W_0^{1,p}(\Omega, \omega)$. By Hölder's inequality, (3.2), (3.7), and (3.4), we find

$$\begin{aligned} |\langle G_\varepsilon u, \varpi \rangle| &= \left| \int_{\Omega} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) \varpi \, dx + \int_{\Omega} a(x, u, \nabla u) \nabla \varpi \, dx + \int_{\Omega} \Phi_\varepsilon(x, u, \nabla u) \varpi \, dx \right| \\ &\leq \left\| \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) \right\|_{q', \sigma^{1-q'}} \|\varpi\|_{q, \sigma} \\ &\quad + \|a(x, u, \nabla u)\|_{p', \omega^{1-p'}} \|\nabla \varpi\|_{p, \omega} + \left\| \frac{1}{\varepsilon} \right\|_{q', \sigma^{1-q'}} \|\varpi\|_{q, \sigma} \\ &\leq C_1 \|\varpi\|_X + \left(\|k\|_{p'} + \|u\|_{q, \sigma}^q + \|\nabla u\|_{p, \omega}^p \right) \|\varpi\|_X + C_2 \|\varpi\|_X \\ &\leq C_3 \|\varpi\|_X, \end{aligned}$$

which shows that G_ε is bounded.

Next, for all $u \in W_0^{1,p}(\Omega, \omega)$, we get

$$\langle G_\varepsilon u, u \rangle = \int_{\Omega} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) u \, dx + \int_{\Omega} a(x, u, \nabla u) \nabla u \, dx + \int_{\Omega} \Phi_\varepsilon(x, u, \nabla u) u \, dx.$$

Noting that the first term on the right-hand side is nonnegative, it follows from (3.6) that

$$\langle G_\varepsilon u, u \rangle \geq \alpha \|u\|_X^p - C_2 \|u\|_X.$$

Then

$$\frac{\langle G_\varepsilon u, u \rangle}{\|u\|_X} \geq \alpha \|u\|_X^{p-1} - C_2.$$

Consequently, G_ε is coercive.

Now, it remains to show that G_ε is pseudo-monotone. Indeed, let $(u_k)_k \subset X$ be such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } X, \\ G_\varepsilon u_k &\rightharpoonup \chi \quad \text{weakly in } W^{-1,p'}(\Omega, \omega^{1-p'}), \end{aligned}$$

$$\limsup_{k \rightarrow +\infty} \langle G_\varepsilon u_k, u_k \rangle \leq \langle \chi, u \rangle.$$

We will prove that

$$\chi = G_\varepsilon u \quad \text{and} \quad \langle G_\varepsilon u_k, u_k \rangle \rightarrow \langle \chi, u \rangle \quad \text{as } k \rightarrow +\infty. \quad (5.4)$$

Since $(u_k)_k$ is a bounded sequence in X , by (3.7) and (3.4), we deduce that $(a(x, u_k, \nabla u_k))_k$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'})$, and hence there exists a function $h_\varepsilon \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'})$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup h_\varepsilon \quad \text{weakly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}) \text{ as } k \rightarrow \infty.$$

Similarly, it is easy to see that $(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_k)) + \Phi_\varepsilon(x, u_k, \nabla u_k))_k$ is bounded in $L^{r'}(\Omega)$, where r is given in (3.5). Hence there exists a function $k_\varepsilon \in L^{r'}(\Omega)$ such that

$$\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_k)) + \Phi_\varepsilon(x, u_k, \nabla u_k) \rightharpoonup k_\varepsilon \quad \text{weakly in } L^{r'}(\Omega) \text{ as } k \rightarrow \infty.$$

It is clear that for all $v \in X$, we have

$$\begin{aligned} \langle \chi, v \rangle &= \lim_{k \rightarrow +\infty} \langle G_\varepsilon u_k, v \rangle \\ &= \lim_{k \rightarrow +\infty} \int_{\Omega} \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_k)) + \Phi_\varepsilon(x, u_k, \nabla u_k) \right) v \, dx \\ &\quad + \lim_{k \rightarrow +\infty} \int_{\Omega} a(x, u_k, \nabla u_k) \nabla v \, dx. \end{aligned}$$

Consequently, we obtain

$$\langle \chi, v \rangle = \int_{\Omega} h_\varepsilon \cdot \nabla v \, dx + \int_{\Omega} k_\varepsilon v \, dx, \quad v \in X. \quad (5.5)$$

On the one hand, we have

$$\int_{\Omega} \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_k)) + \Phi_\varepsilon(x, u_k, \nabla u_k) \right) u_k \, dx \longrightarrow \int_{\Omega} k_\varepsilon u \, dx \quad \text{as } k \rightarrow \infty. \quad (5.6)$$

On the other hand, by the hypotheses, we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \left\{ \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u_k \, dx + \int_{\Omega} \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_k)) + \Phi_\varepsilon(x, u_k, \nabla u_k) \right) u_k \, dx \right\} \\ \leq \int_{\Omega} h_\varepsilon \cdot \nabla u \, dx + \int_{\Omega} k_\varepsilon u \, dx. \end{aligned}$$

Therefore,

$$\limsup_{k \rightarrow \infty} \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u_k \, dx \leq \int_{\Omega} h_\varepsilon \cdot \nabla u \, dx. \quad (5.7)$$

By virtue of (3.8), we have

$$\int_{\Omega} (a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u)) \cdot (\nabla u_k - \nabla u) dx > 0.$$

Consequently,

$$\begin{aligned} \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u_k dx &\geq - \int_{\Omega} a(x, u_k, \nabla u) \cdot \nabla u dx + \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u dx \\ &\quad + \int_{\Omega} a(x, u_k, \nabla u) \cdot \nabla u_k dx. \end{aligned}$$

Hence,

$$\liminf_{k \rightarrow \infty} \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u_k dx \geq \int_{\Omega} h_{\varepsilon} \cdot \nabla u dx, \quad (5.8)$$

which implies, by using (5.7), that

$$\lim_{k \rightarrow \infty} \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u_k dx = \int_{\Omega} h_{\varepsilon} \cdot \nabla u dx. \quad (5.9)$$

By means of (5.5), (5.6) and (5.9), we obtain

$$\langle G_{\varepsilon} u_k, u_k \rangle \rightarrow \langle \chi, u \rangle \quad \text{as } k \rightarrow +\infty.$$

Moreover, by (5.9) and the fact that

$$a(x, u_k, \nabla u) \rightarrow a(x, u, \nabla u) \quad \text{strongly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}),$$

it follows that

$$\lim_{k \rightarrow \infty} \int_{\Omega} (a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u)) \cdot (\nabla u_k - \nabla u) dx = 0.$$

Hence, by Lemma 4.3,

$$\nabla u_k \rightarrow \nabla u \quad \text{a.e. in } \Omega.$$

We then deduce that

$$a(x, u_k, \nabla u_k) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}),$$

and

$$\gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_k)) + \Phi_{\varepsilon}(x, u_k, \nabla u_k) \rightharpoonup \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u)) + \Phi_{\varepsilon}(x, u, \nabla u) \quad \text{weakly in } L^{r'}(\Omega).$$

Thus, it follows that $\chi = G_{\varepsilon} u$.

Therefore, since $f \in W^{-1,p'}(\Omega, \omega^{1-p'})$, there exists at least one weak solution u_{ε} to problem (5.1) (cf. Theorem 2.1 and Remark 2.1 in Chapter 2 of [25]). That is, for all $\varphi \in X$, we have

$$\int_{\Omega} \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon})) \varphi dx + \int_{\Omega} a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \varphi dx + \int_{\Omega} \Phi_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \varphi dx = \langle f, \varphi \rangle, \quad (5.10)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and his dual $W^{-1,p'}(\Omega, \omega^{1-p'})$. The lemma is proved. $\square$

Step 2: A priori estimates and basic convergence results. In this step, we will establish some a priori estimates for the solutions u_ε of the problems defined in (5.1). These estimates will help us to prove Theorem 3.3.

First, let us define the function

$$\theta(s) = \int_0^s \frac{\Phi_1(\tau)}{\alpha} d\tau, \quad (5.11)$$

where Φ_1 refers to the function appearing in (3.9). Recall that $\Phi_1 \in L^1(\mathbb{R})$, then

$$0 \leq \theta(s) \leq \theta(\infty) = \int_0^\infty \frac{\Phi_1(\tau)}{\alpha} d\tau$$

is a finite real number.

Lemma 5.2. *Let u_ε be a weak solution of (5.1) and $\psi \in X \cap L^\infty(\Omega)$ with $\psi \geq 0$. Then the following assertions hold:*

1. *We have*

$$\begin{aligned} \int_{\Omega} e^{\theta(u_\varepsilon)} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \psi dx + \int_{\Omega} e^{\theta(u_\varepsilon)} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi dx \\ \leq \int_{\Omega} e^{\theta(u_\varepsilon)} f \psi dx + \int_{\Omega} e^{\theta(u_\varepsilon)} |\Phi_0| \psi dx, \end{aligned} \quad (5.12)$$

$$\begin{aligned} \int_{\Omega} e^{-\theta(u_\varepsilon)} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \psi dx + \int_{\Omega} e^{-\theta(u_\varepsilon)} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi dx \\ + \int_{\Omega} e^{-\theta(u_\varepsilon)} \Phi_0 \psi dx \geq \int_{\Omega} e^{-\theta(u_\varepsilon)} f \psi dx. \end{aligned} \quad (5.13)$$

2. *If Ψ is a locally Lipschitz continuous, odd and increasing real function such that $\Psi(0) = 0$, then there exists $C > 0$ such that*

$$\int_{\Omega} \omega \Psi'(u_\varepsilon) |\nabla u_\varepsilon|^p dx \leq C \int_{\Omega} |\Psi(u_\varepsilon)| dx. \quad (5.14)$$

Proof. 1. Letting $\psi \geq 0$ and taking $e^{\theta(u_\varepsilon)} \psi$ as a test function in (5.10), we get

$$\begin{aligned} \int_{\Omega} e^{\theta(u_\varepsilon)} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \psi dx + \int_{\Omega} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla (e^{\theta(u_\varepsilon)} \psi) dx \\ + \int_{\Omega} \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) e^{\theta(u_\varepsilon)} \psi dx = \int_{\Omega} e^{\theta(u_\varepsilon)} f \psi dx. \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega} e^{\theta(u_\varepsilon)} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \psi dx + \int_{\Omega} \frac{\Phi_1(u_\varepsilon)}{\alpha} e^{\theta(u_\varepsilon)} \psi a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \\ + \int_{\Omega} e^{\theta(u_\varepsilon)} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi dx \end{aligned}$$

$$\leq \int_{\Omega} e^{\theta(u_{\varepsilon})} f \psi \, dx + \int_{\Omega} e^{\theta(u_{\varepsilon})} |\Phi_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon})| \psi \, dx.$$

By (3.6) and (3.9), we obtain

$$\begin{aligned} \int_{\Omega} e^{\theta(u_{\varepsilon})} \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon})) \psi \, dx + \int_{\Omega} e^{\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \psi \, dx \\ \leq \int_{\Omega} e^{\theta(u_{\varepsilon})} f \psi \, dx + \int_{\Omega} |\Phi_0| e^{\theta(u_{\varepsilon})} \psi \, dx. \end{aligned}$$

Thus (5.12) holds.

Letting $\psi \geq 0$ and taking $e^{-\theta(u_{\varepsilon})} \psi$ as a test function in (5.10), we have after simplification that

$$\begin{aligned} \int_{\Omega} e^{-\theta(u_{\varepsilon})} \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon})) \psi + \int_{\Omega} e^{-\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \psi \\ + \int_{\Omega} e^{-\theta(u_{\varepsilon})} \Phi_0 \psi \, dx \geq \int_{\Omega} e^{\theta(u_{\varepsilon})} f \psi \, dx. \end{aligned}$$

Thus (5.13) holds.

2. Let Ψ be a locally Lipschitz continuous, odd and increasing real function such that $\Psi(0) = 0$. Choosing $\psi = \Psi(u_{\varepsilon}^+)$ in (5.12) and $\psi = \Psi(u_{\varepsilon}^-)$ in (5.13), we derive the following inequalities:

$$\begin{aligned} \int_{\Omega} e^{\theta(u_{\varepsilon})} \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon})) \Psi(u_{\varepsilon}^+) \, dx + \int_{\Omega} e^{\theta(u_{\varepsilon})} \Psi'(u_{\varepsilon}) a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^+ \, dx \\ \leq \int_{\Omega} e^{\theta(u_{\varepsilon})} f \Psi(u_{\varepsilon}^+) \, dx + \int_{\Omega} e^{\theta(u_{\varepsilon})} |\Phi_0| \Psi(u_{\varepsilon}^+) \, dx \quad (5.15) \end{aligned}$$

and

$$\begin{aligned} - \int_{\Omega} e^{-\theta(u_{\varepsilon})} \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon})) \Psi(u_{\varepsilon}^-) \, dx - \int_{\Omega} e^{-\theta(u_{\varepsilon})} \Psi'(u_{\varepsilon}) a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^- \, dx \\ \leq - \int_{\Omega} e^{-\theta(u_{\varepsilon})} f \Psi(u_{\varepsilon}^-) \, dx + \int_{\Omega} e^{-\theta(u_{\varepsilon})} |\Phi_0| \Psi(u_{\varepsilon}^-) \, dx. \quad (5.16) \end{aligned}$$

Neglecting the nonnegative terms in (5.15) and (5.16), and using the inequality $|\Psi(u_{\varepsilon}^{\pm})| \leq |\Psi(u_{\varepsilon})|$, we obtain after adding up both formulas

$$\begin{aligned} \int_{\Omega} e^{\theta(u_{\varepsilon})} \Psi'(u_{\varepsilon}) a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^+ \, dx - \int_{\Omega} e^{-\theta(u_{\varepsilon})} \Psi'(u_{\varepsilon}) a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^- \, dx \\ \leq \int_{\Omega} e^{\theta(u_{\varepsilon})} |f| |\Psi(u_{\varepsilon})| \, dx + \int_{\Omega} e^{\theta(u_{\varepsilon})} |\Phi_0| |\Psi(u_{\varepsilon})| \, dx \\ + \int_{\Omega} e^{-\theta(u_{\varepsilon})} |f| |\Psi(u_{\varepsilon})| \, dx + \int_{\Omega} e^{-\theta(u_{\varepsilon})} |\Phi_0| |\Psi(u_{\varepsilon})| \, dx. \end{aligned}$$

Now, noting that $e^{-\theta(\infty)} \leq e^{\pm\theta(u_{\varepsilon})} \leq e^{\theta(\infty)}$ and $f, \Phi_0 \in L^{\infty}(\Omega)$, we get

$$\int_{\Omega} e^{\theta(u_{\varepsilon})} \Psi'(u_{\varepsilon}) a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^+ \, dx$$

$$- \int_{\Omega} e^{-\theta(u_{\varepsilon})} \Psi'(u_{\varepsilon}) a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^{-} dx \leq C.$$

Using the coercivity condition (3.6), we deduce that

$$\int_{\Omega} \omega \Psi'(u_{\varepsilon}) |\nabla u_{\varepsilon}|^p dx \leq C \int_{\Omega} |\Psi(u_{\varepsilon})| dx. \quad (5.17)$$

The lemma is proved. $\square$

Corollary 5.3. *The sequence $(u_{\varepsilon})_{\varepsilon}$ is bounded in $X \cap L^{\infty}(\Omega)$.*

Proof. Setting $\Psi(s) = G_k(s)$ in (5.17), it follows that

$$\int_{\Omega} \omega |\nabla G_k(u_{\varepsilon})|^p dx \leq C \int_{\Omega(k)} |G_k(u_{\varepsilon})| dx, \quad (5.18)$$

where $\Omega(k) := \{x \in \Omega : |u_{\varepsilon}| > k\}$. By Hölder's inequality together with (3.5), we find

$$\int_{\Omega} \omega |\nabla G_k(u_{\varepsilon})|^p dx \leq C |\Omega(k)|^{\frac{1}{r'}} \|G_k(u_{\varepsilon})\|_X,$$

which gives

$$\|G_k(u_{\varepsilon})\|_X^{p-1} \leq C |\Omega(k)|^{\frac{1}{r'}}.$$

For $1 < k < h$, the continuous embedding (3.5) and the previous inequality imply that

$$\begin{aligned} (h-k)^p |\Omega(h)|^{\frac{p}{r}} &= \left(\int_{\Omega(h)} (h-k)^r dx \right)^{\frac{p}{r}} \leq \left(\int_{\Omega(h)} (|u_{\varepsilon}| - k)^r dx \right)^{\frac{p}{r}} \\ &\leq \left(\int_{\Omega(k)} (|u_{\varepsilon}| - k)^r dx \right)^{\frac{p}{r}} \leq \left(\int_{\Omega(k)} |G_k(u_{\varepsilon})|^r dx \right)^{\frac{p}{r}} \\ &\leq C \int_{\Omega} \omega |\nabla G_k(u_{\varepsilon})|^p dx \leq C_3 |\Omega(k)|^{\frac{p'}{r'}}. \end{aligned}$$

Thus, we deduce that

$$|\Omega(h)| \leq \frac{C_5}{(h-k)^r} |\Omega(k)|^{\frac{rp'}{pr'}}.$$

Since $r > p$ implies $\frac{rp'}{pr'} > 1$, we can proceed as in [30] to conclude that $u_{\varepsilon} \in L^{\infty}(\Omega)$, and there exists a positive constant C_{∞} such that

$$\|u_{\varepsilon}\|_{L^{\infty}(\Omega)} \leq C_{\infty}. \quad (5.19)$$

Letting now $\Psi(s) = s$ in (5.17), we get

$$\int_{\Omega} \omega |\nabla u_{\varepsilon}|^p dx \leq C_6 \int_{\Omega} |u_{\varepsilon}| dx = C_7. \quad (5.20)$$

Hence,

$$\|u_{\varepsilon}\|_X \leq C_8, \quad (5.21)$$

which concludes the proof. $\square$

As a consequence of Corollary 5.3, we can extract a subsequence of u_ε , still denoted by u_ε , such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } X, \quad (5.22)$$

$$u_\varepsilon \rightarrow u \quad \text{strongly in } L^q(\Omega, \sigma) \quad \text{and a.e. in } \Omega, \quad (5.23)$$

$$-C_\infty \leq u_\varepsilon \leq C_\infty. \quad (5.24)$$

Taking into account (5.24), we get that $|\gamma_\varepsilon(T_\frac{1}{\varepsilon}(u_\varepsilon))|$ is uniformly bounded. Consequently, there exists $v \in L^\infty(\Omega)$ such that

$$\gamma_\varepsilon(T_\frac{1}{\varepsilon}(u_\varepsilon)) \xrightarrow{*} v \text{ in } L^\infty(\Omega). \quad (5.25)$$

Moreover,

$$-\gamma^0(-C_\infty) \leq v \leq \gamma^0(C_\infty), \quad (5.26)$$

where γ^0 denotes the minimal selection of the graph of γ . Namely, $\gamma^0(l)$ is the minimal in the norm element of $\beta(l)$, that is,

$$\gamma^0(l) = \inf\{|r| : r \in \mathbb{R} \text{ and } r \in \beta(l)\}.$$

Since $u_\varepsilon \rightarrow u$ strongly in $L^1(\Omega)$, applying [9, Lemma G], it follows that $v(x) \in \gamma(u(x))$ a.e. on Ω .

On top of that, using the growth condition (3.7), we have, for each $i = 1, \dots, N$,

$$\begin{aligned} & \int_{\Omega} \omega_i(x)^{1-p'} |a_i(x, u_\varepsilon, \nabla u_\varepsilon)|^{p'} dx \\ & \leq C_9 \int_{\Omega} \left(k^{p'}(x) + \sigma(x) |u_\varepsilon|^q + \sum_{j=1}^N \omega_j(x) |\partial_j u_\varepsilon|^p \right) dx \\ & \leq C_{10} \int_{\Omega} \left(k^{p'}(x) + \sigma(x) |u_\varepsilon|^q + \sum_{j=1}^N \omega_j(x) |\partial_j u_\varepsilon|^p \right) dx \leq C_{11}. \end{aligned}$$

Thus, from the Hardy-type inequality and Corollary 5.3, the sequence $(a(x, u_\varepsilon, \nabla u_\varepsilon))_\varepsilon$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'})$. Therefore, there exists a function $\vartheta \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'})$ such that

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup \vartheta \quad \text{weakly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}). \quad (5.27)$$

Now we have all the ingredients to demonstrate the strong convergence of the sequence $(u_\varepsilon)_\varepsilon$.

Lemma 5.4. *Suppose that the hypotheses of Theorem 3.3 are satisfied and let u_ε be a weak solution of (5.1). Then*

$$u_\varepsilon \rightarrow u \text{ strongly in } X \text{ as } \varepsilon \rightarrow 0. \quad (5.28)$$

Proof. To establish this lemma, it is enough to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) - a(x, u_{\varepsilon}, \nabla u)) \cdot \nabla(u_{\varepsilon} - u) \, dx = 0 \quad (5.29)$$

and then apply Lemma 4.3.

First, we have

$$\begin{aligned} & e^{-\theta(\infty)} \int_{\Omega} (a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) - a(x, u_{\varepsilon}, \nabla u)) \cdot \nabla(u_{\varepsilon} - u) \, dx \\ & \leq \int_{\{u_{\varepsilon} - u \geq 0\}} e^{\theta(u_{\varepsilon})} (a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) - a(x, u_{\varepsilon}, \nabla u)) \cdot \nabla(u_{\varepsilon} - u) \, dx \\ & \quad + \int_{\{u_{\varepsilon} - u \leq 0\}} e^{-\theta(u_{\varepsilon})} (a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) - a(x, u_{\varepsilon}, \nabla u)) \cdot \nabla(u_{\varepsilon} - u) \, dx \\ & =: A_{\varepsilon} - B_{\varepsilon} + C_{\varepsilon} - D_{\varepsilon}, \end{aligned}$$

where

$$\begin{aligned} A_{\varepsilon} &= \int_{\{u_{\varepsilon} - u \geq 0\}} e^{\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla(u_{\varepsilon} - u) \, dx, \\ B_{\varepsilon} &= \int_{\{u_{\varepsilon} - u \geq 0\}} e^{\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u) \cdot \nabla(u_{\varepsilon} - u) \, dx, \\ C_{\varepsilon} &= \int_{\{u_{\varepsilon} - u \leq 0\}} e^{-\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla(u_{\varepsilon} - u) \, dx, \\ D_{\varepsilon} &= \int_{\{u_{\varepsilon} - u \leq 0\}} e^{-\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u) \cdot \nabla(u_{\varepsilon} - u) \, dx. \end{aligned}$$

For B_{ε} , by (3.7) and (5.23), it follows that

$$e^{\theta(u_{\varepsilon})} a(x, u_{\varepsilon}, \nabla u) \longrightarrow e^{\theta(u)} a(x, u, \nabla u) \text{ strongly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}).$$

Then (5.22) implies that

$$\lim_{\varepsilon \rightarrow 0} B_{\varepsilon} = 0. \quad (5.30)$$

For D_{ε} , by reasoning in the same way as for B_{ε} , we readily obtain that

$$\lim_{\varepsilon \rightarrow 0} D_{\varepsilon} = 0. \quad (5.31)$$

For A_{ε} , choosing $\psi = (u_{\varepsilon} - u)^+$ in (5.12), we obtain

$$\int_{\Omega} e^{\theta(u_{\varepsilon})} \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon}))(u_{\varepsilon} - u)^+ \, dx + A_{\varepsilon} \leq \int_{\Omega} e^{\theta(u_{\varepsilon})} (f + \Phi_0)(u_{\varepsilon} - u)^+ \, dx.$$

Combining (5.23), which yields $u_{\varepsilon} \rightarrow u$ in $L^1(\Omega)$, with the facts that $(f + \Phi_0) \in L^{\infty}(\Omega)$ and $\gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon})) \xrightarrow{*} v$ in $L^{\infty}(\Omega)$, we get

$$\limsup_{\varepsilon \rightarrow 0} A_{\varepsilon} \leq 0. \quad (5.32)$$

For A_ε , choosing $\psi = (u_\varepsilon - u)^-$ in (5.13) and reasoning as above, we can deduce that

$$\limsup_{\varepsilon \rightarrow 0} C_\varepsilon \leq 0. \quad (5.33)$$

Finally, bringing together (5.30)–(5.33) gives us

$$\limsup_{\varepsilon \rightarrow 0} A_\varepsilon - B_\varepsilon + C_\varepsilon - D_\varepsilon \leq 0. \quad (5.34)$$

From (5.34), together with (3.8), we deduce (5.29). The lemma is proved. $\square$

As a direct consequence of the strong convergence ($u_\varepsilon \rightarrow u$ strongly in X), we have for a subsequence, still denoted by u_ε , that

$$\nabla u_\varepsilon \rightarrow \nabla u \quad \text{a.e. in } \Omega. \quad (5.35)$$

Step 3: Strong convergence of $(\Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))_\varepsilon$.

Lemma 5.5. *Let u_ε be a weak solution of (5.1). Then*

$$\Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow \Phi(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \quad (5.36)$$

Proof. On the one hand, by (5.23) and (5.35), we obtain

$$\Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow \Phi(x, u, \nabla u) \quad \text{a.e. in } \Omega. \quad (5.37)$$

On the other hand, having in mind (5.24) and using (3.9), there exists a constant $C_{12} > 0$ such that

$$|\Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \leq \Phi_0(x) + \Phi_1(u_\varepsilon)\omega(x) |\nabla u_\varepsilon|^p \leq \Phi_0(x) + C_{12}\omega(x) |\nabla u_\varepsilon|^p.$$

Then, from the equi-integrability of the sequence $(|\nabla u_\varepsilon|^p)_\varepsilon$, which follows from (5.28), and $\Phi_0 \in L^1(\Omega)$, we conclude that $(\Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))_\varepsilon$ is equi-integrable.

Hence, by virtue of Vitali's theorem, we find (5.36). $\square$

Step 4: Passing to the limit. From (5.23) and (5.35), we have

$$a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow a(x, u, \nabla u) \quad \text{a.e. in } \Omega. \quad (5.38)$$

The previous convergence, together with (5.27), yields

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}). \quad (5.39)$$

Finally, in view of (5.25), (5.36) and (5.39), we can pass to the limit in (5.10) to have

$$\int_{\Omega} v\varphi \, dx + \int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} \Phi(x, u, \nabla u) \varphi \, dx = \int_{\Omega} f\varphi \, dx$$

for any $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

So, this last step finishes the proof of Theorem 3.3.

6. Example

In this final section, we will provide an illustrative example of our problem. To do this, we take $\Omega = B_{\mathbb{R}^N}(0, 1)$, where $B_{\mathbb{R}^N}(0, 1)$ is the unit ball of $\mathbb{R}^N$. Consider the Carathéodory functions:

$$a_i(x, s, \xi) = w_i |\xi_i|^{p-2} \xi_i \quad \text{for } i = 1, \dots, N$$

and

$$\Phi(x, s, \xi) = \frac{1}{s^2 + 1} \sum_{i=1}^N \omega_i(x) |\xi_i|^p,$$

and γ is the maximal monotone graph defined by $\gamma(s) = (s-1)^+ - (s+1)^-$, where $p \geq 2$, w_i and σ are given weight functions strictly positive almost everywhere in Ω . In particular, let us use the special weight functions expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \text{dist}(x, \partial\Omega)$ and set

$$w_i(x) = w(x) = d^\varepsilon(x), \quad \sigma(x) = d^\kappa(x), \quad x \in \Omega, \quad \text{for all } i = 0, \dots, N.$$

In this case, the Hardy-type inequality reads

$$\left(\int_{\Omega} |u(x)|^q d^\kappa(x) dx \right)^{1/q} \leq c \left(\int_{\Omega} |\nabla u(x)|^p d^\varepsilon(x) dx \right)^{1/p}.$$

If $q > 1$ with $\kappa(q' - 1) < 1$, then the corresponding embedding (3.4) is compact (see, for example, [24, pp. 40–41]).

It is straightforward to demonstrate that $a_i(x, s, \xi)$ are the Carathéodory functions that satisfy the growth condition (3.7), the coercivity (3.6) and the strict monotonicity condition (3.8). Also, the Carathéodory function $\Phi(x, s, \xi)$ satisfies the condition (3.9), where $\Phi_1(s) = \frac{1}{1+s^2}$ is a function belonging to $L^1(\mathbb{R})$ and $\Phi_0 = 0$.

Finally, for any L^∞ -function f , the problem

$$\begin{cases} \gamma(u) - \text{div} \left(d^\varepsilon(x) |\nabla u|^{p-2} \nabla u \right) + \Phi(x, u, \nabla u) \ni f \text{ in } D'(B_{\mathbb{R}^N}(0, 1)), \\ u \in W_0^{1,p}(\Omega, d^\varepsilon(x)) \cap L^\infty(\Omega), \quad \Phi(x, u, \nabla u) \in L^1(B_{\mathbb{R}^N}(0, 1)), \end{cases}$$

has at least one solution.

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**Обмежені розв'язки сильно нелінійних еліптичних
задач без обмежень на знак у вагових просторах
Соболева**

Hassan El Hamri, Morad Ouboufettal, and Youssef Akdim

У цій роботі ми досліджуємо існування та L^∞ -регулярність розв'язків для загального класу сильно нелінійних еліптичних задач, пов'язаних із диференціальним включенням

$$\gamma(u) + A(u) + \Phi(x, u, \nabla u) \ni f,$$

де A є оператором Лере–Ліонса з $W_0^{1,p}(\Omega, \omega)$ у його спряжений простір $W^{-1,p'}(\Omega, \omega^*)$, γ є максимальним монотонним відображенням з $0 \in \gamma(0)$, Φ є нелінійним членом, що задовольняє лише умову росту без жодних обмежень на знак щодо s , а права частина f , як передбачається, належить до $L^\infty(\Omega)$.

Ключові слова: задачі про включення, L^∞ -оцінка, p -зростання, слабкий розв'язок, вагові простори Соболева