

Hopf Real Hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ with Anticommutative Structure Jacobi Operator and Structure Lie Operator

Dehe Li

In this paper, we classify Hopf real hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ for which the structure Jacobi operator and structure Lie operator are anticommutative.

Key words: real hypersurface, structure Jacobi operator, structure Lie operator, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$

Mathematical Subject Classification 2020: 53B25, 53B35, 53C42

1. Introduction

The Riemannian product $\mathbb{S}^2 \times \mathbb{S}^2$ of two unit 2-spheres and the Riemannian product $\mathbb{H}^2 \times \mathbb{H}^2$ of two hyperbolic planes are important and interesting Hermitian 4-manifolds. Here, we denote by $\bar{M}^2(\epsilon)$, the unit sphere $\mathbb{S}^2$ when $\epsilon = 1$, or the hyperbolic plane $\mathbb{H}^2$ when $\epsilon = -1$. In recent years, many results about real hypersurfaces in $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ have been obtained, we refer to, e.g., [2–6, 8–11, 13, 15] and the references therein.

Given a real hypersurface M in a Kähler manifold $(\tilde{M}, J, \tilde{g})$ with unit normal vector field N , then M has a standard almost contact metric structure (ϕ, ξ, η, g) induced from $\tilde{M}$, where $\xi = -JN$ is the Reeb vector field of M , ϕ is a $(1, 1)$ -tensor field. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{M}$. From the fact that $\tilde{\nabla}J = 0$ and the Gauss–Weingarten formulae, one can have

$$\nabla_X \xi = \phi AX, \quad X, Y \in TM, \quad (1.1)$$

where ∇ and A are the Levi-Civita connection and the shape operator of M , respectively. M is called a Hopf real hypersurface if the integral curves of ξ are geodesics. A real hypersurface of a Kähler manifold is Hopf if and only if the Reeb vector field is a principal curvature vector field (cf. [1]).

Related to the Reeb vector field ξ , one can consider a $(1, 1)$ -tensor field: the structure Lie operator L_ξ defined by $g(L_\xi X, Y) := (\mathcal{L}_\xi g)(X, Y)$ for any vector fields X, Y on M . From (1.1) we derive that

$$L_\xi X = (\phi A - A\phi)X, \quad X \in TM. \quad (1.2)$$

Let R be the Riemannian curvature tensor of type $(1, 3)$ on M . Then, associating with the the Reeb vector field ξ , one has the so-called structure Jacobi operator R_ξ of M defined by

$$R_\xi X = R(X, \xi)\xi, \quad X \in TM. \quad (1.3)$$

Concerning the structure Lie operator and the structure Jacobi operator of real hypersurfaces, There has been considerable research in various ambient spaces, see e.g. [3, 7, 8, 12, 14, 15]. For example, Gao–Hu–Ma–Yao [3] and Zhang–Gao–Hu–Yao [15] classified real hypersurfaces in both $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ with isometric Reeb flow. By (1.2), the condition of isometric Reeb flow is equivalent to $L_\xi = 0$. Hu–Zhang [9] classified the real hypersurfaces according to various conditions on the structure Jacobi operator of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, such as real hypersurfaces with vanishing structure Jacobi operator and real hypersurfaces with parallel structure Jacobi operator. Hu–Lu [7] classified the real hypersurfaces with parallel or η -parallel structure Lie operators of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$. In this paper, we consider Hopf real hypersurfaces in $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ such that their structure Jacobi operator and structure Lie operator are anticommutative, i.e., $L_\xi R_\xi + R_\xi L_\xi = 0$.

Before stating our results, we recall some typical examples. Notice that for the Riemannian product $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$, there exists a natural product structure P defined by $P(v_1, v_2) = (v_1, -v_2)$ for any tangent vector fields v_1, v_2 on $\bar{M}^2(\epsilon)$. Then, for an orientable real hypersurface M of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ with a unit normal vector field N , we can introduce a function C , defined by $C := \langle PN, N \rangle$. Here, $\langle \cdot, \cdot \rangle$ denotes the standard product metric on $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$. The geometry of real hypersurfaces in $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ is closely related to the function C . For the sake of brevity, we call C the product angle function of M . It is clear that $-1 \leq C \leq 1$. In the case $C^2 \equiv 1$ on M , Urbano [13] and Gao–Ma–Yao [4] proved respectively that there exists a smooth curve Γ in $\bar{M}^2(\epsilon)$ such that M is locally isometric to the real hypersurface M_Γ defined by

$$M_\Gamma := \{(x, y) \in \bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon) \mid x \in \Gamma, y \in \bar{M}^2(\epsilon)\}.$$

In the case $C = 0$, some typical real hypersurfaces of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$, constructed by Urbano [13] and Gao–Ma–Yao [4], are summarized as follows:

- (i) $M_t := \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, q \rangle_{\mathbb{R}^3} = t\}$ for $t \in (-1, 1)$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ is the canonical metric on $\mathbb{R}^3$;
- (ii) $M_\tau := \{(p, q) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid \langle p, q \rangle_{\mathbb{R}_1^3} = \tau\}$ for $\tau < -1$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}_1^3}$ is the Lorentzian metric on the three-dimensional Minkowski space $\mathbb{R}_1^3$;
- (iii) $M_{1,-1}^{1/2}$ is defined by the immersion

$$\begin{aligned} \Phi : \mathbb{R}^3 &\rightarrow \mathbb{H}^2 \times \mathbb{H}^2 \\ (t, r, s) &\mapsto (p(t, r), q(t, s)), \end{aligned}$$

with

$$\begin{cases} p(t, r) = \cosh\left(\frac{t}{\sqrt{2}}\right) \left(\frac{2+r^2}{2}, r, \frac{r^2}{2}\right) + \sinh\left(\frac{t}{\sqrt{2}}\right) \left(-\frac{r^2}{2}, -r, \frac{2-r^2}{2}\right), \\ q(t, s) = \cosh\left(\frac{t}{\sqrt{2}}\right) \left(\frac{2+s^2}{2}, s, \frac{s^2}{2}\right) + \sinh\left(\frac{t}{\sqrt{2}}\right) \left(\frac{s^2}{2}, s, \frac{-2+s^2}{2}\right). \end{cases}$$

From [3] and [15], we know that M_Γ is a Hopf real hypersurface with at most two distinct principal curvatures in $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$; M_t and M_τ are Hopf real

hypersurfaces with three distinct constant principal curvatures in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, respectively; $M_{1,-1}^{1/2}$ is a Hopf real hypersurface with two distinct constant principal curvatures in $\mathbb{H}^2 \times \mathbb{H}^2$.

Now, we state our result as follows:

Theorem 1.1. *Let M be a Hopf real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ that satisfies the condition $L_\xi R_\xi + R_\xi L_\xi = 0$. Then, up to isometries of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$, one of the following cases occurs:*

- (i) M is an open part of M_Γ for a smooth curve Γ in $\bar{M}^2(\epsilon)$;
- (ii) M is an open part of M_t in $\mathbb{S}^2 \times \mathbb{S}^2$ for some $t \in (-1, 1)$;
- (iii) M is an open part of M_τ in $\mathbb{H}^2 \times \mathbb{H}^2$ for some $\tau < -1$;
- (iv) M is an open part of $M_{1,-1}^{1/2}$ in $\mathbb{H}^2 \times \mathbb{H}^2$.

Remark 1.2. For $\mathbb{S}^2 \times \mathbb{S}^2$, the examples M_Γ and M_t first appeared in [13]. For $\mathbb{H}^2 \times \mathbb{H}^2$, the examples M_Γ , M_τ and $M_{1,-1}^{1/2}$ first appeared in [4].

The paper is organized as follows: In Section 2, we review basic materials about $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ and its geometry of real hypersurfaces. In Section 3, we present the proof of Theorem 1.1.

2. Preliminaries

2.1. The Kähler structure of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$. In this subsection, we recall the basic materials of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ from [4, 13]. Let $\bar{M}^2(\epsilon)$ be the unit 2-sphere $\mathbb{S}^2$ endowed with the canonical metric $\langle \cdot, \cdot \rangle$ when $\epsilon = 1$, and the hyperbolic plane $\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1, x_1 > 0\}$ endowed with the canonical metric $\langle \cdot, \cdot \rangle$ when $\epsilon = -1$.

The complex structure J on $\bar{M}^2(\epsilon)$ is defined as follows:

$$\begin{cases} J_p v = p \wedge v & \text{for any } p \in \mathbb{S}^2, v \in T_p \mathbb{S}^2, \\ J_p v = p \boxtimes v & \text{for any } p \in \mathbb{H}^2, v \in T_p \mathbb{H}^2, \end{cases}$$

where $\wedge$ is the cross product in $\mathbb{R}^3$ and $\boxtimes$ is the Lorentzian cross product in $\mathbb{R}_1^3$ defined by

$$(a_1, a_2, a_3) \boxtimes (b_1, b_2, b_3) = (a_3 b_2 - a_2 b_3, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

We endow $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ with the product metric $\langle \cdot, \cdot \rangle$ and the complex structures

$$J_1 = (J, J), \quad J_2 = (J, -J).$$

Then, with respect to J_1 or J_2 , $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ is a Kähler surface. The product structure P on $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ is defined as follows:

$$P(v_1, v_2) = (v_1, -v_2), \quad v_1, v_2 \in T_p(\bar{M}^2(\epsilon)).$$

So, P satisfies the following properties:

$$P = -J_1 J_2 = -J_2 J_1, \quad P^2 = \text{Id}, \quad \langle PX, Y \rangle = \langle X, PY \rangle, \quad X, Y \in T(\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)).$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$, we have $\bar{\nabla}P = 0$.

The curvature tensor $\bar{R}$ of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ is given by

$$\bar{R}(X, Y)Z = \frac{1}{2}\epsilon[\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle PY, Z \rangle PX - \langle PX, Z \rangle PY],$$

$$X, Y, Z \in (\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)). \quad (2.1)$$

2.2. Real hypersurfaces of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$. Let M be a real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ with a unit normal vector field N . We still use $\langle \cdot, \cdot \rangle$ to denote the induced metric on M . The smooth function $C : M \rightarrow \mathbb{R}$ and the tangent vector field V on M are defined by

$$C := \langle PN, N \rangle, \quad V := PN - CN,$$

where C is called the product angle function. It holds that $-1 \leq C \leq 1$ and $\|V\|^2 := \langle V, V \rangle = 1 - C^2$. We put

$$J_1X = \phi X + \eta(X)N, \quad PX = TX + \mu(X)N, \quad X \in M, \quad (2.2)$$

where ϕX and $\eta(X)N$ (resp. TX and $\mu(X)N$) are the tangent and normal parts of J_1X (resp. PX), respectively. So ϕ and T are tensor fields of type $(1, 1)$, η and μ are 1-forms on M . From (2.2), we have

$$\begin{cases} \eta(X) = \langle X, \xi \rangle, & \eta(\phi X) = 0, & \langle TX, Y \rangle = \langle X, TY \rangle, \\ \phi^2 X = -X + \eta(X)\xi, & \langle \phi X, Y \rangle = -\langle X, \phi Y \rangle, \\ \mu(X) = \langle PX, N \rangle, & \langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \end{cases} \quad (2.3)$$

where $\xi := -J_1N$ is the Reeb vector field of M .

Let ∇ be the Levi-Civita connection on M . Then the Gauss and Weingarten formulae are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle N, \quad \bar{\nabla}_X N = -AX, \quad X, Y \in TM, \quad (2.4)$$

where A is the shape operator of M .

Let R be the Riemannian curvature tensor of M . The Gauss and Codazzi equations of M in $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ are given by:

$$R(X, Y)Z = \frac{1}{2}\epsilon\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle TY, Z \rangle TX - \langle TX, Z \rangle TY\}$$

$$+ \langle AY, Z \rangle AX - \langle AX, Z \rangle AY, \quad X, Y, Z \in TM, \quad (2.5)$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{1}{2}\epsilon\{\langle X, V \rangle TY - \langle Y, V \rangle TX\}, \quad X, Y \in TM. \quad (2.6)$$

We also need the following useful formulas (cf. [4, 13]):

$$\nabla C = -2AV, \quad \nabla_X V = CAX - TAX, \quad X \in TM. \quad (2.7)$$

2.3. A canonical frame for real hypersurfaces of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$. Let M be a real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ with $|C| < 1$. Then, following [3], we consider the local orthonormal frame field $\{E_i\}_{i=1}^3$ on M as follows:

$$E_1 = J_1 N, \quad E_2 = \frac{J_2 N - C J_1 N}{\sqrt{1 - C^2}}, \quad E_3 = \frac{V}{\sqrt{1 - C^2}}. \quad (2.8)$$

So, the following relations hold:

$$\begin{cases} P E_1 = C E_1 + \sqrt{1 - C^2} E_2, & P E_2 = \sqrt{1 - C^2} E_1 - C E_2, \\ P E_3 = -C E_3 + \sqrt{1 - C^2} N, & T E_1 = C E_1 + \sqrt{1 - C^2} E_2, \\ T E_2 = \sqrt{1 - C^2} E_1 - C E_2, & T E_3 = -C E_3, \quad \phi E_1 = 0, \\ \phi E_2 = -E_3, \quad \phi E_3 = E_2, & \eta(E_1) = -1, \quad \eta(E_2) = 0, \quad \eta(E_3) = 0. \end{cases} \quad (2.9)$$

Put $A E_i = \sum_{j=1}^3 h_{ij} E_j$ with $h_{ij} = h_{ji}$, $1 \leq i, j \leq 3$. By (2.4)–(2.7), (2.9), and $\bar{\nabla} P = 0$, we get the following lemma.

Lemma 2.1 (cf. Subsection 2.3 of [9]). *Let M be a real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ with $|C| < 1$. Then, in terms of $\{E_i\}_{i=1}^3$ as defined by (2.8), it holds that*

$$E_i(C) = -2h_{i3}\sqrt{1 - C^2}, \quad i = 1, 2, 3. \quad (2.10)$$

$$\begin{cases} \nabla_{E_1} E_1 = -h_{13} E_2 + h_{12} E_3, & \nabla_{E_1} E_2 = h_{13} E_1 + \left(h_{11} - \frac{2C}{\sqrt{1 - C^2}} h_{12} \right) E_3, \\ \nabla_{E_1} E_3 = -h_{12} E_1 - \left(h_{11} - \frac{2C}{\sqrt{1 - C^2}} h_{12} \right) E_2, \\ \nabla_{E_2} E_1 = -h_{23} E_2 + h_{22} E_3, & \nabla_{E_2} E_2 = h_{23} E_1 + \left(h_{12} - \frac{2C}{\sqrt{1 - C^2}} h_{22} \right) E_3, \\ \nabla_{E_2} E_3 = -h_{22} E_1 - \left(h_{12} - \frac{2C}{\sqrt{1 - C^2}} h_{22} \right) E_2, \\ \nabla_{E_3} E_1 = -h_{33} E_2 + h_{23} E_3, & \nabla_{E_3} E_2 = h_{33} E_1 + \left(h_{13} - \frac{2C}{\sqrt{1 - C^2}} h_{23} \right) E_3, \\ \nabla_{E_3} E_3 = -h_{23} E_1 - \left(h_{13} - \frac{2C}{\sqrt{1 - C^2}} h_{23} \right) E_2. \end{cases} \quad (2.11)$$

Moreover, with respect to the frame $\{E_i\}_{i=1}^3$, the Gauss and Codazzi equations yield:

$$E_2(h_{11}) - E_1(h_{12}) = -2h_{11}h_{13} + 3h_{13}h_{22} - 3h_{12}h_{23} + \frac{2C}{\sqrt{1 - C^2}} h_{12}h_{13}, \quad (2.12)$$

$$E_2(h_{12}) - E_1(h_{22}) = -h_{12}h_{13} - h_{11}h_{23} - \frac{2h_{13}h_{22}C}{\sqrt{1 - C^2}} + \frac{4h_{12}h_{23}C}{\sqrt{1 - C^2}}, \quad (2.13)$$

$$E_2(h_{13}) - E_1(h_{23}) = -h_{13}^2 - h_{23}^2 + h_{33} \left(h_{22} - h_{11} + \frac{2h_{12}C}{\sqrt{1 - C^2}} \right), \quad (2.14)$$

$$E_3(h_{11}) - E_1(h_{13}) = 2h_{11}h_{12} + 3h_{13}h_{23} - 3h_{12}h_{33} - \frac{2h_{12}^2C}{\sqrt{1-C^2}} + \frac{\epsilon C\sqrt{1-C^2}}{2}, \quad (2.15)$$

$$E_3(h_{12}) - E_1(h_{23}) = h_{12}^2 + h_{23}^2 + h_{22}(h_{11} - h_{33}) - \frac{2h_{13}h_{23}C}{\sqrt{1-C^2}} + \frac{2h_{12}(h_{33} - h_{22})C}{\sqrt{1-C^2}} + \frac{\epsilon(1-C^2)}{2}, \quad (2.16)$$

$$E_3(h_{13}) - E_1(h_{33}) = h_{11}h_{23} + h_{12} \left(h_{13} - \frac{2h_{23}C}{\sqrt{1-C^2}} \right), \quad (2.17)$$

$$E_3(h_{22}) - E_2(h_{23}) = 2h_{12}h_{22} + \frac{(4h_{22}(h_{33} - h_{22}) - 8h_{23}^2 - \epsilon(1-C^2))C}{2\sqrt{1-C^2}} + h_{13}h_{23} + h_{12}h_{33}, \quad (2.18)$$

$$E_3(h_{23}) - E_2(h_{33}) = h_{13}(h_{22} + 2h_{33}) + h_{23} \left(h_{12} - \frac{2(h_{22} + h_{33})C}{\sqrt{1-C^2}} \right). \quad (2.19)$$

3. Proof of Theorem 1.1

Let M be a Hopf real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ that satisfies the condition $L_\xi R_\xi + R_\xi L_\xi = 0$. Since $0 \leq C^2 \leq 1$ on M , we can put $M = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{p \in M \mid C^2(p) < 1\}, \quad \Omega_2 = \{p \in M \mid C^2(p) = 1\}.$$

If Ω_1 is non-empty, then Ω_1 is an open subset of M .

Proposition 3.1. *Let M be a Hopf real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ that satisfies the condition $L_\xi R_\xi + R_\xi L_\xi = 0$. If Ω_1 is non-empty, then M has constant product angle function C ($C \neq \pm 1$). Moreover, up to isometries of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$, one of the following cases occurs:*

- (i) M is an open part of M_t in $\mathbb{S}^2 \times \mathbb{S}^2$ for some $t \in (-1, 1)$;
- (ii) M is an open part of M_τ in $\mathbb{H}^2 \times \mathbb{H}^2$ for some $\tau < -1$;
- (iii) M is an open part of $M_{1,-1}^{1/2}$ in $\mathbb{H}^2 \times \mathbb{H}^2$.

Proof. Since Ω_1 is non-empty, we can choose the local orthonormal frame field $\{E_i\}_{i=1}^3$ as defined by (2.8). Let $AE_i = \sum_{j=1}^3 h_{ij}E_j$ with $h_{ij} = h_{ji}$, $1 \leq i, j \leq 3$.

3. From (1.2) and (2.9), we obtain

$$\begin{aligned} L_\xi(E_1) &= h_{13}E_2 - h_{12}E_3, \\ L_\xi(E_2) &= h_{13}E_1 + 2h_{23}E_2 + (h_{33} - h_{22})E_3, \\ L_\xi(E_3) &= -h_{12}E_1 + (h_{33} - h_{22})E_2 - 2h_{23}E_3. \end{aligned} \quad (3.1)$$

It follows from (1.3), (2.5), (2.8), and $\xi = -J_1N$ that

$$\begin{aligned} R_\xi(E_1) &= 0, \\ R_\xi(E_2) &= (h_{11}h_{22} - h_{12}^2)E_2 + (h_{11}h_{23} - h_{12}h_{13})E_3, \end{aligned}$$

$$R_\xi(E_2) = (h_{11}h_{23} - h_{12}h_{13})E_2 + \left(\frac{1-C^2}{2}\epsilon + h_{11}h_{33} - h_{13}^2\right)E_3. \quad (3.2)$$

We substitute $X = E_i$ and $Y = E_j$, $1 \leq i, j \leq 3$, into $\langle L_\xi R_\xi X + R_\xi L_\xi X, Y \rangle = 0$, then by applying (3.1) and (3.2), we can derive that

$$h_{11}(h_{13}h_{22} - h_{12}h_{23}) = 0, \quad (3.3)$$

$$h_{11}(h_{13}h_{23} - h_{12}h_{33}) + \frac{\epsilon}{2}h_{12}(C^2 - 1) = 0, \quad (3.4)$$

$$-2h_{12}^2h_{23} + h_{12}h_{13}(h_{22} - h_{33}) + h_{11}h_{23}(h_{22} + h_{33}) = 0, \quad (3.5)$$

$$(h_{22} - h_{33})(h_{12}^2 + h_{13}^2 - h_{11}(h_{22} + h_{33})) + \frac{\epsilon}{2}(h_{22} - h_{33})(C^2 - 1) = 0, \quad (3.6)$$

$$h_{12}h_{13}(h_{22} - h_{33}) + h_{23}(2h_{13}^2 - h_{11}(h_{22} + h_{33}) + \epsilon(C^2 - 1)) = 0. \quad (3.7)$$

Since M is a Hopf real hypersurface, it follows that $h_{12} = h_{13} = 0$.

In the following, we further put $\Omega_1 = \Omega_{11} \cup \Omega_{12} \cup \Omega_{13}$, where

$$\begin{cases} \Omega_{11} = \{p \in M \mid h_{11}(p) \neq 0, h_{22}(p) + h_{33}(p) \neq 0\}, \\ \Omega_{12} = \{p \in M \mid h_{11}(p) = 0\}, \\ \Omega_{13} = \{p \in M \mid h_{11}(p) \neq 0, h_{22}(p) + h_{33}(p) = 0\}. \end{cases}$$

If Ω_{11} is non-empty, then Ω_{11} is an open subset of Ω_1 .

Claim 1. The set Ω_{12} does not contain any non-empty open subset of Ω_1 .

To verify this claim we argue by contradiction. Suppose on the contrary that Ω_{12} contains some non-empty open subset $\hat{\Omega}$ of Ω_1 , then $h_{11} \equiv 0$ on $\hat{\Omega}$. Now, by substituting $h_{11} = h_{13} = 0$ into (3.7), we get $h_{23} = 0$ on $\hat{\Omega}$. From $h_{11} = h_{12} = h_{13} = h_{23} = 0$, (2.14) and (2.16), we get that $h_{22}h_{33} = 0$ and $-h_{22}h_{33} + \frac{\epsilon}{2}(1 - C^2) = 0$ on $\hat{\Omega}$. It follows that $C^2 = 1$ on $\hat{\Omega}$, which is a contradiction. This verifies the assertion of **Claim 1**.

Claim 2. The set Ω_{13} does not contain any non-empty open subset of Ω_1 .

To verify **Claim 2** we argue again by contradiction. Suppose on the contrary that Ω_{13} contains some non-empty open subset $\bar{\Omega}$ of Ω_1 , then $h_{22} + h_{33} = 0$ on $\bar{\Omega}$. From (3.6) and (3.7), we derive that $h_{22} = h_{33} = h_{23} = 0$ on $\bar{\Omega}$. Then, by substituting $h_{12} = h_{13} = h_{23} = h_{22} = 0$ into (2.16), we derive that $C^2 = 1$ on $\bar{\Omega}$, which is a contradiction. This verifies the assertion of **Claim 2**.

From (3.5) and $h_{12} = h_{13} = 0$, we get that $h_{23} = 0$ on Ω_{11} . By **Claim 1** and **Claim 2**, we know that Ω_{11} is dense in Ω_1 .

Next, we further put $\Omega_{11} = \Omega_{111} \cup \Omega_{112}$, where

$$\Omega_{111} = \{p \in M \mid h_{11}(p) \neq 0, h_{22}(p) + h_{33}(p) \neq 0, h_{22}(p) \neq h_{33}(p)\},$$

$$\Omega_{112} = \{p \in M \mid h_{11}(p) \neq 0, h_{22}(p) + h_{33}(p) \neq 0, h_{22}(p) = h_{33}(p)\}.$$

Claim 3. The set Ω_{112} does not contain any non-empty open subset of Ω_{11} .

To verify **Claim 3** we argue again by contradiction. Suppose on the contrary that Ω_{112} contains some non-empty open subset $\check{\Omega}$ of Ω_{11} , then $h_{22} = h_{33}$ on $\check{\Omega}$. By substituting $h_{12} = h_{13} = h_{23} = 0$ and $h_{33} = h_{22}$ into (2.14) and (2.16), we

derive that $h_{22}(h_{22} - h_{11}) = 0$ and $h_{22}(h_{11} - h_{22}) + \frac{\epsilon}{2}(1 - C^2) = 0$. It follows that $C^2 = 1$ on $\check{\Omega}$, which is a contradiction. This verifies the assertion of **Claim 3**.

By **Claim 3**, we know that Ω_{111} is dense in Ω_{11} . Now, on Ω_{111} , according to $h_{12} = h_{13} = 0$, $h_{22} \neq h_{33}$ and (3.6), we deduce that

$$h_{11}(h_{22} + h_{33}) + \frac{\epsilon}{2}(1 - C^2) = 0. \quad (3.8)$$

From $h_{12} = h_{13} = h_{23} = 0$, (2.14) and (2.16), we get that

$$h_{33}(h_{22} - h_{11}) = 0, \quad (3.9)$$

$$h_{22}(h_{11} - h_{33}) + \frac{\epsilon}{2}(1 - C^2) = 0. \quad (3.10)$$

If $h_{33} \neq 0$ at some point p of Ω_{111} , by (3.9) we get that $h_{11} = h_{22}$. It follows from (3.8) and (3.10) that $h_{11}h_{33} = 0$ at p , which is a contradiction. Thus it holds that $h_{33} = 0$ on Ω_{111} . Thus, by (2.10), we derive that C is a constant on Ω_{111} . Since Ω_{111} is dense in Ω_{11} , and Ω_{11} is dense in Ω_1 , we have C is a constant on Ω_1 by continuity of the function C . According to the assumption that Ω_1 is non-empty, then Ω_2 is empty on M . It follows that M is a Hopf real hypersurface with constant product angle function C ($C \neq \pm 1$).

Now, when $\epsilon = 1$, we can apply Theorem 1.5 in [3] to derive that M is an open part of M_t . When $\epsilon = -1$, we can apply Theorem 2.2 in [15] to derive that M is either an open part of $M_{1,-1}^{1/2}$, or an open part of M_τ for $\tau < -1$. On the other hand, it can be checked that real hypersurfaces M_t , M_τ and $M_{1,-1}^{1/2}$ all satisfy the condition of $L_\xi R_\xi + R_\xi L_\xi = 0$. $\square$

Proposition 3.2. *Let M be a Hopf real hypersurface of $\bar{M}^2(\epsilon) \times \bar{M}^2(\epsilon)$ that satisfies the condition $L_\xi R_\xi + R_\xi L_\xi = 0$. If Ω_1 is empty, then M is an open part of M_Γ for a smooth curve Γ of $\bar{M}^2(\epsilon)$.*

Proof. If Ω_1 is empty, then $C^2 \equiv 1$ on M . Now, from Subsection 3.1 of [13] and Lemma 3.3 of [4], we know that M is an open part of M_Γ for a smooth curve Γ of $\bar{M}^2(\epsilon)$. On the other hand, since hypersurfaces M_Γ satisfy $A\phi = \phi A$, i.e., $L_\xi = 0$, they also satisfy the condition of $L_\xi R_\xi + R_\xi L_\xi = 0$. $\square$

Completion of the proof of Theorem 1.1. Based on Proposition 3.1 and Proposition 3.2, we have completed the proof of Theorem 1.1. $\square$

Acknowledgments. This project was supported by the Project of Youth Backbone Teachers of Colleges and Universities of Henan Province (No. 2023GGJS124) and the Program for Science & Technology Innovation Talents in Universities of Henan Province (No. 25HASTIT023).

References

- [1] A.A. Borisenko, *On the global structure of Hopf Hypersurfaces in complex space form*, Illinois journal of Mathematics, **45** (2001), 265–277.

- [2] J.B.M. dos Santos, J.P. dos Santos, *Isoparametric hypersurfaces in product spaces*, Differential Geom. Appl. **88** (2023), Paper No. 102005, 19 pp.
- [3] D. Gao, Z.J. Hu, H. Ma, and Z.K. Yao, *On real hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$* , Proc. Amer. Math. Soc. **150** (2022), 4447–4461.
- [4] D. Gao, H. Ma, and Z.K. Yao, *On hypersurfaces of $\mathbb{H}^2 \times \mathbb{H}^2$* , Sci. China Math. **67** (2024), 339–366.
- [5] D. Gao, H. Ma, and Z.K. Yao, *Isoparametric hypersurfaces in product spaces of space forms*, Differential Geom. Appl. **95** (2024), Paper No. 102155, 8 pp.
- [6] Z.J. Hu, and X.G. Lu, *On real hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ with parallel normal Jacobi operator*, Mediterr. J. Math. **21** (2024), Paper No. 133, 16 pp.
- [7] Z.J. Hu, and X.G. Lu, *Real hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ with parallel operators*, Mathematische Nachrichten **298** (2025), 2888–2900.
- [8] Z.J. Hu, X.G. Lu, Z.K. Yao, and X. Zhang, *On curvature-adapted hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$* , Internat. J. Math. **36** (2025), Paper No. 2450072, 22 pp.
- [9] Z.J. Hu, X. Zhang, *On some hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$* , Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **118** (2024), Paper No. 116, 16 pp.
- [10] H.Z. Li, L. Vrancken, X.F. Wang, and Z.K. Yao, *Hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant sectional curvature*, Calc. Var. Partial Differ. Equ. **63** (2024), Paper No. 167, 33 pp.
- [11] X.G. Lu, P. Wang, and X.Z. Wang, *On the geometry of hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$* , Results Math. **78** (2023), Paper No. 10, 18 pp.
- [12] Y.J. Suh, and G.J. Kim, *Real hypersurfaces in the complex quadric with Lie invariant structure Jacobi operator*, Canad. Math. Bull. **63** (2020), 204–221.
- [13] F. Urbano, *On hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$* , Comm. Anal. Geom. **27** (2019), 1381–1416.
- [14] Y. Wang, *Real hypersurfaces in nonflat complex space forms with cyclic parallel structure Lie operators*, Math. Notes. **116** (2024), 942–948.
- [15] X. Zhang, D. Gao, Z.J. Hu, and Z.K. Yao, *On Hopf hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$* , J. Geom. Phys. **194** (2023), Paper No. 104996, 14 pp.

Received August 20, 2025, revised October 28, 2025.

Dehe Li,

School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China,

E-mail: lidehehe@163.com

Гіперповерхні Гопфа в $\mathbb{S}^2 \times \mathbb{S}^2$ та $\mathbb{H}^2 \times \mathbb{H}^2$ з антикомутативними структурним оператором Якобі та структурним оператором Лі

Dehe Li

У цій роботі ми класифікуємо дійсні гіперповерхні Гопфа в $\mathbb{S}^2 \times \mathbb{S}^2$ та $\mathbb{H}^2 \times \mathbb{H}^2$, для яких структурний оператор Якобі та структурний оператор Лі є антикомутативними.

Ключові слова: дійсна гіперповерхня, структурний оператор Якобі, структурний оператор Лі, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$