

Maximum Principles and the Symmetry of Solutions for Parabolic Fractional $p(x, \cdot)$ -Equations in Exterior Domains

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In this paper, we first establish some maximum principles, including a simple maximum principle, the maximum principle for parabolic fractional $p(x, \cdot)$ -Laplacians and the maximum principle for anti-symmetric functions. Then we consider a Hopf's lemma for parabolic fractional $p(x, \cdot)$ -Laplacians, which is an extension of the results in Wang–Chen [Commun. Pure Appl. Anal. 21 (2022) 3055–3069]. In the end, applying the maximum principle for anti-symmetric functions and the method of moving planes, we obtain the symmetry and monotonicity of solutions to nonlinear parabolic fractional $p(x, \cdot)$ -equations in exterior domains of a ball. With the hope of providing new perspectives and tools for subsequent research.

Key words: parabolic fractional $p(x, \cdot)$ -Laplacians, exterior domains, maximum principle for anti-symmetric functions, Hopf's lemma, symmetry and monotonicity

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1. Introduction

The maximum principle and Hopf's lemma are fundamental properties of solutions to boundary value problems for elliptic or parabolic partial differential equations (PDEs). The roots of Hopf's lemma can be traced all the way back to the groundbreaking discovery of maximum principles for harmonic functions, as documented in [19]. In recent years, with the in-depth and extensive exploration of fractional Laplacians and fractional p -Laplacians, a comprehensive and systematic series of fractional-form Hopf's lemmas have been successfully established, which marks a significant progress in this field of research. For example, Birkner, López-Mimbela, and Wakolbinger [3] put forward a fractional form Hopf's lemma specifically tailored for anti-symmetric functions. This lemma can be directly employed in the method of moving planes, thereby playing a crucial role in establishing qualitative properties of fractional equations. Chen, Li, and Qi [12] proved a Hopf-type lemma for positive weak super-solutions to fractional p -equations. Lu and Zhu [32] used the Hopf-type lemma to prove maximum principles of fully nonlinear equations. Wang and Chen [36] established Hopf's lemmas for parabolic fractional p -equations and an asymptotic Hopf's lemma for anti-symmetric solutions to parabolic fractional Laplacians. These studies have gradually enriched the theoretical framework regarding Hopf-type lemmas

in the context of nonlocal equations. For more additional results, please refer to [14, 21, 24, 28, 30].

The study of solution symmetry is an intriguing topic in the research of partial differential equations. The symmetry of solutions to semilinear elliptic partial differential equations can date back to famous results of Gidas, Ni, and Nirenberg [17]. Their results have been extended to the whole space and exterior domains by Li [27] and Reichel [33] under some suitable assumptions. In 2020, Hao and Dai [18] proved the symmetry and monotonicity results for viscosity solutions of the fully nonlinear parabolic equation in annular and exterior domains. More and more scholars have conducted research on the equations involving the fractional Laplacian operator. In particular, a series of results have been obtained in terms of the existence and symmetry of solutions, such as symmetry and nonexistence of solutions [10], Liouville theorems and classification of solutions to equations and system involving higher-order fractional Laplacians [5, 6], asymptotic symmetry and monotonicity of solutions to fractional parabolic equations [13, 36], maximum principles for fractional Dirichlet problems [4, 29], maximum principles and symmetry to the fractional p -Laplacian [11], existence and symmetry results of Schrödinger equations [15]. Meanwhile, elliptic equations involving either local or nonlocal operators have been extensively studied by many scholars and a number of systematic approaches have been established to explore qualitative properties of solutions [9, 16, 20, 22, 25, 26, 31, 40]. For other results on fractional and fractional p -equations, we refer readers to [5–8, 25, 34, 35, 37, 38].

However, it is worth noting that replacing p with $p(x, \cdot)$ in the fractional p -Laplacian will generate a new fractional $p(x, \cdot)$ -Laplacian, which is a fractional Laplacian with a variable exponent. Currently, Bahrouni et al. [1, 2] obtained comparison principle and the monotonicity of solutions to nonlinear elliptic equations involving the fractional $p(x, \cdot)$ -Laplacian. Crucially, there are very few results available on the fractional $p(x, \cdot)$ -Laplacian. Wang and Wang [37] derived the monotonicity and symmetry of positive solutions to the fractional $p(x, \cdot)$ -Laplacian equation in bounded and unbounded domains.

The purpose of our paper is to extend the results in [18, 36] to the fractional $p(x, \cdot)$ -equations. Compared with general semilinear and nonlinear parabolic equations with local operators, the computations for the fractional $p(x, \cdot)$ -Laplacian are more complex, requiring consideration of more aspects.

In this paper, we investigate the Hopf-type lemma of the following parabolic fractional $p(x, \cdot)$ -equation

$$\frac{\partial u}{\partial t} + (-\Delta)_{p(x, \cdot)}^s u(x, t) = g(t, u(x, t)), \quad (x, t) \in \Omega \times (0, +\infty), \quad (1.1)$$

where $0 < s < 1$, $2 < p(x, \cdot) < +\infty$ and Ω is either a bounded or an unbounded domain in $\mathbb{R}^N$. For each fixed $t > 0$, the fractional $p(x, \cdot)$ -Laplacian is defined by

$$(-\Delta)_{p(x, \cdot)}^s u(x, t) = P.V. \int_{\mathbb{R}^N} \frac{|u(x, t) - u(y, t)|^{p(x, y)-2} (u(x, t) - u(y, t))}{|x - y|^{N+sp(x, y)}} dy, \quad (1.2)$$

where $s \in (0, 1)$ and $P.V.$ stands for the Cauchy principal value.

It is easy to see that for $u \in C_{\text{loc}}^{1,1}(\mathbb{R}^N) \cap \mathcal{L}_{sp(x,\cdot)}$ and $(-\Delta)_{p(x,\cdot)}^s u$ is well defined, where

$$\mathcal{L}_{sp(x,\cdot)} = \left\{ u(\cdot, t) \in L_{\text{loc}}^{p(x,y)-1}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} \frac{|u(x, t)|^{p(x,y)-1}}{1 + |x|^{N+sp(x,y)}} dx < +\infty \right\}.$$

Obviously, the fractional $p(x, \cdot)$ -Laplacian has nonlocality and nonlinearity. When $p(x, \cdot) = p \in (2, +\infty)$, the fractional $p(x, \cdot)$ -Laplacian becomes the usual fractional p -Laplacian. Especially, when $p(x, \cdot) = p = 2$, the fractional $p(x, \cdot)$ -Laplacian becomes to the classical fractional Laplacian. When $s = 1$, the fractional $p(x, \cdot)$ -Laplacian becomes to the $p(x)$ -Laplacian with variable exponents

$$\Delta_{p(x)} u = \text{div}(|\nabla u|^{p(x)-2} \nabla u).$$

When $s = 1$ and $p(x, \cdot) = p \in (2, +\infty)$, the fractional $p(x, \cdot)$ -Laplacian becomes the classical p -Laplacian. When $s = 1$ and $p(x, \cdot) = p = 2$, the fractional $p(x, \cdot)$ -Laplacian becomes the classical Laplacian. We say that u is a classical global solution of (1.1) if

$$u(x, t) \in (C_{\text{loc}}^{1,1}(\Omega) \cap \mathcal{L}_{sp(x,\cdot)}) \times C^1((0, +\infty))$$

and satisfies (1.1) pointwise in $\Omega \times (0, +\infty)$.

For convenience, let's first introduce some relevant background notation. Choose any direction to be the x_1 direction. Let

$$T_\lambda = \{x \in \mathbb{R}^N \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in \mathbb{R}^N \mid x_1 < \lambda\}$$

be the left region of the plane,

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_N)$$

be the reflection of x about the plane T_λ ,

$$u_\lambda(x, t) = u(x^\lambda, t) \quad \text{and} \quad w_\lambda(x, t) = u_\lambda(x, t) - u(x, t).$$

Obviously, w_λ is an anti-symmetric function.

Additionally, like [2], we assume the following assumptions on the exponent function:

(A1) $p(x, y) : \mathbb{R}^{2N} \rightarrow (2, +\infty)$ is a continuous function with

$$\max\{1 - \frac{1}{\ln(m)}, 2\} < p^- := \min_{(x,y) \in \mathbb{R}^{2N}} p(x, y) \leq p(x, y) \leq p^+ := \max_{(x,y) \in \mathbb{R}^{2N}} p(x, y),$$

for some $0 < m < 1$ and $sp^+ < N$, where $s \in (0, 1)$.

(A2) There exists a continuous function $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$p(x, y) = Q(|x - y|) \quad \text{for all } (x, y) \in \mathbb{R}^{2N},$$

Q is nondecreasing, while $z \rightarrow z^{Q(t)}$ is increasing on $\mathbb{R}_+$.

We first obtain the following

Theorem 1.1 (A Hopf's lemma for parabolic fractional $p(x, \cdot)$ -Laplacians).
Assume that $u(x, t) \in \left(C_{\text{loc}}^{1,1}(\Omega) \cap \mathcal{L}_{sp(x, \cdot)}\right) \times C^1((0, T])$ is a positive solution to

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_{p(x, \cdot)}^s u(x, t) = g(t, u(x, t)), & (x, t) \in \Omega \times (0, T], \\ u(x, t) = 0, & (x, t) \in \Omega^c \times (0, T], \end{cases} \quad (1.3)$$

where $0 < s < 1$, $p(x, \cdot)$ satisfies (A1) and (A2) and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Assume that

$$g(t, 0) = 0, \quad g \text{ is Lipschitz continuous in } u \text{ uniformly for } t. \quad (1.4)$$

Then there exists a positive constant C_0 such that for any $\tilde{t} \in (0, T)$ and for all x near the boundary of Ω , we have

$$u(x, \tilde{t}) \geq C_0 d^s(x),$$

where $d(x) = \text{dist}(x, \partial\Omega)$. It follows that

$$\frac{\partial u}{\partial \alpha^s}(\tilde{x}, \tilde{t}) < 0 \quad (\text{may possibly be } -\infty) \quad \text{for all } \tilde{t} \in (0, T),$$

where α is the outward normal of $\partial\Omega$ at $\tilde{x}$ and $\frac{\partial u}{\partial \alpha^s}$ is the normal derivative of fractional order s .

Remark 1.2. When $p(x, \cdot) = p \in [2, +\infty)$, Theorem 1.1 reduces to Theorem 1.1 in [36].

Finally, we also consider the symmetry of solution for the following fully nonlinear parabolic equation in exterior domains of the ball,

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_{p(x, \cdot)}^s u(x, t) = g(t, u(x, t)) & \text{in } \mathcal{D} \times (0, T], \\ 0 < u(x, t) \leq m < 1 & \text{in } \mathcal{D} \times (0, T], \\ u(x, t) = b & \text{on } \bar{B} \times (0, T], \\ u(x, t) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \quad t \in [0, T], \\ u(x, 0) = u_0(x) & \text{on } \bar{\mathcal{D}}, \end{cases} \quad (1.5)$$

where $\mathcal{D} = \mathbb{R}^N \setminus B$ for some ball B centered at the origin, $T > 0$, the definition of m can be found in (A1) and constant $b > m$.

Theorem 1.3 (Symmetry and monotonicity in the exterior domains). *Suppose that (A1) and (A2) hold. Let $u \in (C_{\text{loc}}^{1,1}(\mathcal{D}) \cap \mathcal{L}_{sp(x,\cdot)}) \times C^1((0, T])$ be a positive solution to (1.5). Suppose that g is Lipschitz continuous with respect to u ,*

$$u_0(x_1, x') \leq u_0(y, x'), \quad x_1 \leq y \leq -x_1, \quad x_1 \leq 0, \quad x' = (x_2, \dots, x_N),$$

and u_0 is symmetric with respect to the hyperplane $\{x \in \mathcal{D} \mid x_1 = 0\}$, i.e.,

$$u_0(-x_1, x') = u_0(x_1, x').$$

Then u is symmetric with respect to the hyperplane $\{x \in \mathcal{D} \mid x_1 = 0\}$, i.e.,

$$u(-x_1, x', t) = u(x_1, x', t)$$

and u is non-decreasing for $x_1 < 0$.

Furthermore, if $u_0(x)$ is radially symmetric with respect to the origin and non-increasing in $|x|$, then for each fixed $t \in (0, T]$, $u(x, t)$ is radially symmetric with respect to $(0, t)$ and non-increasing in $|x|$.

Remark 1.4. For $p(x, \cdot) = p \in [2, +\infty)$, our results are still new.

The paper is organized as follows: In Section 2, we present some maximum principles. In Section 3, we present proofs of the Hopf's lemma for parabolic fractional $p(x, \cdot)$ -Laplacians based on the maximum principles and proper construction of sub-solutions. In Section 4, we prove the symmetry and monotonicity results for nonlinear parabolic fractional $p(x, \cdot)$ -equations in exterior domains of a ball.

2. Various maximum principles

In this section, we present some maximum principles, including a *simple maximum principle*, the *maximum principle for parabolic fractional $p(x, \cdot)$ -Laplacians* and the *maximum principle for anti-symmetric functions*.

Theorem 2.1 (Simple maximum principle). *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that $u(x, t) \in (C_{\text{loc}}^{1,1}(\Omega) \cap \mathcal{L}_{sp(x,\cdot)}) \times C^1((0, T])$ is lower semi-continuous on $\bar{\Omega}$ and satisfies*

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) + (-\Delta)_{p(x,\cdot)}^s u(x, t) \geq 0, & (x, t) \in \Omega \times [0, +\infty), \\ u(x, t) \geq 0, & (x, t) \in (\mathbb{R}^N \setminus \Omega) \times [0, +\infty), \\ u(x, 0) \geq 0, & x \in \Omega, \end{cases} \quad (2.1)$$

then for any $T > 0$, we have

$$u(x, t) \geq 0 \quad \text{for all } (x, t) \in \Omega \times [0, T]. \quad (2.2)$$

Furthermore, (2.2) holds for unbounded domain Ω if we further assume that for all $t \in [0, T]$,

$$\lim_{|x| \rightarrow +\infty} u(x, t) \geq 0. \quad (2.3)$$

Under the conclusion (2.2), if $u = 0$ at some point $(\bar{x}, \bar{t}) \in \Omega \times [0, +\infty)$, then

$$u(x, \bar{t}) = 0 \quad \text{almost everywhere in } \mathbb{R}^N.$$

Remark 2.2. As we can see from the proof of (2.2), since $T > 0$ is arbitrary, taking $T \rightarrow +\infty$, we also have

$$u(x, t) \geq 0 \quad \text{for all } (x, t) \in \Omega \times [0, +\infty).$$

Our theorem can be seen as an extension of Theorem 1 of Chen and Li [11] to parabolic fractional $p(x, \cdot)$ -Laplacians.

Proof of Theorem 2.1. Fix an arbitrary $T \in (0, +\infty)$. If (2.2) does not hold, due to the lower semi-continuity of u on $\bar{\Omega}$ and the second to third inequality of (2.1), then there exists $\hat{x} \in \Omega$ and a $\hat{t} \in [0, T]$ such that

$$u(\hat{x}, \hat{t}) := \min_{\Omega \times [0, T]} u(x, t) < 0. \quad (2.4)$$

Thus,

$$\frac{\partial u}{\partial t}(\hat{x}, \hat{t}) \leq 0. \quad (2.5)$$

Using the second inequality of (2.1) and (2.4), we infer

$$\begin{aligned} (-\Delta)_{p(\hat{x}, \cdot)}^s u(\hat{x}, \hat{t}) &= \int_{\mathbb{R}^N} \frac{|u(\hat{x}, \hat{t}) - u(y, \hat{t})|^{p(\hat{x}, y)-2} (u(\hat{x}, \hat{t}) - u(y, \hat{t}))}{|\hat{x} - y|^{N+sp(\hat{x}, y)}} dy \\ &= \int_{\Omega} \frac{|u(\hat{x}, \hat{t}) - u(y, \hat{t})|^{p(\hat{x}, y)-2} (u(\hat{x}, \hat{t}) - u(y, \hat{t}))}{|\hat{x} - y|^{N+sp(\hat{x}, y)}} dy \\ &\quad + \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(\hat{x}, \hat{t}) - u(y, \hat{t})|^{p(\hat{x}, y)-2} (u(\hat{x}, \hat{t}) - u(y, \hat{t}))}{|\hat{x} - y|^{N+sp(\hat{x}, y)}} dy \\ &< 0. \end{aligned} \quad (2.6)$$

Combining (2.5) and (2.6), we arrive at

$$\frac{\partial u}{\partial t}(\hat{x}, \hat{t}) + (-\Delta)_{p(\hat{x}, \cdot)}^s u(\hat{x}, \hat{t}) < 0,$$

which contradicts the first inequality in (2.1). Consequently, (2.2) holds.

If Ω is unbounded, (2.3) guarantees that the negative minimum of $u(x, t)$ must be attained at some point, then one can follow the same discussion as in the proof of (2.2) to arrive at a contradiction.

Based on (2.2), suppose that there exists $(\bar{x}, \bar{t}) \in \Omega \times [0, +\infty)$ such that $u(\bar{x}, \bar{t}) = 0$. It is obvious that $(\bar{x}, \bar{t})$ is a minimum point of $u(x, t)$. Thus,

$$\frac{\partial u}{\partial t}(\bar{x}, \bar{t}) \leq 0. \quad (2.7)$$

By using the second inequality of (2.1) and (2.2), we obtain

$$\begin{aligned} (-\Delta)_{p(\bar{x}, \cdot)}^s u(\bar{x}, \bar{t}) &= \int_{\mathbb{R}^N} \frac{|-u(y, \bar{t})|^{p(\bar{x}, y)-2} (-u(y, \bar{t}))}{|\bar{x} - y|^{N+sp(\bar{x}, y)}} dy \\ &= \int_{\Omega} \frac{|-u(y, \bar{t})|^{p(\bar{x}, y)-2} (-u(y, \bar{t}))}{|\bar{x} - y|^{N+sp(\bar{x}, y)}} dy \\ &\quad + \int_{\mathbb{R}^N \setminus \Omega} \frac{|-u(y, \bar{t})|^{p(\bar{x}, y)-2} (-u(y, \bar{t}))}{|\bar{x} - y|^{N+sp(\bar{x}, y)}} dy \\ &\leq \int_{\Omega} \frac{|-u(y, \bar{t})|^{p(\bar{x}, y)-2} (-u(y, \bar{t}))}{|\bar{x} - y|^{N+sp(\bar{x}, y)}} dy \leq 0. \end{aligned} \quad (2.8)$$

Obviously, we also have

$$(-\Delta)_{p(\bar{x}, \cdot)}^s u(\bar{x}, \bar{t}) = \int_{\mathbb{R}^N} \frac{|-u(y, \bar{t})|^{p(\bar{x}, y)-2} (-u(y, \bar{t}))}{|\bar{x} - y|^{N+sp(\bar{x}, y)}} dy \geq 0, \quad (2.9)$$

where we used that the first inequality in (2.1) and (2.7). Combining (2.8) and (2.9), we deduce

$$\int_{\mathbb{R}^N} \frac{|-u(y, \bar{x})|^{p(\bar{x}, y)-2} (-u(y, \bar{t}))}{|\bar{x} - y|^{N+sp(\bar{x}, y)}} dy = 0.$$

Clearly, $u(x, \bar{t}) = 0$ a. e. in $\mathbb{R}^N$. Therefore, Theorem 2.1 is proved. $\square$

Theorem 2.3 (Maximum principle for parabolic fractional $p(x, \cdot)$ -Laplacians). *Let Ω be a bounded domain in $\mathbb{R}^N$. Suppose that*

$$u(x, t), \underline{u}(x, t) \in \left(C_{\text{loc}}^{1,1}(\Omega) \cap \mathcal{L}_{sp(x, \cdot)} \right) \times C^1([0, +\infty))$$

and $w(x, t) = u(x, t) - \underline{u}(x, t)$ is lower semi-continuous on $\bar{\Omega}$. If

$$\begin{cases} \frac{\partial w}{\partial t}(x, t) + (-\Delta)_{p(x, \cdot)}^s u(x, t) \\ \quad - (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t) \geq 0, & (x, t) \in \Omega \times (0, +\infty), \\ w(x, t) \geq 0, & (x, t) \in (\mathbb{R}^N \setminus \Omega) \times [0, +\infty), \\ w(x, 0) \geq 0, & x \in \Omega, \end{cases} \quad (2.10)$$

then we obtain

$$w(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times [0, +\infty). \quad (2.11)$$

Furthermore, (2.11) holds for unbounded region Ω if we further assume that

$$\lim_{|x| \rightarrow +\infty} w(x, t) \geq 0 \quad \text{for } t \in [0, T] \text{ and } T > 0. \quad (2.12)$$

Under the conclusion (2.11), if $w = 0$ at some point $(x_0, t_0) \in \Omega \times (0, +\infty)$, we have

$$w(x, t_0) = 0 \quad \text{almost everywhere in } \mathbb{R}^N. \quad (2.13)$$

Remark 2.4. When Ω is a narrow region in the x_1 direction, Theorem 2.3 still holds, that is, the narrow region principle of parabolic fractional $p(x, \cdot)$ -Laplacians. When $p(x, \cdot) = p \geq 2$, Theorem 2.3 reduces to Theorem 2.1 in [36]. The equivalent expression of (2.13) is that under the conclusion (2.11), if $w > 0$ at some point $(x_0, t_0) \in \Omega \times (0, +\infty)$, then we deduce

$$w_\lambda(x, t_0) > 0, \quad x \in \Omega.$$

Our Theorem 2.3 can also be seen as an extension of Theorem 2 when $c(x, t) = 0$ in [34] to fractional $p(x, \cdot)$ -Laplacian. It is worth noting that, like fractional p -Laplacian, the fractional $p(x, \cdot)$ -Laplacian is also non-linear, hence we need to use $(-\Delta)_{p(x, \cdot)}^s u(x, t) - (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t)$ instead of $(-\Delta)_{p(x, \cdot)}^s w(x, t)$.

Proof of Theorem 2.3. We first show that

$$w(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times [0, T] \text{ for any fixed } T > 0.$$

Assume that the above inequality is wrong, then the lower semi-continuity of $w(x, t)$ on $\bar{\Omega} \times (0, +\infty)$ and the second to third inequalities of (2.10) imply that there exist $\tilde{x} \in \Omega$ and $\tilde{t} \in (0, T]$, we obtain

$$w(\tilde{x}, \tilde{t}) := \min_{\Omega \times (0, T]} w(x, t) < 0.$$

Hence,

$$\frac{\partial w}{\partial t}(\tilde{x}, \tilde{t}) \leq 0.$$

We denote $H(z) = |z|^{p(x, y)-2}z$. Obviously, $H(z)$ is a strictly increasing function. Through a direct calculation,

$$\begin{aligned} & (-\Delta)_{p(\tilde{x}, \cdot)}^s u(\tilde{x}, \tilde{t}) - (-\Delta)_{p(\tilde{x}, \cdot)}^s \underline{u}(\tilde{x}, \tilde{t}) \\ &= \int_{\mathbb{R}^N} \frac{H(u(\tilde{x}, \tilde{t}) - u(y, \tilde{t})) - H(u(\tilde{x}, \tilde{t}) - \underline{u}(y, \tilde{t}))}{|\tilde{x} - y|^{N+sp(\tilde{x}, y)}} dy \\ &= \int_{\Omega} \frac{H(u(\tilde{x}, \tilde{t}) - u(y, \tilde{t})) - H(u(\tilde{x}, \tilde{t}) - \underline{u}(y, \tilde{t}))}{|\tilde{x} - y|^{N+sp(\tilde{x}, y)}} dy \\ &\quad + \int_{\mathbb{R}^N \setminus \Omega} \frac{H(u(\tilde{x}, \tilde{t}) - u(y, \tilde{t})) - H(u(\tilde{x}, \tilde{t}) - \underline{u}(y, \tilde{t}))}{|\tilde{x} - y|^{N+sp(\tilde{x}, y)}} dy \\ &=: \bar{I}_1 + \bar{I}_2. \end{aligned} \quad (2.14)$$

First, we estimate $\bar{I}_1$, through the minimum point $(\tilde{x}, \tilde{t})$ of w in $\bar{\Omega}$ and the monotonicity of $H(\cdot)$, we have

$$[u(\tilde{x}, \tilde{t}) - u(y, \tilde{t})] - [\underline{u}(\tilde{x}, \tilde{t}) - \underline{u}(y, \tilde{t})] = w(\tilde{x}, \tilde{t}) - w(y, \tilde{t}) \leq 0.$$

Consequently, for $y \in \Omega$, we infer

$$H(u(\tilde{x}, \tilde{t}) - u(y, \tilde{t})) - H(\underline{u}(\tilde{x}, \tilde{t}) - \underline{u}(y, \tilde{t})) \leq 0,$$

Thus,

$$\bar{I}_1 \leq 0. \quad (2.15)$$

For I_2 , noticing that $w(y, \tilde{t}) \geq 0$, $y \in \mathbb{R}^N \setminus \Omega$ and $w(\tilde{x}, \tilde{t}) < 0$, we have

$$[u(\tilde{x}, \tilde{t}) - u(y, \tilde{t})] - [\underline{u}(\tilde{x}, \tilde{t}) - \underline{u}(y, \tilde{t})] = w(\tilde{x}, \tilde{t}) - w(y, \tilde{t}) < 0, \quad y \in \mathbb{R}^N \setminus \Omega.$$

It follows from the strict monotonicity of $H(\cdot)$ that

$$\bar{I}_2 < 0. \quad (2.16)$$

Combining (2.14), (2.15), and (2.16), we deduce

$$(-\Delta)_{p(\tilde{x}, \cdot)}^s u(\tilde{x}, \tilde{t}) - (-\Delta)_{p(\tilde{x}, \cdot)}^s \underline{u}(\tilde{x}, \tilde{t}) < 0.$$

Since

$$\frac{\partial w}{\partial t}(\tilde{x}, \tilde{t}) \leq 0,$$

clearly,

$$\frac{\partial w}{\partial t}(\tilde{x}, \tilde{t}) + (-\Delta)_{p(\tilde{x}, \cdot)}^s u(\tilde{x}, \tilde{t}) - (-\Delta)_{p(\tilde{x}, \cdot)}^s \underline{u}(\tilde{x}, \tilde{t}) < 0,$$

which contradicts the first inequality of (2.10). Hence, we get that

$$w(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times [0, T] \quad \text{for any fixed } T > 0.$$

Due to $T > 0$ is arbitrary, letting $T \rightarrow +\infty$, then (2.11) is valid.

Furthermore, if Ω is unbounded, (2.12) guarantees that the negative minimum of $w(x, t)$ must be attained at some point, then one can follow the same discussion as in the proof of (2.11) to arrive at a contradiction.

Next, we prove (2.13) based on (2.11). Assume that there exists $(x_0, t_0) \in \Omega \times (0, +\infty)$ such that $w(x_0, t_0) = 0$. Obviously, (x_0, t_0) is a minimum point of $w(x, t)$. Thus,

$$\frac{\partial w}{\partial t}(x_0, t_0) \leq 0. \quad (2.17)$$

Then we give the following analysis lemma.

Lemma 2.5 ([37]). *For $H(z) = |z|^{p(x,y)-2}z$, $p(x, y) > 2$, there exists a constant $C > 0$ such that*

$$H(z_2) - H(z_1) \geq C(z_2 - z_1)^{p(x,y)-1} \quad \text{for arbitrary } z_2 \geq z_1.$$

By using (2.11), the second inequality of (2.10), (2.17), and Lemma 2.5, one has

$$\begin{aligned} & \frac{\partial w}{\partial t}(x_0, t_0) + (-\Delta)_{p(x_0, \cdot)}^s u(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s \underline{u}(x_0, t_0) \\ & \leq \int_{\mathbb{R}^N} \frac{H(u(x_0, t_0) - u(y, t_0)) - H(\underline{u}(x_0, t_0) - \underline{u}(y, t_0))}{|x_0 - y|^{N+sp(x_0, y)}} dy \\ & \leq \int_{\mathbb{R}^N} \frac{-c(w(y, t_0))^{p(x_0, y)-1}}{|x_0 - y|^{N+sp(x_0, y)}} dy, \end{aligned}$$

where c is a positive constant. Now combining (2.10), the above inequality and $w(y, t_0) \geq 0$ for $y \in \mathbb{R}^N$, we obtain

$$w(x, t_0) = 0 \quad \text{almost everywhere in } \mathbb{R}^N.$$

This ends the proof of Theorem 2.3. $\square$

Theorem 2.6 (Maximum principle for anti-symmetric functions). *Suppose that (A1) and (A2) hold. Let Ω be a domain in Σ_λ . Assume that $u \in (0, m)$, $m < 1$, $w_\lambda \in (C_{\text{loc}}^{1,1}(\Omega) \cap \mathcal{L}_{sp(x, \cdot)}) \times C^1([0, +\infty))$ is lower semi-continuous in x on $\bar{\Omega}$ and satisfies*

$$\begin{cases} \frac{\partial w_\lambda}{\partial t}(x, t) + (-\Delta)_{p(x, \cdot)}^s u_\lambda(x, t) \\ \quad - (-\Delta)_{p(x, \cdot)}^s u(x, t) \geq c(x, t)w_\lambda(x, t), & (x, t) \in \Omega \times [0, +\infty), \\ w_\lambda(x^\lambda, t) = -w_\lambda(x, t), & (x, t) \in \Sigma_\lambda \times [0, +\infty), \\ w_\lambda(x, t) \geq 0, & (x, t) \in (\Sigma_\lambda \setminus \Omega) \times [0, +\infty), \\ w_\lambda(x, 0) \geq 0, & x \in \Omega, \end{cases} \quad (2.18)$$

where $c(x, t)$ is bounded from above. Then

(i) if Ω is a bounded domain, then we have

$$w_\lambda(x, t) \geq 0 \quad \text{in } \Omega \times [0, +\infty); \quad (2.19)$$

(ii) furthermore, under conclusion (2.19), if w_λ attains 0 at some point $(x_0, t_0) \in \Omega \times (0, +\infty)$, then

$$w_\lambda(x, t_0) = 0 \quad \text{for almost all } x \in \mathbb{R}^N; \quad (2.20)$$

(iii) if Ω is unbounded, then conclusions (2.19) and (2.20) still hold under the additional condition:

$$\lim_{|x| \rightarrow +\infty} w_\lambda(x, t) \geq 0 \quad \text{uniformly for } t \in [0, +\infty). \quad (2.21)$$

Remark 2.7. As we can see from the proof, if $c(x, t) = 0$ or $\Omega \subset \Sigma_\lambda$ is a narrow region, the conclusions of Theorem 2.6 still hold, that is, the narrow region

principle of parabolic fractional $p(x, \cdot)$ -Laplacians for anti-symmetric functions. Compared with the narrow region principle of elliptic fractional $p(x, \cdot)$ -Laplacian (see Theorem 1.1 of [37]), we do not need the domain Ω to be narrow here, because our differential inequality contains the term $\frac{\partial w_\lambda}{\partial t}$, which can offset the influence of the $c(x, t)$ function by constructing an appropriate auxiliary function. So our Theorem 2.6 can also serve as a narrow region principle for parabolic fractional $p(x, \cdot)$ -Laplacian.

When $p(x, \cdot) = p \geq 2$, Theorem 2.6 reduces to Theorem 2.2 in [35]. When $p(x, \cdot) = 2$, Theorem 2.6 reduces to Theorem 2.1 in [39]. We also need to use $(-\Delta)_{p(x, \cdot)}^s u_\lambda(x, t) - (-\Delta)_{p(x, \cdot)}^s u(x, t)$ instead of $(-\Delta)_{p(x, \cdot)}^s w_\lambda(x, t)$, due to the non-linearity of fractional $p(x, \cdot)$ -Laplacians.

Proof of Theorem 2.6. (i) Since Ω is not narrow and $c(x, t)$ does not have a good sign over $\Omega \times (0, +\infty)$, we will introduce some auxiliary functions. Taking $k > 0$ as a fixed constant to be selected later, we construct some auxiliary functions

$$\tilde{w}_\lambda(x, t) = \tilde{u}_\lambda(x, t) - \tilde{u}(x, t) = e^{-kt} u_\lambda(x, t) - e^{-kt} u(x, t) = e^{-kt} w_\lambda(x, t).$$

By using (2.18), we have

$$\begin{aligned} & \frac{\partial \tilde{w}_\lambda}{\partial t}(x, t) + (-\Delta)_{p(x, \cdot)}^s \tilde{u}_\lambda(x, t) - (-\Delta)_{p(x, \cdot)}^s \tilde{u}(x, t) \\ &= -k e^{-kt} w_\lambda(x, t) + e^{-kt} \frac{\partial w_\lambda}{\partial t}(x, t) \\ &+ e^{-k(p(x, y)-1)t} [(-\Delta)_{p(x, \cdot)}^s u_\lambda(x, t) - (-\Delta)_{p(x, \cdot)}^s u(x, t)]. \end{aligned} \quad (2.22)$$

We will obtain (2.19) by first proving

$$\tilde{w}_\lambda(x, t) \geq 0, \quad (x, t) \in \Omega \times [0, T] \quad \text{for any fixed } T > 0. \quad (2.23)$$

If (2.23) is not valid, then the lower semi-continuity of $w_\lambda(x, t)$ with respect to x on $\bar{\Omega}$ indicates that there exists a $(x_0, t_0) \in \bar{\Omega} \times (0, T]$ such that

$$\tilde{w}_\lambda(x_0, t_0) := \min_{\bar{\Omega} \times (0, T]} \tilde{w}_\lambda(x, t) < 0.$$

Due to the same sign for $\tilde{w}_\lambda$ and w_λ , one can further deduce from the third and fourth inequalities of (2.18) that (x_0, t_0) is in the interior of $\Omega \times (0, T]$. Clearly,

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x_0, t_0) \leq 0.$$

We denote

$$H(z) = |z|^{p(x, y)-2} z, \quad H_*(z) = |z|^{p(x, y^\lambda)-2} z \quad \text{and} \quad h(z) = H(z) - H_*(z).$$

Noting that H and H_* are two increasing functions in z ,

$$H'(z) = (p(x, y) - 1)|z|^{p(x, y)-2} \geq 0 \quad \text{and} \quad H'_*(z) = (p(x, y^\lambda) - 1)|z|^{p(x, y^\lambda)-2} \geq 0.$$

By a direct computation, we have

$$\begin{aligned}
& (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}(x_0, t_0) \\
&= \int_{\mathbb{R}^N} \frac{H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y|^{N+sp(x_0, y)}} dy \\
&= \int_{\Sigma_\lambda} \frac{H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y|^{N+sp(x_0, y)}} dy \\
&\quad + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
&= \int_{\Sigma_\lambda} \left[\frac{1}{|x_0 - y|^{N+sp(x_0, y)}} - \frac{1}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} \right] \\
&\quad \times [H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))] dy \\
&\quad + \left\{ \int_{\Sigma_\lambda} \frac{H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \right. \\
&\quad \left. + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \right\} \\
&=: \tilde{I}_1 + \tilde{I}_2. \tag{2.24}
\end{aligned}$$

First, we estimate $\tilde{I}_1$, through the monotonicity of H and the fact that $[\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)] - [\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0)] = \tilde{w}_\lambda(x_0, t_0) - \tilde{w}_\lambda(y, t_0) \leq 0$, but $\neq 0$, then we get

$$H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0)) \leq 0, \quad \text{but } \neq 0.$$

Let us apply the following lemma.

Lemma 2.8 ([2]). *Let Ω be a bounded domain in Σ_λ . Suppose that conditions (A1) and (A2) are satisfied. Then for $x_0 \in \bar{\Omega}$, we have*

$$\frac{1}{|x_0 - y|^{N+sp(x_0, y)}} > \frac{1}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} \quad \text{for all } y \in \Sigma_\lambda.$$

Hence, we obtain

$$\tilde{I}_1 < 0. \tag{2.25}$$

Now we estimate $\tilde{I}_2$,

$$\begin{aligned}
\tilde{I}_2 &= \int_{\Sigma_\lambda} \frac{H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
&\quad + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
&= \int_{\Sigma_\lambda} \frac{H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy
\end{aligned}$$

$$\begin{aligned}
& + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& - \int_{\Sigma_\lambda} \frac{[H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))]}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& - \int_{\Sigma_\lambda} \frac{-H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0)) + H_*(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& = \int_{\Sigma_\lambda} \frac{H(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& + \int_{\Sigma_\lambda} \frac{-[H(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))]}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& = \int_{\Sigma_\lambda} \frac{h(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - h(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& + \int_{\Sigma_\lambda} \frac{H_*(\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0)) - H_*(\tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& = \int_{\Sigma_\lambda} \frac{[\tilde{w}_\lambda(x_0, t_0) - \tilde{w}_\lambda(y, t_0)]h'(z_0)}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \\
& + \tilde{w}_\lambda(x_0, t_0) \int_{\Sigma_\lambda} \frac{H'_*(z_1) + H'_*(z_2)}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy, \tag{2.26}
\end{aligned}$$

where $z_0 \in (\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0), \tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))$, $z_1 \in (\tilde{u}_\lambda(x_0, t_0) - \tilde{u}_\lambda(y, t_0), \tilde{u}(x_0, t_0) - \tilde{u}_\lambda(y, t_0))$, and $z_2 \in (\tilde{u}_\lambda(x_0, t_0) - \tilde{u}(y, t_0), \tilde{u}(x_0, t_0) - \tilde{u}(y, t_0))$. Since $\tilde{w}_\lambda(x_0, t_0) < 0$ and $H'_*(z) \geq 0$, we infer

$$\tilde{I}_2 \leq \int_{\Sigma_\lambda} \frac{[\tilde{w}_\lambda(x_0, t_0) - \tilde{w}_\lambda(y, t_0)]h'(z_0)}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy. \tag{2.27}$$

Next we prove that

$$h'(z_0) \geq 0. \tag{2.28}$$

From $h(z) = H(z) - H_*(z)$, it is obvious that

$$h'(z_0) = (p(x, y) - 1)|z_0|^{p(x, y)-2} - (p(x, y^\lambda) - 1)|z_0|^{p(x, y^\lambda)-2}.$$

Now we consider the function

$$f(\theta) = (\theta - 1)|z_0|^{\theta-2}, \quad \theta > 0,$$

thus,

$$f'(\theta) = |z_0|^{\theta-2}(1 + (\theta - 1) \ln(|z_0|)).$$

Recalling that $u \in (0, m)$ and $m < 1$, then $|z_0| \in (0, m)$. It follows that

$$f'(\theta) < 0 \quad \text{for all } \theta \geq 1 - \frac{1}{\ln(|z_0|)}.$$

Consequently, f is decreasing for all $\theta \geq \max \left\{ 1 - \frac{1}{\ln(m)}, 2 \right\}$. According to (A2), we obtain

$$\max \left\{ 1 - \frac{1}{\ln(m)}, 2 \right\} < p(x, y) \leq p(x, y^\lambda).$$

Obviously, $f(p(x, y)) \geq f(p(x, y^\lambda))$. Hence,

$$h'(z_0) = (p(x, y) - 1)|z_0|^{p(x, y)-2} - (p(x, y^\lambda) - 1)|z_0|^{p(x, y^\lambda)-2} \geq 0.$$

By using (2.27), (2.28), and the fact $\tilde{w}_\lambda(x_0, t_0) - \tilde{w}_\lambda(y, t_0) \leq 0$, we infer that

$$\tilde{I}_2 \leq \int_{\Sigma} \frac{[\tilde{w}_\lambda(x_0, t_0) - \tilde{w}_\lambda(y, t_0)]h'(z_0)}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \leq 0. \quad (2.29)$$

Combining (2.24), (2.25), and (2.29), we deduce

$$(-\Delta)_{p(x_0, \cdot)}^s \tilde{u}_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}(x_0, t_0) < 0. \quad (2.30)$$

According to the above inequality and $\frac{\partial \tilde{w}_\lambda}{\partial t}(x_0, t_0) \leq 0$, we have

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x_0, t_0) + (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}(x_0, t_0) < 0. \quad (2.31)$$

Due to $\tilde{w}_\lambda$ and w_λ have the same sign, one has

$$w_\lambda(x_0, t_0) := \min_{\bar{\Omega} \times (0, T]} w_\lambda(x, t) < 0.$$

Similar to the proof of (2.30), it is clear that

$$(-\Delta)_{p(x_0, \cdot)}^s u_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s u(x_0, t_0) < 0. \quad (2.32)$$

On the other hand, through (2.22), (2.32), the first inequality of (2.18), and $e^{-kt} > e^{-k(p(x, \cdot)-1)t}$ with $t > 0$ and $p(x, \cdot) > 2$, we derive

$$\begin{aligned} & \frac{\partial \tilde{w}_\lambda}{\partial t}(x_0, t_0) + (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}(x_0, t_0) \\ &= -ke^{-kt}w_\lambda(x_0, t_0) + e^{-kt}\frac{\partial w_\lambda}{\partial t}(x_0, t_0) \\ & \quad + e^{-kt(p(x_0, y)-1)} \left[(-\Delta)_{p(x_0, \cdot)}^s u_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s u(x_0, t_0) \right] \\ & \geq -ke^{-kt}w_\lambda(x_0, t_0) \\ & \quad + e^{-kt} \left[\frac{\partial w_\lambda}{\partial t}(x_0, t_0) + (-\Delta)_{p(x_0, \cdot)}^s u_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s u(x_0, t_0) \right] \end{aligned}$$

$$= (-k + c(x_0, t_0))\tilde{w}_\lambda(x_0, t_0).$$

Due to $c(x, t)$ is bounded from above, selecting $k = |c(x, t)| + 1$ to make $-k + c(x, t) < 0$, then

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x_0, t_0) + (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s \tilde{u}(x_0, t_0) > 0,$$

which contradicts with (2.31). Thus, it is clear that (2.23) is right. Since $T > 0$ is arbitrary, letting $T \rightarrow +\infty$, (2.19) holds.

(ii) Now we prove (2.20) based on (2.19). Assume that there exists $(x_0, t_0) \in \Omega \times (0, +\infty)$ such that

$$w_\lambda(x_0, t_0) = 0.$$

It is obvious that (x_0, t_0) is a minimum point of $w_\lambda(x, t)$. Therefore,

$$\frac{\partial w_\lambda}{\partial t}(x_0, t_0) \leq 0. \quad (2.33)$$

Similar to (2.24) and (2.26), we obtain

$$\begin{aligned} & (-\Delta)_{p(x_0, \cdot)}^s u_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s u(x_0, t_0) \\ &= \int_{\Sigma_\lambda} \left[\frac{1}{|x_0 - y|^{N+sp(x_0, y)}} - \frac{1}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} \right] \\ & \quad \times [H(u_\lambda(x_0, t_0) - u_\lambda(y, t_0)) - H(u(x_0, t_0) - u(y, t_0))] dy \\ & \quad + \left\{ \int_{\Sigma_\lambda} \frac{[w_\lambda(x_0, t_0) - w_\lambda(y, t_0)]h'(z_0)}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \right. \\ & \quad \left. + w_\lambda(x_0, t_0) \int_{\Sigma_\lambda} \frac{H'_*(z_1) + H'_*(z_2)}{|x_0 - y^\lambda|^{N+sp(x_0, y^\lambda)}} dy \right\} \\ &=: \check{I}_1 + \check{I}_2. \end{aligned} \quad (2.34)$$

Referring to (2.29) and using $w_\lambda(x_0, t_0) = 0$, $w_\lambda(y, t_0) \geq 0$, $y \in \Sigma_\lambda$, we get

$$\check{I}_2 \leq 0. \quad (2.35)$$

Through the first inequality of (2.18) and (2.33)–(2.35), we have

$$\begin{aligned} 0 &\leq -\frac{\partial w_\lambda}{\partial t}(x_0, t_0) \\ &\leq (-\Delta)_{p(x_0, \cdot)}^s u_\lambda(x_0, t_0) - (-\Delta)_{p(x_0, \cdot)}^s u(x_0, t_0) + c(x_0, t_0)w_\lambda(x_0, t_0) \\ &= \check{I}_1 + \check{I}_2 \leq \check{I}_1, \end{aligned}$$

that is, $\check{I}_1 \geq 0$. Consequently, according to (2.34) and Lemma 2.8,

$$H(u_\lambda(x_0, t_0) - u_\lambda(y, t_0)) - H(u(x_0, t_0) - u(y, t_0)) \geq 0,$$

then due to the monotonicity of H , for almost all $y \in \Sigma_\lambda$, we derive

$$[u_\lambda(x_0, t_0) - u_\lambda(y, t_0)] - [u(x_0, t_0) - u(y, t_0)] = -w_\lambda(y, t_0) \geq 0.$$

Therefore, we have $w_\lambda(x, t_0) \equiv 0$, a.e. in Σ_λ and $t_0 \in (0, +\infty)$.

Recalling $w_\lambda(x^\lambda, t) = -w_\lambda(x, t)$, we infer

$$w_\lambda(x, t_0) = 0 \quad \text{for almost all } x \in \mathbb{R}^N, \quad t_0 \in (0, +\infty).$$

(iii) If Ω is unbounded, then (2.21) ensures that the negative minimum of $w_\lambda(x, t)$ must be reached at some point. Then we can come up with a contradiction based on the same discussion as (i). From the proof of (ii), it can be seen that when the domain Ω is unbounded, the conclusion of (ii) still holds with the addition of condition (2.21). This completes the proof of Theorem 2.6. $\square$

3. Hopf's lemmas for parabolic fractional $p(x, \cdot)$ -equations

In this section, before starting to prove Theorem 1.1, we first provide a useful lemma.

Lemma 3.1. *Suppose that (A1) and (A2) hold. Let $s \in (0, 1)$ and $u(x) = (1 - |x|^2)_+^s$, then $(-\Delta)_{p(x, \cdot)}^s u(x)$ is uniformly bounded in the unit ball $B_1 \subset \mathbb{R}^N$.*

Proof. Since (A1),

$$p^- \leq p(x, y) \leq p^+,$$

by appropriately scaling $p(x, y)$ and using the same method as in Theorem 1.1 of [30], one can prove this lemma. The proof process is omitted here. $\square$

Now we start to prove Theorem 1.1.

Proof of Theorem 1.1. In the Fig. 3.1, we consider the cylinder. Let $\partial B_\delta(\tilde{x})$ be a sphere in Ω that is tangent to $\tilde{x}$. By this, we construct a small circular cylinder $B_\delta(\tilde{x}) \times [\tilde{t} - \rho, \tilde{t} + \rho] \subset \Omega \times (0, T)$ for $\tilde{x} \in \Omega$, $\tilde{t} \in (0, T)$, where δ and ρ are small positive constants. Through translation and scaling, we may assume that $\tilde{B} := B_1(0) \times [\tilde{t} - 1, \tilde{t} + 1]$, for simplicity of notation. Then let $\hat{B} = D \times [\tilde{t} - 1, \tilde{t} + 1]$ be a compact subset of E , and it has a positive distance from B . Since u is positive and continuous, for some constant C , we have

$$u(x, t) \geq C > 0, \quad (x, t) \in \hat{B}.$$

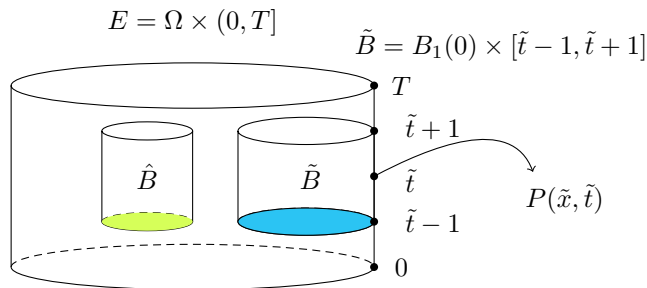


Fig. 3.1: Cylinder.

To obtain a lower bound of u in B , we construct a subsolution. Denote

$$\underline{u}(x, t) = \chi_D(x)u(x, t) + \delta\vartheta(x, t),$$

where δ is a positive constant and $\vartheta(x, t) = (1 - |x|^2)_+^s \eta(t)$,

$$\chi_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D^c. \end{cases}$$

Meanwhile, $\eta(t) \in C_0^\infty([0, T])$, satisfies

$$\eta(t) = \begin{cases} 1, & t \in \left[\tilde{t} - \frac{1}{2}, \tilde{t} + \frac{1}{2}\right], \\ 0, & t \notin (\tilde{t} - 1, \tilde{t} + 1). \end{cases} \quad (3.1)$$

By the definition of $(-\Delta)_{p(x, \cdot)}^s$ and the Lemma 2.5, for any $x \in B_1(0)$ and each fixed $t \in [\tilde{t} - 1, \tilde{t} + 1]$, we infer

$$\begin{aligned} (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t) &= (-\Delta)_{p(x, \cdot)}^s [\chi_D(x)u(x, t) + \delta\vartheta(x, t)] \\ &= \int_{\mathbb{R}^N} \frac{H(\delta\vartheta(x, t) - \chi_D(y)u(y, t) - \delta\vartheta(y, t))}{|x - y|^{N+sp(x, y)}} dy \\ &= \int_{B_1(0)} \frac{H(\delta\vartheta(x, t) - \chi_D(y)u(y, t) - \delta\vartheta(y, t))}{|x - y|^{N+sp(x, y)}} dy \\ &\quad + \int_D \frac{H(\delta\vartheta(x, t) - \chi_D(y)u(y, t) - \delta\vartheta(y, t))}{|x - y|^{N+sp(x, y)}} dy \\ &\quad + \int_{\mathbb{R}^N \setminus (B_1(0) \cup D)} \frac{H(\delta\vartheta(x, t) - \chi_D(y)u(y, t) - \delta\vartheta(y, t))}{|x - y|^{N+sp(x, y)}} dy \\ &= \int_{B_1(0)} \frac{H(\delta\vartheta(x, t) - \delta\vartheta(y, t))}{|x - y|^{N+sp(x, y)}} dy + \int_D \frac{H(\delta\vartheta(x, t) - u(y, t))}{|x - y|^{N+sp(x, y)}} dy \\ &\quad + \int_{\mathbb{R}^N \setminus (B_1(0) \cup D)} \frac{H(\delta\vartheta(x, t))}{|x - y|^{N+sp(x, y)}} dy \\ &\quad + \int_D \frac{H(\delta\vartheta(x, t))}{|x - y|^{N+sp(x, y)}} dy - \int_D \frac{H(\delta\vartheta(x, t))}{|x - y|^{N+sp(x, y)}} dy \\ &= (-\Delta)_{p(x, \cdot)}^s (\delta\vartheta(x, t)) + \int_D \frac{H(\delta\vartheta(x, t) - u(y, t)) - H(\delta\vartheta(x, t))}{|x - y|^{N+sp(x, y)}} dy \\ &\leq (-\Delta)_{p(x, \cdot)}^s (\delta\vartheta(x, t)) - \int_D \frac{c(u(y, t))^{p(x, y)-1}}{|x - y|^{N+sp(x, y)}} dy \\ &\leq a\delta^{p(x, y)-1} \eta^{p(x, y)-1}(t) - c_1, \end{aligned} \quad (3.2)$$

where both c and c_1 are positive constants, $0 \leq \eta(t) \leq 1$ and the last inequality is valid due to (see Lemma 3.1)

$$(-\Delta)_{p(x, \cdot)}^s (1 - |x|^2)_+^s \leq a, \quad x \in B_1(0),$$

where a is a positive constant. For $(x, t) \in B_1^c(0) \times [\tilde{t} - 1, \tilde{t} + 1]$, we get

$$\underline{u}(x, t) = \chi_D(x)u(x, t) \leq u(x, t),$$

and it follows from (3.1) that $\underline{u}(x, \tilde{t} - 1) \leq u(x, \tilde{t} - 1)$ for $x \in B_1(0)$.

Denote $w(x, t) = u(x, t) - \underline{u}(x, t)$. For $(x, t) \in B_1(0) \times [\tilde{t} - 1, \tilde{t} + 1]$, through (1.3), (3.2), and

$$\underline{u} = \delta \vartheta(x, t) = \delta(1 - |x|^2)_+^s \eta(t),$$

we infer

$$\begin{aligned} & \frac{\partial w}{\partial t}(x, t) + (-\Delta)_{p(x, \cdot)}^s u(x, t) - (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t) \\ &= \frac{\partial u}{\partial t}(x, t) + (-\Delta)_{p(x, \cdot)}^s u(x, t) - \frac{\partial \underline{u}}{\partial t}(x, t) - (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t) \\ &\geq g(t, u) - \delta(1 - |x|^2)_+^s \eta'(t) - a\delta^{p(x, y)-1} \eta^{p(x, y)-1}(t) + c_1 \\ &= C(x, t)u(x, t) - \delta \eta'(t)(1 - |x|^2)_+^s - a\delta^{p(x, y)-1} \eta^{p(x, y)-1}(t) + c_1, \end{aligned}$$

where $p(x, y) > 2$ and

$$C(x, t) = \frac{g(t, u) - g(t, 0)}{u(x, t) - 0} = \frac{g(t, u)}{u(x, t)}$$

is bounded by (1.4). From $u(x, t) = 0$ for $(x, t) \in \Omega^c \times (0, T)$ and the continuity of $u(x, t)$, we infer $u(x, t)$ is small, when $x \in \Omega$ is sufficiently close to $\partial\Omega$ and $t \in [\tilde{t} - \epsilon, \tilde{t} + \epsilon]$. If δ is sufficiently small, for $(x, t) \in B_1(0) \times [\tilde{t} - 1, \tilde{t} + 1]$, we have

$$\frac{\partial w}{\partial t} + (-\Delta)_{p(x, \cdot)}^s u(x, t) - (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t) \geq 0.$$

Consequently,

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)_{p(x, \cdot)}^s u(x, t) - (-\Delta)_{p(x, \cdot)}^s \underline{u}(x, t) \geq 0, & (x, t) \in B_1(0) \times [\tilde{t} - 1, \tilde{t} + 1], \\ w(x, t) \geq 0, & (x, t) \in B_1^c(0) \times [\tilde{t} - 1, \tilde{t} + 1], \\ w(x, \tilde{t} - 1) \geq 0, & x \in B_1(0). \end{cases}$$

Through Theorem 2.3, we obtain

$$w(x, t) \geq 0 \quad \text{in } B_1(0) \times [\tilde{t} - 1, \tilde{t} + 1]$$

and

$$u(x, t) \geq \delta \vartheta(x, t) = \delta(1 - |x|^2)_+^s \eta(t) \quad \text{in } B_1(0) \times [\tilde{t} - 1, \tilde{t} + 1].$$

In particular, fixed δ , by (3.1), for $x \in B_1(0)$, it is clear that

$$u(x, \tilde{t}) \geq \delta(1 - |x|^2)_+^s = \delta((1 - |x|)(1 + |x|))_+^s = C_0 d^s(x),$$

where $d(x) = \text{dist}(x, \partial\Omega) = 1 - |x|$ and C_0 is a positive constant. Hence,

$$\lim_{x \rightarrow \partial\Omega} \frac{u(x, \tilde{t})}{d^s(x)} \geq C_0 > 0 \quad \text{for all } \tilde{t} \in (0, T).$$

Therefore, if α is the outward normal of $\partial\Omega$ at $\tilde{x}$, due to $u(\tilde{x}, \tilde{t}) = 0$ for $\tilde{x} \in \partial\Omega$, then we infer

$$\frac{\partial u(x, \tilde{t})}{\partial \alpha^s} < 0$$

for all $x \in \partial\Omega$ and for all $\tilde{t} \in (0, T)$. This completes the proof of Theorem 1.1. $\square$

Next, we briefly explain how to apply *Hopf's lemma for parabolic $p(x, \cdot)$ -Laplacian* in the first step of the method of moving planes.

We take $\Omega = B_1(0)$ as an example. Let $\Sigma_\lambda^\Omega = \{x \in \Omega | x_1 < \lambda\}$. To obtain the radial symmetry of positive solutions to

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_{p(x, \cdot)}^s u(x, t) = g(t, u(x, t)), & (x, t) \in \Omega \times (0, T], \\ u(x, t) = 0, & (x, t) \in \Omega^c \times (0, T], \end{cases}$$

where $0 < s < 1$, $p(x, \cdot)$ satisfies **(A1)** and **(A2)**. The first step is to show that for the plane T_λ sufficiently close to the left end of Ω , that is, λ is sufficiently close to $\inf\{x_1 | x \in \Omega\}$, we have

$$u(x^\lambda, t) > u(x, t), \quad x \in \Sigma_\lambda^\Omega, \quad t \in (0, T). \quad (3.3)$$

This will provide a starting point for the method of moving planes.

By applying *Hopf's lemma for parabolic fractional $p(x, \cdot)$ -Laplacians*, that is, Theorem 1.1, we deduce

$$\frac{\partial u}{\partial x_1^s}(\tilde{x}, t) > 0, \quad \tilde{x} \in \partial\Omega, \quad t \in (0, T).$$

Consequently,

$$\frac{\partial u}{\partial x_1}(\tilde{x}, t) = +\infty, \quad \tilde{x} \in \partial\Omega, \quad t \in (0, T).$$

Then using the continuity of $\frac{\partial u}{\partial x_1^s}$ in some proper sense, it is natural to expect that

$$\frac{\partial u}{\partial x_1}(x, t) > 0 \quad \text{for } x \in \Omega \text{ sufficiently close to } \tilde{x}, \quad t \in (0, T),$$

which proves (3.3).

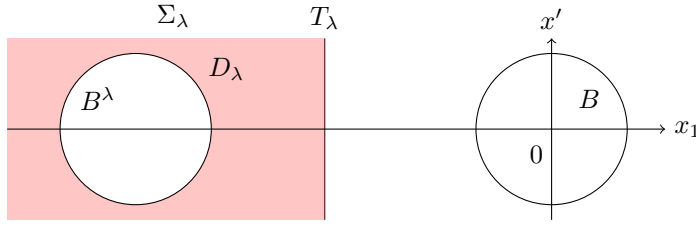


Fig. 4.1: Domain.

4. Symmetry and monotonicity in the exterior domain of the ball

In this section, we focus on the proof of Theorem 1.3.

Proof of Theorem 1.3. For any $\lambda < 0$, the definition of x^λ , T_λ , Σ_λ , and $u_\lambda(x, t)$ can be found in Section 1. Denote $x = (x_1, x') \in \mathbb{R}^N$ with $x' = (x_2, \dots, x_n)$, $B^\lambda = \{x^\lambda \mid x \in B\}$ be the reflection of the ball B about T_λ and $D_\lambda = \Sigma_\lambda \setminus B^\lambda$, see the following Fig. 4.1.

Define

$$w_\lambda(x, t) := u_\lambda(x, t) - u(x, t) = u(x^\lambda, t) - u(x, t).$$

We will divide our proof into two steps.

Step 1. We start the plane from negative infinity, i.e., we prove that for sufficiently negative λ , we have

$$w_\lambda(x, t) \geq 0, \quad (x, t) \in D_\lambda \times (0, T]. \quad (4.1)$$

If (4.1) is not true. Suppose $w_\lambda < 0$ somewhere in $D_\lambda \times (0, T]$. Using the fourth line in (1.5), we know

$$\lim_{|x| \rightarrow \infty} w_\lambda(x, t) = 0, \quad (x, t) \in D_\lambda \times (0, T].$$

It follows that w_λ attains its minimum in $\bar{D}_\lambda \times (0, T]$. Let $(x^0, t^0) \in \bar{D}_\lambda \times (0, T]$ be the minimum point of w_λ on $\bar{D}_\lambda \times (0, T]$, i.e.,

$$w_\lambda(x^0, t^0) := \min_{\bar{D}_\lambda \times (0, T]} w_\lambda < 0.$$

Now we claim that $(x^0, t^0) \in D_\lambda \times (0, T]$.

Indeed, for $x \in \partial D_\lambda \cap T_\lambda$ and $0 \leq t \leq T$, one has $w_\lambda(x, t) = 0$. For $x \in \overline{B^\lambda}$ and $0 < t \leq T$, by the second and third lines in (1.5), we infer

$$w_\lambda(x, t) = u(x^\lambda, t) - u(x, t) = b - u(x, t) > 0.$$

So we deduce $w_\lambda \geq 0$ for $x \in \partial D_\lambda$ and $0 < t \leq T$. For $x \in \bar{D}_\lambda$ and $t = 0$, by the monotonicity assumption of $u_0(x)$, we obtain $w_\lambda(x, 0) = u_0(2\lambda - x_1, x') - u_0(x_1, x') \geq 0$. Subsequently, we get that $(x^0, t^0) \in D_\lambda \times (0, T]$. The claim is proved.

Through the first equation in (1.5), it is easy to verify that

$$\frac{\partial w_\lambda}{\partial t} + (-\Delta)_{p(x,\cdot)}^s u_\lambda(x, t) - (-\Delta)_{p(x,\cdot)}^s u(x, t) + c(x, t)w_\lambda(x, t) = 0, \quad x \in D_\lambda \times (0, T],$$

where

$$c(x, t) = \frac{g(t, u(x, t)) - g(t, u_\lambda(x, t))}{u_\lambda(x, t) - u(x, t)}$$

is uniformly bounded by the Lipschitz continuity of g . Since $w_\lambda(x, 0) \geq 0$, $x \in D_\lambda$ and $w_\lambda(x, t) = u_\lambda(x, t) - u(x, t) \geq 0$, $x \in B^\lambda \times [0, +\infty)$. Choosing $\Omega = D_\lambda = \Sigma_\lambda \setminus B^\lambda$, for sufficiently negative λ , by applying *the maximum principle for anti-symmetric functions* (Theorem 2.6), we know that there is no negative minimum point within $D_\lambda \times (0, T]$, this leads to a contradiction.

Therefore, (4.1) must be true.

Step 2. The inequality (4.1) provides a starting point to move the plane. Now we move the plane T_λ towards the right as long as the inequality holds. We will show that T_λ can be moved all the way to its limiting position. More precisely, define

$$\lambda_0 = \sup_{\mu \in (-\infty, \lambda)} \{\lambda < 0 \mid w_\mu(x, t) \geq 0, (x, t) \in \Sigma_\mu \times (0, T]\},$$

then we show that $\lambda_0 = 0$.

Otherwise, suppose that $\lambda_0 < 0$, by the definition of w_{λ_0} , for $x \in \partial D_{\lambda_0} \cap \partial B^{\lambda_0}$ and $t \in (0, T]$, one has

$$w_{\lambda_0}(x, t) = u(x^{\lambda_0}, t) - u(x, t) = b - u(x, t) > 0.$$

Hence, using *the maximum principle for anti-symmetric functions* (Theorem 2.6), we derive

$$w_{\lambda_0} > 0 \quad \text{in } D_{\lambda_0} \times (0, T] \text{ for } \lambda_0 < 0.$$

Next, when $\lambda_0 < 0$, there exists a sufficiently small $\varepsilon > 0$ such that $\lambda_0 + \varepsilon < 0$. If $w_{\lambda_0 + \varepsilon} \geq 0$ in $D_{\lambda_0 + \varepsilon} \times (0, T]$, we obtain a contradiction to the definition of λ_0 . If $w_{\lambda_0 + \varepsilon} \leq 0$ somewhere in $D_{\lambda_0 + \varepsilon} \times (0, T]$, by the asymptotic behavior of u and the initial boundary condition, we know that the minimum point is achieved in the interior of $D_{\lambda_0 + \varepsilon}$. By using the same argument as that in **Step 1**, we know that it is impossible. Thus, we deduce $\lambda_0 = 0$.

Since the x_1 direction can be chosen arbitrarily, this means that $w_0(x, t) = 0$ in $D_0 \times (0, T]$. It follows that

$$u(x_1, x_2, \dots, x_N, t) = u(-x_1, x_2, \dots, x_N, t),$$

i.e., u is symmetric about the hyperplane $\{x \in \mathcal{D} \mid x_1 = 0\}$.

Since $w_\lambda(x, t) \geq 0$ in $D_\lambda \times (0, T]$ and $w_\lambda(x, t) = 0$ in $T_\lambda \times (0, T]$, which imply that w_λ is non-increasing in x_1 for $x_1 < 0$ and $(x_1, x', t) \in D_{\lambda_0} \times (0, T]$. This shows that

$$\frac{\partial w_\lambda}{\partial x_1}(x, t) = \frac{\partial u_\lambda}{\partial x_1}(x, t) - \frac{\partial u}{\partial x_1}(x, t) = -2 \frac{\partial u}{\partial x_1}(x, t) \leq 0.$$

And thus, there is

$$\frac{\partial u}{\partial x_1}(x, t) \geq 0,$$

then u is non-decreasing in x_1 for $x_1 < 0$ and $(x_1, x', t) \in D_{\lambda_0} \times (0, T]$.

If the initial value u_0 is radially symmetric and non-increasing with respect to $|x|$. Since the x_1 direction can be chosen arbitrarily, then through the same argument as above, we will obtain $\lambda_0 = 0$ again, and this leads to the symmetry of the solution at $x_1 = 0$. Hence, for any fixed $t \in (0, T]$, we find that u is radially symmetric with respect to $(0, T]$ and non-increasing in $|x|$. This completes the proof of Theorem 1.3. $\square$

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Принципи максимуму та симетрія розв'язків для параболічних дробових $p(x, \cdot)$ -рівнянь у зовнішніх областях

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У цій роботі ми спочатку встановлюємо деякі принципи максимуму, зокрема простий принцип максимуму, принцип максимуму для параболічних дробових $p(x, \cdot)$ -лапласіанів та принцип максимуму для антисиметричних функцій. Далі ми розглядаємо лему Гопфа для параболічних дробових $p(x, \cdot)$ -лапласіанів, яка є узагальненням результатів із праці Ванга–Ченя [Commun. Pure Appl. Anal. 21 (2022) 3055–3069]. Наприкінці, застосовуючи принцип максимуму для антисиметричних функцій та метод рухомих площин, ми отримуємо симетрію та монотонність розв'язків нелінійних параболічних дробових $p(x, \cdot)$ -рівнянь у зовнішніх областях кулі. З сподіванням надати нові перспективи та інструменти для подальших досліджень.

Ключові слова: параболічні дробові $p(x, \cdot)$ -лапласіани, зовнішні області, принцип максимуму для антисиметричних функцій, лема Гопфа, симетрія та монотонність