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Existence Study of Solutions for a System of n Nonlinear Fractional Differential Equations with Integral Conditions

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This paper offers a thorough discussion and study of the existence and uniqueness of solutions proposed for a class of new systems of n nonlinear fractional differential equations and their main properties using the fractional derivative of Katugampola with n integral conditions. Schauder's fixed point theorem, the Banach contraction principle and Leray-Schauder type nonlinear alternative are applied to attain the desired goal. In order to exhibit the usefulness of our main results, several examples are also presented in the paper.

Key words: system, fractional differential equation, integral conditions, existence, uniqueness

Mathematical Subject Classification 2010: 26A33, 34A08, 34A12, 34A34

1. Introduction

Fractional calculus, a mathematical branch, is known to be implemented in such fields as fluid flow, theory of dynamical systems control, diffusive transport akin to diffusion, probability and statistics, etc. It studies the properties of integrals and derivatives of non-integer order. For further reading on the subject, readers can refer to the following books (Samko et al. 1993 [19], Podlubny 1999 [18], Kilbas et al. 2006 [15], Diethelm 2010 [10]).

The existence and uniqueness of solutions for a single or a system of fractional differential equations have been investigated in recent years. For a small sample of such works, we refer readers to [1–9, 11, 15–20].

In [2], Ahmed et al. studied the coupled system of fractional differential equations supplemented with coupled nonlocal and integral boundary conditions:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}x(t)=f\left(t,x(t),y(t),{}^{C}\mathcal{D}^{\gamma}y(t)\right), & 1<\alpha\leq 2,\ 0<\gamma<1, \quad t\in[0,T], \\ {}^{C}\mathcal{D}^{\beta}y(t)=g\left(t,x(t),{}^{C}\mathcal{D}^{\delta}x(t),y(t)\right), & 1<\beta\leq 2,\ 0<\delta<1, \quad t\in[0,T], \\ x(0)=h(y), \quad \displaystyle\int_0^T y(s)\,ds=\mu_1 x(\eta), & \\ y(0)=\phi(x), \quad \displaystyle\int_0^T x(s)\,ds=\mu_2 y(\xi), & \eta,\xi\in(0,T), \end{cases}$$

where ${}^{C}\mathcal{D}$ denotes the Caputo fractional derivative, $f, g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ are given continuous functions, and μ_1, μ_2 are real constants.

In [20], Zhai and Jiang considered a new coupled system of fractional differential equations with the following integral boundary conditions:

$$\begin{cases} \mathcal{D}^{\alpha} u(t) + f(t, v(t)) = a, & 0 < t < 1, \\ \mathcal{D}^{\beta} v(t) + g(t, u(t)) = b, & 0 < t < 1, \\ u(0) = 0, \ u(1) = \displaystyle\int_0^1 \phi(s) u(s)\, ds, & \\ v(0) = 0, \ v(1) = \displaystyle\int_0^1 \psi(s) v(s)\, ds, & \end{cases}$$

where $1 < \alpha, \beta \le 2$, a, b are constants, $\mathcal{D}$ denotes the usual Riemann–Liouville fractional derivative, $f, g \in C([0,1] \times \mathbb{R})$, $\phi, \psi \in L^1[0,1]$.

In this work, our objective is to study the existence and uniqueness of solutions of a system of n nonlinear fractional differential equations

$$\begin{cases} {}^{\rho}\mathcal{D}_{0^+}^{\alpha_1} u_1(t) = f_1\left(t, u(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta_1} u_1(t)\right), & t \in [0,T], \\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha_2} u_2(t) = f_2\left(t, u(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta_2} u_2(t)\right), & t \in [0,T], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha_n} u_n(t) = f_n\left(t, u(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta_n} u_n(t)\right), & t \in [0,T], \end{cases} \tag{1.1}$$

with the integral conditions

$$\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha_1} u_1\right)(0^+) = \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha_2} u_2\right)(0^+) = \cdots = \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha_n} u_n\right)(0^+) = 0, \tag{1.2}$$

where $u = (u_1, u_1, \dots, u_n) \in \mathbb{R}^n$, for $n \in \mathbb{N}^* := \{1, 2, 3, \dots\}$. Also $\rho, T > 0$, $0 < \beta_i < \alpha_i \le 1$ and $f_i : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous functions for every $i \in \overline{1,n} := \{1, 2, \dots, n\}$. The symbol ${}^{\rho}\mathcal{D}_{0^+}^{\alpha}$ (respectively, ${}^{\rho}\mathcal{I}_{0^+}^{\alpha}$) presents the Katugampola fractional derivative (respectively, integral) of order $\alpha > 0$.

2. Preliminaries

In this section, some of the necessary definitions from fractional calculus theory are given. As in [15], we consider the space $X_c^p([0,T], \mathbb{R})$ (with $c \in \mathbb{R}$, $1 \le p \le \infty$) of those real-valued Lebesgue measurable functions y on $[0,T]$ for which $\|y\|_{X_c^p} < \infty$, where the norm is defined by

$$\|y\|_{X_c^p} = \left(\int_0^T |s^c y(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}}, \text{ for } 1 \le p < \infty \text{ and } \|y\|_{X_c^\infty} = \operatorname*{ess\,sup}_{0 \le t \le T} [t^c |y(t)|].$$

By $C([0,T], \mathbb{R})$, we denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{0 \le t \le T} |y(t)|.$$

Remark 2.1 ([3–6]). Let $p, c, T \in \mathbb{R}_{+}^{*}$, be such that $p \geq 1$ and $T \leq (pc)^{\frac{1}{pc}}$. It is clear that for all $y \in C([0,T],\mathbb{R})$,

$$\|y\|_{X_c^p} \leq \frac{T^c}{(pc)^{\frac{1}{p}}} \|y\|_{\infty} \text{ and } \|y\|_{X_c^{\infty}} \leq T^c \|y\|_{\infty},$$

which implies that $C([0,T],\mathbb{R}) \hookrightarrow X_c^p([0,T],\mathbb{R})$ and $\|y\|_{X_c^p} \leq \|y\|_{\infty}$ for all $T \leq (pc)^{\frac{1}{pc}}$.

Definition 2.2 (Katugampola's fractional integral [13])**.** *Katugampola's* fractional integral of order $\alpha \in \mathbb{R}_{+}$ of a function $y \in X_c^p([0,T],\mathbb{R})$ is defined by

$${}^{\rho}\mathcal{I}_{0+}^{\alpha} y(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} (t^{\rho} - s^{\rho})^{\alpha-1} y(s)\, ds, \ t \in [0,T], \tag{2.1}$$

for $\rho > 0$. This integral is a left-sided integral.

In a similar way, we can define a right-sided integral [13–15]. We also have:

Definition 2.3 (Katugampola's fractional derivative [14])**.** The generalized fractional derivative of order $\alpha \in \mathbb{R}_{+}$, corresponding to *Katugampola's* fractional integral (2.1), for any $t \in [0,T]$, is defined by

$$\begin{aligned} {}^{\rho}\mathcal{D}_{0+}^{\alpha} y(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^m \left({}^{\rho}\mathcal{I}_{0+}^{m-\alpha} y\right)(t) \\ &= \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^m \int_0^t s^{\rho-1} (t^{\rho} - s^{\rho})^{m-\alpha-1} y(s)\, ds, \end{aligned} \tag{2.2}$$

$m = [\alpha] + 1$, $[\alpha]$ denote the integer part of α and $\rho > 0$ if the integral exists.

Throughout this paper, T, p and c are real constants such that

$$p \geq 1, \quad c > 0, \quad \text{and} \quad T \leq (pc)^{\frac{1}{pc}}.$$

Lemma 2.4 ([6])**.** *Let $0 < \beta < \alpha \leq 1$, $\rho > 0$ and $y, {}^{\rho}\mathcal{D}_{0+}^{\alpha} y \in C([0,T],\mathbb{R})$. We define*

$$P := \left\{ y \in C([0,T],\mathbb{R}) \middle| \left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha} y\right)(0^{+}) = 0 \right\}.$$

Then $(P, \|\cdot\|_{\infty})$ is a Banach space and $y \in P$. We have for every $t \in [0,T]$ that

$$\left|{}^{\rho}\mathcal{D}_{0+}^{\beta} y(t)\right| \leq \frac{T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)} \|{}^{\rho}\mathcal{D}_{0+}^{\alpha} y\|_{\infty}.$$

Lemma 2.5 ([6])**.** *Let $\alpha, \beta, \rho > 0$, be such that $\beta < \alpha \leq 1$. Let $y, {}^{\rho}\mathcal{D}_{0+}^{\alpha} y \in C([0,T],\mathbb{R})$, and $f\left(t, y(t), {}^{\rho}\mathcal{D}_{0+}^{\beta} y(t)\right)$ is a continuous function. Then the problem*

$$\begin{cases} {}^{\rho}\mathcal{D}_{0+}^{\alpha} y(t) = f\left(t, y(t), {}^{\rho}\mathcal{D}_{0+}^{\beta} y(t)\right), & t \in [0,T], \\ \left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha} y\right)(0^{+}) = 0, \end{cases}$$

is equivalent to the integral equation

$$y(t) = \int_0^t G_\alpha(t,s) f\left(s, y(s), {}^\rho\mathcal{D}_{0+}^{\beta} y(t)\right) ds,$$

where G_α is a continuous function of $s \in [0,t)$ and $t \in [0,T]$, which is given by

$$G_\alpha(t,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} (t^\rho - s^\rho)^{\alpha-1}.$$

Lemma 2.6 ([6]). *Let $\mathcal{T} : P \to C([0,T], \mathbb{R})$ be an integral operator defined by*

$$\mathcal{T}y(t) = \int_0^t G_\alpha(t,s) f\left(s, y(s), {}^\rho\mathcal{D}_{0+}^{\beta} y(t)\right) ds \tag{2.3}$$

equipped with the norm

$$\|\mathcal{T}y\|_\infty = \sup_{0\le t\le T} |\mathcal{T}y(t)|.$$

Then $\mathcal{T}(P) \subset P$.

3. Main results

In what follows, we present some significant lemmas to clarify the principal theorems.

Lemma 3.1. *Let $u_i, {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i \in C([0,T], \mathbb{R})$ for every $i \in \overline{1,n}$. Then the solution of problem (1.1), (1.2) is equivalent to the n fractional integral equations*

$$\begin{cases} u_1(t) = \int_0^t G_{\alpha_1}(t,s) f_1\left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_1} u_1(s)\right) ds, \\ u_2(t) = \int_0^t G_{\alpha_2}(t,s) f_2\left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_2} u_2(s)\right) ds, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_n(t) = \int_0^t G_{\alpha_n}(t,s) f_n\left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_n} u_n(s)\right) ds, \end{cases} \tag{3.1}$$

with G_{α_i} being a continuous function of $s \in [0,t)$ and $t \in [0,T]$, which is given by

$$G_{\alpha_i}(t,s) = \frac{\rho^{1-\alpha_i} s^{\rho-1}}{\Gamma(\alpha_i)} (t^\rho - s^\rho)^{\alpha_i-1}. \tag{3.2}$$

Proof. We replace α and β by $\alpha_{i\in\overline{1,n}}$ and $\beta_{i\in\overline{1,n}}$, and use the same argument as that of the proof of Lemma 2.5. For more details, see [6]. □

Let us introduce the space $E = P_1 \times P_2 \times \cdots \times P_n$, where

$$P_i := \left\{u_i \in C([0,T], \mathbb{R}) \middle| \left({}^\rho\mathcal{I}_{0+}^{1-\alpha_i} u_i\right)(0^+) = 0\right\}, \quad i \in \overline{1,n}. \tag{3.3}$$

with the norm
$$\|u\|_E = \sup_{1\leq i\leq n} \|u_i\|_\infty .$$
In view of Lemma 2.4, it is clear that $(P_i, \|\cdot\|_\infty)_{i\in\overline{1,n}}$ are Banach spaces, and $\forall u_i \in P_i$. For every $t\in[0,T]$, we have
$$\left|{}^\rho\mathcal{D}_{0^+}^{\beta_i} u_i(t)\right| \leq \frac{T^{\rho(\alpha_i-\beta_i)}}{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)} \left\|{}^\rho\mathcal{D}_{0^+}^{\alpha_i} u_i\right\|_\infty , \quad i\in\overline{1,n}. \tag{3.4}$$

Lemma 3.2. *Let the integral operator $\mathcal{A}_i : E\to C([0,T],\mathbb{R})$ be defined by*
$$\mathcal{A}_i u(t) = \int_0^t G_{\alpha_i}(t,s) f_i\left(s, u(s), {}^\rho\mathcal{D}_{0^+}^{\beta_i} u_i(s)\right) ds, \tag{3.5}$$
where
$$G_{\alpha_i}(t,s) = \frac{\rho^{1-\alpha_i} s^{\rho-1}}{\Gamma(\alpha_i)} (t^\rho - s^\rho)^{\alpha_i-1},$$
equipped with the norm $\|\mathcal{A}_i u\|_\infty = \sup_{0\leq t\leq T} |\mathcal{A}_i u(t)|$. Then $\mathcal{A}_i(E)\subset P_i$, $i\in\overline{1,n}$.

Proof. By the same arguments as those used in the proof of Lemma 2.6, we are to replace α and β by $\alpha_{i\in\overline{1,n}}$ and $\beta_{i\in\overline{1,n}}$ to get the required result. For more details, see [6]. □

We define an operator $\mathcal{A}: E\to E$ by
$$\mathcal{A}u(t) = \begin{pmatrix} \mathcal{A}_1 u(t) \\ \mathcal{A}_2 u(t) \\ \vdots \\ \mathcal{A}_n u(t)\end{pmatrix}, \tag{3.6}$$
where $(\mathcal{A}_i u(t))_{i\in\overline{1,n}}$ are integral operators, which are given by (3.5), with the norm
$$\|\mathcal{A}u\|_E = \sup_{1\leq i\leq n} \|\mathcal{A}_i u\|_\infty .$$
We propose the following hypotheses:

(H1) For $1\leq k\leq n$, there exist two families of constants $\lambda_{i,k}, \gamma_i > 0$, where $\gamma_i < \frac{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)}{T^{\rho(\alpha_i-\beta_i)}}$ such that
$$|f_i(t,u,x_i) - f_i(t,v,w_i)| \leq \sum_{k=1}^n \lambda_{i,k} |u_k - v_k| + \gamma_i |x_i - w_i|,$$
for any $u,v\in\mathbb{R}^n$, $x_i, w_i\in\mathbb{R}$, with $i\in\overline{1,n}$ and $t\in[0,T]$.

(H2) For $1\leq k\leq n$, there exist three families of positive functions $a_i, b_{i,k}, c_i \in C([0,T]\,\mathbb{R}_+)$ such that
$$|f_i(t,u,x_i)| \leq a_i(t) + c_i(t)|x_i| + \sum_{k=1}^n b_{i,k}(t)|u_k| .$$
for any $u\in\mathbb{R}^n$, $x_i\in\mathbb{R}$, with $i\in\overline{1,n}$ and $t\in[0,T]$.

We denote $\lambda_i = \max_{1\leq k\leq n} \{\lambda_{i,k}\}$, $b_i(t) = \max_{1\leq k\leq n} \{b_{i,k}(t)\}$,

$$M_i = \frac{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)\, a_i^*}{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i) - c_i^* T^{\rho(\alpha_i-\beta_i)}},$$

and

$$N_i = \frac{n\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)\, b_i^*}{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i) - c_i^* T^{\rho(\alpha_i-\beta_i)}},$$

where $0 < \beta_i < \alpha_i \leq 1$ and

$$a_i^* = \sup_{0\leq t\leq T} a_i(t),\ b_i^* = \sup_{0\leq t\leq T} b_i(t),\ c_i^* = \sup_{0\leq t\leq T} c_i(t),$$

with

$$c_i^* < \frac{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)}{T^{\rho(\alpha_i-\beta_i)}}, \quad i \in \overline{1,n}.$$

In what follows, we present the principal theorems.

Theorem 3.3. *Assume* **(H1)** *holds. If*

$$\sup_{1\leq i\leq n} \left\{ \frac{n\lambda_i T^{\rho\alpha_i}\Gamma(1+\alpha_i-\beta_i)}{\Gamma(\alpha_i+1)\left[\rho^{\alpha_i}\Gamma(1+\alpha_i-\beta_i) - \gamma_i\rho^{\beta_i}T^{\rho(\alpha_i-\beta_i)}\right]} \right\} < 1, \tag{3.7}$$

then problem (1.1), (1.2) *admits a unique solution on* $[0,T]$.

Proof. To begin the proof, we transform problem (1.1), (1.2) into a fixed point problem $\mathcal{A}u(t) = u(t)$, with $\mathcal{A} : E \to E$ being defined by (3.6),

$$\mathcal{A}u(t) = \begin{pmatrix} \mathcal{A}_1 u(t) \\ \vdots \\ \mathcal{A}_n u(t) \end{pmatrix}, \tag{3.8}$$

with $(\mathcal{A}_i u(t))_{i\in\overline{1,n}}$ being the integral operators given by (3.5),

$$\mathcal{A}_i u(t) = \int_0^t G_{\alpha_i}(t,s)\, f_i\left(s, u(s), {}^{\rho}\mathcal{D}_{0^+}^{\beta_i} u_i(s)\right) ds,$$

where

$$G_{\alpha_i}(t,s) = \frac{\rho^{1-\alpha_i}s^{\rho-1}}{\Gamma(\alpha_i)}(t^\rho - s^\rho)^{\alpha_i-1}, \quad i \in \overline{1,n}.$$

Because problem (1.1), (1.2) is equivalent to the system of n fractional integral equations (3.8), the fixed point of $\mathcal{A}$ is a solution of problem (1.1), (1.2).

Let $u, v \in E$ be two functions that satisfy (1.1), (1.2), Then we get

$$\mathcal{A}_i u(t) - \mathcal{A}_i v(t) = \int_0^t G_{\alpha_i}(t,s)\left[f_i\left(s, u(s), {}^{\rho}\mathcal{D}_{0^+}^{\beta_i} u_i(s)\right) - f_i\left(s, v(s), {}^{\rho}\mathcal{D}_{0^+}^{\beta_i} v_i(s)\right)\right] ds,$$

for each $i \in \overline{1,n}$, which implies that

$$\left|\mathcal{A}_i u(t) - \mathcal{A}_i v(t)\right| \leq \int_0^t G_{\alpha_i}(t,s) \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i} u_i(s) - {}^{\rho}\mathcal{D}_{0+}^{\alpha_i} v_i(s)\right| ds, \quad i \in \overline{1,n}. \tag{3.9}$$

By **(H1)**, we have

$$\begin{aligned}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i} u_i(t) - {}^{\rho}\mathcal{D}_{0+}^{\alpha_i} v_i(t)\right| &= \left|f_i\left(t, u(t), {}^{\rho}\mathcal{D}_{0+}^{\beta_i} u_i(t)\right) - f_i\left(t, v(t), {}^{\rho}\mathcal{D}_{0+}^{\beta_i} v_i(t)\right)\right| \\ &\leq \sum_{k=1}^{n} \lambda_{i,k} \left|u_k(t) - v_k(t)\right| + \gamma_i \left|{}^{\rho}\mathcal{D}_{0+}^{\beta_i} u_i(t) - {}^{\rho}\mathcal{D}_{0+}^{\beta_i} v_i(t)\right|.\end{aligned}$$

By using (3.4), we get

$$\begin{aligned}\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i} u_i - {}^{\rho}\mathcal{D}_{0+}^{\alpha_i} v_i\right\|_{\infty} &\leq n\lambda_i \left\|u - v\right\|_E \\ &\quad + \frac{\gamma_i T^{\rho(\alpha_i - \beta_i)}}{\rho^{\alpha_i - \beta_i}\Gamma(1 + \alpha_i - \beta_i)} \left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i} u_i - {}^{\rho}\mathcal{D}_{0+}^{\alpha_i} v_i\right\|_{\infty}.\end{aligned}$$

Thus

$$\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i} u_i - {}^{\rho}\mathcal{D}_{0+}^{\alpha_i} v_i\right\|_{\infty} \leq \frac{n\lambda_i \rho^{\alpha_i - \beta_i}\Gamma(1 + \alpha_i - \beta_i)}{\rho^{\alpha_i - \beta_i}\Gamma(1 + \alpha_i - \beta_i) - \gamma_i T^{\rho(\alpha_i - \beta_i)}} \left\|u - v\right\|_E$$

for each $i \in \overline{1,n}$. From (3.9), we have

$$\left\|\mathcal{A}_i u - \mathcal{A}_i v\right\|_{\infty} \leq \frac{n\lambda_i T^{\rho\alpha_i}\Gamma(1 + \alpha_i - \beta_i)}{\Gamma(\alpha_i + 1)\left[\rho^{\alpha_i}\Gamma(1 + \alpha_i - \beta_i) - \gamma_i \rho^{\beta_i} T^{\rho(\alpha_i - \beta_i)}\right]} \left\|u - v\right\|_E.$$

Consequently,

$$\begin{aligned}&\left\|\mathcal{A}u - \mathcal{A}v\right\|_E \\ &\quad \leq \sup_{1\leq i\leq n} \left\{\frac{n\lambda_i T^{\rho\alpha_i}\Gamma(1 + \alpha_i - \beta_i)}{\Gamma(\alpha_i + 1)\left[\rho^{\alpha_i}\Gamma(1 + \alpha_i - \beta_i) - \gamma_i \rho^{\beta_i} T^{\rho(\alpha_i - \beta_i)}\right]}\right\} \left\|u - v\right\|_E.\end{aligned}$$

This implies that by (3.7), $\mathcal{A}$ is a contraction operator.

As a consequence of the Banach fixed-point theorem, using Banach's contraction principle [12], we deduce that $\mathcal{A}$ has a unique fixed point, which is the unique solution of problem (1.1), (1.2) on $[0, T]$. □

Theorem 3.4. *Assume that hypotheses* **(H1)** *and* **(H2)** *hold. If we put*

$$\frac{N_i T^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i + 1)} < 1 \quad \text{for every } i \in \overline{1,n},$$

then problem (1.1), (1.2) *has at least one solution on* $[0, T]$.

Proof. In the proof of the previous Theorem 3.3, we already transformed problem (1.1), (1.2) into a fixed points problem (3.8).

We show that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem. This will be proved through three steps.

Step 1. $\mathcal{A}$ is a continuous operator. Let $(u_m)_{m\in\mathbb{N}} = (u_1^m, u_2^m, \ldots, u_n^m)$ be n real sequences such that $\lim\limits_{m\to\infty} u_m = u$ in E. Then, for each $t \in [0,T]$ and $i \in \overline{1,n}$, we have

$$|\mathcal{A}_i u_m(t) - \mathcal{A}_i u(t)| \le \int_0^t G_{\alpha_i}(t,s) \times \left| f_i\left(s, u_m(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i^m(s)\right) - f_i\left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(s)\right)\right| ds, \quad (3.10)$$

where

$$\begin{aligned} {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m(t) &= f_i\left(t, u_m(t), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i^m(t)\right), \\ {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(t) &= f_i\left(t, u(t), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(t)\right). \end{aligned}$$

As a consequence of **(H1)**, we easily find that ${}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m \to {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i$ in P_i, $i \in \overline{1,n}$. In fact, we get

$$\begin{aligned} \left|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m - {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(t)\right| &= \left| f_i\left(t, u_m(t), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i^m(t)\right) - f_i\left(t, u(t), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(t)\right)\right| \\ &\le \sum_{k=1}^n \lambda_{i,k} |u_k^m(t) - u_k(t)| + \gamma_i \left|{}^\rho\mathcal{D}_{0+}^{\beta_i} u_i^m(t) - {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(t)\right|. \end{aligned}$$

By using (3.4), we have

$$\left\|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m - {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i\right\|_\infty \le \frac{n\lambda_i \rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)}{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i) - \gamma_i T^{\rho(\alpha_i-\beta_i)}} \|u_m - u\|_E.$$

Since $u_m \to u$ in E, then, for each $i \in \overline{1,n}$, we get ${}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m(t) \to {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(t)$ as $m \to \infty$ for any $t \in [0,T]$.

Now, let $K > 0$ be such that for each $t \in [0,T]$, we have

$$\left|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m(t)\right| \le K, \quad \left|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(t)\right| \le K, \quad i \in \overline{1,n}.$$

Then we obtain

$$\begin{aligned} |\mathcal{A}_i u_m(t) - \mathcal{A}_i u(t)| &\le \int_0^t G_{\alpha_i}(t,s) \\ &\quad \times \left| f_i(s, u_m(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i^m(s)) - f_i\left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(s)\right)\right| ds, \\ &\le \int_0^t G_{\alpha_i}(t,s) \left|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m(s) - {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(s)\right| ds \\ &\le \int_0^t G_{\alpha_i}(t,s) \left[\left|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i^m(s)\right| + \left|{}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(s)\right|\right] ds \\ &\le \int_0^t 2K G_{\alpha_i}(t,s)\, ds. \end{aligned}$$

For each $i \in \overline{1,n}$, the function $s \to 2KG_{\alpha_i}(t,s)$ is integrable $\forall t \in [0,T]$. Then the Lebesgue dominated convergence theorem and (3.10) imply that

$$|\mathcal{A}_i u_m(t) - \mathcal{A}_i u(t)| \to 0 \text{ as } m \to \infty,$$

and hence
$$\lim_{m\to\infty}\left\|\mathcal{A}u_m-\mathcal{A}u\right\|_E=0.$$
Consequently, $\mathcal{A}$ is continuous.

Step 2. Let
$$r\geq\frac{M_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)-N_iT^{\rho\alpha_i}},\quad i\in\overline{1,n},$$
and we define
$$E_r:=\left\{u\in E\mid\left\|u\right\|_E\leq r\right\}.$$
It is clear that E_r is a bounded, closed and convex subset of E. Let $\mathcal{A}:E_r\to E$ be the integral operator defined in (3.8). Then $\mathcal{A}\left(E_r\right)\subset E_r$. In fact, by using (3.4) and **(H2)**, for each $t\in[0,T]$, we have

$$\begin{aligned}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i}u_i\left(t\right)\right|&=\left|f_i\left(t,u\left(t\right),{}^{\rho}\mathcal{D}_{0+}^{\beta_i}u_i\left(t\right)\right)\right|\\&\leq a_i\left(t\right)+c_i\left(t\right)\left|{}^{\rho}\mathcal{D}_{0+}^{\beta_i}u_i\left(t\right)\right|+\sum_{k=1}^{n}b_{i,k}\left(t\right)\left|u_k\left(t\right)\right|\\&\leq a_i^*+nb_i^*\left\|u\right\|_E+\frac{c_i^*T^{\rho(\alpha_i-\beta_i)}}{\rho^{\alpha_i-\beta_i}\Gamma\left(1+\alpha_i-\beta_i\right)}\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i}u_i\right\|_\infty,\quad i\in\overline{1,n}.\end{aligned}$$

Then

$$\begin{aligned}\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha_i}u_i\right\|_\infty&\leq\frac{\rho^{\alpha_i-\beta_i}\Gamma\left(1+\alpha_i-\beta_i\right)a_i^*}{\rho^{\alpha_i-\beta_i}\Gamma\left(1+\alpha_i-\beta_i\right)-c_i^*T^{\rho(\alpha_i-\beta_i)}}\\&\quad+\frac{n\rho^{\alpha_i-\beta_i}\Gamma\left(1+\alpha_i-\beta_i\right)b_i^*}{\rho^{\alpha_i-\beta_i}\Gamma\left(1+\alpha_i-\beta_i\right)-c_i^*T^{\rho(\alpha_i-\beta_i)}}r\\&\leq M_i+N_ir,\quad i\in\overline{1,n}.\end{aligned}\tag{3.11}$$

Thus,

$$\begin{aligned}\left|\mathcal{A}_iu\left(t\right)\right|&\leq\int_0^tG_{\alpha_i}\left(t,s\right)\left|f_i\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta_i}u_i\left(s\right)\right)\right|ds\\&\leq\frac{M_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)}+\frac{N_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)}r\\&\leq\frac{\left[\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)-N_iT^{\rho\alpha_i}\right]\frac{M_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)-N_iT^{\rho\alpha_i}}+N_iT^{\rho\alpha_i}r}{\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)}\\&\leq\frac{\left[\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)-N_iT^{\rho\alpha_i}\right]r+N_iT^{\rho\alpha_i}r}{\rho^{\alpha_i}\Gamma\left(\alpha_i+1\right)}\leq r,\quad i\in\overline{1,n}.\end{aligned}$$

Hence, $\left\|\mathcal{A}_iu\right\|_\infty\leq r$, $i\in\overline{1,n}$, also $\left\|\mathcal{A}u\right\|_E\leq r$. Consequently, $\mathcal{A}\left(E_r\right)\subset E_r$.

Step 3. $\mathcal{A}\left(E_r\right)$ is relatively compact. Let $t_1,t_2\in\left[0,T\right]$, $t_1<t_2$ and $u\in E_r$. Then, for every $i\in\overline{1,n}$, we get

$$\left|\mathcal{A}_iu\left(t_2\right)-\mathcal{A}_iu\left(t_1\right)\right|=\left|\int_0^{t_2}G_{\alpha_i}\left(t_2,s\right)f_i\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta_i}u_i\left(s\right)\right)ds\right.$$

$$
\begin{aligned}
&\quad - \int_0^{t_1} G_{\alpha_i}(t_1,s) f_i\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta_i} u_i(s)\right) ds \Bigg| \\
&\leq \int_0^{t_1} \left| \left[G_{\alpha_i}(t_2,s) - G_{\alpha_i}(t_1,s)\right] f_i\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta_i} u_i(s)\right) \right| ds \\
&\quad + \int_{t_1}^{t_2} G_{\alpha_i}(t_2,s) \left| f_i\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta_i} u_i(s)\right) \right| ds \\
&\leq (M_i + N_i r) \left[\int_0^{t_1} \left| G_{\alpha_i}(t_2,s) - G_{\alpha_i}(t_1,s) \right| ds \int_{t_1}^{t_2} G_{\alpha_i}(t_2,s)\, ds \right]. \qquad (3.12)
\end{aligned}
$$

As $t_2 > t_1$, we obtain

$$
\begin{aligned}
\left| G_{\alpha_i}(t_2,s) - G_{\alpha_i}(t_1,s) \right| &= \frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)} s^{\rho-1} \left| \left(t_2^{\rho} - s^{\rho}\right)^{\alpha_i - 1} - \left(t_1^{\rho} - s^{\rho}\right)^{\alpha_i - 1} \right| \\
&= \frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)} s^{\rho-1} \left[\left(t_1^{\rho} - s^{\rho}\right)^{\alpha_i - 1} - \left(t_2^{\rho} - s^{\rho}\right)^{\alpha_i - 1} \right] \\
&= \frac{-1}{\alpha_i \rho^{\alpha_i} \Gamma(\alpha_i)} \frac{d}{ds} \left[\left(t_1^{\rho} - s^{\rho}\right)^{\alpha_i} - \left(t_2^{\rho} - s^{\rho}\right)^{\alpha_i} \right].
\end{aligned}
$$

Then

$$
\int_0^{t_1} \left| G_{\alpha_i}(t_2,s) - G_{\alpha_i}(t_1,s) \right| ds \leq \frac{1}{\rho^{\alpha_i} \Gamma(\alpha_i + 1)} \left[\left(t_2^{\rho} - t_1^{\rho}\right)^{\alpha_i} + \left(t_2^{\rho\alpha_i} - t_1^{\rho\alpha_i}\right) \right].
$$

We also have

$$
\begin{aligned}
\int_{t_1}^{t_2} G_{\alpha_i}(t_2,s)\, ds &= \frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} s^{\rho-1} \left(t_2^{\rho} - s^{\rho}\right)^{\alpha_i - 1} ds \\
&= \frac{-1}{\alpha_i \rho^{\alpha_i} \Gamma(\alpha_i)} \left[\left(t_2^{\rho} - s^{\rho}\right)^{\alpha_i} \right]_{t_1}^{t_2} \leq \frac{1}{\rho^{\alpha_i} \Gamma(\alpha_i + 1)} \left(t_2^{\rho} - t_1^{\rho}\right)^{\alpha_i}.
\end{aligned}
$$

Then (3.12) gives

$$
\left| \mathcal{A}_i u(t_2) - \mathcal{A}_i u(t_1) \right| \leq \frac{M_i + N_i r}{\rho^{\alpha_i} \Gamma(\alpha_i + 1)} \left[2\left(t_2^{\rho} - t_1^{\rho}\right)^{\alpha_i} + \left(t_2^{\rho\alpha_i} - t_1^{\rho\alpha_i}\right) \right].
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero for every $i \in \overline{1,n}$.

As a consequence of steps 1 to 3 together, and by means of the Ascoli–Arzelà theorem, we deduce that $\mathcal{A} : E_r \to E_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem. Then $\mathcal{A}$ has a fixed point which is a solution of problem (1.1), (1.2) on $[0, T]$. □

Theorem 3.5. *Assume* **(H1)** *and* **(H2)** *hold. Then problem* (1.1), (1.2) *has at least one solution on* $[0, T]$.

Proof. Let $\alpha_i, \beta_i, \rho > 0$, be such that $\beta_i < \alpha_i \leq 1$ for every $i \in \overline{1,n}$.

We shall show that the operator $\mathcal{A}$, defined in (3.8), satisfies the assumption of the Leray–Schauder fixed point theorem (see [12]). The proof will be given in several steps.

Step 1. Clearly, $\mathcal{A}$ is continuous.

Step 2. $\mathcal{A}$ maps bounded sets into bounded sets in E. Indeed, it is enough to show that for any $\omega > 0$ there exists a positive constant ℓ such that for each $u \in B_\omega := \{u \in E : \|u\|_E \leq \omega\}$ we have $\|\mathcal{A}u\|_E \leq \ell$. For $u \in B_\omega$, we have, for each $i \in \overline{1,n}$ and $t \in [0,T]$,

$$|\mathcal{A}_i u(t)| \leq \int_0^t G_{\alpha_i}(t,s) \left| f_i \left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(s) \right) \right| ds. \tag{3.13}$$

By **(H2)**, similarly to (3.11), for each $t \in [0,T]$, we have

$$\left| f_i \left(t, u(t), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(t) \right) \right| \leq M_i + N_i \omega, \ \forall i \in \overline{1,n}.$$

Thus, (3.13) implies that

$$\|\mathcal{A}_i u\|_\infty \leq \frac{M_i T^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)} + \frac{N_i T^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}\omega, \quad i \in \overline{1,n}$$

and

$$\|\mathcal{A}u\|_E \leq \sup_{1\leq i\leq n} \left\{ \frac{M_i T^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)} + \frac{N_i T^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}\omega \right\} = \ell.$$

Step 3. Clearly, $\mathcal{A}$ maps bounded sets into equicontinuous sets of P. We conclude that $\mathcal{A} : P \to P$ is continuous and completely continuous.

Step 4. A priori bounds. We now show that there exists an open set $U \subset E$ with $u \neq \mu\mathcal{A}(u)$ for $\mu \in (0,1)$ and $u \in \partial U$. Let $u \in E$ and $u = \mu\mathcal{A}(u)$ for some $0 < \mu < 1$. Thus, for each $i \in \overline{1,n}$ and $t \in [0,T]$, we have

$$u_i(t) \leq \mu \int_0^t G_{\alpha_i}(t,s) \left| f_i \left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(s) \right) \right| ds.$$

By **(H2)**, for all solutions $u \in E$ of problem (1.1), (1.2) for all $i \in \overline{1,n}$, we have

$$\begin{aligned} |u_i(t)| &= \left| \int_0^t G_{\alpha_i}(t,s) f_i \left(s, u(s), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(s) \right) ds \right| \\ &\leq \int_0^t G_{\alpha_i}(t,s) \left| {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(s) \right| ds. \end{aligned}$$

Then for each $t \in [0,T]$ and for all $i \in \overline{1,n}$, we have

$$\begin{aligned} \left| {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(t) \right| &= \left| f_i \left(t, u(t), {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(t) \right) \right| \\ &\leq a_i(t) + c_i(t) \left| {}^\rho\mathcal{D}_{0+}^{\beta_i} u_i(t) \right| + \sum_{k=1}^n b_{i,k}(t) |u_k(t)| \\ &\leq a_i^* + n b_i^* \|u\|_E + \frac{c_i^* T^{\rho(\alpha_i-\beta_i)}}{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)} \sup_{0\leq t\leq T} \left| {}^\rho\mathcal{D}_{0+}^{\alpha_i} u_i(t) \right|. \end{aligned}$$

Then

$$\sup_{0\leq t\leq T}\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha_i}u_i(t)\right| \leq \frac{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)}{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)-c_i^* T^{\rho(\alpha_i-\beta_i)}}\left(a_i^*+nb_i^*\|u\|_E\right)$$
$$\leq M_i+N_i\sup_{0\leq i\leq n}\|u_i\|_\infty \quad \text{for every } i\in\overline{1,n}.$$

Hence

$$\sup_{0\leq t\leq T}|u_i(t)|\leq \sup_{0\leq i\leq n}\left\{\frac{M_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}+\int_0^t N_iG_{\alpha_i}(t,s)\sup_{0\leq s\leq T}|u_i(s)|\right\}ds.$$

By the Gronwall Lemma, we have

$$\sup_{0\leq t\leq T}|u_i(t)|\leq \sup_{0\leq i\leq n}\left\{\frac{M_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}\exp\left(\frac{N_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}\right)\right\}.$$

Thus

$$\|u\|_E\leq \sup_{0\leq i\leq n}\left\{\frac{M_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}\exp\left(\frac{N_iT^{\rho\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}\right)\right\}=\kappa.$$

Let

$$U:=\left\{u\in E:\|u\|_E<\kappa+1\right\}.$$

By choosing U, there is no $u\in\partial U$ such that $u=\mu\mathcal{A}(u)$ for $\mu\in(0,1)$.

As a consequence of Leray–Schauder's theorem (see [12]), $\mathcal{A}$ has a fixed point u in U which is a solution to (1.1), (1.2). □

4. Illustrative Examples

Example 4.1. For $t\in\left[0,\frac{\pi}{4}\right]$, consider the following problem:

$$\begin{aligned}
{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}u_1(t)&=\frac{\cos(t)\left(\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\right)^{-1}}{\left[1+\frac{1}{2}\left(|u_1(t)|+|u_2(t)|+|u_3(t)|\right)+\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}u_1(t)\right|\right]},\\
{}^1\mathcal{D}_{0^+}^{\frac{2}{3}}u_2(t)&=\frac{1}{\sqrt{7}+|u_1(t)|}+\frac{1}{\sqrt{8}+|u_2(t)|}+\frac{1}{3+|u_3(t)|}+\frac{{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}u_2(t)}{\pi+t^4},\\
{}^1\mathcal{D}_{0^+}^{\frac{3}{4}}u_3(t)&=\frac{\tan(t)}{1+\frac{1}{5}|u_1(t)|+\frac{1}{10}|u_2(t)|+\frac{1}{15}|u_3(t)|+\frac{1}{\pi}\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}u_3(t)\right|},\\
\left({}^1\mathcal{I}_{0^+}^{\frac{1}{2}}u_1\right)(0^+)&=\left({}^1\mathcal{I}_{0^+}^{\frac{2}{3}}u_2\right)(0^+)=\left({}^1\mathcal{I}_{0^+}^{\frac{3}{4}}u_3\right)(0^+)=0.
\end{aligned}\tag{4.1}$$

Set:

$$f_1\left(t,u_1,u_2,u_3,{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}u_1\right)=\frac{\cos(t)\left(\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\right)^{-1}}{\left[1+\frac{1}{2}\left(|u_1|+|u_2|+|u_3|\right)+\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}u_1\right|\right]},$$

$$f_2\left(t,u_1,u_2,u_3,{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}u_2\right)=\frac{1}{\sqrt{7}+|u_1|}+\frac{1}{\sqrt{8}+|u_2|}+\frac{1}{3+|u_3|}+\frac{{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}u_2}{\pi+t^4},$$

$$f_3\left(t,u_1,u_2,u_3,{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}u_3\right)=\frac{\tan(t)}{1+\frac{1}{5}|u_1|+\frac{1}{10}|u_2|+\frac{1}{15}|u_3|+\frac{1}{\pi}\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}u_3\right|}.$$

Because $\sin(t)$, $\cos(t)$ and $\tan(t)$ are continuous positive functions on $\left[0,\frac{\pi}{4}\right]$, the functions f_i are jointly continuous for all $u_i\mathbb{R}$ with $i=1,2,3$. For any $t\in\left[0,\frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2}\leq\cos(t)\leq 1$, $0\leq\sin(t)\leq\frac{\sqrt{2}}{2}$ and $0\leq\tan(t)\leq 1$, and thus

$$\left|f_1\left(t,u_1,u_2,u_3,{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}u_1\right)-f_1\left(t,v_1,v_2,v_3,{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}v_1\right)\right|$$
$$\leq\sum_{k=1}^{3}\frac{1}{2\pi}|u_k-v_k|+\frac{1}{\pi}\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}u_1-{}^1\mathcal{D}_{0^+}^{\frac{1}{4}}v_1\right|,$$

$$\left|f_2\left(t,u_1,u_2,u_3,{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}u_2\right)-f_2\left(t,v_1,v_2,v_3,{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}v_2\right)\right|$$
$$\leq\sum_{k=1}^{3}\frac{1}{6+k}|u_k-v_k|+\frac{1}{\pi}\left|{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}u_2-{}^1\mathcal{D}_{0^+}^{\frac{5}{12}}v_2\right|,$$

$$\left|f_3\left(t,u_1,u_2,u_3,{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}u_3\right)-f_3\left(t,v_1,v_2,v_3,{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}v_3\right)\right|$$
$$\leq\sum_{k=1}^{3}\frac{1}{5k}|u_k-v_k|+\frac{1}{\pi}\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}u_3-{}^1\mathcal{D}_{0^+}^{\frac{1}{2}}v_3\right|.$$

Hence, condition **(H1)** is satisfied with $\lambda_1=\frac{1}{2\pi}$, $\lambda_2=\frac{1}{7}$, $\lambda_3=\frac{1}{5}$, and $\alpha_i-\beta_i=\frac{1}{4}$ for any $i=1,2,3$ and

$$\gamma_1=\gamma_2=\ \gamma_3=\frac{1}{\pi}\simeq 0.31831$$
$$<\frac{\rho^{\alpha_i-\beta_i}\Gamma(1+\alpha_i-\beta_i)}{T^{\rho(\alpha_i-\beta_i)}}=\left(\frac{\pi}{4}\right)^{-\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)\simeq 0.9628.$$

Also, we have:

$$\frac{n\lambda_1T^{\rho\alpha_1}\Gamma(1+\alpha_1-\beta_1)}{\Gamma(\alpha_1+1)\left[\rho^{\alpha_1}\Gamma(1+\alpha_1-\beta_1)-\gamma_1\rho^{\beta_1}T^{\rho(\alpha_1-\beta_1)}\right]}$$
$$=\frac{\frac{3}{2\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{2}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)\left(\Gamma\left(\frac{5}{4}\right)-\frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right)}\simeq 0.71327,$$

$$\frac{n\lambda_2T^{\rho\alpha_2}\Gamma(1+\alpha_2-\beta_2)}{\Gamma(\alpha_2+1)\left[\rho^{\alpha_2}\Gamma(1+\alpha_2-\beta_2)-\gamma_2\rho^{\beta_2}T^{\rho(\alpha_2-\beta_2)}\right]}$$
$$=\frac{\frac{3}{7}\left(\frac{\pi}{4}\right)^{\frac{2}{3}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{3}\right)\left(\Gamma\left(\frac{5}{4}\right)-\frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right)}\simeq 0.60371,$$

$$\frac{n\lambda_3 T^{\rho\alpha_3}\Gamma(1+\alpha_3-\beta_3)}{\Gamma(\alpha_3+1)\left[\rho^{\alpha_3}\Gamma(1+\alpha_3-\beta_3)-\gamma_3\rho^{\beta_3}T^{\rho(\alpha_3-\beta_3)}\right]}$$
$$=\frac{\frac{3}{5}\left(\frac{\pi}{4}\right)^{\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)\left(\Gamma\left(\frac{5}{4}\right)-\frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right)}\simeq 0.81365.$$

It remains to show that the condition (3.7),

$$\sup_{1\le i\le 3}\left\{\frac{n\lambda_i T^{\rho\alpha_i}\Gamma(1+\alpha_i-\beta_i)}{\Gamma(\alpha_i+1)\left[\rho^{\alpha_i}\Gamma(1+\alpha_i-\beta_i)-\gamma_i\rho^{\beta_i}T^{\rho(\alpha_i-\beta_i)}\right]}\right\}\simeq 0.81365<1,$$

is satisfied. It follows from Theorem 3.3 that problem (4.1) has a unique solution.

Example 4.2. For $t\in\left[0,\frac{\pi}{4}\right]$, consider the following problem:

$$\begin{aligned}
&{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}u_1(t)=\frac{\cos(t)\left(2+\frac{1}{2}\left(|u_1(t)|+|u_2(t)|\right)+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1(t)\right|\right)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left(1+\frac{1}{2}\left(|u_1(t)|+|u_2(t)|\right)+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1(t)\right|\right)},\\
&{}^{1}\mathcal{D}_{0^+}^{\frac{3}{4}}u_2(t)=\tan(t)\left(1+\frac{|u_1(t)|}{5}+\frac{|u_2(t)|}{10}+\frac{1}{\pi}\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}u_2(t)\right|\right),\\
&\left({}^{1}\mathcal{I}_{0^+}^{\frac{1}{2}}u_1\right)(0^+)=\left({}^{1}\mathcal{I}_{0^+}^{\frac{3}{4}}u_2\right)(0^+)=0.
\end{aligned}\tag{4.2}$$

Set:

$$\begin{aligned}
&f_1\left(t,u_1,u_2,{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1\right)\\
&\quad=\frac{\cos(t)\left(2+\frac{1}{2}\left(|u_1(t)|+|u_2(t)|\right)+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1(t)\right|\right)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left(1+\frac{1}{2}\left(|u_1(t)|+|u_2(t)|\right)+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1(t)\right|\right)},\\
&f_2\left(t,u_1,u_2,{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}u_2\right)\\
&\quad=\tan(t)\left(1+\frac{|u_1(t)|}{5}+\frac{|u_2(t)|}{10}+\frac{1}{\pi}\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}u_2(t)\right|\right).
\end{aligned}$$

Clearly, for each $t\in\left[0,\frac{\pi}{4}\right]$, the functions f_i are jointly continuous for all $u_i\in\mathbb{R}$ with $i=1,2$. We have

$$\begin{aligned}
&\left|f_1\left(t,u_1,u_2,{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1\right)-f_1\left(t,v_1,v_2,{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}v_1\right)\right|\\
&\qquad\le\sum_{k=1}^{2}\frac{1}{2\pi}|u_k-v_k|+\frac{1}{\pi}\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u_1-{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}v_1\right|,\\
&\left|f_2\left(t,u_1,u_2,{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}u_2\right)-f_2\left(t,v_1,v_2,{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}v_2\right)\right|
\end{aligned}$$

$$\leq \sum_{k=1}^{2} \frac{1}{5k} |u_k - v_k| + \frac{1}{\pi} \left| {}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u_2 - {}^1\mathcal{D}_{0^+}^{\frac{1}{2}} v_2 \right|.$$

Therefore, condition **(H1)** is satisfied with $\lambda_1 = \frac{1}{2\pi}$, $\lambda_2 = \frac{1}{5}$ and $\alpha_i - \beta_i = \frac{1}{4}$ for any $i = 1, 2$ and

$$\gamma_1 = \gamma_2 = \frac{1}{\pi} < \frac{\rho^{\alpha_i - \beta_i} \Gamma(1 + \alpha_i - \beta_i)}{T^{\rho(\alpha_i - \beta_i)}} \simeq 0.9628.$$

Also, we have

$$\left| f_1\left(t, u_1, u_2, {}^1\mathcal{D}_{0^+}^{\frac{1}{4}} u_1\right)\right| \leq \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)} \left(2 + \frac{1}{2}\left(|u_1(t)| + |u_2(t)|\right) + \left|{}^1\mathcal{D}_{0^+}^{\frac{1}{4}} u_1(t)\right|\right),$$

$$\left| f_2\left(t, u_1, u_2, {}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u_2\right)\right| \leq \tan(t)\left(1 + \frac{1}{5}|u_1(t)| + \frac{1}{10}|u_2(t)| + \frac{1}{\pi}\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u_2(t)\right|\right).$$

Thus, condition **(H2)** is satisfied with

$$a_1(t) = \frac{2\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}, \qquad a_2(t) = \tan(t),$$

$$b_1(t) = \frac{\cos(t)}{2\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}, \qquad b_2(t) = \frac{1}{5}\tan(t),$$

$$c_1(t) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}, \qquad c_2(t) = \frac{1}{\pi}\tan(t).$$

We also have $a_1^* = \frac{2}{\pi}$, $a_2^* = 1$, $b_1^* = \frac{1}{2\pi}$ and $b_2^* = \frac{1}{5}$. Thus $\alpha_i - \beta_i = \frac{1}{4}$ for any $i = 1, 2$ and

$$c_1^* = c_2^* = \frac{1}{\pi} < \frac{\rho^{\alpha_i - \beta_i} \Gamma(1 + \alpha_i - \beta_i)}{T^{\rho(\alpha_i - \beta_i)}} \simeq 0.9628$$

and

$$N_1 = \frac{n\rho^{\alpha_1 - \beta_1}\Gamma(1 + \alpha_1 - \beta_1)\, b_1^*}{\rho^{\alpha_1 - \beta_1}\Gamma(1 + \alpha_1 - \beta_1) - c_1^* T^{\rho(\alpha_1 - \beta_1)}} = \frac{\frac{1}{\pi}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{4}\right) - \frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}} \simeq 0.47551.$$

Additionally,

$$N_2 = \frac{n\rho^{\alpha_2 - \beta_2}\Gamma(1 + \alpha_2 - \beta_2)\, b_2^*}{\rho^{\alpha_2 - \beta_2}\Gamma(1 + \alpha_2 - \beta_2) - c_2^* T^{\rho(\alpha_2 - \beta_2)}} = \frac{\frac{2}{5}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{4}\right) - \frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}} \simeq 0.59755$$

and the condition

$$\frac{N_1 T^{\rho\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1 + 1)} \simeq 0.50511 < \frac{N_2 T^{\rho\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2 + 1)} \simeq 0.54243 < 1.$$

It follows from Theorems 3.4 and 3.5, that problem (4.2) has at least one solution.

5. Conclusion

Using Schauder's fixed point theorem, the Banach contraction principle and the Leray–Schauder type nonlinear alternative, this paper explored the existence and main properties of at least one solution and its uniqueness for a class of a new system of n nonlinear fractional differential equations with n integral conditions, with Katugampola's fractional derivative being used as the differential operator, and which is crucial for generalizing Hadamard's and Riemann–Liouville's fractional derivatives into a single form.

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Дослідження існування розв'язків системи n нелінійних дробових диференціальних рівнянь з інтегральними умовами

Bilal Basti and Yacine Arioua

У цій роботі обговорено і досліджено існування і єдиність розв'язків для нового класу систем n нелінійних диференціальних рівнянь з дробовими похідними та їх основні властивості, використовуючи дробову похідну Катуґамроли з n інтегральними умовами. Для досягнення бажаної мети застосовано теореми Шаудера і Банаха про нерухому точку та нелінійну альтернативу типу Лере–Шаудера. Для того, щоб продемонструвати корисність наших основних результатів, у роботі надано декілька прикладів.

Ключові слова: система, дробове диференціальне рівняння, інтегральні умови, існування, єдиність