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Uniform Regularity of the Magnetic Bénard Problem in a Bounded Domain

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In this paper, we prove the uniform regularity of the magnetic Bénard problem in a bounded domain.

Key words: magnetic Bénard problem, bounded domain, uniform regularity

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1. Introduction

In this paper, we consider the following 3D magnetic Bénard problem [9]:

$$\partial_t u + (u\cdot\nabla)u + \nabla(\pi + \frac{1}{2}|b|^2) - \mu\Delta u = (b\cdot\nabla)b + \theta e_3, \tag{1.1}$$

$$\partial_t b + (u\cdot\nabla)b = (b\cdot\nabla)u + \eta\Delta b, \tag{1.2}$$

$$\partial_t \theta + (u\cdot\nabla)\theta - k\Delta\theta = ue_3, \tag{1.3}$$

$$\operatorname{div} u = \operatorname{div} b = 0 \quad \text{in} \quad \Omega\times(0,\infty), \tag{1.4}$$

$$u = 0, b\cdot n = 0, \operatorname{rot} b\times n = 0, \frac{\partial\theta}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty), \tag{1.5}$$

$$(u,b,\theta)(\cdot,0) = (u_0,b_0,\theta_0)(\cdot) \quad \text{in} \quad \Omega\subset\mathbb{R}^3. \tag{1.6}$$

Here u, the fluid velocity field, π, the pressure, b, the magnetic field, and θ, the temperature, are the unknowns, $e_3 := (0,0,1)^t$, μ is the viscosity coefficient, η is the resistivity coefficient, k is the heat conductivity coefficient, Ω is a bounded and simply connected domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, n is the unit outward normal vector to $\partial\Omega$.

When $b\equiv 0$, the system reduces to the well-known Boussinesq system. Lai–Pan–Zhao [12] and K. Zhao [20] showed the global well-posedness of smooth solutions with $\mu = 0, k = 1$ or $\mu = 1, k = 0$. Jin–Fan–Nakamura–Zhou [11] studied the partial vanishing viscosity limit.

Zhou–Fan–Nakamura [21] showed the global well-posedness of smooth solution to the problem (1.1)–(1.6) when $k = 0$ and $\Omega := \mathbb{R}^2$ for large initial data b_0 but with positive resistivity. For other studies of magnetic Bénard problem, we refer readers to [4–8, 16, 18, 19].

The aim of this paper is to prove some uniform regularity estimates. We will prove the following.

Theorem 1.1. *Let* $0 < \eta, k < 1, \theta_0 \in W^{1,6}, u_0 \in H_0^1 \cap H^2, b_0 \in H^2$ *with* $b_0 \cdot n = 0$, $\operatorname{rot} b_0 \times n = 0$ *on* $\partial\Omega$ *and* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ *in* Ω.

Then there exists a small time T *independent of* $\eta, k > 0$ *and a unique strong solution* u, θ, b *to the initial boundary value problem* (1.1)–(1.6) *such that*

$$u \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,6}), \quad u_t \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1),$$
$$b, \theta \in L^\infty(0,T;W^{1,6}),\ b_t, \theta_t \in L^\infty(0,T;L^2),\ \sqrt{\eta} b, \sqrt{k}\theta \in L^\infty(0,T;H^2), \tag{1.7}$$

with the corresponding norms that are uniformly bounded with respect to η *and* k*.*

Theorem 1.1 will be proved by using the Banach fixed point theorem. We denote the nonempty set by

$$\mathcal{A} := \{\tilde{u} \in \mathcal{A};\ \tilde{u}(\cdot,0) = u_0,\ \operatorname{div} \tilde{u} = 0,\ \|\tilde{u}\|_{\mathcal{A}} \le A\}$$

with the norm

$$\|\tilde{u}\|_{\mathcal{A}} := \|\tilde{u}\|_{L^\infty(0,T;H^2)} + \|\tilde{u}\|_{L^2(0,T;W^{2,6})} + \|\partial_t \tilde{u}\|_{L^\infty(0,T;L^2)} + \|\partial_t \tilde{u}\|_{L^2(0,T;H^1)}.$$

Let $\tilde{u} \in \mathcal{A}$ be given, we consider the following linear problems:

$$\partial_t b + \tilde{u} \cdot \nabla b - b \cdot \nabla \tilde{u} = \eta \Delta b, \tag{1.8}$$
$$\operatorname{div} b = 0, \tag{1.9}$$
$$b(\cdot,0) = b_0, \tag{1.10}$$
$$b \cdot n = 0, \operatorname{rot} b \times n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.11}$$
$$\partial_t \theta + \tilde{u} \cdot \nabla \theta - k \Delta \theta = \tilde{u} e_3, \tag{1.12}$$
$$\theta(\cdot,0) = \theta_0, \tag{1.13}$$
$$\frac{\partial \theta}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,T); \tag{1.14}$$
$$\partial_t u + \tilde{u} \cdot \nabla u + \nabla \pi - \mu \Delta u = b \cdot \nabla b - \frac{1}{2} \nabla |b|^2 + \theta e_3, \tag{1.15}$$
$$u(\cdot,0) = u_0, \tag{1.16}$$
$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{1.17}$$

Let u be a unique strong solution to the above problem, we define the fixed point map $F : \tilde{u} \in \mathcal{A} \to u \in \mathcal{A}$ with $\tilde{u}(\cdot,0) = u_0$ and $\tilde{u} = 0$ on $\partial\Omega \times (0,T)$. We are to prove that the map F maps $\mathcal{A}$ into $\mathcal{A}$ for a suitable constant A and a small T and F is a contraction mapping on $\mathcal{A}$. Thus F has a unique fixed point in $\mathcal{A}$. This proves Theorem 1.1.

2. Preliminaries

In this section, we will collect some lemmas which will be used in the proof.

Lemma 2.1 (Poincaré inequality). *Let Ω be a bounded simple connected domain with smooth boundary and let w be a smooth vector satisfying $w \cdot n = 0$ on the boundary $\partial\Omega$. Then*

$$\|w\|_{L^p} \le C\|\nabla w\|_{L^p} \tag{2.1}$$

holds for $2 \le p < \infty$.

Proof. For $p = 2$, the proof was given in Lions [13, (6.47), page 75]. We assume $2 < p < \infty$. Using the Gagliardo–Nirenberg inequality and the case $p = 2$, we see that

$$\begin{aligned}\|w\|_{L^p} &\le C\|w\|_{L^2}^{1-\theta}\|\nabla w\|_{L^p}^{\theta} + C\|w\|_{L^2} \le C\|\nabla w\|_{L^2}^{1-\theta}\|\nabla w\|_{L^p}^{\theta} + C\|\nabla w\|_{L^2} \\ &\le C\|\nabla w\|_{L^p}^{1-\theta}\|\nabla w\|_{L^p}^{\theta} + C\|\nabla w\|_{L^p} \le C\|\nabla w\|_{L^p}.\end{aligned}$$

This completes the proof. □

Lemma 2.2 ([17]). *There holds*

$$\|\nabla w\|_{L^p} \le C(\|\operatorname{div} w\|_{L^p} + \|\operatorname{rot} w\|_{L^p}) \tag{2.2}$$

for any smooth vector w satisfying $w \cdot n = 0$ or $w \times n = 0$ on $\partial\Omega$ and $1 < p < \infty$.

Lemma 2.3 ([3]). *There holds*

$$\begin{aligned}-\int_\Omega \Delta f \cdot f|f|^{p-2}\,\mathrm{d}x &= \int_\Omega |f|^{p-2}|\nabla f|^2\,\mathrm{d}x \\ &\quad + 4\frac{p-2}{p^2}\int_\Omega |\nabla|f|^{\frac{p}{2}}|^2\,\mathrm{d}x - \int_{\partial\Omega} |f|^{p-2}(n\cdot\nabla)f\cdot f\,\mathrm{d}S\end{aligned} \tag{2.3}$$

for any smooth vector f and $1 < p < \infty$.

Lemma 2.4 ([2, Lemma 2.2]). *Assume that b is sufficiently smooth and satisfies the boundary condition $b \cdot n = 0, \operatorname{rot} b \times n = 0$ on $\partial\Omega$. Then the following identity holds for $J := \operatorname{rot} b$:*

$$-\frac{\partial J}{\partial n}\cdot J = (\epsilon_{1jk}\epsilon_{1\beta\gamma} + \epsilon_{2jk}\epsilon_{2\beta\gamma} + \epsilon_{3jk}\epsilon_{3\beta\gamma})J_j J_\beta \partial_k n_\gamma \tag{2.4}$$

on $\partial\Omega$, where ϵ_{ijk} denotes the totally anti-symmetric tensor such that $(a\times b)_i = \epsilon_{ijk}a_j b_k$.

Lemma 2.5 ([1, Lemma 7.44] and [14, Corollary 1.7]). *There holds*

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \tag{2.5}$$

for any smooth f and $1 < p < \infty$.

Proof. We have

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{W^{\frac{1}{p},p}(\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}}.$$ □

Lemma 2.6 ([10]). *Let b be a solution to the Poisson equation*

$$-\Delta b = f \quad \text{in} \quad \Omega$$

with the boundary condition

$$b \cdot n = 0, \quad \operatorname{rot} b \times n = 0 \quad \text{on} \quad \partial\Omega.$$

Then it holds

$$\|b\|_{H^2} \leq C\|f\|_{L^2} + C\|\nabla b\|_{L^2}. \tag{2.6}$$

Lemma 2.7 ([15]). *For the bounded domain Ω and $\theta \in C^2(\overline{\Omega})$, satisfying $\frac{\partial\theta}{\partial n} = 0$ on $\partial\Omega$, we have*

$$\frac{\partial}{\partial n}|\nabla\theta|^2 \leq 2K|\nabla\theta|^2 \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

where $K = K(\Omega)$ is an upper bound for the curvatures of $\partial\Omega$.

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Lemma 3.1. *Let $\tilde{u} \in \mathcal{A}$ be given. Then the problem (1.8)–(1.11) has a unique solution b satisfying*

$$\|\operatorname{rot} b\|_{L^6(\Omega)} \leq C, \tag{3.1}$$

$$\|\partial_t b(\cdot, t)\|_{L^2} \leq C, \tag{3.2}$$

$$\sqrt{\eta}\|\operatorname{rot} \partial_t b\|_{L^2(0,T;L^2)} \leq C, \tag{3.3}$$

$$\sqrt{\eta}\|b(\cdot, t)\|_{H^2} \leq C + CA. \tag{3.4}$$

for some small $0 < T \leq 1$.

Proof. Since equations (1.8)–(1.11) are linear with regular $\tilde{u}$, the existence and uniqueness are well known, we only need to show a priori estimates.

Denoting

$$J := \operatorname{rot} b,$$

applying rot to (1.8), we observe that

$$\partial_t J + \tilde{u} \cdot \nabla J - \eta\Delta J = g := -\sum_i \nabla\tilde{u}_i \times \partial_i b + \operatorname{rot}(b \cdot \nabla\tilde{u}). \tag{3.5}$$

Testing (3.5) by $|J|^4 J$, using (2.3), (2.4), (2.2), and (2.5), we have

$$\begin{aligned}
&\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |J|^6 \mathrm{d}x + \eta\int_\Omega |J|^4|\nabla J|^2 \mathrm{d}x + \frac{4}{9}\eta\int_\Omega |\nabla|J|^3|^2\,\mathrm{d}x \\
&\quad = \eta\int_{\partial\Omega} |J|^4(\epsilon_{1jk}\epsilon_{1\beta\gamma} + \epsilon_{2jk}\epsilon_{2\beta\gamma} + \epsilon_{3jk}\epsilon_{3\beta\gamma})J_j J_\beta \partial_k n_\gamma\,\mathrm{d}S + \int_\Omega g|J|^4 J\,\mathrm{d}x
\end{aligned}$$

$$\leq C\eta \int_{\partial\Omega} |J|^6 \mathrm{d}S + C\|\nabla \tilde{u}\|_{L^\infty}\|J\|^6_{L^6(\Omega)} + C\|b\|_{L^\infty(\Omega)}\|\nabla^2\tilde{u}\|_{L^6(\Omega)}\|J\|^5_{L^6(\Omega)}$$
$$\leq C\eta \||J|^3\|_{L^2(\Omega)}\|\nabla|J|^3\|_{L^2(\Omega)} + C(\|\nabla\tilde{u}\|_{L^\infty(\Omega)} + \|\nabla^2\tilde{u}\|_{L^6(\Omega)})\|J\|^6_{L^6(\Omega)}$$
$$\leq \frac{1}{9}\eta\|\nabla|J|^3\|^2_{L^2(\Omega)} + C(1 + \|\nabla\tilde{u}\|_{L^\infty(\Omega)} + \|\nabla^2\tilde{u}\|_{L^6(\Omega)})\|J\|^6_{L^6(\Omega)},$$

which gives (3.1):

$$\|J\|_{L^6(\Omega)} \leq \|J_0\|_{L^6(\Omega)} \exp\left(C\int_0^T (1 + \|\nabla\tilde{u}\|_{L^\infty} + \|\nabla^2\tilde{u}\|_{L^6})\mathrm{d}t\right)$$
$$\leq C\exp(C\sqrt{T}A) \leq C \tag{3.6}$$

if $\sqrt{T}A \leq 1$.

Here we have used the estimate

$$\|b\|_{L^\infty} \leq C(\|b\|_{L^6} + \|\nabla b\|_{L^6}) \leq C\|\nabla b\|_{L^6} \leq C\|J\|_{L^6}. \tag{3.7}$$

Taking ∂_t to (1.8), testing by $\partial_t b$, using (3.6) and (3.7), we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t b|^2\,\mathrm{d}x + \eta\int |\mathrm{rot}\,\partial_t b|^2\,\mathrm{d}x = -\int \tilde{u}\cdot\nabla\partial_t b\cdot\partial_t b\,\mathrm{d}x + \int \partial_t b\cdot\nabla\tilde{u}\cdot\partial_t b\,\mathrm{d}x$$
$$-\int \partial_t\tilde{u}\cdot\nabla b\cdot\partial_t b\,\mathrm{d}x + \int b\cdot\nabla\partial_t\tilde{u}\cdot\partial_t b\,\mathrm{d}x$$
$$\leq C\|\nabla\tilde{u}\|_{L^\infty}\|\partial_t b\|^2_{L^2} + C\|\partial_t\tilde{u}\|_{L^6}\|\nabla b\|_{L^3}\|\partial_t b\|_{L^2} + C\|b\|_{L^\infty}\|\nabla\partial_t\tilde{u}\|_{L^2}\|\partial_t b\|_{L^2},$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t b\|_{L^2} \leq C\|\nabla\tilde{u}\|_{L^\infty}\|\partial_t b\|_{L^2} + C\|\nabla\partial_t\tilde{u}\|_{L^2}. \tag{3.8}$$

Whence we obtain (3.2):

$$\|\partial_t b(\cdot,t)\|_{L^2} \leq \left(\|\partial_t b(\cdot,0)\|_{L^2} + \int_0^T \|\nabla\partial_t\tilde{u}\|_{L^2}\mathrm{d}t\right)\exp\left(C\int_0^T \|\nabla\tilde{u}\|_{L^\infty}\mathrm{d}t\right)$$
$$\leq C(1+\sqrt{T}A)\exp(C\sqrt{T}A) \leq C \tag{3.9}$$

if $\sqrt{T}A \leq 1$. Now it is obvious that (3.3) and (3.4) hold. The lemma is proved. □

Lemma 3.2. *Let $\tilde{u} \in \mathcal{A}$ be given. Then the problem (1.12)–(1.14) has a unique solution θ satisfying*

$$\int |\nabla\theta|^6\,\mathrm{d}x \leq C, \tag{3.10}$$
$$\|\theta_t(\cdot,t)\|_{L^2} \leq C, \tag{3.11}$$
$$\sqrt{k}\|\nabla\theta_t\|_{L^2(0,T;L^2)} + \sqrt{k}\|\theta\|_{L^\infty(0,T;H^2)} \leq C \tag{3.12}$$

for some small $0 < T \leq 1$.

Proof. Since equation (1.12) is linear with regular $\tilde{u}$, the existence and uniqueness are well known, we only need to establish a priori estimates.

Since

$$\tilde{u}(x,t) = u_0(x) + \int_0^t \partial_t \tilde{u} \mathrm{d}s,$$

we have

$$\|\tilde{u}\|_{L^\infty(0,T;L^6)} \le \|u_0\|_{L^6} + \int_0^T \|\partial_t \tilde{u}\|_{L^6} \,\mathrm{d}t \le C + C\sqrt{T}A \le C \tag{3.13}$$

if $A\sqrt{T} \le 1$.

Testing (1.12) by θ and using (3.13), we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \theta^2 \,\mathrm{d}x + k\int |\nabla\theta|^2 \,\mathrm{d}x = \int \tilde{u}e_3\theta \,\mathrm{d}x \le \|\tilde{u}\|_{L^2}\|\theta\|_{L^2} \le C\|\theta\|_{L^2},$$

which yields

$$\int \theta^2 \,\mathrm{d}x + k\int_0^T \int |\nabla\theta|^2 \,\mathrm{d}x \,\mathrm{d}t \le C.$$

Taking ∇ to (1.12), testing by $|\nabla\theta|^4\nabla\theta$, using (2.3), (2.5), and (2.7), we derive

$$\begin{aligned}
&\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\theta|^6 \,\mathrm{d}x + k\int |\nabla\theta|^4|\nabla^2\theta|^2 \,\mathrm{d}x + \frac{4}{9}k\int |\nabla|\nabla\theta|^3|^2 \,\mathrm{d}x \\
&\quad \le Ck\int_{\partial\Omega} |\nabla\theta|^6 \,\mathrm{d}S + C\|\nabla\tilde{u}\|_{L^\infty}(1+\|\nabla\theta\|_{L^6})\|\nabla\theta\|_{L^6}^5 \\
&\quad \le Ck\|\nabla\theta\|_{L^6(\Omega)}^3\|\nabla|\nabla\theta|^3\|_{L^2(\Omega)} + C\|\nabla\tilde{u}\|_{L^\infty}(1+\|\nabla\theta\|_{L^6})\|\nabla\theta\|_{L^6}^5 \\
&\quad \le \frac{1}{9}k\|\nabla|\nabla\theta|^3\|_{L^2}^2 + C\|\nabla\theta\|_{L^6}^6 + C\|\nabla\tilde{u}\|_{L^\infty}(1+\|\nabla\theta\|_{L^6})\|\nabla\theta\|_{L^6}^5,
\end{aligned}$$

which implies (3.10):

$$\int |\nabla\theta|^6 \,\mathrm{d}x \le C\exp(C\sqrt{T}A) \le C \tag{3.14}$$

if $\sqrt{T}A \le 1$.

Applying ∂_t to (1.12), testing by θ_t and using (3.14), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \theta_t^2 \,\mathrm{d}x + k\int |\nabla\theta_t|^2 \,\mathrm{d}x &= \int \tilde{u}_t e_3\theta_t \,\mathrm{d}x - \int \tilde{u}_t\cdot\nabla\theta\cdot\theta_t \,\mathrm{d}x \\
&\le \|\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^2} + \|\tilde{u}_t\|_{L^6}\|\nabla\theta\|_{L^3}\|\theta_t\|_{L^2} \\
&\le A\|\theta_t\|_{L^2} + C\|\nabla\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^2}.
\end{aligned}$$

Whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_{L^2} \le A + C\|\nabla\tilde{u}_t\|_{L^2},$$

which implies (3.11):

$$\|\theta_t(\cdot,t)\|_{L^2} \le C + AT + C\sqrt{T}A \le C \tag{3.15}$$

if $AT \le 1$ and $A\sqrt{T} \le 1$.

Similarly to (3.3) and (3.4), we have (3.12). The lemma is proved. □

Lemma 3.3. *Let $\tilde{u} \in \mathcal{A}$ be given. Then the problem (1.15)–(1.17) has a unique solution u satisfying*

$$\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;W^{2,6})} + \|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le C_1 \quad (3.16)$$

for some small $0 < T \le 1$. Here C_1 is a positive constant independent of η, k and A.

Proof. Since equation (1.15) is linear with regular $\tilde{u}, b, \theta$, the existence and uniqueness are well known, we only need to establish (3.16).

Testing (1.15) by u and using Lemmas 3.1 and 3.2, we see that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\,\mathrm{d}x + \int \mu|\nabla u|^2\,\mathrm{d}x &= \int b\cdot\nabla b\cdot u\,\mathrm{d}x + \int \theta e_3 u\,\mathrm{d}x\\ &\le \|\nabla b\|_{L^6}\|b\|_{L^3}\|u\|_{L^2} + \|\theta\|_{L^2}\|u\|_{L^2} \le C\|u\|_{L^2},\end{aligned}$$

which gives

$$\int |u|^2\,\mathrm{d}x + \int_0^T\int |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le C. \quad (3.17)$$

Testing (1.15) by u_t and using Lemmas 3.1 and 3.2, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \mu|\nabla u|^2\,\mathrm{d}x + \int |u_t|^2\,\mathrm{d}x\\ &= -\int \tilde{u}\cdot\nabla u\cdot u_t\,\mathrm{d}x + \int b\cdot\nabla b\cdot u_t\,\mathrm{d}x + \int \theta e_3 u_t\,\mathrm{d}x\\ &\le \frac{1}{4}\int |u_t|^2 dx + C\|\tilde{u}\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + C\|\nabla b\|_{L^2}\|b\|_{L^\infty}\|u_t\|_{L^2} + \|\theta\|_{L^2}\|u_t\|_{L^2}\\ &\le \frac{1}{2}\int |u_t|^2 dx + CA^2\|\nabla u\|_{L^2}^2 + C + CA^2,\end{aligned}$$

which implies

$$\int |\nabla u|^2\,\mathrm{d}x + \int_0^T\int |u_t|^2\,\mathrm{d}x\,\mathrm{d}t \le C \quad (3.18)$$

if $A^2T \le 1$.

Applying ∂_t to (1.15), testing by u_t, using Lemmas 3.1, 3.2 and (3.18), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u_t|^2\,\mathrm{d}x + \int \mu|\nabla u_t|^2\,\mathrm{d}x\\ &= -\int \tilde{u}_t\cdot\nabla u\cdot u_t\,\mathrm{d}x - \int \partial_t(b\otimes b) : \nabla u_t\,\mathrm{d}x + \int \theta_t e_3 u_t\,\mathrm{d}x\\ &\le \|\tilde{u}_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6} + C\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u_t\|_{L^2} + \|\theta_t\|_{L^2}\|u_t\|_{L^2}\\ &\le CA\|u_t\|_{L^3} + C\|\tilde{u}_t\|_{L^3}\|u_t\|_{L^6} + C\|\nabla u_t\|_{L^2}\\ &\le \frac{\mu}{2}\int |\nabla u_t|^2\,\mathrm{d}x + C\|\tilde{u}_t\|_{L^2}\|\nabla\tilde{u}_t\|_{L^2} + C,\end{aligned}$$

which gives

$$\int |u_t|^2 \,\mathrm{d}x + \int_0^T \int |\nabla u_t|^2 \,\mathrm{d}x \,\mathrm{d}t \le C_1 \tag{3.19}$$

if $A^2\sqrt{T} \le 1$.

We rewrite (1.15) as

$$-\mu\Delta u + \nabla \pi = f := \operatorname{rot} b \times b - \partial_t u - \tilde{u}\cdot\nabla u + \theta e_3. \tag{3.20}$$

By the H^2-theory of elliptic systems, we get

$$\begin{aligned}\|u\|_{H^2} &\le C\|f\|_{L^2} \le C\|\operatorname{rot} b\|_{L^6}\|b\|_{L^3} + C\|\partial_t u\|_{L^2} + C\|\tilde{u}\|_{L^6}\|\nabla u\|_{L^3} + C\|\theta\|_{L^2} \\ &\le C + C\|\nabla u\|_{L^3},\end{aligned}$$

which yields

$$\|u\|_{L^\infty(0,T;H^2)} \le C_1. \tag{3.21}$$

In a similar way, by the $W^{2,6}$-theory of elliptic systems, we obtain

$$\begin{aligned}\|u\|_{W^{2,6}} &\le C\|f\|_{L^6} \\ &\le C\|\operatorname{rot} b\|_{L^6}\|b\|_{L^\infty} + C\|u_t\|_{L^6} + C\|\tilde{u}\|_{L^6}\|\nabla u\|_{L^\infty} + C\|\theta\|_{L^6} \\ &\le C + C\|\nabla u\|_{L^\infty} + CA^2 + C\|u_t\|_{L^6} \\ &\le C + C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|u\|_{W^{2,6}}^{\frac{3}{4}} + CA^2 + C\|\nabla u_t\|_{L^2}.\end{aligned}$$

Whence

$$\|u\|_{W^{2,6}} \le C + CA^2 + C\|\nabla u_t\|_{L^2},$$

which yields

$$\|u\|_{L^2(0,T;W^{2,6})} \le C_1 \tag{3.22}$$

if $A^4 T \le 1$. This completes the proof. □

Due to the above Lemmas 2.1–2.3, we can take $A := C_1$, and thus F maps $\mathcal{A}$ into $\mathcal{A}$. The following lemma tells us that F is a contraction mapping in the sense of a weaker norm.

Lemma 3.4. *There is a constant $0 < \delta < 1$ such that for any $\tilde{u}_i$ $(i = 1, 2)$,*

$$\|F(\tilde{u}_1) - F(\tilde{u}_2)\|_{L^2(0,T;H^1)} \le \delta\|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T;H^1)} \tag{3.23}$$

for some small $0 < T \le 1$.

Proof. Suppose u_i, π_i, b_i, θ_i, $i = 1, 2$, are the solutions to the problem (1.8), (1.17) corresponding to $\tilde{u}_i$ $(i = 1, 2)$. Denote

$$u := u_1 - u_2, \quad b := b_1 - b_2, \quad \theta := \theta_1 - \theta_2, \quad \tilde{u} := \tilde{u}_1 - \tilde{u}_2.$$

Then we have

$$\partial_t b - \eta\Delta b = -\tilde{u}_1\cdot\nabla b - \tilde{u}\cdot\nabla b_2 + b_1\cdot\nabla\tilde{u} + b\cdot\nabla\tilde{u}_2, \tag{3.24}$$

$$\partial_t\theta + \tilde{u}_1\cdot\nabla\theta + \tilde{u}\cdot\nabla\theta_2 - k\Delta\theta = \tilde{u}e_3, \tag{3.25}$$

$$\begin{aligned}&\partial_t u + \tilde{u}_1\cdot\nabla u + \nabla(\pi_1-\pi_2) - \mu\Delta u + \tilde{u}\cdot\nabla u_2\\ &\qquad = \operatorname{div}(b_1\otimes b_1 - b_2\otimes b_2) + \theta e_3.\end{aligned} \tag{3.26}$$

Testing (3.24) by b, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|b^2|\mathrm{d}x + \eta\int|\mathrm{rot}\, b|^2\mathrm{d}x &\le C\|\nabla\tilde{u}_2\|_{L^\infty}\|b\|_{L^2}^2 + C\|\nabla\tilde{u}\|_{L^2}\|b\|_{L^2}\\ &\le \epsilon_1\|\nabla\tilde{u}\|_{L^2}^2 + C\|b\|_{L^2}^2 + C\|\nabla\tilde{u}_2\|_{L^\infty}\|b\|_{L^2}^2\end{aligned} \tag{3.27}$$

for any $0<\epsilon_1<1$.

Testing (3.25) by θ, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\theta^2\mathrm{d}x + k\int|\nabla\theta|^2\mathrm{d}x &= -\int\tilde{u}\cdot\nabla\theta_2\cdot\theta\mathrm{d}x + \int\tilde{u}e_3\theta\mathrm{d}x\\ &\le \|\tilde{u}\|_{L^6}\|\nabla\theta_2\|_{L^3}\|\theta\|_{L^2} + \|\tilde{u}\|_{L^2}\|\theta\|_{L^2} \le C\|\nabla\tilde{u}\|_{L^2}\|\theta\|_{L^2}\\ &\le \epsilon_2\|\nabla\tilde{u}\|_{L^2}^2 + C\|\theta\|_{L^2}^2\end{aligned} \tag{3.28}$$

for any $0<\epsilon_2<1$.

Testing (3.26) by u, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^2\mathrm{d}x + \int\mu|\nabla u|^2\mathrm{d}x\\ &\quad= -\int\tilde{u}\cdot\nabla u_2\cdot u dx - \int(b_1\otimes b_1 - b_2\otimes b_2):\nabla u\mathrm{d}x + \int\theta e_3 u\mathrm{d}x\\ &\quad\le \|\tilde{u}\|_{L^2}\|\nabla u_2\|_{L^6}\|u\|_{L^3} + C\|b\|_{L^2}(\|b_1\|_{L^\infty} + \|b_2\|_{L^\infty})\|\nabla u\|_{L^2} + \|\theta\|_{L^2}\|u\|_{L^2}\\ &\quad\le \frac{\mu}{8}\|\nabla u\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}\|u\|_{L^3} + C\|b\|_{L^2}^2 + C\|\theta\|_{L^2}^2\\ &\quad\le \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \epsilon_3\|\tilde{u}\|_{L^2}^2 + C\|u\|_{L^2}^2 + C\|b\|_{L^2}^2 + C\|\theta\|_{L^2}^2\end{aligned} \tag{3.29}$$

for any $0<\epsilon_3<1$.

Combining (3.27)–(3.29) and taking ϵ_i, $i=1,2,3$, small enough, by using the Gronwall inequality, we arrive at (3.23) for small $0<T\le 1$. This completes the proof. □

Proof of Theorem 1.1. By Lemmas 3.1–3.4 and the Banach fixed point theorem, we finish the proof. □

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References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev spaces*, **140**, Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, 2003.

[2] H. Beirão da Veiga and L. C. Berselli, *Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary*, J. Differential Equations **246** (2009), 597–628.

[3] H. Beirão da Veiga and F. Crispo, *Sharp inviscid limit results under Navier type boundary conditions. An L^p theory*, J. Math. Fluid Mech. **12** (2010), 397–411.

[4] J. Cheng and L. Du, *On two-dimensional magnetic Bénard problem with mixed partial viscosity*, J. Math. Fluid Mech. **17** (2015), 769–797.

[5] J. Fan, D. Liu, and Y. Zhou, *Uniform global strong solutions of the 2D magnetic Bénard problem in a bounded domain*, Appl. Math. Lett. **86** (2018), 166–172.

[6] J. Fan, L. Wang, and Y. Zhou, *Global strong solutions of the 2D density-dependent incompressible magnetic Bénard problem*, Bull. Malays. Math. Sci. Soc. **44** (2021), 1749–1769.

[7] J. Fan and Y. Zhou, *Uniform regularity of the density-dependent incompressible MHD system in a bounded domain*, Math. Phys. Anal. Geom. **23** (2020), 39.

[8] J. Fan and Y. Zhou, *Uniform regularity of fully compressible Hall-MHD systems*, Electron. J. Differential Equations (2021), 17.

[9] G.P. Galdi and M. Padula, *A new approach to energy theory in the stability of fluid motion*, Arch. Rational Mech. Anal. **110** (1990), 187–286.

[10] T. Huang, C. Wang, and H. Wen, *Strong solutions of the compressible nematic liquid crystal flow*, J. Differential Equations, **252** (2012), 2222–2265.

[11] L. Jin, J. Fan, G. Nakamura, and Y. Zhou, *Partial vanishing viscosity limit for the 2D Boussinesq system with a slip boundary condition*, Bound. Value Probl. **2012** (2012), 20.

[12] M.-J. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for two-dimensional viscous Boussinesq equations*, Arch. Ration. Mech. Anal. 199 (2011), 739–760.

[13] P.-L. Lions, *Mathematical topics in fluid mechanics,* **2**: Compressible Models, Clarendon Press, Oxford, 1998.

[14] A. Lunardi, *Interpolation theory*, Edizioni della Normale, Pisa, 2009.

[15] N. Mizoguchi and P. Souplet, *Nondegeneracy of blow-up points for the parabolic Keller-Segel system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 851–875.

[16] G. Mulone and S. Rionero, *Necessary and sufficient conditions for nonlinear stability in the magnetic Bénard problem*, Arch. Ration. Mech. Anal. **166** (2003), 197–218.

[17] W. von Wahl, *Estimating ∇u by $\operatorname{div} u$ and $\operatorname{curl} u$*, Math. Methods Appl. Sci. **15** (1992), 123–143.

[18] K. Yamazaki, *Global regularity of generalized magnetic Benard problem*, Math. Methods Appl. Sci. **40** (2017), 2013–2033.

[19] Z. Ye, *Global regularity of the 2D magnetic Bénard system with partial dissipation*, Adv. Differential Equations 23 (2018), 193–238.

[20] K. Zhao, *2D inviscid heat conductive Boussinesq equations on a bounded domain*, Michigan Math. J. **59** (2010), 329–352.

[21] Y. Zhou, J. Fan, and G. Nakamura, *Global Cauchy problem for a 2D magnetic Bénard problem with zero thermal conductivity*, Appl. Math. Lett. **26** (2013), 627–630.

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Однорідна регулярність магнітної проблеми Бернарда в обмеженій області

Shengqi Lu and Miaochao Chen

У цій роботі ми доводимо однорідну регулярність магнітної проблеми Бернарда в обмеженій області.

Ключові слова: магнітна проблема Бернарда, обмежена область, однорідна регулярність