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Dynamic to Quotients of Hyperspaces

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Let (X, d) be a compact metric space and let n be a positive integer. Let $\mathcal{C}_n(X)$ be the space of all nonempty closed subsets of X with at most n components and let $\mathcal{F}_1(X)$ be the space of singletons of X. Given a map $f\colon X \to X$, we consider the induced map $\mathcal{C}_n(f) : \mathcal{C}_n(X) \to \mathcal{C}_n(X)$ given by $\mathcal{C}_n(f)(A) = f(A)$, for each $A \in \mathcal{C}_n(X)$. The discrete dynamical system (X, f) induces the discrete dynamic system $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$, where $\mathcal{PHS}_n(X)$ is the quotient space $\mathcal{C}_n(X)/\mathcal{F}_1(X)$ topologized with the quotient topology. In this paper, we generalize some results from [22] and study some relationships between the discrete dynamical systems (X, f), $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ and $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$.

Key words: chaotic map, exact map, mixing map, totally transitive map, transitive map, weakly mixing map, hyperspace, induced map, n-fold pseudo hyperspace suspension

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1. Introduction

Let (X, d) be a compact metric space. Given a positive integer n, we consider the hyperspaces 2^X, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ of X, where 2^X consists of all nonempty and closed subsets of X, $\mathcal{C}_n(X)$ consists of all elements of 2^X with at most n components and $\mathcal{F}_n(X)$ consists of all elements of 2^X with at most n points. All these hyperspaces are considered with the Hausdorff metric.

Let $f\colon X \to X$ be a map and let n be a positive integer. It can be seen that f induces a map on the hyperspace 2^X denoted by $2^f\colon 2^X \to 2^X$ and defined by $2^f(A) = f(A)$, for each $A \in 2^X$. The induced maps to other hyperspaces mentioned, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$, are simply the restriction of 2^f to each of these hyperspaces denoted by $\mathcal{C}_n(f)$ and $\mathcal{F}_n(f)$, respectively. In 2008, J. C. Macías introduced the notion of the n-fold pseudo-hyperspace suspension of a continuum X, denoted by $\mathcal{PHS}_n(X)$, as the quotient space $\mathcal{C}_n(X)/\mathcal{F}_1(X)$ topologized with the quotient topology [20]. Later, in [21], J. C. Macías and S. Macías considered the induced map $\mathcal{PHS}_n(f) : \mathcal{PHS}_n(X) \to \mathcal{PHS}_n(X)$, which they called the induced map of f on the n-fold pseudo-hyperspace suspension of X. In the same article, some topological properties of $\mathcal{PHS}_n(f)$ were studied. Recently, in [29], the authors proved more results on the n-fold pseudo-hyperspace suspension of continua and on the induced map $\mathcal{PHS}_n(f)$.

Furthermore, given a discrete dynamical system (X, f), one can obtain associated systems induced by (X, f), some are $(2^X, 2^f)$, $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ and $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$, given that it is important to study the connection of dynamical properties among (X, f) and its hyperspaces. In recent years, various interesting results have been obtained in this line of investigation (see, for example, [1–7, 12–14, 19, 26–28, 31]).

In this paper, we generalize some results from [22] and study some connections between dynamical properties of f and dynamical properties of the induced maps $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$. The paper is organized as follows. In Section 2, we recall basic definitions, introduce some notation and give some basic results. Section 3 is devoted to studying dynamical properties of the induced function $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$, on $(\mathcal{C}_n(X), \tau_L)$ and on $(\mathcal{C}_n(X), \tau)$. Some results given in this section generalize those obtained by D. Massod and P. Singh in [22]. Section 4 is divided into two subsections. In the first subsection, some dynamical properties of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ and $(\mathcal{PHS}_n(X), \tau)$ are considered. For instance, we prove that the exactness of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$ is equivalent. In the second subsection, some dynamical properties of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ and $(\mathcal{PHS}_n(X), \tau_U)$ are considered. More preciously, we study the relationships between a continuous function f on X in relation to the transitivity of its extensions $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$.

2. Definitions, notation and basic result

Throughout the paper, (X, f) denotes a discrete dynamical system, where X is a compact metric space and $f : X \to X$ is a map. A map is a continuous function and if we need to indicate the topology τ, used on the space, we will write the dynamical system as $((X, \tau), f)$. The symbol $\mathbb{N}$ denotes the set of positive integers. Given a compact metric space (X, d), a point $a \in X$ and $\epsilon > 0$, $B_d(a, \epsilon)$ denotes the open ball with center a and radius ϵ. Moreover, $N_d(A, \epsilon) = \bigcup_{a\in A} B_d(a, \epsilon)$.

For a compact metric space (X, d) and a positive integer n, we consider the following hyperspaces of X:

$$\begin{aligned}
2^X &= \{A \subseteq X \mid A \text{ is closed and nonempty}\},\\
\mathcal{C}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},\\
\mathcal{F}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}.
\end{aligned}$$

We agree that $\mathcal{C}(X) = \mathcal{C}_1(X)$. For $A, B \in 2^X$, let

$$\mathcal{H}(A, B) = \inf\{\epsilon > 0 \colon A \subseteq N_d(B, \epsilon) \text{ and } B \subseteq N_d(A, \epsilon)\}.$$

Then $\mathcal{H}$ is know as the Hausdoff metric in 2^X [23, (0.1)].

Let n be a positive integer. The quotient space

$$\mathcal{PHS}_n(X) = \mathcal{C}_n(X)/\mathcal{F}_1(X),$$

topologized with the quotient topology and known as the *n-fold pseudo-hyperspace suspension* of X, was introduced in 2008 by J. Macías [20]. Notice that $\mathcal{PHS}_1(X)$ corresponds to the hyperspace suspension $\mathcal{HS}(X)$ defined by S. Nadler in [24].

Remark 2.1. A compact metric space X, $q_X^n \colon \mathcal{C}_n(X) \to \mathcal{PHS}_n(X)$ denotes the quotient map. Also, F_X^n and T_X^n denote the points $q_X^n(\mathcal{F}_1(X))$ and $q_X^n(X)$.

Remark 2.2. Note that $\mathcal{PHS}_n(X)\setminus\{F_X^n\}$ and $\mathcal{PHS}_n(X)\setminus\{T_X^n, F_X^n\}$ are homeomorphic to $\mathcal{C}_n(X) \setminus \mathcal{F}_1(X)$ and $\mathcal{C}_n(X) \setminus (\{X\} \cup \mathcal{F}_1(X))$, respectively, using the appropriate restriction of q_X^n.

A map $f \colon X \to Y$ between compact metric spaces and a positive integer n, the function $\mathcal{C}_n(f) : \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ defined by $\mathcal{C}_n(f)(A) = f(A)$, for all $A \in \mathcal{C}_n(X)$, is the *induced map by f between the n-fold hyperspaces of X and Y*. Note that $\mathcal{C}_n(f)$ is continuous [15, 13.3]. Also, we consider the function $\mathcal{PHS}_n(f) : \mathcal{PHS}_n(X) \to \mathcal{PHS}_n(Y)$ given by

$$\mathcal{PHS}_n(f)(\chi) = \begin{cases} q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(\chi))) & \text{if } \chi \neq F_X^n \\ F_Y^n & \text{if } \chi = F_X^n \end{cases}. \tag{2.1}$$

Note that, by [10, 4.3, p. 126], $\mathcal{PHS}_n(f)$ is continuous and it is called *the induced map* of f on *the n-fold pseudo-hyperspace suspensions* of X and Y. In addition, the diagram

$$\begin{array}{ccc} \mathcal{C}_n(X) & \xrightarrow{\mathcal{C}_n(f)} & \mathcal{C}_n(Y) \\ \downarrow{\scriptstyle q_X^n} & & \downarrow{\scriptstyle q_Y^n} \\ \mathcal{PHS}_n(X) & \xrightarrow[\mathcal{PHS}_n(f)]{} & \mathcal{PHS}_n(Y) \end{array}$$

is commutative, that is, $q_Y^n \circ \mathcal{C}_n(f) = \mathcal{PHS}_n(f) \circ q_X^n$ (both pathways around the diagram give the same result).

In addition, on 2^X, we define some topologies.

(1) **Upper Vietoris topology**: for $m \in \mathbb{N}$ and any finite collection of non-empty open sets $\{U_1, U_2, \ldots, U_m\}$, define

$$\langle U_1, U_2, \ldots, U_m\rangle = \left\{ A \in 2^X \mid A \subseteq \bigcup_{i=1}^{m} U_i \right\}.$$

The upper Vietoris topology, denoted by τ_U, has

$$\mathcal{B}_U = \{\langle U_1, U_2, \ldots, U_m\rangle \colon m \in \mathbb{N} \text{ and } U_1, \ldots, U_m \text{ are open subsets of } X\}$$

as a basis.

(2) Lower Vietoris topology: for $m \in \mathbb{N}$ and any finite collection of non-empty open sets $\{U_1, U_2, \ldots, U_m\}$, define

$$\langle U_1, U_2, \ldots, U_m \rangle' = \left\{ A \in 2^X \mid A \cap U_i \neq \varnothing, \text{ for each } i \in \{1, \ldots, m\} \right\}.$$

The lower Vietoris topology, denoted by τ_L, has

$$\mathcal{B}_L = \{\langle U, U_2, \ldots, U_m \rangle' \colon m \in \mathbb{N} \text{ and } U_1, \ldots, U_m \text{ are open subsets of } X\}$$

as a basis.

(3) Vietoris topology: for $m \in \mathbb{N}$ and any finite collection of non-empty open sets $\{U_1, U_2, \ldots, U_m\}$, define

$$\langle U_1, U_2, \ldots, U_m \rangle'' = \left\{ A \in 2^X \mid A \subseteq \bigcup_{i=1}^{m} U_i \right. \\ \left. \text{and } A \cap U_i \neq \varnothing \text{ for each } i \in \{1, \ldots, m\} \right\}.$$

The Vietoris topology, denoted by τ, has

$$\mathcal{B} = \left\{ \langle U_1, U_2, \ldots, U_m \rangle'' \mid m \in \mathbb{N} \text{ and } U_1, U_2, \ldots, U_m \text{ are open subsets of } X \right\}$$

as a basis.

It can be seen that the Vietoris topology equals the join of upper and lower Vietoris topologies [30]. The upper and lower Vietoris topologies on $\mathcal{C}_n(X)$ are specified by $(\mathcal{C}_n(X), \tau_U)$ and $(\mathcal{C}_n(X), \tau_L)$.

Remark 2.3. Let X be a compact metric space, n be a positive integer, and let $U_1, U_2, \ldots, U_m$ be a finite family of open subsets of X. Then

- $\langle U_1, U_2, \ldots, U_m \rangle_n$ denotes the set $\langle U_1, U_2, \ldots, U_m \rangle \cap (\mathcal{C}_n(X), \tau_U)$;
- $\langle U_1, U_2, \ldots, U_m \rangle'_n$ denotes the set $\langle U_1, U_2, \ldots, U_m \rangle' \cap (\mathcal{C}_n(X), \tau_L)$;
- $\langle U_1, U_2, \ldots, U_m \rangle''_n$ denotes the set $\langle U_1, U_2, \ldots, U_m \rangle'' \cap (\mathcal{C}_n(X), \tau)$.

On the other hand, given a discrete dynamical system (X, f) and $x \in X$, we define $f^0 = Id_X$, where Id_X denotes the identity map on X, and for each $k \in \mathbb{N}$, let $f^k = f \circ f^{k-1}$. The *orbit of x under f* is the set $\mathrm{Orb}(x, f) = \{f^k(x) \mid k \in \mathbb{N} \cup \{0\}\}$. The set of all limit points of $\mathrm{Orb}(x, f)$ is called *the ω-limit set of x under f*, and it is denoted by $\omega(x, f)$. A subset K of X is said to be *invariant under f* if $f(K) \subseteq K$ and *strongly invariant under f* if $f(K) = K$.

Definition 2.4. Let (X, f) be a discrete dynamical system. A point x of X is said to be

1) a *fixed* point of f if $f(x) = x$;
2) a *periodic* point of f if there exists $k \in \mathbb{N}$ such that $f^k(x) = x$;
3) a *recurrent* point of f if for every neighborhood U of x there is $k \in \mathbb{N}$ such that $f^k(x) \in U$;

4) a *nonwandering* point of f if for every neighborhood U of x there is $k \in \mathbb{N}$ such that $f^k(U) \cap U \neq \varnothing$.

The sets of fixed points, periodic points, recurrent points and nonwandering points of (X, f) will be denoted by $\mathrm{Fix}(f)$, $\mathrm{Per}(f)$, $\mathrm{Rec}(f)$ and $\mathrm{NW}(f)$, respectively.

Remark 2.5. Let (X, f) be a discrete dynamical system. As an immediate consequence from the above definitions, we infer that

$$\mathrm{Fix}(f) \subseteq \mathrm{Per}(f) \subseteq \mathrm{Rec}(f) \subseteq \mathrm{NW}(f).$$

As an immediate consequence of Remark 2.5, we infer that

Proposition 2.6. *Let (X, f) be a discrete dynamical system. If $\mathrm{Per}(f)$ is dense in X, then $\mathrm{Rec}(f)$ and $NW(f)$ are dense in X.*

Now we recall the following definition.

Definition 2.7 ([18, 25])**.** Let (X, f) be a discrete dynamical system. A subset A is called a transitive subset of (X, f) if for any choice of nonempty open subset V_A of A and nonempty open subset U of X with $A \cap U \neq \varnothing$ there exists $n \in \mathbb{N}$ such that $f^n(V_A) \cap U \neq \varnothing$.

By [29], we have that:

Proposition 2.8. *Let (X, f) be a discrete dynamical system. Then, for each $k, s \in \mathbb{N}$, the following statements hold:*

1) $(\mathcal{C}_n(f))^k(A) = f^k(A)$ *for every* $A \in \mathcal{C}_n(X)$;
2) $q^n_Y \circ (\mathcal{C}_n(f))^k = (\mathcal{PHS}_n(f))^k \circ q^n_X$;
3) $((\mathcal{C}_n(f))^s)^k = (\mathcal{C}_n(f))^{sk}$;
4) $q^n_Y \circ ((\mathcal{C}_n(f))^s)^k = ((\mathcal{PHS}_n(f))^s)^k \circ q^n_X$.

Given a discrete dynamical system (X, f) and a positive integer n, observe that $\mathcal{F}_1(X)$ is a subset of $\mathcal{C}_n(X)$ such that $\mathcal{F}_1(X)$ is strongly invariant under $\mathcal{C}_n(f)$. However, we know that the dynamical system (X, f) induces the dynamical system $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$. Thus, we conclude this section with the following observations.

Remark 2.9. Let (X, f) be a discrete dynamical system and let n be a positive integer. Consider the discrete dynamical system $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$. Note that $F^n_X \in (\mathcal{PHS}_n(f))^{-1}(F^n_X)$. Moreover, since F^n_X is a fix point of $\mathcal{PHS}_n(X)$, we have that $\omega(F^n_X, \mathcal{PHS}_n(f)) = \{F^n_X\}$.

Now we recall the classes of dynamical systems used in this paper.

Definition 2.10. Let (X, f) be a discrete dynamical system. We say that f is

1) *exact* if for each nonempty open subset U of X, there exists $k \in \mathbb{N}$ such that $f^k(U) = X$;

2) *mixing* if for every pair of nonempty open subsets U and V of X, there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$ for every $k \geq N$;
3) *weakly mixing* if for all nonempty open subsets U_1, U_2, V_1 and V_2 of X, there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \varnothing$ for each $i \in \{1, 2\}$;
4) *transitive* if for every pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$;
5) *totally transitive* if f^s is transitive for all $s \in \mathbb{N}$;
6) *chaotic* if it is transitive and $\mathrm{Per}(f)$ is dense in X;
7) *minimal* if there is no proper subset $M \subseteq X$ which is nonempty, closed and M is invariant under f, i.e., if the orbit of every point of X is dense in X.

Remark 2.11 ([17]). Let (X, f) be a discrete dynamical system. The exactness of f implies that f is mixing, weakly mixing, totally transitive, transitive and surjective.

Remark 2.12 ([18]). Let (X, f) be a discrete dynamical system. Then f is transitive if and only if X is a transitive subset of (X, f).

Remark 2.13. Let (X, f) be a discrete dynamical system and let $k \in \mathbb{N}$. If f is isometric, then for any $x, y \in X$, $d(x, y) = d(f^k(x), f^k(y))$.

The following example will be important for our work.

Example 2.14. Let $I = [-1, 1]$ be the unit interval and let $f : I \to I$ be given by

$$f(x) = \begin{cases} -2(x+1) & \text{if } -1 \leq x \leq -\frac{1}{2} \\ 2x & \text{if } \quad |x| < \frac{1}{2} \\ 2(1-x) & \text{if } \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

This function is an extension of the tent map. Observe that

$$f^2(x) = \begin{cases} -4(x+1) & \text{if } -1 \leq x < -\frac{3}{4} \\ 2(2x+1) & \text{if } -\frac{3}{4} \leq x \leq -\frac{1}{2} \\ -2(2x+1) & \text{if } -\frac{1}{2} < x \leq -\frac{1}{4} \\ 4x & \text{if } -\frac{1}{4} < x < \frac{1}{4} \\ 2(1-2x) & \text{if } \quad \frac{1}{4} \leq x < \frac{1}{2} \\ -2(1-2x) & \text{if } \quad \frac{1}{2} \leq x \leq \frac{3}{4} \\ 4(1-x) & \text{if } \quad \frac{3}{4} < x \leq 1 \end{cases}.$$

It can be proved that f is not transitive. Thus, by Remark 2.11, we obtain that f is not exact, not mixing, not totally transitive and not weakly mixing.

By [9, p. 791], we know that $f|_{[0,1]}$ is transitive, but the induced map $\mathcal{C}(f|_{[0,1]})$ on $(\mathcal{C}_n(X), \tau)$ is not transitive (see [1, p. 1015]). Therefore, $\mathcal{C}(f|_{[0,1]})$ on $(\mathcal{C}_n([0, 1]), \tau)$ is not exact, not mixing, not weakly mixing and not totally transitive.

On the other hand, by [11], we know that $f|_{[0,1]}$ is exact. Thus, by Remark 2.11, we deduce that $f|_{[0,1]}$ is mixing, totally transitive and weakly mixing.

Now we will prove the following result.

Theorem 2.15. *Let (X, f) be a discrete dynamical system and let A be a nonempty set of X. If $\mathcal{C}(A)$ is a transitive subset of $((\mathcal{C}(X), \tau), \mathcal{C}(f))$, then A is a transitive subset of (X, f).*

Proof. Suppose that $\mathcal{C}(A)$ is a transitive subset of $((\mathcal{C}(X), \tau), \mathcal{C}(f))$. We prove that A is a transitive subset of (X, f). For this, let V_A be a nonempty open subset of A and let U be a nonempty open subset of X such that $A \cap U \neq \varnothing$. We prove that there exists $n \in \mathbb{N}$ such that $f^n(V_A) \cap U \neq \varnothing$. If V_A is a nonempty open subset of A, then there exists a nonempty open subset V of X such that $V_A = V \cap A$. Define $\mathcal{V}_A = \langle V \rangle_1'' \cap \mathcal{C}(A)$ and $\mathcal{U} = \langle U \rangle_1''$. Note that $\mathcal{V}_A$ is a nonempty open subset of $(\mathcal{C}(A), \tau)$ and $\mathcal{U}$ is a nonempty open subset of $(\mathcal{C}(X), \tau)$. Moreover, since $U \cap A \neq \varnothing$, we have that $\mathcal{U} \neq \varnothing$. If $\mathcal{C}(A)$ is a transitive subset of $((\mathcal{C}(X), \tau), \mathcal{C}(f))$, then there exists $n \in \mathbb{N}$ such that $(\mathcal{C}(f))^n(\mathcal{V}_A) \cap \mathcal{U} \neq \varnothing$. This implies that there is $B \in \mathcal{V}_A$ and $C \in \mathcal{U}$ such that $(\mathcal{C}(f))^n(B) = C$. In consequence, $B \subseteq V \cap A$ and $C \subseteq U$. Let $V_A = V \cap A$ and let $c \in C$. There is $b \in B$ such that $f^n(b) = c$. Thus $c \in U$ and $b \in V_A$. This implies that $f^n(b) \in f^n(V_A)$. Therefore $f^n(V_A) \cap U \neq \varnothing$. This completes the proof of the theorem. □

Theorem 2.16. *Let (X, f) be a discrete dynamical system and let A be a nonempty set of X. If A is a transitive subset of (X, f), then $\mathcal{C}(A)$ is a transitive subset of $((\mathcal{C}(X), \tau_U), \mathcal{C}(f))$.*

Proof. Suppose that A is a transitive subset of (X, f). We prove that $\mathcal{C}(A)$ is a transitive subset of $((\mathcal{C}(X), \tau_U), \mathcal{C}(f))$. Let $\mathcal{V}_A$ be a nonempty open subset of $\mathcal{C}(A)$ and let $\mathcal{U}$ be a nonempty open subset of $\mathcal{C}(X)$. We prove that there exists $n \in \mathbb{N}$ such that $(\mathcal{C}(f))^n(\mathcal{V}_A) \cap \mathcal{U} \neq \varnothing$ and there exists a nonempty open subset $\mathcal{V}$ of $\mathcal{C}(X)$ such that $\mathcal{V}_A = \mathcal{V} \cap \mathcal{C}(A)$. Moreover, there exist nonempty open subsets $V_1, V_2, \ldots, V_k$, of X such that $\langle V_1, V_2, \ldots, V_k \rangle_1 \subseteq \mathcal{V}$ and there exist nonempty open subsets $U_1, U_2, \ldots, U_m$, of X such that $\langle U_1, U_2, \ldots, U_m \rangle_1 \subseteq \mathcal{U}$. Let $U = \bigcup_{i=1}^m U_i$ and let $V = \bigcup_{i=1}^k V_i$. Note that U and V are open subsets of X. Let $V_A = V \cap A$. Since V_A is a nonempty open subset of A, U is a nonempty open subset of X and A is a transitive subset of X, there exists $n \in \mathbb{N}$ such that $f^n(V_A) \cap U \neq \varnothing$. It follows that there is $u \in U$ and $v \in V_A$ such that $f^n(v) = u$. Observe that $\{u\} \subseteq U$, $\{v\} \subseteq V$ and $\{v\} \subseteq A$. Further, $\{u\} \in \langle U_1, U_2, \ldots, U_m \rangle_1$ and $\{v\} \in \langle V_1, V_2, \ldots, V_m \rangle_1$. Finally, by part 1) of Proposition 2.8, we have that $(\mathcal{C}_n(f))^n(\{v\}) = \{u\}$. This implies that $(\mathcal{C}_n(f))^n(\mathcal{V}_A) \cap \mathcal{U} \neq \varnothing$. Therefore, $\mathcal{C}(A)$ is a transitive subset of $((\mathcal{C}(X), \tau_U), \mathcal{C}(f))$. □

As a consequence of Theorem 2.16 and [18, p. 2], we obtain the following result.

Corollary 2.17. *Let (X, f) be a discrete dynamical system and let A be a nonempty set of X. If A is a transitive subset of (X, f), then $\mathcal{C}\left(\overline{A}\right)$ is a transitive subset of $((\mathcal{C}(X), \tau_U), \mathcal{C}(f))$.*

As an application of Theorem 2.16, we present the following example.

Example 2.18. Consider the function $f|_{[0,1]}$ given in Example 2.14. We know that $f|_{[0,1]}$ is transitive. Moreover, by [18, p. 2, Example 4], we know that $\left[\frac{1}{4}, \frac{3}{4}\right]$ is a transitive subset of (X, f). Thus, by Theorem 2.15, we obtain that $\mathcal{C}(\left[\frac{1}{4}, \frac{3}{4}\right])$ is a transitive subset of $((\mathcal{C}(X), \tau_U), \mathcal{C}(f))$.

3. Results on $\mathcal{C}_n(X)$

3.1. Dynamical properties of the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$. This section is devoted to studying dynamical properties of a discrete dynamical system $((\mathcal{C}_n(X), \tau_U), \mathcal{C}_n(f))$. We begin this section with the following result.

Theorem 3.1. *Let (X, f) be a discrete dynamical system. Then the following statements are equivalent:*

(1) *f is exact;*
(2) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is exact for some $n \in \mathbb{N}$;*
(3) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is exact for each $n \in \mathbb{N}$.*

Proof. The proof of this result is similar to that of [16, 5, p. 5]. □

As an immediate consequence of Theorem 3.1 and Remark 2.11, we obtain the following result.

Corollary 3.2. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If f is exact, then*

(a) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is mixing;*
(b) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing;*
(c) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is totally transitive;*
(d) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive;*
(e) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is surjective.*

Theorem 3.3. *Let (X, f) be a dinamical system. Then the following statements are equivalent:*

(1) *f is weakly mixing;*
(2) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing for some $n \in \mathbb{N}$;*
(3) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing for each $n \in \mathbb{N}$.*

Proof. It is clear that (3) implies (2). Therefore, to complete the proof of the theorem it suffices to prove that (1) implies (3) and (2) implies (1).

Suppose that f is weakly mixing and $n \in \mathbb{N}$. We prove that the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing. For this, we consider nonempty open subsets $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{C}_n(X)$. We see that there exists $k \in \mathbb{N}$ such that $((\mathcal{C}_n(f)))^k(\mathcal{U}_i) \cap \mathcal{V}_i \neq \varnothing$ for each $i \in \{1, 2\}$. There exist nonempty open subsets $U_{i_1}, U_{i_2}, \ldots, U_{i_m}$, $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ of X, for each $i \in \{1, 2\}$, such that $\langle U_{1_1}, U_{1_2}, \ldots, U_{1_m}\rangle_n \subseteq \mathcal{U}_1$, $\langle U_{2_1}, U_{2_2}, \ldots, U_{2_m}\rangle_n \subseteq \mathcal{U}_2$, $\langle V_{1_1}, V_{1_2}, \ldots, V_{1_m}\rangle_n \subseteq \mathcal{V}_1$ and $\langle V_{2_1}, V_{2_2}, \ldots, V_{2_m}\rangle_n \subseteq \mathcal{V}_2$. Let $U_1 = \bigcup_{i=1}^m U_{1_i}$, $U_2 = \bigcup_{i=1}^m U_{2_i}$, $V_1 = \bigcup_{i=1}^m V_{1_i}$ and $V_2 = \bigcup_{i=1}^m V_{2_i}$. Note that U_1, U_2, V_1 and V_2 are open subsets of X. Since

f is weakly mixing, there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \varnothing$ for each $i \in \{1,2\}$. Let $y_1 \in V_1$ and $y_1 \in f^k(U_1)$. Moreover, $y_2 \in V_2$ and $y_2 \in f^k(U_2)$. Thus, there exists $u_1 \in U_1$ such that $f^k(u_1) = y_1$ and $u_2 \in U_2$ such that $f^k(u_2) = y_2$. Note that $\{y_1\} \subseteq V_1$, $\{y_2\} \subseteq V_2$, $\{u_1\} \subseteq U_1$ and $\{u_2\} \subseteq U_2$. Hence, $\{y_i\} \in \langle V_{i_1}, V_{i_2}, \dots, V_{i_m}\rangle_n$ and $\{u_i\} \in \langle U_{i_1}, U_{i_2}, \dots, U_{i_m}\rangle_n$, for each $i \in \{1,2\}$. Moreover, $f^k(\{u_i\}) = \{y_i\}$ for each $i \in \{1,2\}$. By part 1) of Proposition 2.8, we have that $(\mathcal{C}_n(f))^k(\{u_i\}) = \{y_i\}$ for each $i \in \{1,2\}$. Hence, $((\mathcal{C}_n(f)))^k(\mathcal{U}_i) \cap \mathcal{V}_i \neq \varnothing$ for each $i \in \{1,2\}$. Therefore, the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing.

Assume that $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing, for some $n \in \mathbb{N}$. We see that f is weakly mixing. For this end, let U_1, U_2, V_1 and V_2 be nonempty open subsets of X. We see that there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \varnothing$ for each $i \in \{1,2\}$. It follows that $\langle U_1\rangle_n$, $\langle U_2\rangle_n$, $\langle V_1\rangle_n$ and $\langle V_2\rangle_n$ are nonempty open subsets of $(\mathcal{C}_n(X), \tau_U)$. Since $\mathcal{C}_n(f)$ is weakly mixing on $(\mathcal{C}_n(X), \tau_U)$, there exists $k \in \mathbb{N}$ such that $(\mathcal{C}_n(f))^k(\langle U_i\rangle_n) \cap \langle V_i\rangle_n \neq \varnothing$ for each $i \in \{1,2\}$. Thus, for each $i \in \{1,2\}$ there exits $B_i \in \langle U_i\rangle_n$ such that $(\mathcal{C}_n(f))^k(B_i) \in \langle V_i\rangle_n$. For each $i \in \{1,2\}$, let $b_i \in B_i$ such that $(\mathcal{C}_n(f))^k(b_i) \in (\mathcal{C}_n(f))^k(B_i) \in \langle V_i\rangle_n$. By part 1) of Proposition 2.8, we deduce that $f^k(b_i) \in V_i$. In consequence, $f^k(U_i) \cap V_i \neq \varnothing$. Therefore, f is weakly mixing. □

As an application of Theorem 3.3, we present the following example.

Example 3.4. Consider the function $f|_{[0,1]}$ given in Example 2.14. We know that $f|_{[0,1]}$ is weakly mixing, Thus, by Theorem 3.3, we obtain that the induced map $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_U)$ is weakly mixing for each $n \in \mathbb{N}$.

As an immediate consequence of Theorem 3.3, we obtain the following result.

Corollary 3.5. *Let (X, f) be a discrete dynamical system and let n be an integer. If f is weakly mixing, then*

(a) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is totally transitive;*
(b) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive;*
(c) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is surjective.*

The technique we use to prove the result below is similar to that used in Theorem 3.3. This result generalizes the theorem of D. Masood and P. Singh [22, 3.1, p. 178].

Theorem 3.6. *Let (X, f) be a discrete dynamical system. Then the following statements are equivalent:*

(1) *f is transitive;*
(2) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive for some $n \in \mathbb{N}$;*
(3) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive for each $n \in \mathbb{N}$.*

As an application of Theorem 3.6, we present the following example.

Example 3.7. Consider the function $f|_{[0,1]}$ given in Example 2.14. We know that $f|_{[0,1]}$ is transitive, but the induced map $\mathcal{C}(f|_{[0,1]})$ on $(\mathcal{C}_n(X), \tau)$ is not transitive (see Example 2.14). However, by Theorem 3.6, we obtain that the induced map $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_U)$ is transitive for each $n \in \mathbb{N}$.

The proof of the next result is similar to that of Theorem 3.3. This result generalizes the theorem of D. Masood and P. Singh [22, 3.3, p. 178].

Theorem 3.8. *Let (X, f) be a discrete dynamical system. Then the following statements are equivalent:*

(1) *f is totally transitive;*
(2) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is totally transitive for some $n \in \mathbb{N}$;*
(3) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is totally transitive for each $n \in \mathbb{N}$.*

As an application of Theorem 3.8, we present the following example.

Example 3.9. Consider the function $f|_{[0,1]}$ given in Example 2.14. We know that $f|_{[0,1]}$ is totally transitive, Thus, by Theorem 3.6, we obtain that the induced map $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_U)$ is totally transitive for each $n \in \mathbb{N}$.

As an immediate consequence of Theorem 3.8, we obtain the following result.

Corollary 3.10. *Let (X, f) be a discrete dynamical system and let n be an integer. If f is totally transitive, then*

(a) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive;*
(b) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is surjective.*

The proof of the next result is similar to that of Theorem 3.3.

Theorem 3.11. *Let (X, f) be a dynamical system. Then the following statements are equivalent:*

(1) *f is mixing;*
(2) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is mixing for some $n \in \mathbb{N}$;*
(3) *the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is mixing for each $n \in \mathbb{N}$.*

Remark 3.12. Theorem 3.11 is a generalization of Theorem 3.7 of [22].

As an application of Theorem 3.11, we present the following example.

Example 3.13. Consider the function $f|_{[0,1]}$ given in Example 2.14. We know that $f|_{[0,1]}$ is mixing. Thus, by Theorem 3.11, we obtain that the induced map $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_U)$ is mixing for each $n \in \mathbb{N}$.

As an immediate consequence of Theorem 3.11, we obtain the following result.

Corollary 3.14. *Let (X, f) be a discrete dynamical system and let n be an integer. If f is mixing, then*

(a) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing;*
(b) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is totally transitive;*
(c) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive;*
(d) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is surjective.*

Theorem 3.15. *Let (X, f) be a discrete dynamical system. Then the following statements are equivalent:*

(1) $\mathrm{Per}(f)$ *is dense in* X*;*
(2) $\mathrm{Per}(\mathcal{C}_n(f))$ *on* $(\mathcal{C}_n(X), \tau_U)$ *is dense for some* $n \in \mathbb{N}$*;*
(3) $\mathrm{Per}(\mathcal{C}_n(f))$ *on* $(\mathcal{C}_n(X), \tau_U)$ *is dense for each* $n \in \mathbb{N}$*.*

Proof. It is clear that (3) implies (2). Therefore, to complete the proof of the theorem, it suffices to prove that (1) implies (3) and (2) implies (1).

Suppose that $\mathrm{Per}(f)$ is dense on X and $n \in \mathbb{N}$. We prove that $\mathrm{Per}(\mathcal{C}_n(f))$ on $(\mathcal{C}_n(X), \tau_U)$ is dense. For this, let $\mathcal{U}$ be a nonempty open subset of $\mathcal{C}_n(X)$. We see that $\mathcal{U} \cap \mathrm{Per}(\mathcal{C}_n(f)) \neq \varnothing$. There exist nonempty open subsets $U_1, \ldots, U_m$ of X such that $\langle U_1, U_2, \ldots, U_m\rangle_n \subseteq \mathcal{U}$. Let $U = \bigcup_{i=1}^{m} U_i$. Note that U is an open subset of X. Since $\mathrm{Per}(f)$ is dense, we have that $\mathrm{Per}(f) \cap U \neq \varnothing$. In consequence, there exists $x \in U$ and $k \in \mathbb{N}$ such that $f^k(x) = x$. Note that $\{x\} \subseteq U$. Hence $\{x\} \in \langle U_1, U_2, \ldots, U_m\rangle_n$. Moreover, $f^k(\{x\}) = \{x\}$. By part 1) of Proposition 2.8, we have that $(\mathcal{C}_n(f))^k(\{x\}) = \{x\}$. This implies that $\mathcal{U} \cap \mathrm{Per}(\mathcal{C}_n(f)) \neq \varnothing$. Therefore $\mathrm{Per}(\mathcal{C}_n(f))$ is dense on $(\mathcal{C}_n(X), \tau_U)$.

Assume that $\mathrm{Per}(\mathcal{C}_n(f))$ on $(\mathcal{C}_n(X), \tau_U)$ is dense for some $n \in \mathbb{N}$. We prove that $\mathrm{Per}(f)$ is dense. For this, let U be a nonempty open subset of X. It follows that $\langle U\rangle_n$ is a nonempty open subset of $(\mathcal{C}_n(X), \tau_U)$. Since $\mathrm{Per}(\mathcal{C}_n(f))$ on $(\mathcal{C}_n(X), \tau_U)$ is dense, $\mathrm{Per}(\mathcal{C}_n(f)) \cap \langle U\rangle_n \neq \varnothing$. It follows that there exits $B \in \langle U\rangle_n$ and $k \in \mathbb{N}$ such that $(\mathcal{C}_n(f))^k(B) = B$. Let $b \in B$. Observe that $(\mathcal{C}_n(f))^k(b) \in (\mathcal{C}_n(f))^k(B) = B \subseteq U$. By part 1) of Proposition 2.8, we have that $f^k(b) \in B \subseteq X$. Therefore $\mathrm{Per}(f) \cap X \neq \varnothing$. □

Using Theorem 3.15 and Proposition 2.6, we obtain the following.

Corollary 3.16. *Let* (X, f) *be a discrete dynamical system and let* n *be an integer. If* $\mathrm{Per}(f)$ *is dense on* X*, then the sets* $\mathrm{Rec}(\mathcal{C}_n(f))$ *and* $\mathrm{NW}(\mathcal{C}_n(f))$ *on* $(\mathcal{C}_n(X), \tau_U)$ *are dense.*

We finish this section with the following result.

Theorem 3.17. *Let* (X, f) *be a discrete dynamical system and let* n *be an integer. Then the following statements are equivalent:*

(1) f *is chaotic;*
(2) $\mathcal{C}_n(f)$ *on* $(\mathcal{C}_n(X), \tau_U)$ *is chaotic.*

Proof. The result is a consequence of Theorems 3.6 and 3.15. □

3.2. Dynamical properties of the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$.

Theorem 3.18. *Let* (X, f) *be a discrete dynamical system such that* X *is pathconnected. Then the following statements are equivalent:*

(1) f *is mixing; the induced map* $\mathcal{C}_n(f)$ *on* $(\mathcal{C}_n(X), \tau_L)$ *is mixing for some* $n \in \mathbb{N}$*; the induced map* $\mathcal{C}_n(f)$ *on* $(\mathcal{C}_n(X), \tau_L)$ *is mixing for each* $n \in \mathbb{N}$*.*

Proof. It is clear that (1) implies (1). Therefore, to complete the proof of the theorem, it suffices to prove that (1) implies (1) and (1) implies (1).

Suppose that f is mixing and $n \in \mathbb{N}$. We prove that $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is mixing. We consider the nonempty open subsets $\mathcal{U}$, $\mathcal{V}$ of $(\mathcal{C}_n(X), \tau_L)$. We see that there exists $N \in \mathbb{N}$ such that $((\mathcal{C}_n(f)))^k(\mathcal{U}) \cap \mathcal{V} \neq \varnothing$ for each $k \geq N$. There exist nonempty open subsets $U_1, U_2, \ldots, U_m$, $V_1, V_2, \ldots, V_m$ of X such that $\langle U_1, U_2, \ldots, U_m\rangle_n' \subseteq \mathcal{U}$ and $\langle V_1, V_2, \ldots, V_m\rangle_n' \subseteq \mathcal{V}$. Note that the pairs (U_i, V_i), for $i \in \{1, \ldots, m\}$, consist of nonempty open sets of X. Since f is mixing in X for each $i \in \{1, \ldots, m\}$, there exists $N_i \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \varnothing$ for each $k \geq N_i$. Let $N = \max\{N_i \colon i \in \{1, \ldots, m\}\}$. Thus, $f^k(U_i) \cap V_i \neq \varnothing$, for $i \in \{1, \ldots, m\}$ and $k \geq N$. For each $i \in \{1, \ldots, m\}$ and $k \geq N$, there are $x_{ik} \in f^k(U_i) \cap V_i$. Define $\alpha_i \colon [0,1] \to X$ given by $\alpha_i(0) = x_{ik}$ and $\alpha_i(1) = x_{(i+1)_k}$. Let $A = \bigcup_{i=1}^m \alpha_i([0,1])$. Observe that $A \in \langle V_1, V_2, \ldots, V_m\rangle_n'$. Moreover, $(\mathcal{C}_n(f))^k(A) \in \mathcal{C}_n(f))^k(\langle U_1, U_2, \ldots, U_{2m}\rangle_n')$. Therefore $((\mathcal{C}_n(f)))^k(\mathcal{U}) \cap \mathcal{V} \neq \varnothing$ for each $k \geq N$. In consequence, $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is mixing.

Assume that $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is mixing for some $n \in \mathbb{N}$. We see that f is mixing. For this end, let U and V be nonempty open subsets of X. We see that there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$ for every $k \geq N$. It follows that $\langle U\rangle_n'$ and $\langle V\rangle_n'$ are nonempty open subsets of $(\mathcal{C}_n(X), \tau_L)$. Since $\mathcal{C}_n(f)$ is mixing on $(\mathcal{C}_n(X), \tau_L)$, there exists $N \in \mathbb{N}$ such that $(\mathcal{C}_n(f))^k(\langle U\rangle_n') \cap \langle V\rangle_n' \neq \varnothing$ for every $k \geq N$. Fix $k \geq \mathbb{N}$ and let $B \in \langle U\rangle_n'$ be such that $(\mathcal{C}_n(f))^k(B) \in \langle V\rangle_n'$. Let $b \in B$. Note that $b \in U$ and $(\mathcal{C}_n(f))^k(b) \in (\mathcal{C}_n(f))^k(B) \in \langle V\rangle_n$. By part 1) of Proposition 2.8, we deduce that $f^k(b) \in V$. In consequence, $f^k(U) \cap V \neq \varnothing$. Therefore f is mixing. □

Remark 3.19. Theorem 3.18 is a generalization of Theorem 3.8 from [22].

As an immediate consequence of Theorem 3.18, we obtain the following result.

Corollary 3.20. *Let (X, f) be a discrete dynamical system and let n be an integer. If X is pathconnected and f is mixing, then*

(a) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is weakly mixing;*
(b) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is totally transitive;*
(c) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is transitive;*
(d) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_L)$ is surjective.*

As an application of Theorem 3.18, we present the following example.

Example 3.21. Consider the function $f|_{[0,1]}$ given in Example 2.14. We know that $f|_{[0,1]}$ is mixing. Thus, by Theorem 3.18, we obtain that the induced map $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_L)$ is mixing for each $n \in \mathbb{N}$.

As an application of Corollary 3.20, we present the following example.

Example 3.22. Consider the function $f|_{[0,1]}$ given in Example 2.14. Let n be an integer. We know that $f|_{[0,1]}$ is mixing, and thus, by Theorem 3.20, we obtain that:

(a) $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1]), \tau_L)$ is weakly mixing;
(b) $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_L)$ is totally transitive;

(c) $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_L)$ is transitive;
(d) $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1])), \tau_L)$ is surjective.

3.3. Dynamical properties of the induced map $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$. This section is devoted to studying dynamical properties of the discrete dynamical system $((\mathcal{C}_n(X), \tau), \mathcal{C}_n(f))$. We begin this section with the result that can be proved similarly to that from [3].

Theorem 3.23. *Let (X, f) be a dynamical system and let n be an integer. If f is an isometry, then $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is not transitive.*

As a consequence of Theorem 3.23, we have the following result.

Theorem 3.24. *Let (X, f) be a discrete dynamical system and let n be an integer. Let $\mathcal{M}$ be one of the following classes of maps: exact, mixing, weakly mixing, totally transitive, chaotic and minimal. If f is an isometry, then $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is not in the class $\mathcal{M}$.*

Now we introduce the following definition.

Definition 3.25. Let $f\colon X \to X$ and $g\colon Y \to Y$ be two maps. Then f and g are said to be topologically conjugate if there exists a homeomorphism $h\colon X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism h is a topological conjugation, and thus we write $f \sim g$.

Therefore, if two maps are topologically conjugate, and we want to understand the dynamics of one of them, we can study the other one as its dynamics will be qualitatively the same.

Theorem 3.26. *Let (X, f) and (Y, g) be discrete dynamical systems and let n be a positive integer. If the dynamical systems (X, f) and (Y, g) are topologically conjugate, then the same holds for the induced dynamical systems $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ and $(\mathcal{C}_n(Y), \mathcal{C}_n(g))$ on $(\mathcal{C}_n(X), \tau)$.*

Proof. Let h be a conjugacy between the pairs (X, f) and (Y, g). In consequence, $h\colon X \to Y$ is a homeomorphisms such that $h \circ f = g \circ h$. Define $\mathcal{C}_n(h)\colon \mathcal{C}_n(X) \to \mathcal{C}_n(X)$, given by $\mathcal{C}_n(h)(A) = h(A)$, for each $A \in \mathcal{C}_n(X)$. Since $h \circ f = g \circ h$, by part 2) of Proposition 2.8, we have that $\mathcal{C}_n(h) \circ \mathcal{C}_n(f) = \mathcal{C}_n(g) \circ \mathcal{C}_n(h)$. On the other hand, since h is a homeomorphism by [8, 4.6, p. 801], it follows that $\mathcal{C}_n(h)$ on $(\mathcal{C}_n(X), \tau)$ is a homeomorphism. Therefore, the induced dynamical systems $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ and $(\mathcal{C}_n(Y), \mathcal{C}_n(g))$ on $(\mathcal{C}_n(X), \tau)$ are topologically conjugate. □

4. Results on $\mathcal{PHS}_n(X)$

4.1. Dynamical properties of the map $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$. For readers' convenience, we give the following result in detail. The proofs of other results of this section are similar to those of Section 4 from [3].

Theorem 4.1. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is exact, then $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is exact.*

Proof. Suppose that $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is exact. We see that $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is also exact. Let $\mathcal{U}$ be a nonempty open subset of $(\mathcal{PHS}_n(X), \tau)$. Since q_X^n is continuous, we have that $(q_X^n)^{-1}(\mathcal{U})$ is a nonempty open subset of $(\mathcal{C}_n(X), \tau)$. On the other hand, since $\mathcal{C}_n(f)$ is exact, there exists $k \in \mathbb{N}$ such that $(\mathcal{C}_n(f))^k((q_X^n)^{-1}(\mathcal{U})) = \mathcal{C}_n(X)$. Since q_X^n is surjective, we have that $q_X^n(\mathcal{C}_n(f)^k((q_X^n)^{-1}(\mathcal{U}))) = \mathcal{PHS}_n(X)$. Using (2.1), we obtain that $(\mathcal{PHS}_n(f))^k(\mathcal{U}) = \mathcal{PHS}_n(X)$. Therefore $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is exact. □

Theorem 4.2 is used in the proof of Corollary 4.4.

Theorem 4.2. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is exact, then f is exact.*

Proof. The proof of this result is similar to that given in [3, 4.7, p. 462]. □

As an application of Theorems 4.1 and 4.2, we present the following example.

Example 4.3. Consider the function given in Example 2.14. By Theorem 4.2, we obtain that $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(I), \tau)$ is not exact and by Theorem 4.1, we deduce that $\mathcal{C}_n(f)$ on $(\mathcal{PHS}_n(I), \tau)$ is not exact.

In the following corollary we establish the relationships between the exactness of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$.

Corollary 4.4. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) *f is exact;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is exact;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is exact.*

Proof. By [22, 3.1, p. 178], we obtain that (1) and (2) are equivalent. Now, by Theorems 4.1, 4.2, we have that (2) and (3) are equivalent. □

As an application of Corollary 4.4, we present the following example.

Example 4.5. Consider the function given in Example 2.14. By Corollary 4.2, we obtain that $\mathcal{PHS}_n(f|_{[0,1]})$ on $(\mathcal{PHS}_n([0,1]), \tau)$ is exact.

Lemma 4.6 is used in the proof of Corollary 4.10.

Theorem 4.6. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is transitive if and only if $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is transitive.*

Proof. The proof of this result is similar to that from [3, 4.10, p. 464]. □

As the applications of Theorem 4.6 and Example 3.7, we present the following.

Example 4.7. Consider the function given in Example 2.14. By Example 3.7, we know that $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1]),\tau)$ is not transitive. Therefore, by Theorem 4.6, we obtain that $\mathcal{PHS}_n(f|_{[0,1]})$ on $(\mathcal{PHS}_n([0,1]),\tau)$ is not transitive.

Lemma 4.8 is used in the proof of Corollary 4.10.

Theorem 4.8. *Let (X,f) be a discrete dynamical system and let n be a positive integer. If $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X),\tau)$ is transitive, then f is transitive.*

Proof. Supposing that $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X),\tau)$ is transitive, we prove that f is transitive. For this end, let U and V be nonempty open subsets of X. Moreover, let U_1, U_2, V_1, V_2 be nonempty open subsets of X such that $U_1\cup U_2\subseteq U$, $V_1\cup V_2\subseteq V$, $U_1\cap U_2=\varnothing$ and $V_1\cap V_2=\varnothing$. Note that $\langle U_1,U_2\rangle_n$ and $\langle V_1,V_2\rangle_n$ are nonempty open subsets of $(\mathcal{C}_n(X),\tau)$ such that $\langle U_1,U_2\rangle_n\cap\mathcal{F}_1(X)=\varnothing$ and $\langle V_1,V_2\rangle_n\cap\mathcal{F}_1(X)=\varnothing$. Thus, by Remark 2.2, we have that $q_X^n(\langle U_1,U_2\rangle_n)$ and $q_X^n(\langle U_1,U_2\rangle_n)$ are nonempty open subsets of $(\mathcal{PHS}_n(X),\tau)$. Since $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X),\tau)$ is transitive, there exists $k\in\mathbb{N}$ such that $(\mathcal{PHS}_n(f))^k(q_X^n(\langle U_1,U_2,\rangle_n))\cap q_X^n(\langle V_1,V_2,\rangle_n)\neq\varnothing$. In consequence, there exists $A\in q_X^n(\langle U_1,U_2,\rangle_n)$ such that $(\mathcal{PHS}_n(f))^k(A)\in q_X^n(\langle V_1,V_2,\rangle_n)$. It follows that there exists $B\in\langle U_1,U_2,\rangle_n$ such that $q_X^n(B)=A$, and $(\mathcal{PHS}_n(f))^k(q_X^n(B))\in q_X^n(\langle V_1,V_2,\rangle_n)$. By part 2) of Proposition 2.8, we obtain that $q_X^n((\mathcal{C}_n(f))^k(B))\in q_X^n(\langle V_1,V_2\rangle_n)$. This implies that $(\mathcal{C}_n(f))^k(B)\in\langle V_1,V_2\rangle_n$. Therefore $f^k(B)\subseteq V$. Let $b\in B$. Note that $f^k(b)\in V$ and $b\in U$. In consequence, $f^k(b)\in f^k(U)$. Thus, $f^k(U)\cap V\neq\varnothing$, and therefore f is transitive. □

As an application of Theorem 4.8, we present the following example.

Example 4.9. Consider the function given in Example 2.14. By Theorem 4.8, we obtain that $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(I),\tau)$ is not transitive.

In the following corollary we establish relationships between the transitivity of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$.

Corollary 4.10. *Let (X,f) be a discrete dynamical system and let n be a positive integer. Consider the following statements:*

(1) *f is transitive;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X),\tau)$ is transitive;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X),\tau)$ is transitive.*

Then (2) and (3) are equivalent, (3) implies (1), (2) implies (1), (1) does not imply (2), and (1) does not imply (3).

Proof. By Theorem 4.6, we have that (2) and (3) are equivalent. By Theorem 4.8, we obtain that (3) implies (1). Consequently, we deduce that (2) implies (1).

On the other hand, by Example 2.14, it follows that (1) does not imply (2). Finally, by Examples 2.14 and 4.7, we deduce that (1) does not imply (3). □

Theorem 4.11. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is totally transitive if and only if $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is totally transitive.*

Proof. The proof of this result is similar to that given in [3, 4.12, p. 465]. □

As an application of Theorem 4.11 and Example 3.7, we present the following.

Example 4.12. Consider the function given in Example 2.14. By Example 3.7, we know that $\mathcal{C}_n(f|_{[0,1]})$ on $(\mathcal{C}_n([0,1]), \tau)$ is not totally transitive. Therefore, by Theorem 4.11, we obtain that $\mathcal{PHS}_n(f|_{[0,1]})$ on $(\mathcal{PHS}_n([0,1]), \tau)$ is not totally transitive.

Theorem 4.13. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is totally transitive, then f is totally transitive.*

Proof. The proof of this result is similar to that given in [3, 4.12, p. 465]. □

As an application of Theorem 4.13, we present the following example.

Example 4.14. Consider the function given in Example 2.14. By Theorem 4.13, we obtain that $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(I), \tau)$ is not totally transitive.

In the following corollary, we establish relationships between the total transitivity of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$.

Corollary 4.15. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Consider the following statements:*

(1) *f is totally transitive;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is totally transitive;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is totally transitive.*

Then (2) and (3) are equivalent, (3) implies (1), (2) implies (1), (1) does not imply (2), and (1) does not imply (3).

Proof. By Theorem 4.11, we have that (2) and (3) are equivalent. By Theorem 4.13, we obtain that (3) implies (1), and thus (2) implies (1).

Moreover, by Example 2.14, it follows that (1) does not imply (2). Finally, by Examples 2.14 and 4.14, it follows that (1) does not imply (3). □

Theorem 4.16. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is mixing if and only if $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is mixing.*

Proof. The proof of this result is similar to that given in [3, 4.9, p. 463]. □

As an application of Theorem 4.16, we present the following example.

Example 4.17. Consider the function given in Example 2.14. By Theorem 4.16, we obtain that $\mathcal{PHS}_n(f|_{[0,1]})$ on $(\mathcal{PHS}_n([0,1]), \tau)$ is not mixing.

Theorem 4.18. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is mixing, then f is mixing.*

Proof. The proof of this result is similar to that given in [3, 4.9, p. 463]. □

As an application of Theorem 4.18, we present the following example.

Example 4.19. Consider the function given in Example 2.14. By Theorem 4.18, we obtain that $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(I), \tau)$ is not mixing.

In the following corollary, we establish relationships between the mixing of f, $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$.

Corollary 4.20. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Consider the following statements:*

(1) *f is mixing;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is mixing;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is mixing.*

Then (2) and (3) are equivalent, (3) implies (1), (2) implies (1), (1) does not imply (2), and (1) does not imply (3).

Proof. By Theorem 4.16, we have that (2) and (3) are equivalent. By Theorem 4.18, we obtain that (3) implies (1) and we can deduce that (2) implies (1).

Furthermore, by Example 2.14, it follows that (1) does not imply (2). Finally, by Examples 2.14 and 4.17, it follows that (1) does not imply (3). □

Theorem 4.21. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) $\mathrm{Per}(f)$ *on $(\mathcal{C}_n(X), \tau)$ is dense;*
(2) $\mathrm{Per}(\mathcal{C}_n(f))$ *on $(\mathcal{C}_n(X), \tau)$ is dense.*

Proof. The proof of this result is similar to that of Theorem 3.15. □

Theorem 4.22. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) $\mathrm{Per}(\mathcal{C}_n(f))$ *on $(\mathcal{C}_n(X), \tau)$ is dense;*
(2) $\mathrm{Per}(\mathcal{PHS}_n(f))$ *on $(\mathcal{PHS}_n(X), \tau)$ is dense.*

Proof. The proof of this result is similar to that given in [3, 4.16, p. 469]. □

Theorem 4.23. *If discrete dynamical systems (X, f) and (Y, g) are topologically conjugate, then the same holds for the induced dynamical systems $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$ and $(\mathcal{PHS}_n(Y), \mathcal{PHS}_n(f))$ for each $n \in \mathbb{N}$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{PHS}_n(X) & \xrightarrow{\mathcal{PHS}_n(f)} & \mathcal{PHS}_n(X) \\ {\scriptstyle \mathcal{PHS}_n(h)} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{PHS}_n(h)} \\ \mathcal{PHS}_n(Y) & \xrightarrow[\mathcal{PHS}_n(g)]{} & \mathcal{PHS}_n(Y) \end{array}$$

Proof. Let h be a conjugacy between the pairs (X, f) and (Y, g) and $n \in \mathbb{N}$. By Theorem 3.26, the dynamical system $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ and $(\mathcal{C}_n(Y), \mathcal{C}_n(g))$ are topologically conjugate by the induced map $\mathcal{C}_n(h)$ given by $\mathcal{C}_n(h) = h(A)$, for each $A \in \mathcal{C}_n(X)$. Define $\mathcal{PHS}_n(h) \colon \mathcal{PHS}_n(X) \to \mathcal{PHS}_n(Y)$ given by

$$\mathcal{PHS}_n(h)(\chi) = \begin{cases} q_Y^n(\mathcal{C}_n(h)((q_X^n)^{-1}(\chi))) & \text{if } \chi \neq F_X^n \\ F_Y^n & \text{if } \chi = F_X^n \end{cases}. \tag{4.1}$$

By [21, p. 31, 4.1], we have that $\mathcal{PHS}_n(h)$ is a homeomorphism. Then $\mathcal{PHS}_n(h)$ is the required topological conjugacy between the discrete dynamical systems $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$ and $(\mathcal{PHS}_n(Y), \mathcal{PHS}_n(g))$. □

4.2. Dynamical properties of the map $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$. This subsection is dedicated to more applications of the results obtained in Section 4.1. We begin this subsection with a consequence of Theorem 3.1 and Corollary 4.4 to obtain the following.

Corollary 4.24. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) *f is exact;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is exact;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is exact.*

As a consequence of Corollary 4.24, we obtain the following result.

Corollary 4.25. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If f is exact, then $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is mixing, totally transitive, weakly mixing and transitive.*

Corollary 4.26. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) *f is transitive;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is transitive.*

Proof. By Theorem 3.6, we have that (1) and (2) are equivalent, and by Theorem 4.6, we obtain that (2) and (3) are equivalent. □

Corollary 4.27. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) *f is totally transitive;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is totally transitive;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is totally transitive.*

Proof. By Theorem 3.8, we have that (1) and (2) are equivalent and, by Theorem 4.11, we obtain that (2) and (3) are equivalent. □

As a consequence of Corollary 4.27, we obtain the following results.

Corollary 4.28. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If f is totally transitive, then*

(1) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is transitive and surjective;*
(2) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is transitive and surjective.*

Corollary 4.29. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent:*

(1) *f is mixing;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is mixing;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is mixing.*

Proof. By Theorem 3.11, we have that (1) and (2) are equivalent, and by Theorem 4.16, we obtain that (2) and (3) are equivalent. □

As a consequence of Corollary 4.29, we obtain the following results.

Corollary 4.30. *Let (X, f) be a discrete dynamical system and let n be a positive integer. If f is mixing, then*

(1) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is weakly mixing, transitive and surjective;*
(2) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is weakly mixing, transitive and surjective.*

Corollary 4.31. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following are equivalent :*

(1) $\mathrm{Per}(f)$ *on X is dense;*
(2) $\mathrm{Per}(\mathcal{C}_n(f))$ *on $(\mathcal{C}_n(X), \tau_U)$ is dense;*
(3) $\mathrm{Per}(\mathcal{PHS}_n(f))$ *on $(\mathcal{PHS}_n(X), \tau_U)$ is dense.*

Proof. By Theorem 3.15, we have that (1) and (2) are equivalen,t and by Theorem 4.22, we obtain that (2) and (3) are equivalent. □

Lastly, we have a direct consequence of Corollary 4.26 and Corollary 4.31 and obtain the following result.

Theorem 4.32. *Let (X, f) be a discrete dynamical system and let n be a positive integer. Then the following statements are equivalent:*

(1) *f is chaotic;*
(2) *$\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau_U)$ is chaotic;*
(3) *$\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau_U)$ is chaotic.*

Conclusion

In this paper, we have shown connections between some dynamical properties of a discrete dynamical system and dynamical properties of a set-valued discrete dynamical system associated to it. Specifically, we introduce the dynamical system $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$ and study connections between dynamical properties of f and dynamical properties of the induced maps $\mathcal{C}_n(f)$ and $\mathcal{PHS}_n(f)$. We obtain the relationships between the following statements:

(1) $f \in \mathcal{M}$,

(2) $\mathcal{C}_n(f) \in \mathcal{M}$ and

(3) $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f)) \in \mathcal{M}$, when $\mathcal{M}$ is one of the following classes of maps: exact, mixing, totally transitive, transitive, weakly mixing.

Among other results, we obtain that if $\mathcal{C}_n(f)$ on $(\mathcal{C}_n(X), \tau)$ is exact, then $\mathcal{PHS}_n(f)$ on $(\mathcal{PHS}_n(X), \tau)$ is also exact. Moreover, in [22], D. Masood and P. Singh obtained the results on the induced dynamical system on the discrete dynamical system $((\mathcal{C}(X), \tau_U), \mathcal{C}(f))$. In this paper, we generalized some of these results.

This investigation should be used in further studying connections between individual and collective dynamics.

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Динамика фактор-гиперпросторів

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Нехай (X, d) є компактним метричним простором і нехай and n є цілим додатним числом. Нехай $\mathcal{C}_n(X)$ є простором усіх непорожніх замкнених підмножин X з не більше ніж n компонентами і нехай $\mathcal{F}_1(X)$ є простором одно-елементних множин X. Для заданого відображення $f\colon X \to X$ ми розглядаємо індуковане відображення $\mathcal{C}_n(f) : \mathcal{C}_n(X) \to \mathcal{C}_n(X)$, що задається співвідношенням $\mathcal{C}_n(f)(A) = f(A)$ для кожного $A \in \mathcal{C}_n(X)$. Дискретна динамічна система (X, f) індукує дискретну динамічну систему $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$, де $\mathcal{PHS}_n(X)$ є фактор-простором $\mathcal{C}_n(X)/\mathcal{F}_1(X)$ з відповідною топологією фактор-простору. У цій роботі ми узагальнюємо деякі результати роботи [22] і вивчаємо співвідношення між дискретними динамічними системами (X, f), $(\mathcal{C}_n(X), \mathcal{C}_n(f))$ і $(\mathcal{PHS}_n(X), \mathcal{PHS}_n(f))$.

Ключові слова: хаотичне відображення, точне відображення, перемішувальне відображення, тотально транзитивне відображення, транзитивне відображення, слабко перемішувальне відображення, гіперповерхня, індуковане відображення, n-кратна псевдогіперпросторова надбудова