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The Korteweg–De Vries Equation with Forcing Involving Products of Eigenfunctions

A.S. Fokas and A. Latifi

A new methodology has been recently introduced, which starting with an integrable evolution equation constructs an integrable *forced* version of this equation. The forcing consists of terms involving quadratic products of certain eigenfunctions of the associated Lax pair. We implement this methodology starting with the celebrated Kortewg–de Vries equation. The initial value problem of the associated integrable forced equation can be formulated as a Riemann–Hilbert problem with a "jump matrix" that has explicit x and t dependence that can be computed in terms of the initial data. Thus, this equation can be solved as efficiently as the Kortewg–de Vries equation itself. It is also shown that this forced equation together with the x-part of its Lax pair, appear in the modelling of important physical phenomena. Specifically, in the context of laser-plasma interaction, as well as in the description of resonant gravity-capillary waves.

Key words: force KdV equation, Lax pair, integrability, Riemann–Hilbert problem

Mathematical Subject Classification 2020: 37K10, 35B35, 35B34

1. Introduction

An algorithmic approach was presented in [4], which starting with a given integrable evolution equation in one spatial variable, constructs an integrable forced version of this equation. The forcing consists of nonlinear terms involving the eigenfunctions of the associated Lax pair.

For the particular case that the Lax pair of the starting integrable PDE involves a 2×2 matrix, μ, this new approach constructs a Lax pair of the new integrable PDE by following two steps:

(i) Modify the t-part of the Lax pair of the given PDE by the addition of the operator Δ defined by

$$\Delta f = -\frac{1}{2\mathrm{i}}(Hg)f\sigma_3 + \frac{1}{2\mathrm{i}}\left(Hgf\sigma_3 f^{-1}\right)f,$$
$$(Hf)(k) = \frac{p}{\pi}\int_{\mathbb{R}} \frac{f(l)}{l-k}\,dl, \quad k \in \mathbb{R}, \tag{1.1}$$

where H denotes the Hilbert transform (see the second equation of (1.1)), p denotes the principle value integral, σ_3 is the third Pauli matrix (see equation (2.5)), and $g(t, k)$ is a scalar function whose Hilbert transform exists and which satisfies a particular analyticity constraint defined later. Depending on the specific situation, the addition of the above operator may or may not necessitate the inclusion of more terms in the t-part of the Lax pair.

(ii) By requiring that the terms of the t-part of the Lax pair which are of order $1/k$ as $k \to \infty$ are consistent with the value of μ obtained from the x-part of the Lax pair, obtain an integrable forced version of the given equation. The fact that this equation is indeed the compatibility condition of the new Lax pair can be verified directly.

Having constructed the t-part of new Lax pair, the initial value problem on the infinite line of the new integrable forced PDE can be solved following two steps:

(i) Using the x-part of the Lax pair of the starting integrable PDE and following the usual approach of the inverse scattering method, formulate a Riemann-Hilbert problem (RH) for appropriate eigenfunctions, μ^- and μ^+, satisfying this Lax pair. These eigenfunctions have a "jump" across the real k axis defined via the matrix $J(x, t, k)$. The x-dependence of this matrix is the one obtained via the usual inverse scattering method.

(ii) Using the t-part of the new Lax pair, compute the t-evolution of the jump matrix J. Surprisingly, despite the occurrence of nonlinear terms in the definition of Δ, the time evolution of J can be written explicitly. Hence, these forced equations can be solved as efficiently as the original equations from which they arose.

It is well known that starting with a RH problem, and using the "dressing method" introduced in the pioneering works of Zakharov and Shabat [23], it is possible to identify algorithmically the Lax pair $M_j\mu = 0$, $j = 1, 2$, associated with this RH problem, as well as the x and t dependence of its jump matrix J. This is achieved by requiring that: first, the two operators M_j commute with J, which means that the functions $M_j\mu = 0$, $j = 1, 2$, satisfy the same jump condition as μ (this requirement fixes the x and t dependence of J); second, the coefficients of these two operators satisfy appropriate relations so that $M_j\mu = 0$, $j = 1, 2$, are of order $1/k$ as k tends to ∞. These requirements imply that the functions $M_j\mu$, satisfy the homogeneous version of the given RH problem, and hence the assumption of the unique solvability of this RH problem implies $M_j\mu = 0$.

The main novelty of the forced integrable PDEs constructed via the new methodology is due to the fact that the operator defined in (1.1) does not commute with J. Hence, it is not possible to obtain directly the t-dependence of the J matrix, nor to conclude immediately that the equation $M_2\mu = 0$ is compatible with $M_1\mu = 0$. In this sense, the methodology introduced here goes beyond the dressing method.

The implementation of the new algorithmic approach starting with the nonlinear Schrödinger and the Kortewg–de Vries (KdV) equations was briefly sketched in [4]. Remarkably, the forced versions of both the nonlinear Schrödinger and of the KdV are physically significant. Details of the solution of the forced integrable version associated with the nonlinear Schrödinger equation and physical applications of this equation are discussed in [5]. In what follows we present analogues results for the forced integrable extension associated with the KdV.

The rest of this paper is organised as follows: It is shown in Section 2 that the forced integrable extension of the KdV, namely the equation

$$u_t + \alpha(u_{xxx} + 6uu_x) = d_x(x,t) + 2h_x(x,t), \tag{1.2}$$

where α is introduced in order to consider the $\alpha = 0$ limit and d and h are defined by

$$d(x,t) = \frac{1}{\pi}\int_{\mathbb{R}} g(t,l)(v_{11}v_{22} + v_{12}v_{21})(x,t,l)\,dl, \tag{1.3a}$$

$$h(x,t) = \frac{1}{\pi}\int_{\mathbb{R}} g(t,l)v_{21}v_{22}(x,t,l)\,dl, \tag{1.3b}$$

is integrable. The functions v_{ij}, $i,j = 1,2$ are the ij components of the matrix v given by

$$v(x,t,k) = \frac{1}{2}\begin{pmatrix} \phi(x,t,-k) - \frac{1}{\mathrm{i}k}\phi_x(x,t,-k) & \phi(x,t,k) - \frac{1}{\mathrm{i}k}\phi_x(x,t,k) \\ \phi(x,t,-k) + \frac{1}{\mathrm{i}k}\phi_x(x,t,-k) & \phi(x,t,k) + \frac{1}{\mathrm{i}k}\phi_x(x,t,k) \end{pmatrix}, \tag{1.4}$$

where ϕ is an appropriate solution of the associated Lax pair. Namely, ϕ satisfies

$$\phi_{xx} + (u + k^2)\phi = 0, \tag{1.5}$$

and

$$\begin{aligned} &\phi_t + \alpha\left(4\mathrm{i}k^3 - u_x\right)\phi + \alpha(2u - 4k^2)\phi_x + \frac{1}{2\mathrm{i}}(Hg)\phi - \frac{1}{4\mathrm{i}}\left[Hg\left(\phi\widehat{\phi} + \frac{1}{k^2}\phi_x\widehat{\phi}_x\right)\right]\frac{\phi_x}{\mathrm{i}k} \\ &\quad + \frac{1}{4\mathrm{i}}\left[Hg\left(\phi\widehat{\phi} - \frac{1}{k^2}\phi_x\widehat{\phi}_x\right)\right]\phi - \frac{1}{4\mathrm{i}}\left[Hg\left(\frac{1}{\mathrm{i}k}\big(\phi\widehat{\phi}\big)_x\right)\right]\frac{\phi_x}{\mathrm{i}k} = 0, \end{aligned} \tag{1.6}$$

where the hat denotes evaluation at $-k$.

It is shown in Section 3 that u can be obtained via the formula

$$u(x,t) = -\mathrm{i}\partial_x V_{22}^{(1)}(x,t), \quad V_{22}^{(1)}(x,t) = \lim_{k\to\infty}\left[\mathrm{e}^{\mathrm{i}kx}\phi(x,t,k) - 1\right], \tag{1.7}$$

where $\phi = \Phi\mathrm{e}^{\mathrm{i}kx}$ satisfies the scalar RH problem

$$\frac{\Phi(x,t,k)}{a(t,k)} = \Psi(x,t,k) + r(t,k)\Psi(x,t-k)\mathrm{e}^{-2\mathrm{i}kx}, \quad k \in \mathbb{R}, \tag{1.8}$$

where $a(t,k)$ and $r(t,k)$ are given by

$$a(t,k) = a_0(k)\mathrm{e}^{\mathrm{i}(HG(|a_0(k)|^2)}, \tag{1.9a}$$

$$r(t,k) = r_0(k)\mathrm{e}^{-8\mathrm{i}\alpha k^3 t - \mathrm{i}\left[HG|a_0(k)|^2(1+|r_0(k)|^2)\right]}, \tag{1.9b}$$

and G(t,k) is defined by

$$G(t,k) = \int_0^t g(\tau,k)\,d\tau. \tag{1.10}$$

In Section 4 we present two important physical applications of equation (1.2). First, the set of equations

$$\frac{\partial}{\partial\tau}q(\xi,\tau) = \frac{\partial}{\partial\xi}\int_{-\infty}^{+\infty}\mathfrak{g}(\omega)|\widetilde{E}(\xi,\tau,\omega)|^2\,d\omega, \tag{1.11a}$$

$$\widetilde{E}_{\xi\xi}(\xi,\tau) + (k^2+q)\widetilde{E}(\xi,\tau) = 0, \tag{1.11b}$$

which appears in the context of laser-plasma interaction. In these equations, q represents the cavitation in the ion density, while $\widetilde{E}$ is the slowly varying envelope of a high frequency electrostatic field. In the co-moving frame with the electrostatic wave, τ and ξ are slowly varying time and space, respectively. The set of equations (1.11) fits equations (1.2) and (1.5) by setting $\alpha = 0$, $\phi = \widetilde{E}$, $u = q$ and $g = 2\mathfrak{g}$.

Second, the set of equations

$$\frac{\partial\widetilde{B}}{\partial\tau_2} = -\int_{-\infty}^{+\infty}\widetilde{g}_2(k)\frac{\partial|\widetilde{A}|^2}{\partial x_1}\,dk, \tag{1.12a}$$

$$\lambda^2\widetilde{A} + \frac{\partial^2\widetilde{A}}{\partial x_1^2} = \widetilde{B}\widetilde{A}, \tag{1.12b}$$

which appears in the context of resonant gravity-capillary waves. In these equations, $\widetilde{A}$ is the scaled complex amplitude of the fluid free surface elevation with a Gaussian profile $(\widetilde{g}_2)^{1/2}$, λ is related to the phase velocity of the gravity wave, and $\widetilde{B}$ is the scaled group velocity of the capillary waves. The independent variables x_1 and τ_2 are scaled space and time, respectively. The set of equations (1.12) fits equations (1.2) and (1.5) by setting $\alpha = 0$, $\phi = \widetilde{A}$, $u = \widetilde{B}$ and $g = 2\widetilde{g}_2$.

In the appendix we verify directly that the equations (1.5) and (1.6) constitute a Lax pair of equation (1.2).

We hope that this work together with the results of [4], and [5] will motivate several researches to construct integrable forced extensions of a variety of integrable PDEs in one and two spatial dimensions. In this connection it is noted that the correct analogue of the operator Δ needed for evolution PDEs in two space dimensions is presented in [4].

2. The Lax pair

The KdV equation

$$u_t + u_{xxx} + 6uu_x = 0, \quad u\in\mathbb{R},\ x\in\mathbb{R},\ t>0, \tag{2.1}$$

possesses the Lax pair

$$\phi_{xx} + (u + k^2)\phi = 0, \tag{2.2a}$$

$$\phi_t = (-4\mathrm{i}k^3 + u_x)\phi + (4k^2 - 2u)\phi_x, \quad k \in \mathbb{C}, \tag{2.2b}$$

where $\phi(x, t, k)$ is a scalar function.

A matrix form of the Lax pair. An alternative Lax pair is given by the following equations:

$$v_x + \mathrm{i}k\sigma_3 v = \frac{u}{2k}\sigma v, \tag{2.3a}$$

$$v_t + 4\mathrm{i}k^3\sigma_3 v = (2ku\sigma_2 + u_x\sigma_1)v - \frac{q}{k}\sigma v, \quad k \in \mathbb{C}, \tag{2.3b}$$

where

$$\sigma = (\sigma_2 - \mathrm{i}\sigma_3), \quad q = \frac{u_{xx}}{2} + u^2, \tag{2.4}$$

and σ_j , $j = 1, 2, 3$, denote the Pauli matrices, which are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

Hence,

$$\sigma = \mathrm{i}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

It is straightforward to verify that if ϕ satisfies (2.2), then v satisfies (2.3), where the 2×2 matrix v is defined by

$$v(x, t, k) = \frac{1}{2}\begin{pmatrix} \phi(x, t, -k) - \frac{1}{\mathrm{i}k}\phi_x(x, t, -k) & \phi(x, t, k) - \frac{1}{\mathrm{i}k}\phi_x(x, t, k) \\ \\ \phi(x, t, -k) + \frac{1}{\mathrm{i}k}\phi_x(x, t, -k) & \phi(x, t, k) + \frac{1}{\mathrm{i}k}\phi_x(x, t, k) \end{pmatrix}. \tag{2.6}$$

For example, the "12"component of (2.3a) yields the identity

$$\phi_x + \frac{1}{\mathrm{i}k}(u + k^2)\phi + \mathrm{i}k(\phi - \frac{1}{\mathrm{i}k}\phi_x) = \frac{u}{\mathrm{i}k}\phi,$$

where we used (2.2a) to replace ϕ_{xx}.

The matrix v has a pole singularity at $k = 0$. It can be shown [6] that if ϕ is a solution of (2.2a) normalized by the condition that ϕ tends to $\exp(-\mathrm{i}kx)$ as $x \to -\infty$, then ϕ and ϕ_x have the following behaviour as k tends to 0:

$$\begin{aligned} &\phi(x, t, \pm k) = f(x, t) + O(k), && k \to 0, \\ &\phi_x(x, t, \pm k) = \widetilde{f}(x, t) \mp \mathrm{i}k\widetilde{f}_1(x, t) + O(k^2), && k \to 0, \end{aligned} \tag{2.7}$$

where, since

$$\phi(x, t, k) = \mathrm{e}^{-\mathrm{i}kx} + \int_{-\infty}^{x} K(x, y, t)\mathrm{e}^{-\mathrm{i}ky}\, dy,$$

$$f(x,t) = 1 + \int_{-\infty}^{x} K(x,y,t)\,dy,$$
$$\widetilde{f}(x,t) = K(x,x,t) + \int_{-\infty}^{x} K'_x(x,y,t)\,dy,$$

and

$$\widetilde{f}_1(x,t) = 1 + xK(x,x,t) + \int_{-\infty}^{x} K'_x(x,y,t)y\,dy.$$

Then

$$v = -\frac{\widetilde{f}}{2k}\sigma + \frac{1}{2}\begin{pmatrix} f-\widetilde{f}_1 & f+\widetilde{f}_1 \\ f+\widetilde{f}_1 & f-\widetilde{f}_1 \end{pmatrix} + O(k), \quad k \to 0, \tag{2.8}$$

and

$$v_x = \frac{uf}{2k}\sigma + O(1), \quad k \to 0. \tag{2.9}$$

Substituting these expressions in equation (1.3a) we obtain,

$$\begin{aligned} \frac{uf}{2k}\sigma + O(1) &= \frac{u}{2k}\sigma\left(-\frac{\widetilde{f}}{2k}\sigma + \frac{1}{2}\begin{pmatrix} f-\widetilde{f}_1 & f+\widetilde{f}_1 \\ f+\widetilde{f}_1 & f-\widetilde{f}_1 \end{pmatrix} + O(k)\right) + O(1) \\ &= -\frac{u\widetilde{f}}{4k^2}\sigma^2 + \frac{u}{2k}\sigma\frac{1}{2}\begin{pmatrix} f-\widetilde{f}_1 & f+\widetilde{f}_1 \\ f+\widetilde{f}_1 & f-\widetilde{f}_1 \end{pmatrix} + O(1). \end{aligned} \tag{2.10}$$

That confirms that the singularity at $k = 0$ cancels because $\sigma^2 = 0$.

The transformation

$$v = V\mathrm{e}^{-\mathrm{i}kx\sigma_3},$$

maps (2.3a) to the equation

$$V_x + \mathrm{i}k[\sigma_3, V] - \frac{u}{2k}\sigma V = 0, \quad k \in \mathbb{C}. \tag{2.11}$$

The dressing method for the x-part of the Lax pair. Suppose that V satisfies the RH problem

$$V^-(x,t,k) = V^+(x,t,k)J(x,t,k), \qquad k \in \mathbb{R}, \tag{2.12a}$$
$$V(x,t,k) = I + O\left(\frac{1}{k}\right), \qquad k \to \infty, \tag{2.12b}$$

where I denotes the identity 2×2 matrix.

It is straight forward to verify that if V^+ and V^- satisfy the jump condition (2.12a), then the function defined by the LHS of (2.11) also satisfies the same jump condition, provided that J satisfies the equation

$$J_x + \mathrm{i}k[\sigma_3, J] = 0. \tag{2.13}$$

Let us denote the LHS of (2.11) by $M_1(x,t,k)$. Since M_1 satisfies the jump condition (2.12a), the assumption of the unique solvability of (2.12) implies that

$M_1 = 0$ provided that: first, M_1 is sectionally analytic in the complex variable k with respect to the real k-axis, i.e. M_1 is analytic in the upper and lower halfs of the complex k-plane; second,

$$M_1 = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.14}$$

The first condition was verified in (2.10). Regarding the second condition, we expand V in the form

$$V(x,t,k) = I + \frac{V^{(1)}(x,t)}{k} + \frac{V^{(2)}(x,t)}{k^2} + \frac{V^{(3)}(x,t)}{k^3} + \frac{V^{(4)}(x,t)}{k^4} + O\left(\frac{1}{k^5}\right), \quad k \to \infty. \tag{2.15}$$

In the analysis that follows we use the identities,

$$\sigma_2\sigma_3 = \mathrm{i}\sigma_1, \quad \sigma_3\sigma_1 = \mathrm{i}\sigma_2,, \quad \sigma_1\sigma_2 = \mathrm{i}\sigma_3, \quad \sigma_3^2 = I. \tag{2.16}$$

For large k, the term of equation (2.11) which is of order 1 vanishes if and only if

$$V_O^{(1)}(x,t) = 0, \tag{2.17}$$

where the subscripts D and O will denote the diagonal and off-diagonal parts of a matrix.

The terms of equation (2.11) which are of order $1/k$ yield

$$V_{Dx}^{(1)} + \mathrm{i}\left[\sigma_3, V_O^{(2)}\right] = \frac{u\sigma_2}{2} - \frac{\mathrm{i}u\sigma_3}{2}.$$

Hence,

$$V_{Dx}^{(1)} = -\frac{\mathrm{i}u}{2}\sigma_3, \quad \mathrm{i}\left[\sigma_3, V_O^{(2)}\right] = \frac{u}{2}\sigma_2. \tag{2.18}$$

Using that the solution of the first equation below is the second equation,

$$\mathrm{i}[\sigma_3, V_O] = A, \quad V_O = \frac{\mathrm{i}}{2}A\sigma_3, \tag{2.19}$$

we find

$$V_O^{(2)} = -\frac{u}{4}\sigma_1. \tag{2.20}$$

For large k, the terms of equation (2.11) which are of order $1/k^2$ yield

$$V_{Dx}^{(2)} = -\mathrm{i}\frac{u}{2}\sigma_3 V_D^{(1)}, \quad V_{Ox}^{(2)} + \mathrm{i}\left[\sigma_3, V_O^{(3)}\right] = \frac{u}{2}\sigma_2 V_D^{(1)}. \tag{2.21}$$

The second of these equations yields

$$\mathrm{i}\left[\sigma_3, V_O^{(3)}\right] - \frac{u}{2}\sigma_2 V_D^{(1)} = \frac{u_x}{4}\sigma_1. \tag{2.22}$$

Then, (2.19) together with the first two of(2.16) imply

$$V_O^{(3)} = \frac{u_x}{8}\sigma_2 - \frac{u}{4}\sigma_1 V_D^{(1)}. \tag{2.23}$$

For large k, the terms of equation (2.11) which are of order $1/k^3$ yield

$$V_{Dx}^{(3)} = \frac{u}{2}\sigma_2 V_O^{(2)} - \mathrm{i}\frac{u}{2}\sigma_3 V_D^{(2)}, \quad V_{Ox}^{(3)} + \mathrm{i}\left[\sigma_3, V_O^{(4)}\right] = \frac{u}{2}\sigma_2 V_D^{(2)} - \mathrm{i}\frac{u}{2}\sigma_3 V_O^{(2)}. \tag{2.24}$$

These equations with the aid of equation (2.20) become

$$V_{Dx}^{(3)} = -\mathrm{i}\frac{u^2}{8}\sigma_3 - \mathrm{i}\frac{u}{2}\sigma_3 V_D^{(2)} \tag{2.25}$$

and

$$\mathrm{i}\left[\sigma_3, V_O^{(4)}\right] - \frac{u}{2}\sigma_2 V_D^{(2)} = -\frac{1}{8}(u^2 + u_{xx})\sigma_2 + \frac{\sigma_1}{4}(uV_{Dx}^{(1)} + u_x V_D^{(1)}). \tag{2.26}$$

The dressing method for the t-part of the matrix Lax pair. We postulate the form

$$\begin{aligned} &V_t + 4\mathrm{i}\alpha k^3[\sigma_3, V] - 2\alpha k u\sigma_2 V - \alpha u_x \sigma_1 V \\ &\qquad - (Cg)V\sigma_3 + (CgV\sigma_3 V^{-1})V + \frac{1}{k}A\sigma V = 0, \end{aligned} \tag{2.27}$$

where the operator C is defined by

$$Cf(k) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}} \frac{f(l)}{l-k}\,dl, \quad k \in \mathbb{C}.$$

The occurrence of the specific matrix σ in the last term is dictated by the requirement that the term of order $1/k^2$ as k tends to 0 must cancel. The parameter α is introduced for convenience in order to analyse the particular case of $\alpha = 0$, and the scalar function $A(x,t)$ is to be determined.

As explained in the discussion of the NLS, the terms containing the Hilbert transform do not commute with J. However, we still expect the equation defined by (2.27) to be compatible with the x-part of the Lax pair by demanding that the solution μ of both equations has the same large k behaviour.

It is noted that as with the usual KdV, the third and fourth terms in (2.27) were determined by analysing the terms of order k and order 1 (for large k). Indeed, the former terms yield

$$4\mathrm{i}\left[\sigma_3, V_O^{(2)}\right] = 2u\sigma_2,$$

where we used the second of equations (2.18). Similarly, the terms of order 1 (for large k) yield

$$4\mathrm{i}\left[\sigma_3, V_O^{(3)}\right] - 2u\sigma_2 V_D^{(1)} = u_x\sigma_1,$$

where we used the second of equations (2.24).

The diagonal part of the terms of (2.27) which are of order $1/k$ (for large k) yields

$$V_{Dt}^{(1)} - 2\alpha u\sigma_2 V_O^{(2)} + \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}} g(l,t)\,dl\sigma_3$$

$$-\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}} g(l,t)(V\sigma_3V^{-1})_D(x,t,l)\,dl - \mathrm{i}A\sigma_3 = 0. \tag{2.28}$$

The transformation $V = ve^{\mathrm{i}kx\sigma_3}$ implies

$$V\sigma_3V^{-1} = v\sigma_3v^{-1} = (v_{11}v_{22} + v_{12}v_{21})\sigma_3 + 2\begin{pmatrix} 0 & -v_{11}v_{12} \\ v_{21}v_{22} & 0 \end{pmatrix}. \tag{2.29}$$

In what follows, a *hat* will denote changing k to $-k$. Thus

$$\begin{aligned}\widehat{v}_{21}(k) &= v_{21}(-k) = v_{12}(k) = \phi - \frac{1}{\mathrm{i}k}\phi_x,\\ \widehat{v}_{11}(k) &= v_{11}(-k) = v_{22}(k) = \phi + \frac{1}{\mathrm{i}k}\phi_x.\end{aligned} \tag{2.30}$$

Assuming that $g(t,k)$ is even in k, it follows that

$$\frac{1}{\pi}\int_{\mathbb{R}} gv_{21}v_{22}\,dl = \frac{1}{\pi}\int_{\mathbb{R}} gv_{12}v_{11}\,dl.$$

Hence,

$$\frac{1}{\pi}\int_{\mathbb{R}} gV\sigma_3V^{-1}\,dl = d\sigma_3 - 2\mathrm{i}h\sigma_2, \tag{2.31}$$

where

$$d(x,t) = \frac{1}{\pi}\int_{\mathbb{R}} g(t,l)(v_{11}v_{22} + v_{12}v_{21})(x,t,l)\,dl, \tag{2.32a}$$

$$h(x,t) = \frac{1}{\pi}\int_{\mathbb{R}} g(t,l)v_{21}v_{22}(x,t,l)\,dl. \tag{2.32b}$$

Replacing in (2.28) $V_D^{(1)}$ and $V_O^{(2)}$ via the first of equations (2.18) and (2.20) respectively, and then simplifying, we find

$$\partial_x^{-1}u_t + \alpha u^2 + \frac{1}{\pi}\int_{\mathbb{R}} g(l)dl - d + 2A = 0. \tag{2.33}$$

The off-diagonal part of the terms of (2.27) which are of order $1/k$ (for large k) yields

$$4\mathrm{i}\alpha[\sigma_3, V_O^{(4)}] - 2\alpha u\sigma_2V_D^{(2)} - \alpha u_x\sigma_1V_O^{(1)} + h\sigma_2 + A\sigma_2 = 0. \tag{2.34}$$

Using (2.26), the first three terms in this equation yield

$$-\alpha\left(\frac{u_{xx}}{2} + u^2\right)\sigma_2.$$

Hence, equation (2.34) yields

$$A = \alpha\left(\frac{u_{xx}}{2} + u^2\right) - h(x,t). \tag{2.35}$$

Replacing in (2.33) the function $A(x,t)$ via (2.35), and then differentiating the resulting equation with respect to x we find

$$u_t + \alpha(u_{xxx} + 6uu_x) = d_x(x,t) + 2h_x(x,t). \tag{2.36}$$

It is worth nothing that the identity

$$\sigma_1(\sigma_2 - \mathrm{i}\sigma_3) = -(\sigma_2 - \mathrm{i}\sigma_3)$$

ensures that the pole at $k = 0$ of the function defined by the LHS of equation (2.27) vanishes.

3. The inverse scattering transform

Let $\phi(x, k)$ be the solution of (2.2a) defined by the condition that it tends to $\exp(\mathrm{i}kx)$ as x tends to $-\infty$. It is straightforward to verify that the function ϕ satisfies the linear Volterra integral equation

$$\phi(x, k) = \mathrm{e}^{\mathrm{i}kx} - \int_{-\infty}^{x} \frac{\sin[k(x-\xi)]}{k} u(\xi)\phi(\xi, k)\, d\xi, \quad k \in \mathbb{R}. \tag{3.1}$$

Let *bar* denote complex conjugated. Let $\overline{\psi}$ be the solution of (2.2a) which satisfies the condition that it tends to $\exp(-\mathrm{i}kx)$ as x tends to ∞. Thus, $\overline{\psi(\overline{k})}$ also satisfies (2.2a) and tends to $\exp(\mathrm{i}kx)$ as x tends to ∞. Hence, if we define Φ and Ψ by

$$\Phi(x, k) = \phi(x, k)\mathrm{e}^{-\mathrm{i}kx}, \quad \Psi(x, k) = \overline{\psi(x, \overline{k})}\mathrm{e}^{-\mathrm{i}kx}, \quad k \in \mathbb{R}, \tag{3.2}$$

it follows that Φ and Ψ satisfy

$$\Phi(x, t) = 1 - \int_{-\infty}^{x} \frac{1 - \mathrm{e}^{-2\mathrm{i}k(x-\xi)}}{2\mathrm{i}k} u(\xi)\Phi(\xi, k)\, d\xi, \quad \operatorname{Im} k \le 0. \tag{3.3}$$

and

$$\Psi(x, t) = 1 + \int_{x}^{\infty} \frac{1 - \mathrm{e}^{-2\mathrm{i}k(x-\xi)}}{2\mathrm{i}k} u(\xi)\Psi(\xi, k)\, d\xi, \quad \operatorname{Im} k \ge 0, \tag{3.4}$$

Since $x - \xi$ is non-negative in (3.3) and non-positive in (3.4), the function Φ is analytic in k for $\operatorname{Im} k < 0$. Similarly, the function Ψ is analytic in k for $\operatorname{Im} k > 0$.

The function $\overline{\Psi(-\overline{k})}$ also satisfies equation (2.2a). Hence, this equation is simply related with the functions Ψ and $\overline{\Psi}$:

$$\phi(x, t) = a(k)\overline{\psi}(x, k) + b(k)\overline{\psi}(x, -k), \quad k \in \mathbb{R}. \tag{3.5}$$

Using in this equation the definitions (3.2) and then dividing the resulting equation by a we find

$$\frac{\Phi(x, k)}{a(k)} = \Psi(x, k) + r(k)\Psi(x, -k)\mathrm{e}^{-2\mathrm{i}kx}, \quad r(k) = \frac{b(k)}{a(k)}, \quad k \in \mathbb{R}. \tag{3.6}$$

As x tends to ∞, $\overline{\psi}$ tends to $\exp(\mathrm{i}kx)$. Hence, (3.5) implies

$$\phi(x, k) \to a(k)\mathrm{e}^{\mathrm{i}kx} + b(k)\mathrm{e}^{-\mathrm{i}kx}, \quad x \to \infty. \tag{3.7}$$

On the other hand, equation (3.1) implies

$$\phi(x, k) = \left[1 - \frac{1}{2\mathrm{i}k} \int_{-\infty}^{\infty} e^{-\mathrm{i}k\xi} u(\xi)\phi(\xi, k) d\xi\right] e^{\mathrm{i}kx}$$

$$+\left[\frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} e^{\mathrm{i}k\xi}u(\xi)\phi(\xi,k)d\xi\right]e^{-\mathrm{i}kx}, \quad x\to\infty, \quad k\in\mathbb{R}. \tag{3.8}$$

Hence, a and b are given by the expressions

$$a(k) = 1+\frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} u(\xi)\Phi(\xi,k)\,d\xi, \quad \operatorname{Im} k \le 0, \tag{3.9a}$$

and

$$b(k) = \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} e^{\mathrm{i}k\xi}u(\xi)\phi(\xi,k)\,d\xi, \quad k\in\mathbb{R}. \tag{3.9b}$$

The function a inherits its analyticity properties from the function Φ, hence it is also analytic in k for $\operatorname{Im} k<0$.

The t-part of the scalar Lax pair. Writing the "12" and "22" components of equation (2.27) and then adding the resulting equations we find the following equation:

$$\begin{aligned}(v_{12}+v_{22})_t &+ 8\mathrm{i}\alpha k^3 v_{12} + 2\mathrm{i}\alpha k u(v_{22}-v_{12}) - \alpha u_x(v_{12}+v_{22})\\ &+\frac{1}{2\mathrm{i}}(Hg)(v_{12}+v_{22}) + \frac{1}{2\mathrm{i}}\big[Hg(v_{11}v_{22}+v_{12}v_{21})\big](v_{12}-v_{22})\\ &+\frac{1}{\mathrm{i}}(Hgv_{11}v_{12})v_{22} + \frac{1}{\mathrm{i}}(Hgv_{21}v_{22})v_{12} = 0.\end{aligned}$$

Using in this equation the expressions

$$\begin{aligned} v_{12} &= \frac{1}{2}(\phi-\frac{1}{\mathrm{i}k}\phi_x), & v_{22} &= \frac{1}{2}(\phi+\frac{1}{\mathrm{i}k}\phi_x),\\ v_{21} &= \frac{1}{2}(\widehat{\phi}+\frac{1}{\mathrm{i}k}\widehat{\phi}_x), & v_{11} &= \frac{1}{2}(\widehat{\phi}-\frac{1}{\mathrm{i}k}\widehat{\phi}_x), \end{aligned} \tag{3.10}$$

we find the following t-part of the scalar Lax pair:

$$\begin{aligned}\phi_t &+ \alpha(4\mathrm{i}k^3-u_x)\phi + \alpha(2u-4k^2)\phi_x + \frac{1}{2\mathrm{i}}(Hg)\phi - \frac{1}{4\mathrm{i}}[Hg(\phi\widehat{\phi}+\frac{1}{k^2}\phi_x\widehat{\phi}_x)]\frac{\phi_x}{\mathrm{i}k}\\ &+\frac{1}{4\mathrm{i}}[Hg(\phi\widehat{\phi}-\frac{1}{k^2}\phi_x\widehat{\phi}_x)]\phi - \frac{1}{4\mathrm{i}}[Hg(\frac{1}{\mathrm{i}k}(\phi\widehat{\phi})_x)]\frac{\phi_x}{\mathrm{i}k} = 0.\end{aligned} \tag{3.11}$$

Letting in this equation x tend to ∞ and employing (3.5) we find the following evolution equations for $a(t,k)$ and $b(t,k)$:

$$a_t + \frac{1}{2\mathrm{i}}(Hg)a + \frac{1}{2\mathrm{i}}[Hg(|a|^2+|b|^2)]a = 0, \tag{3.12a}$$

$$b_t + 8\mathrm{i}\alpha k^3 b + \frac{1}{2\mathrm{i}}(Hg)b - \frac{1}{2\mathrm{i}}[Hg(|a|^2+|b|^2)]b = 0, \tag{3.12b}$$

where we used the identities

$$\widehat{a} = \bar{a}, \quad \widehat{b} = \bar{b}. \tag{3.13}$$

Equations (3.12) imply that the absolute value of both a and b are conserved. Hence,

$$a(t,k) = a_0(k)\mathrm{e}^{\mathrm{i}(HG(|a_0|^2)}, \tag{3.14a}$$

$$\frac{b(t,k)}{a(t,k)} = \frac{b_0(k)}{a_0(k)} e^{-8i\alpha k^3 t - i[HG(|a_0|^2+|b_0|^2)]}, \tag{3.14b}$$

where

$$G(t,k) = \int_0^t g(\tau,k)\, d\tau, \tag{3.15}$$

and we made use of relation

$$|a|^2 - |b|^2 = 1, \tag{3.16}$$

to simplify (3.14a).

In order for $a(t,k)$ to be analytic for $\operatorname{Im} k > 0$ it is necessary for the function $HG|b_0|^2$ to admit an analytic continuation in the upper half k-complex plane. In this connection we note that

$$iHG|b_0|^2 = \left(iHG|b_0|^2 - G|gb_0|^2\right) + HG|b_0|^2,$$

and the first term in the RHS of the above equation admits an analytic continuation in $\operatorname{Im} k > 0$. Thus, we require that

$$G(k,t)|b_0(k)|^2 = \widetilde{g}(t,k), \quad k \in \mathbb{R},$$

where $\widetilde{g}(t,k)$ is analytic in $\operatorname{Im} k > 0$.

In the absence of solitons, the first of equations (3.6) implies the following linear integral equation for $\Psi(x,t,-k)$:

$$\Psi(x,t,-k) = 1 - \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{b_0(k)}{a_0(k)} \frac{\Psi(x,t,-l)}{l+k} e^{-2ilx - 8i\alpha l^3 t - i\Gamma(t,l)}\, dl, \ \operatorname{Im} k < 0, \tag{3.17a}$$

where Γ is defined by

$$\Gamma(t,k) = HG(|a_0|^2 + |b_0|^2). \tag{3.17b}$$

4. Related physical models

4.1. Linear approximation of ion acoustic waves in the laser plasma interaction. In the last decades, laser-plasma interactions, have been investigated extensively, both experimentally and theoretically. The Euler–Maxwell equations or the Vlasov–Maxwell equations [19] provide the basis of most models accounting for these interactions. The complexity of these models makes it difficult to achieve an analytical formulation or even to obtain effective numerical simulations. This motivates the search for simpler models.

In the seventies, starting with the Euler–Maxwell equations, Zakharov and al., introduced the following equations, known as the Zakharov equations [22] [20]:

$$n_{tt} - n_{xx} - 2\left(|E(x,t)|^2\right)_{xx} = 0, \tag{4.1a}$$

$$iE_t(x,t) + \frac{1}{2}E_{xx}(x,t) - n(x,t)E(x,t) = 0, \tag{4.1b}$$

where $n(x,t)$ is the fractional change in the plasma density and $E(x,t)$ is a high frequency electrostatic field.

Assuming a one-directional ion-sound wave propagation $n_t \approx -n_x$, with a speed of propagation $v \approx 1$, Yajima and Oikawa [21] applied the following transformation to Zakharov equations:

$$n_{tt} - n_{xx} = (\partial_t - \partial_x)(n_t + n_x) = -2\partial_x(n_t + n_x). \tag{4.2}$$

This led to the Yajima–Oikawa (YO) equations,

$$n_t(x,t) + n_x(x,t) = -\left(|E(x,t)|^2\right)_x, \tag{4.3a}$$

$$\mathrm{i}E_t(x,t) + \frac{1}{2}E_{xx}(x,t) - n(x,t)E(x,t) = 0. \tag{4.3b}$$

Remarkably, these equations are integrable [21]. The YO equations, in addition to describing resonant interactions between high-frequency electrostatic field oscillations and low-frequency ion density perturbations, they also describe the resonance interaction between internal gravity-wave packet and mean motion [7].

In the context of laser-plasma interaction, taking into consideration the basic assumption $n_t \approx -n_x$ [21], equation (4.3a) implies that $E(x,t)$ must have a small scale amplitude. Therefore, considering the field $E(x,t)$ as a wave packet with profile $g_1(\omega)$, ω being the frequency of electrostatic field, $E(x,t)$ can be written in the form

$$E(x,t) = \varepsilon^{1/2}\int_{-\infty}^{+\infty} g_1(\omega)\widetilde{E}(x,\tau,\omega)\mathrm{e}^{-\mathrm{i}\omega t}\, d\omega, \tag{4.4}$$

where $\varepsilon = \sqrt{m_e/m_i}$, m_e and m_i are the electron and ion masses, respectively, $\widetilde{E}(x,\tau,\omega)$ is the slowly varying envelope of $E(x,t)$, and $\tau = \varepsilon t$ is the slow time.

Introducing a frame co-moving with the electrostatic field, defined by

$$\xi = x - t, \quad \tau = \varepsilon t, \tag{4.5}$$

and setting

$$\omega = \frac{1}{2}k^2, \quad q = -\frac{1}{2}n, \quad \mathfrak{g}(\omega) = |g_1(\omega)|^2, \tag{4.6}$$

the YO equations (4.3) become

$$\sigma\frac{\partial}{\partial\tau}q(\xi,\tau) = \frac{\partial}{\partial\xi}\int_{-\infty}^{+\infty}\mathfrak{g}(\omega)|\widetilde{E}(\xi,\tau,\omega)|^2\, d\omega, \tag{4.7a}$$

$$\widetilde{E}_{\xi\xi}(\xi,\tau) + (k^2 + q)\widetilde{E}(\xi,\tau) = 0, \tag{4.7b}$$

where $\sigma = +1$ if the *cavitation* $q(\xi,\tau)$ travels faster than the envelope of the electrostatic field, and $\sigma = -1$, if it travels slower (due to its negative sign, q accounts for a cavitation in the electron density).

Under the assumption of a weak nonlinearity approximation of the ion acoustic wave, (4.7) yield the coupled anti-dissipative version of the kdV equation [10] [13],

$$\sigma q_\tau(\xi,\tau) + 6qq_\xi + q_{\xi\xi\xi} = \frac{\partial}{\partial\xi}\int_{-\infty}^{+\infty}|g_1(\omega)|^2|\widetilde{E}(\xi,\tau,\omega)|^2 d\omega, \tag{4.8a}$$

$$\widetilde{E}_{\xi\xi}(\xi,\tau)(k^2+q)\widetilde{E}(\xi,\tau)=0, \tag{4.8b}$$

which is not integrable.

4.2. Resonant capillary-gravity waves. The nonlinear coupling of gravity-capillary waves (GCW), due to its importance in fundamental wave physics, has been extensively studied in recent years. Examples include, resonances in GCW turbulence [1] [8], consequences of the nonlinearity of GCW on wind-wave interactions [9], the nonlinear resonance interactions between three GCW [17] [3], instability of the nonlinear GCW based on the Zakharov nonlinear equation with weak viscous dissipation [16], the impact of the dissipation on the nonlinear interactions of GCW [2], and the nonlinear parametric coupling of GCW under the radiation pressure of ultrasound waves [12].

In what fallows we focus on the two-dimensional GCW in an inviscid, incompressible liquid layer of uniform finite depth h. For such a fluid, the irrotational motion is harmonic,

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0, \tag{4.9}$$

where $\varphi(x,y,t)$ is the velocity potential, $x\in[-\infty,+\infty]$ is the direction of wave propagation along the unperturbed liquid free surface, $y\in[-1,\eta(x,t)]$, where h is normalized, and $\eta(x,t)$ is the free surface elevation from the unperturbed level. The harmonic equation (4.9) is completed with the following boundary conditions [11]:

$$\frac{\partial\eta}{\partial t}-\frac{\partial\varphi}{\partial y}+\frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x}=0 \quad \text{at } y=\eta(x,t), \tag{4.10a}$$

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right] +\eta-\frac{1}{W}\frac{\partial^2\eta}{\partial x^2}\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2\right]^{-\frac{3}{2}}=0 \quad \text{at } y=\eta(x,t), \tag{4.10b}$$

$$\frac{\partial\varphi}{\partial y}=0 \quad \text{at} \quad y=-1. \tag{4.10c}$$

In equations (4.10a)–(4.10c) the phase velocity is normalized, i.e. $\sqrt{gh}=1$, where g is the gravitational acceleration. W is the Weber number, $W=\rho gh^2/T$, where ρ is the fluid density and T is the surface tension.

The resonance occurs when the group velocity of the capillary wave (short wave), is sufficiently close to the phase velocity of the gravity wave (long wave). In order to analyse the nonlinear resonance interaction between the two waves with different time and amplitude scales, it is necessary to apply the multiple scales expansion method [15]:

$$x_n=\varepsilon^n x, \quad t_n=\varepsilon^n t, \quad n=0,1,2,\dots, \tag{4.11}$$

where ε is an arbitrary small parameter. Using these scales we expand η and φ as follows:

$$\eta(x,t,\varepsilon)=\sum_{n=2}^{\infty}\varepsilon^n\eta_n(x_0,x_1,x_2,\dots,t_0,t_1,t_2,\dots), \tag{4.12a}$$

$$\varphi(x,y,t,\varepsilon)=\sum_{n=1}^{\infty}\varepsilon^n\varphi_n(x_0,x_1,x_2,\dots,y,t_0,t_1,t_2,\dots). \tag{4.12b}$$

Interaction equations are derived by choosing the following superposition of long and short waves for η_2 and ϕ_2 [11]:

$$\begin{aligned}\eta_2 = A(x_1,t_1,x_2,t_2,\dots)e^{i\theta}+\overline{A}(x_1,t_1,x_2,t_2,\dots)e^{-i\theta}&\\ +\widetilde{\eta}_2(x_1,t_1,x_2,t_2,\dots),&\end{aligned} \tag{4.13a}$$

$$\begin{aligned}\varphi_2=-\frac{i\omega\cosh\left[k(y+1)\right]}{k\sinh(k)}\left[A(x_1,t_1,x_2,t_2,\dots)e^{i\theta}+\overline{A}(x_1,t_1,x_2,t_2,\dots)e^{-i\theta}\right]&\\ +\widetilde{\varphi}_2(x_1,t_1,x_2,t_2,\dots),&\end{aligned} \tag{4.13b}$$

where $\theta=kx_0-\omega t_0$, k and ω are the wave number and the frequency, respectively. In equation (4.13a), the expression $Ae^{i\theta}$ represents a train of high frequency short waves (A is the complex amplitude), whereas $\widetilde{\eta}_2$ and $\widetilde{\varphi}_2$, represent slowly varying low frequency long waves. The first order potential $\widetilde{\varphi}_1$ is responsible for producing the slowly varying long wave $\widetilde{\eta}_2$. For this reason, in equation (4.13a), the series starts by $n=1$, while in equation (4.13b), the series starts by $n=2$.

Non trivial solutions can be obtained provided that ω and k satisfy the dispersion relation

$$\omega^2=\left(k+\frac{k^3}{W}\right)\tanh(k). \tag{4.14}$$

The group velocity is given by

$$V_g=\frac{d\omega}{dk}=\frac{1}{2\omega}\left[\left(1+\frac{3k^2}{W}\right)\tanh(k)+\left(1+\frac{k^2}{W}\right)\frac{k}{\cosh^2(k)}\right]. \tag{4.15}$$

Substituting equations (4.11), (4.12a), (4.12b), (4.13a), and (4.13b) in equations (4.9), (4.10a), (4.10b), and (4.10c), we obtain at forth order, the following set of equations:

$$B\equiv\frac{\partial\widetilde{\phi}_1}{\partial x_1}=\widetilde{\eta}_2, \tag{4.16a}$$

$$i\left(\frac{\partial A}{\partial t_2}+V_g\frac{\partial A}{\partial x_2}\right)+L\frac{\partial^2 A}{\partial x_1^2}=MBA, \tag{4.16b}$$

$$\frac{\partial B}{\partial t_3}+\frac{\partial B}{\partial x_3}+\frac{3}{2}B\frac{\partial B}{\partial x_1}+N\frac{\partial^3 B}{\partial x_1^3}=-P\frac{\partial |A|^2}{\partial x_1}, \tag{4.16c}$$

where the coefficients L, M, N and P are defined as follows:

$$L=\frac{1}{2}\frac{dV_g}{dk}, \tag{4.17a}$$

$$M = \frac{k}{2}\left[2 - \omega\big(\tanh(k) - \coth(k)\big)\right], \tag{4.17b}$$

$$N = \frac{1}{6}\left(1 - \frac{3}{W}\right), \tag{4.17c}$$

$$P = \frac{\omega\coth(k)}{2}\left[2 - \omega\big(\tanh(k) - \coth(k)\big)V_g\right]. \tag{4.17d}$$

It is possible to write equations (4.16b) and (4.16c) in terms of x_1 and t_1:

$$\frac{\mathrm{i}}{\varepsilon}\left(\frac{\partial A}{\partial t_1} + V_g\frac{\partial A}{\partial x_1}\right) + L\frac{\partial^2 A}{\partial x_1^2} = MBA, \tag{4.18a}$$

$$\frac{1}{\varepsilon^2}\left(\frac{\partial B}{\partial t_1} + \frac{\partial B}{\partial x_1}\right) + \frac{3}{2}B\frac{\partial B}{\partial x_1} + N\frac{\partial^3 B}{\partial x_1^3} = -P\frac{\partial |A|^2}{\partial x_1}. \tag{4.18b}$$

Next we introduce two new frames. The first one is the frame moving at the speed V_g with the scaled time τ_1, namely,

$$\xi_1 = x_1 - V_g t_1, \quad \tau_1 = \varepsilon t_1. \tag{4.19}$$

The second one, is a frame moving at the speed 1 with the scaled time τ_2, namely:

$$\xi_2 = x_1 - t_1, \quad \tau_2 = \varepsilon^2 t_1. \tag{4.20}$$

Rewriting (4.18a) in the frame (4.19), and (4.18b) in the frame (4.20), and taking into consideration that $\partial_{\xi_1} = \partial_{\xi_2} = \partial_{x_1}$ we obtain:

$$\mathrm{i}\frac{\partial A}{\partial \tau_1} + L\frac{\partial^2 A}{\partial x_1^2} = MBA, \tag{4.21a}$$

$$\frac{\partial B}{\partial \tau_2} + \frac{3}{2}B\frac{\partial B}{\partial x_1} + N\frac{\partial^3 B}{\partial x_1^3} = -P\frac{\partial |A|^2}{\partial x_1}. \tag{4.21b}$$

In equation (4.21b), B represents a *slowly varying* long wave, its derivative with respect to the scaled time $\tau_2 = \varepsilon^2 t_1$, contributes to derivatives with respect to the lower space scale.

Let us concentrate on the case $W < 3$, which implies $N < 0$. In what fallows, we choose the physically interesting value of $W = 2.9$. Indeed, at 20°, the surface tension for water is $\approx 72 \times 10^{-3}\ N/m$ [18]. and hence, $W = 2.9$ corresponds to a flat surface of depth $\approx 20\ cm$. In coastal engineering, this is know as *shallow water*, and is a domain of intense activity [14]. A resonance occurs when the group velocity of the long wave is close to the phase velocity of the short wave, i.e. $V_g \approx 1$. In Fig. 1, we plot V_g against the wave number k, for $W = 2.9$. Fig. 1 shows that for any value $0 < k < 1$ resonance does occur. Hence, instead of a monochromatic wave, a wave packet centred in this interval, such as $g_2(k) = \exp\left[-20(k-0.5)^2\right]$ also, allows resonance to take place. Figure 2, shows the variations of coefficients L, M and P, within the interval $0 < k < 1$. The numerical range of values of these parameters, are as follows: for $0 < k < 1$ and $W = 2.9$, we have $0 < L < 0.06$, $0 < M < 1.28$, $1.15 < P < 1.72$ and $N =$

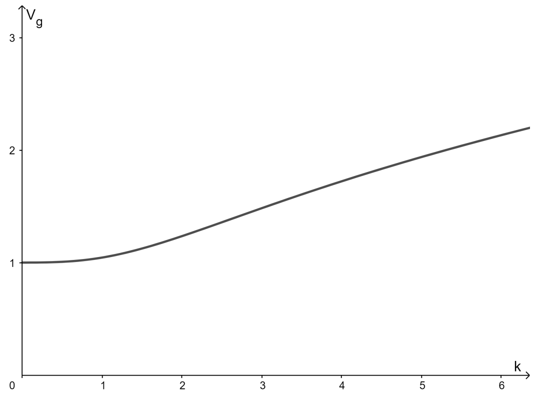

Fig. 4.1: Group velocity V_g versus wavenumber k for $W = 2.9$.

-5.75×10^{-3}. The value of N is of order ε, and the dispersive term $N\dfrac{\partial^3 B}{\partial x_1^3}$ in (4.21b), can be neglected. Setting $\widetilde{B} = \frac{L}{M}B$, $A = \widetilde{A}e^{-i\widetilde{\varphi}}$ and $\lambda^2 = \widetilde{\varphi}/L$, the set of equations (4.21a)–(4.21b) becomes,

$$\lambda^2\widetilde{A} + \frac{\partial^2 \widetilde{A}}{\partial x_1^2} = \widetilde{B}\widetilde{A}, \tag{4.22a}$$

$$\frac{M}{L}\frac{\partial \widetilde{B}}{\partial \tau_2} + \frac{3}{2}\left(\frac{M}{L}\right)^2 \widetilde{B}\frac{\partial \widetilde{B}}{\partial x_1} = -P\frac{\partial|\widetilde{A}|^2}{\partial x_1}. \tag{4.22b}$$

Taking into account that $0.01 < \frac{M}{L} < 0.04$, we have $\left(\frac{M}{L}\right)^2 \ll \frac{M}{L}$. Hence, the nonlinear term in (4.22b), can also be neglected. Thus, equation (4.22a) and (4.22b) yield,

$$\lambda^2\widetilde{A} + \frac{\partial^2 \widetilde{A}}{\partial x_1^2} = \widetilde{B}\widetilde{A}, \tag{4.23a}$$

$$\frac{\partial \widetilde{B}}{\partial \tau_2} = -\int_{-\infty}^{+\infty} \widetilde{g}_2(k)\frac{\partial|\widetilde{A}|^2}{\partial x_1}dk, \tag{4.23b}$$

where $\widetilde{g}_2(k) = \frac{M}{L}\exp\left[-20(k-0.5)^2\right]$.

Appendix

In what follows we will show that the compatibility of the new Lax pair yields the forced KdV equation (2.36). For this purpose we will use Proposition A1 of [5], which states that the compatibility condition of the Lax pair

$$\psi_x + N_1\psi = 0, \tag{A.1a}$$

$$\psi_t + N_2\psi + N\psi = 0, \tag{A.1b}$$

is the equation

$$\left(N_{1t} - N_{2x} - [N_1, N_2]\right)\psi = \left(N_x + [N_1, N]\right)\psi. \tag{A.2}$$

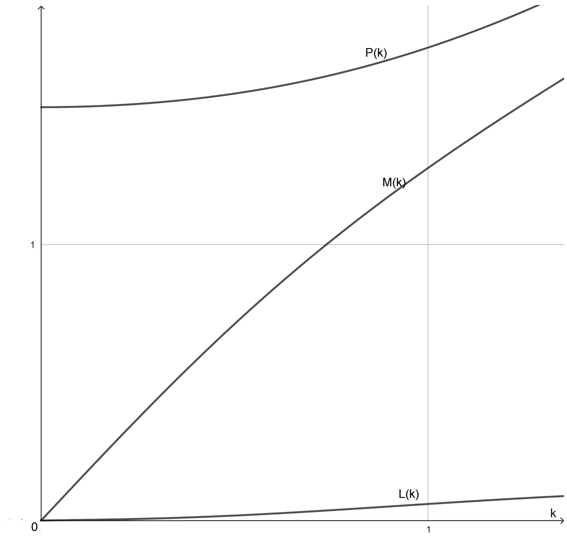

Fig. 4.2: Variations of coefficients L, M and P, versus the wavenumber k in the interval $[0,1]$, for $W = 2.9$.

We will also use the identity

$$(\psi\sigma_3\psi)_x^{-1} = -[N_1, \psi\sigma_3\psi^{-1}]. \tag{A.3}$$

For our case,

$$N_1 = \mathrm{i}k\sigma_3 - \frac{u\sigma}{2\mathrm{i}k}, \quad N = \frac{1}{2\mathrm{i}}Hg\psi\sigma_3\psi^{-1} - \frac{h}{k}\sigma. \tag{A.4}$$

Hence,

$$\begin{aligned} N_x &= -\frac{1}{2\mathrm{i}}Hg\left[\mathrm{i}k\sigma_3 - \frac{u\sigma}{2k}, \psi\sigma_3\psi^{-1}\right] - \frac{h_x}{k}\sigma \\ &= -\frac{1}{2}\left[\sigma_3, 2\mathrm{i}Hkg\psi\sigma_3\psi^{-1}\right] + \frac{u}{4}\left[\sigma, H\frac{g}{k}\psi\sigma_3\psi^{-1}\right] - \frac{h_x}{k}\sigma. \end{aligned}$$

Using the identities

$$Hkf(k) = kHf(k) + \frac{1}{\pi}\int_{\mathbb{R}} f(k)\,dk, \quad H\frac{f(k)}{k} = \frac{1}{k}Hf(k) - \frac{1}{\pi k}\int_{\mathbb{R}} \frac{f(k)}{k}\,dk,$$

we find further

$$\begin{aligned} N_x = -\left[N_1, \frac{1}{2\mathrm{i}}Hg\psi\sigma_3\psi^{-1}\right] &- \frac{1}{2\pi}\left[\sigma_3, \int_{\mathbb{R}} g\left(\psi\sigma_3\psi^{-1}\right)_O\,dl\right] \\ &- \frac{u}{4\pi k}\left[\mathrm{i}\sigma, \int_{\mathbb{R}} \frac{g}{l}\psi\sigma_3\psi^{-1}\right]dl - \frac{h_x}{k}\sigma. \end{aligned}$$

Also,

$$[N_1, N] = \left[N_1, \frac{1}{2\mathrm{i}}Hg\psi\sigma_3\psi^{-1}\right] + \left[\mathrm{i}k\sigma_3 - \frac{u\sigma}{2k}, \frac{h}{k}\sigma\right].$$

Thus, using the identities (see (2.30))

$$\sigma^2 = 0, \quad -\frac{1}{2\pi}\int_{\mathbb{R}} g\psi\sigma_3\psi^{-1}\, dl = \mathrm{i}h\sigma_2,$$

we find

$$N_x + [N_1, N] = \mathrm{i}h[\sigma_3, \sigma_2] - \frac{u}{4k}\left[\mathrm{i}\sigma, \frac{1}{\pi}\int_{\mathbb{R}} \frac{g}{l}\psi\sigma_3\psi^{-1}dl\right] - \frac{h_x}{k}\sigma + h[\sigma_3, \mathrm{i}\sigma].$$

The first and the last terms of the RHS of the above equation cancel. Furthermore, the diagonal of $\psi\sigma_3\psi^{-1}$ involves only even functions, hence the integral of these functions time the odd function $g(l)/l$ vanishes. Also the transformation $k \to -k$ maps the "12" element of $\psi\sigma_3\psi^{-1}$ to the negative of the "21" element. Hence,

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{g}{l}\psi\sigma_3\psi^{-1}\, dl = F(x,t)\sigma_1, \quad F = \frac{2}{\pi}\int_{\mathbb{R}} \frac{g}{l} v_{21}v_{22}\, dl. \tag{A.5}$$

Using the identity

$$[\sigma, \sigma_1] = 2\sigma,$$

we find

$$N_x + [N_1, N] = -\frac{h_x}{k}\sigma - \frac{uF}{k}\mathrm{i}\sigma.$$

The above forcing corresponds to the LHS of the $-u_t\sigma/2k$, thus the LHS of u_t corresponds to the forcing $2h_x - 2\mathrm{i}uF$. We next verify that $-2\mathrm{i}uF = d_x$. Indeed,

$$\begin{aligned} d_x + 2\mathrm{i}uF = \frac{\mathrm{i}}{4}\int_{\mathbb{R}} g\Bigg\{\left[\left(\widehat{\phi} - \frac{1}{\mathrm{i}k}\widehat{\phi}_x\right)\left(\phi + \frac{1}{\mathrm{i}k}\phi_x\right) + \left(\widehat{\phi} + \frac{1}{\mathrm{i}k}\widehat{\phi}_x\right)\left(\phi - \frac{1}{\mathrm{i}k}\phi_x\right)\right]_x \\ + 2\mathrm{i}\frac{u}{k}\left(\widehat{\phi} + \frac{1}{\mathrm{i}k}\widehat{\phi}_x\right)\left(\phi + \frac{1}{\mathrm{i}k}\phi_x\right)\Bigg\}\, dk. \end{aligned}$$

Using the differential equations (2.3a) for ϕ and $\widehat{\phi}$, the above equation simplifies to

$$d_x + 2\mathrm{i}uf = \frac{\mathrm{i}u}{2}\int_{\mathbb{R}} \frac{g}{k}\left(\phi\widehat{\phi} - \frac{1}{k^2}\phi_x\widehat{\phi}_x\right) dk,$$

which vanishes since g is an odd function.

It is interesting to note that

$$2h_x + d_x = \frac{1}{4}\int_{\mathbb{R}} g\left\{\left(2 - \frac{u}{k^2}\right)\left(\phi\widehat{\phi}\right)_x + \frac{4}{\mathrm{i}k}\left[\phi_x\widehat{\phi}_x + (u + k^2)\phi\widehat{\phi}\right]\right\} dk.$$

Thus, the forcing formally simplifies to

$$\frac{1}{2}\partial_x\int_{\mathbb{R}} g\phi\widehat{\phi}\, dl - \frac{u}{4}\partial_x\int_{\mathbb{R}} \frac{g}{l^2}\phi\widehat{\phi}\, dl. \tag{A.6}$$

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A.S. Fokas,

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, CB3 0WA Cambridge, UK,
Viterbi School of Engineering, USC, Los Angeles, 90089 CA, USA,
E-mail: tf227@cam.ac.uk

A. Latifi,

Department of Mechanics, Faculty of Physics, Qom University of Technology, Qom, Iran,
E-mail: latifi@qut.ac.ir

Рівняння Кортевега–де Фріза з форсуванням, що містить добуток власних функцій

A.S. Fokas and A. Latifi

Нещодавно було запроваджено нову методологію, яка, починаючи з інтегровного еволюційного рівняння будує інтегровну форсовану (з правою частиною) версію цього рівняння. Форсування складається з доданків, що включають квадратичні добутки певних власних функцій асоційованої пари Лакса.

Ми застосовуємо цю методологію, починаючи зі знакового рівняння Кортевга–де Фріза. Задача з початковими значеннями для асоційованого інтегровного форсованого рівняння може бути сформульована як задача Рімана–Гільберта з “матрицею стрибка”, шо має явну залежність від x і t, яку можна обчислити за початковими даними. Таким чином, це рівняння можна розв’язати так само ефективно, як і саме рівняння Кортевега–де Фріза.

Також показано, що це форсоване рівняння разом з x-частиною його пари Лакса, з’являються в моделюванні важливих фізичних явищ. Зокрема, в контексті лазерно-плазмової взаємодії, а також в описі резонансних гравітаційно-капілярних хвиль.

Ключові слова: форсоване рівняння КдФ, пара Лакса, інтегровність, проблема Рімана–Гільберта