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The degenerate Carathéodory problem and the elementary multiple factor

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The degenerate matricial interpolation Carathéodory problem is solved. To solve this problem we use the V.P. Potapov's approach based on the theory of J-expansive matrix-functions. The $\mathcal{K}$-type subspace technique also plays an important role in these investigations.

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1. The setting of the problem. The basic matricial inequalities connected with the problem

Let $\mathbb{D} = \{\zeta \in \mathbb{C} : |\ \zeta\ |< 1\}$ be the opened unit disk of the complex plane, $q \in \mathbb{N}$, and let $\mathbb{C}^{q\times q}$ be the set of square matrices of order q with complex entries. Denote by $\mathcal{C}_q$ the set of matrix-valued functions $\mathcal{F}(\zeta)$ analytical in $\mathbb{D}$ with values in $\mathbb{C}^{q\times q}$ and satisfying the inequality

$$\operatorname{Re}\mathcal{F}(\zeta) = \frac{1}{2}\left(\mathcal{F}(\zeta) + \mathcal{F}^*(\zeta)\right) \geq 0$$

for all $\zeta \in \mathbb{D}$. The Carathéodory problem generalized to the matrix case (see, e.g., [1, 2]) is formulated in the following way.

Assume that $c_0, c_1, \ldots, c_n \in \mathbb{C}^{q\times q}$. Problem:

a) *to find necessary and sufficient conditions of existence of a matrix-valued function $\mathcal{F}(\zeta) \in \mathcal{C}_q$ such that $c_0, c_1, \ldots, c_n$ are the first coefficients of its Maclaurin series:*

$$\mathcal{F}(\zeta) = c_0 + c_1\zeta + \ldots + c_n\zeta^n + \ldots; \tag{1.1}$$

b) *to describe all function $\mathcal{F}(\zeta) \in \mathcal{C}_q$ of the form (1.1).*

The Carathéodory problem in the scalar case (i.e., in the case $q = 1$) was investigated in the papers [3, 4].

Set $A_n = C_n + C_n^*$, where

$$C_n = \begin{bmatrix} c_0 & 0 & \dots & 0 \\ c_1 & c_0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_n & c_{n-1} & \dots & c_0 \end{bmatrix}. \tag{1.2}$$

The matrix A_n plays an important role when we study this problem. Namely, the following theorems are true (see, e.g., [2, 5, 6]).

Theorem 1.1. *Let $\{c_k\}_{k=0}^{\infty} \in \mathbb{C}^{q\times q}$ and a function $\mathcal{F}(\zeta)$ is of the form (1.1). Then $\mathcal{F}(\zeta) \in \mathcal{C}_q$ if and only if*

$$A_n = \begin{bmatrix} c_0 + c_0^* & c_1^* & \dots & c_n^* \\ c_1 & c_0 + c_0^* & \dots & c_{n-1}^* \\ \dots & \dots & \dots & \dots \\ c_n & c_{n-1} & \dots & c_0 + c_0^* \end{bmatrix} \geq 0 \tag{1.3}$$

for all $n \in \mathbb{N} \cup \{0\}$.

This theorem is supplemented by the following

Theorem 1.2. *Let $A_n \geq 0$ for $\{c_k\}_{k=0}^{n} \in \mathbb{C}^{q\times q}$. Then there exists $\mathcal{F}(\zeta) \in \mathcal{C}_q$ such that its expansion in the Maclaurin series has the form (1.1).*

Theorems 1.1, 1.2 give us the answer to the first question of the Carathéodory problem. To give an answer to the second question V.P. Potapov proposed a special approach (see, e.g., [1, 2, 5, 7, 8]). According to this approach the basic matricial inequality (BMI) and the dual one are corresponded to each interpolation problem. The solution of each of these inequalities gives us a description of all solutions of the problem. The BMI and the dual one for the matricial Carathéodory problem have the form (1.4) and (1.4′) respectively.

Theorem 1.3. ([1, 2]) *Let $\{c_k\}_{k=0}^{n} \in \mathbb{C}^{q\times q}$ and let $\mathcal{F}(\zeta)$ be a matrix-valued function analytical in the unite disk $\mathbb{D}$. Then $\mathcal{F}(\zeta) \in \mathcal{C}_q$ and can be represented in the form (1.1) if and only if for the function $\mathcal{F}(\zeta)$ the inequality*

$$\left[\begin{array}{cccc|c} c_0 + c_0^* & c_1^* & \dots & c_n^* & \mathcal{F}^*(\zeta) + c_0 \\ c_1 & c_0 + c_0^* & \dots & c_{n-1}^* & \bar{\zeta}[\mathcal{F}^*(\zeta) + c_0 + \frac{c_1}{\zeta}] \\ \dots & \dots & \dots & \dots & \dots\dots\dots\dots \\ c_n & c_{n-1} & \dots & c_0 + c_0^* & \bar{\zeta}^{n+1}[\mathcal{F}^*(\zeta) + c_0 + \frac{c_1}{\zeta} + \ldots + \frac{c_n}{\zeta^n}] \\ \hline & & * & & \frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\zeta\bar{\zeta}} \end{array}\right] \geq 0 \tag{1.4}$$

holds everywhere in $\mathbb{D}$ *or the following inequality*

$$\left[\begin{array}{cccc|c} c_0+c_0^* & c_1^* & \ldots & c_n^* & \frac{1}{\zeta}[\mathcal{F}(\zeta)-c_0] \\ c_1 & c_0+c_0^* & \ldots & c_{n-1}^* & \frac{1}{\zeta^2}[\mathcal{F}(\zeta)-c_0-c_1\zeta] \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots \\ c_n & c_{n-1} & \ldots & c_0+c_0^* & \frac{1}{\zeta^{n+1}}[\mathcal{F}(\zeta)-c_0-c_1\zeta-\ldots-c_n\zeta^n] \\ \hline & & * & & \frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\zeta\bar{\zeta}} \end{array}\right] \geq 0 \tag{1.4'}$$

holds everywhere in $\mathbb{D}$.

Here and further the block denoted by $*$ in the inequations of the form (1.4), (1.4′) is with the block which is adjoint to the upper right block.

Note that the matrix A_n of the form (1.3) is the upper left block in inequalities (1.4) and (1.4′). It turns out that the method of solving of these inequalities depends on the property of the matrix A_n to be degenerate or not. In the case of the nondegenerate matrix A_n the corresponding Carathéodory problem is called nondegenerate, otherwise it is called degenerate. First the nondegenerate matricial Carathéodory problem was solved constructively (i.e., directly in the terms of the interpolation data) in [1]. In this paper V.P. Potapov's approach of solving of the matricial interpolation problems was used (see, e.g., [7–11]. It is based on the theory of analytical J-expansive matrix-valued functions. Note that the nondegenerate matricial Carathéodory problem is solved in [6] in the different way. First the constructive method of solving of the degenerate matricial interpolation problems was obtained by investigating of the Schur problem [12]. The methods of this paper play an important role in the Sect. 3. This section is main in the present paper. There a constructive method of solving of the degenerate matricial interpolation Carathéodory problem is obtained. The main results of the paper are formulated in Theorems 3.1 and 3.2 of this section.

In V.P. Potapov's approach the elementary multiple factor corresponding to the BMI and the dual one plays a very important role. The parametrization of the elementary multiple factor of the full rank connected with the nondegenerate Carathéodory problem is given in [1]. This parametrization is directly connected with the parametrization of the elementary multiple factor of the full rank corresponding to the nondegenerate Schur problem (see [13]). The parametrization of an arbitrary elementary multiple factor corresponding to the Carathéodory problem is obtained in the diploma work of L.V. Mihailova "The parametrization of the elementary multiple factor of the nonfull rank in the Carathéodory problem" (Kharkov National University, 1981). There the results of the paper [14] were used substantially. The proof of this parametrization (see the proof of Theorem 2.2) is given in the present paper to deal with the main results formulated in Theorems 3.1 and 3.2.

2. The parametrization of an arbitrary elementary multiple factor

Let j be a constant hermitian involutive matrix of order N, i.e., $j^* = j$, $j^2 = I$.

Definition 2.1. (See, e.g., [1, 9]) *Let $\mathcal{B}(\zeta)$ be an analytical matrix-valued function of the order N. And let it have a single pole of an arbitrary multiplicity on the extended complex plane. $\mathcal{B}(\zeta)$ is called a j-elementary multiple factor if it is j-expansive in the unit disk and j-unitary on its boundary, i.e.,*

$$\mathcal{B}(\zeta) j \mathcal{B}^*(\zeta) - j \geq 0, \quad |\, \zeta \,| < 1 \tag{2.1}$$

and

$$\mathcal{B}(\zeta) j \mathcal{B}^*(\zeta) - j = 0, \quad |\, \zeta \,| = 1 \tag{2.2}$$

or, equivalently,

$$\mathcal{B}^*(\zeta) j \mathcal{B}(\zeta) - j \geq 0, \quad |\, \zeta \,| < 1$$

and

$$\mathcal{B}^*(\zeta) j \mathcal{B}(\zeta) - j = 0, \quad |\, \zeta \,| = 1.$$

The Carathéodory problem is connected with the matrix j of the form:

$$j = J = \begin{bmatrix} 0 & I_q \\ I_q & 0 \end{bmatrix}. \tag{2.3}$$

It can be explained, e.g., by the fact, that the matrix block standing in the lower right angle of the left part of inequalities (1.4) and (1.4′) can be presented in the following form:

$$\frac{\mathcal{F}(\zeta) + \mathcal{F}^*(\zeta)}{1 - \zeta\bar{\zeta}} = \frac{1}{1 - \zeta\bar{\zeta}} [\mathcal{F}(\zeta), I_q] \begin{bmatrix} 0 & I_q \\ I_q & 0 \end{bmatrix} \begin{bmatrix} \mathcal{F}^*(\zeta) \\ I_q \end{bmatrix}.$$

Consider a J-elementary multiple factor $\mathcal{B}(\zeta)$ of the order $N = 2q$. Assume that $\mathcal{B}(\zeta)$ has the single pole at the point $\zeta = 0$ and

$$\mathcal{B}(\zeta) = d_0 + \frac{d_1}{\zeta} + \ldots + \frac{d_{n+1}}{\zeta^{n+1}}, \quad d_i \in \mathbb{C}^{2q \times 2q},\ i = 0, 1, \ldots, n+1. \tag{2.4}$$

From conditions (2.1), (2.2) we conclude that $\mathcal{B}(\zeta)$ is determined up to a J-unitary multiplier u ($uJu^* = J$). Further we shall assume that the following normalization condition is fulfilled: $\mathcal{B}(1) = I$.

It follows from conditions (2.1), (2.2) that $\operatorname{rank} d_{n+1} \leq q$ (see, e.g., [10]). If $\operatorname{rank} d_{n+1} = q$ then $\mathcal{B}(\zeta)$ is called the J-elementary multiple factor of the full rank.

Theorem 2.1. (On a parametrization [1]) *Let $\mathcal{B}(\zeta)$ be a matrix-valued function of the form (2.4). It is a J-elementary multiple factor of the full rank if and only if*

$$\mathcal{B}(\zeta) = I + \frac{1-\zeta}{\zeta} J \begin{bmatrix} \Lambda_{q,n}(1) & 0 \\ 0 & \Lambda_{q,n}(1) \end{bmatrix} H \begin{bmatrix} \Lambda^*_{q,n}(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda^*_{q,n}(\frac{1}{\zeta}) \end{bmatrix},$$

where

$$\Lambda_{q,n}(\zeta) = [I_q, \zeta I_q, \ldots, \zeta^n I_q], \tag{2.5}$$

$$H = \begin{bmatrix} C^* \\ I \end{bmatrix} (C + C^*)^{-1} [C, I]. \tag{2.6}$$

Here C is the matrix satisfying the conditions

$$C + C^* > 0, \tag{2.7}$$

$$CV_{q,n} = V_{q,n} C \tag{2.8}$$

and $V_{q,n}$ is the square matrix of the $(n+1)q$-th order having the form

$$V_{q,n} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ I_q & 0 & \ldots & \ldots & 0 \\ 0 & I_q & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & I_q & 0 \end{bmatrix}. \tag{2.9}$$

Moreover, the matrix C is defined by $\mathcal{B}(\zeta)$ uniquely.

Remark 2.1. One can easily see that (2.8) holds if and only if the matrix C is a lower-triangle matrix of the form (1.2). This fact determines the connection of the J-elementary multiple factor of the full rank with the interpolation Carathéodory problem.

Let Q be an arbitrary hermitian matrix of the p-th order. Consider the canonical basis $e_k = (\delta_{ik})_{i=1}^p$, $k = 1, 2, \ldots, p$ in $\mathbb{C}^p$, where δ_{ik} is the Kronecker symbol, i.e., $\delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$. We identify the matrix Q and the operator in $\mathbb{C}^p$ defined on the basis $(e_k)_{k=1}^p$ by this matrix (we denote it also by Q). Let Δ_Q and $\operatorname{Ker} Q$ are the range and the kernel of the operator Q respectively. It is well-known (see, e.g., [5]) that the operator (matrix) of the form

$$Q^{[-1]} f = \begin{cases} (Q\,|_{\Delta_Q})^{-1} f, & \text{if } f \in \Delta_Q \\ 0, & \text{if } f \in \operatorname{Ker} Q \end{cases} \tag{2.10}$$

is called the Moore–Penrose inversion of the operator (matrix) Q.

Theorem 2.2. *1) Let $\mathcal{B}(\zeta)$ be a J-elementary multiple factor of the form (2.4). There exist unique matrices P, $C \in \mathbb{C}^{q\times q}$ such that*

$$P^* = P, \quad P^2 = P, \tag{2.11}$$

$$V_{q,n}^* P = P V_{q,n}^* P, \tag{2.12}$$

$$C V_{q,n} = P V_{q,n} C, \tag{2.13}$$

$$C = PC, \tag{2.14}$$

$$PC^* + CP \geq 0, \quad \operatorname{rank}(PC^* + CP) = \operatorname{rank} P. \tag{2.15}$$

Moreover, $\mathcal{B}(\zeta)$ can be represented in the following form:

$$\mathcal{B}(\zeta) = I + \frac{1-\zeta}{\zeta} J \begin{bmatrix} \Lambda_{q,n}(1) & 0 \\ 0 & \Lambda_{q,n}(1) \end{bmatrix} H \begin{bmatrix} \Lambda_{q,n}^*(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda_{q,n}^*(\frac{1}{\zeta}) \end{bmatrix}, \tag{2.16}$$

where

$$H = \begin{bmatrix} C^* \\ P \end{bmatrix} (PC^* + CP)^{[-1]} [C, P], \tag{2.17}$$

$\Lambda_{q,n}(\zeta)$ and $V_{q,n}$ are matrices of the form (2.5) and (2.9) respectively.

2) Let conditions (2.11)–(2.15) be satisfied and let $\mathcal{B}(\zeta)$ be a matrix-valued function of the form (2.16). Then $\mathcal{B}(\zeta)$ is a J-elementary multiple factor. Moreover,

$$\mathcal{B}^*(\zeta) J \mathcal{B}(\zeta) - J$$

$$= \frac{1 - |\zeta|^2}{|\zeta|^2} \begin{bmatrix} \Lambda_{q,n}(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda_{q,n}(\frac{1}{\zeta}) \end{bmatrix} H \begin{bmatrix} \Lambda_{q,n}^*(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda_{q,n}^*(\frac{1}{\zeta}) \end{bmatrix}. \tag{2.18}$$

R e m a r k 2.2. Conditions (2.11) mean that P is an orthoprojector. In addition, (2.12) implies that orthoprojectors projecting in $\mathbb{C}^{(n+1)q}$ on invariant with respect to operator $V_{q,n}^*$ subspaces are admissible. From (2.15) we conclude that the case of the full rank (see Theorem 2.1) is characterized by the condition $P = I_{(n+1)q}$. In fact equalities (2.11), (2.12) and (2.14) are automatically fulfilled and relations (2.13), (2.15) transfer into relations (2.7), (2.8) in this case.

P r o o f o f T h e o r e m 2.2. Let $\mathcal{B}(\zeta)$ be a J-elementary multiple factor of the form (2.4) satisfying the normalization condition $\mathcal{B}(1) = I$. This condition allows us to represent $\mathcal{B}(\zeta)$ in the form

$$\mathcal{B}(\zeta) = I + \frac{1-\zeta}{\zeta} \Lambda_{2q,n}(1) D \Lambda_{2q,n}^*(\frac{1}{\zeta}),$$

where

$$D = \begin{bmatrix} d_1 & d_2 & \dots & d_{n+1} \\ d_2 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ d_{n+1} & 0 & \dots & 0 \end{bmatrix},$$

and $\Lambda_{2q,n}(\zeta)$ has the form analogical to (2.5). As we know ([11]), $\mathcal{B}(\zeta)$ satisfies the BMI of splitting off

$$\begin{bmatrix} D\widetilde{J}D^* & \frac{1}{\zeta}D\Lambda^*_{2q,n}(\frac{1}{\zeta}) \\ * & \frac{\mathcal{B}^*(\zeta)J\mathcal{B}(\zeta)-J}{1-|\zeta|^2} \end{bmatrix} \geq 0, \quad \zeta \in \mathbb{D}, \tag{2.19}$$

and also the dual inequality

$$\begin{bmatrix} D^*\widetilde{J}D & \frac{1}{\zeta}D^*\Lambda^*_{2q,n}(\frac{1}{\zeta}) \\ * & \frac{\mathcal{B}(\zeta)J\mathcal{B}^*(\zeta)-J}{1-|\zeta|^2} \end{bmatrix} \geq 0, \quad \zeta \in \mathbb{D}\backslash\{0\}, \tag{2.20}$$

where

$$\widetilde{J} = \begin{bmatrix} J & 0 & \dots & 0 \\ 0 & J & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & J \end{bmatrix},$$

and J has the form (2.3). Solving inequalities (2.19) or (2.20), we obtain the factor coinciding with $\mathcal{B}(\zeta)$.

Due to [13, 14] let us simplify these inequalities before to solve them. Introduce unitary matrix S of the form

$$S = \begin{bmatrix} I_q & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & I_q & 0 & \dots & 0 \\ 0 & I_q & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & I_q & \dots & 0 \\ 0 & 0 & I_q & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_q & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & I_q \end{bmatrix},$$

such that

$$J_1 = S^*\widetilde{J}S = \begin{bmatrix} 0 & I_{(n+1)q} \\ I_{(n+1)q} & 0 \end{bmatrix}. \tag{2.21}$$

Let $\Gamma = S^*DS$. Then $\Gamma J_1\Gamma^* = S^*D\widetilde{J}D^*S$. Taking into account inequality (2.19), we have $\Gamma J_1\Gamma^* \geq 0$. Denote $\Gamma J_1\Gamma^* = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$, where $a, b, c \in \mathbb{C}^{(n+1)q\times(n+1)q}$. By analogy with [13, p. 214] we obtain the following representation:

$$\Gamma J_1\Gamma^* = \begin{bmatrix} I_{(n+1)q} & X_0^* \\ 0 & I_{(n+1)q} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{C} \end{bmatrix} \begin{bmatrix} I_{(n+1)q} & 0 \\ X_0 & I_{(n+1)q} \end{bmatrix}, \tag{2.22}$$

where X_0 is a solution of the equation $cX_0 = b^*$. Then

$$\operatorname{rank} D = \operatorname{rank} \Gamma = \operatorname{rank} \widetilde{C}, \quad \widetilde{C} \geq 0. \tag{2.23}$$

In accordance to decomposition (2.21) of matrix J_1 let us decompose matrix Γ into blocks

$$\Gamma = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}.$$

Then from (2.22) we obtain

$$WZ^* + ZW^* = \widetilde{C}. \tag{2.24}$$

This equality implies

$$\operatorname{rank} WZ^* = \operatorname{rank} ZW^* = \operatorname{rank} Z = \operatorname{rank} W = \operatorname{rank} \widetilde{C}. \tag{2.25}$$

It follows from (2.24) and (2.25) that

$$\Delta_W = \Delta_Z = \Delta_{\widetilde{C}}. \tag{2.26}$$

Let $P_{\mathcal{L}}$ be the orthoprojector in $\mathbb{C}^{(n+1)q\times(n+1)q}$ onto a subspace $\mathcal{L}$. Consider the nondegenerate operator $W_0 = W\,|_{\Delta_{W^*}}\colon \Delta_{W^*} \to \Delta_W$ and a nondegenerate transformation $W_1 : \operatorname{Ker} W \to \operatorname{Ker} W^*$. Put

$$Q = W_0^{-1}P_{\Delta_W} + W_1^{-1}P_{\operatorname{Ker} W^*}.$$

Obviously,

$$Q^{-1} = W_0P_{\Delta_{W^*}} + W_1P_{\operatorname{Ker} W}.$$

Now define the matrices C and P required in the conditions of Theorem 2.2 in the following way:

$$C = QZ, \quad P = P_{\Delta_{W^*}}.$$

Then equalities (2.11) are fulfilled because P is a orthoprojector.

In contrast to the Schur problem (see [14]) the matrix C corresponding the Carathéodory problem satisfies the following condition

$$\begin{aligned} PC^* + CP &= PZ^*Q^* + QZP = Q(Q^{-1}PZ^* + ZPQ^{*\,-1})Q^* \\ &= Q(WZ^* + ZW^*)Q^* = Q\widetilde{C}Q^*. \end{aligned}$$

Taking into account (2.23) and (2.26), we get relations (2.15) of Theorem 2.2. Condition (2.14) immediately follows from the definition of matrix C. Properties (2.12), (2.13) are established as in [14, p. 61].

To prove necessity of the conditions of Theorem 2.2 it remains to obtain (2.16). Let

$$R = \begin{bmatrix} I_{(n+1)q} & -YQ \\ 0 & Q \end{bmatrix}.$$

According to [14], we obtain that splitting off inequality (2.19) is equivalent to the inequality

$$\begin{bmatrix} RS^*D\widetilde{J}D^*SR^* & \frac{1}{\zeta}RS^*D\Lambda^*_{2q,n}(\frac{1}{\zeta}) \\ * & \frac{\mathcal{B}^*(\zeta)J\mathcal{B}(\zeta)-J}{1-|\zeta|^2} \end{bmatrix} \geq 0. \qquad (2.27)$$

We have $R\Gamma = \begin{bmatrix} 0 & 0 \\ C & P \end{bmatrix}$. It follows from here that

$$RS^*D\widetilde{J}D^*SR^* = R\Gamma J_1\Gamma^* R^* = \begin{bmatrix} 0 & 0 \\ 0 & PC^* + CP \end{bmatrix},$$

$$RS^*D\Lambda^*_{2q,n}(\frac{1}{\zeta}) = \begin{bmatrix} 0 & 0 \\ C & P \end{bmatrix} \begin{bmatrix} \Lambda^*_{q,n}(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda^*_{q,n}(\frac{1}{\zeta}) \end{bmatrix}.$$

Then (2.27) can be rewritten in the form

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & PC^* + CP \end{bmatrix} & \frac{1}{\zeta}\begin{bmatrix} 0 & 0 \\ C & P \end{bmatrix} \begin{bmatrix} \Lambda^*_{q,n}(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda^*_{q,n}(\frac{1}{\zeta}) \end{bmatrix} \\ * & \frac{\mathcal{B}^*(\zeta)J\mathcal{B}(\zeta)-J}{1-|\zeta|^2} \end{bmatrix} \geq 0. \qquad (2.28)$$

Put

$$X = \begin{bmatrix} 0 & 0 \\ (PC^* + CP)^{[-1]}C & (PC^* + CP)^{[-1]}P \end{bmatrix}.$$

It is evident that X is a solution of

$$\begin{bmatrix} 0 & 0 \\ 0 & PC^* + CP \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ C & P \end{bmatrix}.$$

Now, using V.P. Potapov's approach, we solve by the standard method inequality (2.28) (see, e.g., [13]) and obtain representation (2.16). The necessity is proved.

Let $\mathcal{B}(\zeta)$ is of the form (2.16) and conditions (2.11)–(2.15) be satisfied. It follows from (2.11)–(2.14) that

$$H \begin{bmatrix} \Lambda^*_{q,n}(1) & 0 \\ 0 & \Lambda^*_{q,n}(1) \end{bmatrix} J \begin{bmatrix} \Lambda_{q,n}(1) & 0 \\ 0 & \Lambda_{q,n}(1) \end{bmatrix} H$$

$$= H \begin{bmatrix} D_{q,n} & 0 \\ 0 & D_{q,n} \end{bmatrix} + \begin{bmatrix} D^*_{q,n} & 0 \\ 0 & D^*_{q,n} \end{bmatrix} H - H,$$

where $D_{q,n} = I_q + V_{q,n} + V^2_{q,n} + \ldots + V^n_{q,n}$. Hence (2.18) is true. Relations (2.15) and (2.17) implies that $H \geq 0$. With regard to (2.18) we obtain that $\mathcal{B}(\zeta)$ is a J-elementary multiple factor. Using again (2.18), we conclude that the matrix H is determined uniquely by $\mathcal{B}(\zeta)$. Therefore C and P are also determined uniquely by $\mathcal{B}(\zeta)$. Theorem 2.2 is proved.

Let $\widehat{j} = \begin{bmatrix} -I_q & 0 \\ 0 & I_q \end{bmatrix}$. Define

$$\widehat{\mathcal{B}}(\zeta) = \widehat{j}\mathcal{B}^* \left(\frac{1}{\bar{\zeta}}\right) \widehat{j} = \widehat{j} \left(d^*_0 + d^*_1\zeta + \ldots + d^*_{n+1}\zeta^{n+1}\right) \widehat{j} = \widehat{d}_0 + \widehat{d}_1\zeta + \ldots + \widehat{d}_{n+1}\zeta^{n+1}. \tag{2.29}$$

Since $\mathcal{B}(\zeta)$ is a J-elementary multiple factor with the pole of multiplicity $n+1$ at the point $\zeta = 0$, then $\widehat{\mathcal{B}}(\zeta)$ is a J-elementary multiple factor with the pole of multiplicity $n+1$ at the point $\zeta = \infty$. In terms of the J-elementary multiple factor $\widehat{\mathcal{B}}(\zeta)$ Theorem 2.2 can be reformulated in the following way.

Theorem 2.3. *1) Let $\widehat{\mathcal{B}}(\zeta)$ be a J-elementary multiple factor of the form (2.29). There exist unique matrices P, $C \in \mathbb{C}^{q\times q}$ satisfying conditions (2.11)–(2.15). Moreover, $\widehat{\mathcal{B}}(\zeta)$ can be represented in the following form:*

$$\widehat{\mathcal{B}}(\zeta) = I + (1-\zeta)\widehat{j} \begin{bmatrix} \Lambda_{q,n}(\zeta) & 0 \\ 0 & \Lambda_{q,n}(\zeta) \end{bmatrix} H \begin{bmatrix} \Lambda^*_{q,n}(1) & 0 \\ 0 & \Lambda^*_{q,n}(1) \end{bmatrix} \widehat{j}J, \tag{2.30}$$

where H has the form (2.17).

2) Let conditions (2.11)–(2.15) be satisfied and let $\widehat{\mathcal{B}}(\zeta)$ be the matrix-valued function of the form (2.30). Then $\widehat{\mathcal{B}}(\zeta)$ is a J-elementary multiple factor. Moreover,

$$\widehat{\mathcal{B}}(\zeta)J\widehat{\mathcal{B}}^*(\zeta) - J$$

$$= (1- \mid \zeta \mid^2)\widehat{j} \begin{bmatrix} \Lambda_{q,n}(\zeta) & 0 \\ 0 & \Lambda_{q,n}(\zeta) \end{bmatrix} H \begin{bmatrix} \Lambda^*_{q,n}(\zeta) & 0 \\ 0 & \Lambda^*_{q,n}(\zeta) \end{bmatrix} \widehat{j}, \tag{2.31}$$

where $\Lambda_{q,n}(\zeta)$ has the form (2.5).

3. Solving of the basic matricial inequalities

Let E be a unitary space of dimension q and $[E]$ be the set of linear operators acting on E. Denote by $\mathcal{C}[E]$ the class of operator-valued functions $\mathcal{F}(\zeta)$ analytical in $\mathbb{D}$, such that for all $\zeta \in \mathbb{D}$ we have $\mathcal{F}(\zeta) \in [E]$ and $\operatorname{Re}\mathcal{F}(\zeta) = \frac{1}{2}(\mathcal{F}(\zeta) + \mathcal{F}^*(\zeta)) \geq 0$. Let an orthonormal basis in E be fixed. We identify matrix-valued functions $\mathcal{F}(\zeta) \in \mathcal{C}_q$ of the form (1.1) and corresponding to them operator-valued functions from $\mathcal{C}[E]$.

Taking into account the block structure of the matrix C_n (see (1.2)), we conclude that the block matrix A_n of the form (1.3) can be considered as an operator acting in the space

$$E^{(n)} = \underbrace{E \oplus E \oplus \ldots \oplus E}_{n+1}.$$

We embed $E^{(k-1)}$ into $E^{(k)}$ in the following way $E^{(k)} = E^{(k-1)} \oplus E$, $k = 1, 2, \ldots, n$, $E^{(0)} = E$.

Subspace of the type $\mathcal{K}$ introduced in [12] plays an important role when we solve the degenerate Carathéodory problem. In the case of the Carathéodory problem this subspace is defined in the following way.

Definition 3.1. *A subspace $L \subset E^{(n)}$ is said to be a subspace of the type $\mathcal{K}$ if:*
1) L is the complement to the kernel of A_n, i.e., $L \dot{+} \operatorname{Ker} A_n = E^{(n)}$;

2) L is an invariant with respect to $V^*_{q,n}$, where $V_{q,n}$ is defined by equality (2.9).

Note that in the case of the degenerate Carathéodory problem the existence of a subspace of the type $\mathcal{K}$ for the matrix A_n of the form (1.3) is proved in the same way as in the case of the degenerate matricial Schur problem [12].

Let L be an arbitrary subspace of the type $\mathcal{K}$, let $P = P_L$ be the orthoprojector onto L and $C = PC_n$, where C_n has the form (1.2). With regard to definition 3.1, taking into account properties of the orthoprojector P and the equality $PC^* + CP = PA_nP$, we obtain, that conditions (2.11)–(2.15) are satisfied for the matrices P and C.

The operator $\widetilde{A}_n = (PC^* + CP)|_L = PA_nP|_L : \ L \to L$ is a nondegenerate operator. The orthogonal decomposition $E^{(n)} = L \oplus L^{\perp}$ allows us to consider the block representation

$$A_n = \begin{bmatrix} \widetilde{A}_n & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ X^* & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \tag{3.1}$$

where X is a solution of the equation $\widetilde{A}_n X = B$. Using this decomposition $E^{(n)}$, we conclude that the operator $PC^* + CP$ has the form $\begin{bmatrix} \widetilde{A}_n & 0 \\ 0 & 0 \end{bmatrix}$. Then

$(PC^*+CP)^{[-1]}$ has the following block representation $\begin{bmatrix} \widetilde{A}_n^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. It follows from (3.1) that $\operatorname{Ker} A_n = \Delta \begin{bmatrix} -X \\ I \end{bmatrix}$. Hence $\operatorname{Ker}[-X^*, I] = \Delta_{A_n}$.

Now let us solve the matricial inequalities (1.4) and (1.4′). Rewritten them in the form

$$\left[\begin{array}{c|c} A_n & B(\zeta,n) \\ \hline * & \frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\zeta\bar{\zeta}} \end{array}\right] \geq 0, \quad \zeta \in \mathbb{D} \tag{3.2}$$

and

$$\left[\begin{array}{c|c} A_n & \widetilde{B}(\zeta,n) \\ \hline * & \frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\zeta\bar{\zeta}} \end{array}\right] \geq 0, \quad \zeta \in \mathbb{D}\backslash\{0\} \tag{3.2′}$$

respectively. Here

$$B(\zeta,n) = \Lambda^*_{q,n}(\zeta)F^*(\zeta) + C_n\Lambda^*_{q,n}(\zeta),$$

$$\widetilde{B}(\zeta,n) = \frac{1}{\zeta}\Lambda^*_{q,n}\left(\frac{1}{\bar{\zeta}}\right)F(\zeta) - \frac{1}{\zeta}C_n\Lambda^*_{q,n}\left(\frac{1}{\bar{\zeta}}\right).$$

With regard to [12, p. 48, 49] we can show that inequality (3.2) holds if and only if

$$[-X^*, I]B(\zeta,n) = 0, \quad \zeta \in \mathbb{D}, \tag{3.3}$$

$$\frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\mid\zeta\mid^2} - B^*(\zeta,n)\left(PC^*+CP\right)^{[-1]}B(\zeta,n) \geq 0, \quad \zeta \in \mathbb{D}, \tag{3.4}$$

and that inequality (3.2′) holds if and only if

$$[-X^*, I]\widetilde{B}(\zeta,n) = 0, \quad \zeta \in \mathbb{D}\backslash\{0\}, \tag{3.3′}$$

$$\frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\mid\zeta\mid^2} - \widetilde{B}^*(\zeta,n)\left(PC^*+CP\right)^{[-1]}\widetilde{B}(\zeta,n) \geq 0, \quad \zeta \in \mathbb{D}\backslash\{0\}. \tag{3.4′}$$

Consider (3.3), (3.4). Inequality (3.4) may be solved as in the nondegenerate case (see [2, 12]). Since

$$\frac{\mathcal{F}(\zeta)+\mathcal{F}^*(\zeta)}{1-\mid\zeta\mid^2} = [\mathcal{F}(\zeta), I]\frac{J}{1-\mid\zeta\mid^2}\begin{bmatrix} \mathcal{F}^*(\zeta) \\ I \end{bmatrix},$$

$$B(\zeta,n) = [I, C_n]\begin{bmatrix} \Lambda^*_{q,n}(\zeta) & 0 \\ 0 & \Lambda^*_{q,n}(\zeta) \end{bmatrix}\begin{bmatrix} \mathcal{F}^*(\zeta) \\ I \end{bmatrix},$$

then (3.4) can be represented in the form

$$[\mathcal{F}(\zeta), I]\left\{\frac{J}{1-\mid\zeta\mid^2}\right.$$

$$-J\begin{bmatrix} \Lambda_{q,n}(\zeta) & 0 \\ 0 & \Lambda_{q,n}(\zeta) \end{bmatrix} H \begin{bmatrix} \Lambda^*_{q,n}(\zeta) & 0 \\ 0 & \Lambda^*_{q,n}(\zeta) \end{bmatrix} J \Big\} \begin{bmatrix} \mathcal{F}^*(\zeta) \\ I \end{bmatrix} \geq 0, \quad \zeta \in \mathbb{D}, \tag{3.5}$$

where

$$H = \begin{bmatrix} C^* \\ P \end{bmatrix} (PC^* + CP)^{[-1]} [C, P].$$

Since for the matrices P and C conditions (2.11)–(2.15) are fulfilled, then Theorem 2.2 implies that

$$\mathcal{B}(\zeta) = I + \frac{1-\zeta}{\zeta} J \begin{bmatrix} \Lambda_{q,n}(1) & 0 \\ 0 & \Lambda_{q,n}(1) \end{bmatrix} H \begin{bmatrix} \Lambda^*_{q,n}(\frac{1}{\zeta}) & 0 \\ 0 & \Lambda^*_{q,n}(\frac{1}{\zeta}) \end{bmatrix}$$

is a J-elementary multiple factor. In addition, equality (2.18) take place. We have

$$\mathcal{B}^{-1}(\zeta) = J\mathcal{B}^*(\frac{1}{\bar{\zeta}})J, \quad \zeta \neq 0,$$

because $\mathcal{B}(\zeta)$ is a J-unitary on the boundary of the unit disk. It follows from (2.18) we get

$$\frac{J - \mathcal{B}^{-1}(\zeta) J \mathcal{B}^{*\,-1}(\zeta)}{1 - \mid \zeta \mid^2} = J \begin{bmatrix} \Lambda_{q,n}(\zeta) & 0 \\ 0 & \Lambda_{q,n}(\zeta) \end{bmatrix} H \begin{bmatrix} \Lambda^*_{q,n}(\zeta) & 0 \\ 0 & \Lambda^*_{q,n}(\zeta) \end{bmatrix} J.$$

By substitution the last expression into (3.5), we obtain

$$[\mathcal{F}(\zeta), I]\mathcal{B}^{-1}(\zeta) J \mathcal{B}^{*\,-1}(\zeta) \begin{bmatrix} \mathcal{F}^*(\zeta) \\ I \end{bmatrix} \geq 0, \quad \zeta \in \mathbb{D}. \tag{3.6}$$

Let us define the pair of the matrix-functions $[u(\zeta), v(\zeta)]$ in the following way:

$$[u(\zeta), v(\zeta)] = [\mathcal{F}(\zeta), I]\mathcal{B}^{-1}(\zeta), \quad \zeta \in \mathbb{D}. \tag{3.7}$$

In a way analogous to that used in the nondegenerate case (see [2, 12]) we prove that the pair $[u(\zeta), v(\zeta)]$ satisfies the following conditions:

(1) the matrix-functions $u(\zeta)$, $v(\zeta)$ are analytical in $\mathbb{D}$;

(2) the pair $[u(\zeta), v(\zeta)]$ is a J-*nonexpansive* pair in $\mathbb{D}$, i.e., for each $\zeta \in \mathbb{D}$

$$[u(\zeta), v(\zeta)] J \begin{bmatrix} u^*(\zeta) \\ v^*(\zeta) \end{bmatrix} \geq 0; \tag{3.8}$$

(3) for all $\zeta \in \mathbb{D}$ the following inequality holds

$$[u(\zeta), v(\zeta)] \begin{bmatrix} u^*(\zeta) \\ v^*(\zeta) \end{bmatrix} > 0, \tag{3.9}$$

i.e., the pair $[u(\zeta), v(\zeta)]$ is *nonsingular* in $\mathbb{D}$.

Let us decompose $\mathcal{B}(\zeta)$ into the blocks according to the block representation (2.3) of matrix J

$$\mathcal{B}(\zeta) = \begin{bmatrix} a(\zeta) & b(\zeta) \\ c(\zeta) & d(\zeta) \end{bmatrix}.$$

Then using (3.7), we obtain

$$\mathcal{F}(\zeta) = u(\zeta)a(\zeta) + v(\zeta)c(\zeta), \quad I = u(\zeta)b(\zeta) + v(\zeta)d(\zeta).$$

Hence,

$$\mathcal{F}(\zeta) = [u(\zeta)b(\zeta) + v(\zeta)d(\zeta)]^{-1}[u(\zeta)a(\zeta) + v(\zeta)c(\zeta)]. \tag{3.10}$$

The converse is also true. Let $[u(\zeta), v(\zeta)]$ be an arbitrary nonsingular J-nonexpansive pair of analytic matrix-functions in $\mathbb{D}$. Then the matrix $u(\zeta)b(\zeta) + v(\zeta)d(\zeta)$ is invertible in $\mathbb{D}$ and $\mathcal{F}(\zeta)$ satisfies condition (3.6). Hence it also satisfies inequality (3.4). Thus, the following lemma is proved.

Lemma 3.1. *The general solution $\mathcal{F}(\zeta)$ of inequality (3.4) is represented in the form of the linear fractional transformation (3.10), where the parameter $[u(\zeta), v(\zeta)]$ is a nonsingular J-nonexpansive pair of analytical matrix-functions in $\mathbb{D}$. The J-elementary multiple factor*

$$\mathcal{B}(\zeta) = \begin{bmatrix} a(\zeta) & b(\zeta) \\ c(\zeta) & d(\zeta) \end{bmatrix}$$

of the form (2.16) with the pole of multiplicity $n+1$ at the point $\zeta = 0$ is the matrix of coefficients of the linear fractional transformation. In (2.16) P is the orthoprojector onto one of subspaces of the type $\mathcal{K}$ and $C = PC_n$.

Recall, that nonsingularity (J-nonexpansibility respectively) in $\mathbb{D}$ of the pair of matrix-functions $\begin{bmatrix} \widehat{u}(\zeta) \\ \widehat{v}(\zeta) \end{bmatrix}$ means nonsingularity (J-nonexpansibility respectively) in $\mathbb{D}$ of the pair $[\widehat{u}^*(\zeta), \widehat{v}^*(\zeta)]$.

Analogously to Lemma 3.1 due to Theorem 2.3 we obtain

Lemma 3.2. *The general solution $\mathcal{F}(\zeta)$ of inequality* (3.4′) *is represented in the form of the linear fractional transformation:*

$$\mathcal{F}(\zeta) = [\widehat{a}(\zeta)\widehat{u}(\zeta) + \widehat{b}(\zeta)\widehat{v}(\zeta)][\widehat{c}(\zeta)\widehat{u}(\zeta) + \widehat{d}(\zeta)\widehat{v}(\zeta)]^{-1}, \tag{3.10′}$$

where the parameter $\begin{bmatrix} \widehat{u}(\zeta) \\ \widehat{v}(\zeta) \end{bmatrix}$ is a nonsingular J-nonexpansive pair of analytical matrix-functions in $\mathbb{D}$. The J-elementary multiple factor

$$\widehat{\mathcal{B}}(\zeta) = \begin{bmatrix} \widehat{a}(\zeta) & \widehat{b}(\zeta) \\ \widehat{c}(\zeta) & \widehat{d}(\zeta) \end{bmatrix}$$

of the form (2.30) with the pole of multiplicity $n+1$ at the point $\zeta=\infty$ is a the matrix of coefficients of the linear fractional transformation. In (2.30) P is the orthoprojector onto one of subspaces of the type $\mathcal{K}$ and $C = PC_n$.

Now let us choose solutions satisfying condition (3.3) ((3.3′) respectively) in the set of solutions of the form (3.10) ((3.10′) respectively) of inequality (3.4) ((3.4′) respectively) .

Let P_0 be an orthoprojector in $E^{(n)}$ onto $\operatorname{Ker} A_n$. Repeating the reasonings of the paper [12, p. 51, 52], we get that condition (3.3) is equivalent to the condition

$$P_0[I, C_n] \begin{bmatrix} \Lambda^*_{q,n}(1) & 0 \\ 0 & \Lambda^*_{q,n}(1) \end{bmatrix} \begin{bmatrix} u^*(\zeta) \\ v^*(\zeta) \end{bmatrix} = 0, \quad \zeta \in \mathbb{D}, \tag{3.11}$$

and (3.3′) is equivalent to the condition

$$P_0[I, -C_n] \begin{bmatrix} \Lambda^*_{q,n}(1) & 0 \\ 0 & \Lambda^*_{q,n}(1) \end{bmatrix} \begin{bmatrix} \widehat{u}(\zeta) \\ \widehat{v}(\zeta) \end{bmatrix} = 0, \quad \zeta \in \mathbb{D}, \tag{3.11′}$$

respectively. Equalities (3.11) and (3.11′) we can rewrite in the form

$$u(\zeta)\Lambda_{q,n}(1)P_0 - v(\zeta)\Lambda_{q,n}(1)C_nP_0 = 0, \quad \zeta \in \mathbb{D}, \tag{3.12}$$

$$\widehat{u}^*(\zeta)\Lambda_{q,n}(1)P_0 + \widehat{v}^*(\zeta)\Lambda_{q,n}(1)C_nP_0 = 0, \quad \zeta \in \mathbb{D}, \tag{3.12′}$$

respectively.

Note, that $\Lambda^*_{q,n}(1)\Lambda_{q,n}(1) = F_{q,n} + F^*_{q,n} + I$, where

$$F_{q,n} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ I_q & 0 & \dots & \dots & 0 \\ I_q & I_q & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ I_q & I_q & \dots & I_q & 0 \end{bmatrix}.$$

Taking into account $F_{q,n}C_n = C_nF_{q,n}$ and $A_n = C_n + C^*_n$, we obtain

$$\begin{aligned} &P_0C^*_n\Lambda^*_{q,n}(1)\Lambda_{q,n}(1)P_0 + P_0\Lambda^*_{q,n}(1)\Lambda_{q,n}(1)C_nP_0 \\ &= P_0\left(C^*_n(F_{q,n} + F^*_{q,n} + I) + (F_{q,n} + F^*_{q,n} + I)C_n\right)P_0 \\ &= P_0(A_nF_{q,n} + F^*_{q,n}A_n + A_n)P_0 = 0, \end{aligned}$$

i.e.,

$$P_0C^*_n\Lambda^*_{q,n}(1)\Lambda_{q,n}(1)P_0 + P_0\Lambda^*_{q,n}(1)\Lambda_{q,n}(1)C_nP_0 = 0. \tag{3.13}$$

Consider the operators $r = -P_0 C_n^* \Lambda_{q,n}^*(1)$ and $s = P_0 \Lambda_{q,n}^*(1)$ acting in E on $\operatorname{Ker} A_n$. Then (3.13) can be rewritten in the form

$$[r, s] J \begin{bmatrix} r^* \\ s^* \end{bmatrix} = 0, \tag{3.14}$$

where J has the form (2.3).

Let $\widetilde{J} = \begin{bmatrix} -I_q & 0 \\ 0 & I_q \end{bmatrix}$ and $\sigma = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_q & I_q \\ I_q & I_q \end{bmatrix}$. Note that $\sigma = \sigma^* = \sigma^{-1}$ and $\sigma J \sigma^* = \widetilde{J}$. It follows from here that

$$[r, s] J \begin{bmatrix} r^* \\ s^* \end{bmatrix} = [r, s] \sigma^* \widetilde{J} \sigma \begin{bmatrix} r^* \\ s^* \end{bmatrix}$$

$$= \frac{1}{2} [-r + s, r + s] \widetilde{J} \begin{bmatrix} -r^* + s^* \\ r^* + s^* \end{bmatrix}$$

$$= \frac{1}{2} \left(-(-r + s)(-r^* + s^*) + (r + s)(r^* + s^*) \right) = 0,$$

i.e., condition (3.14) is equivalent to the equality

$$(-r + s)(-r^* + s^*) = (r + s)(r^* + s^*). \tag{3.15}$$

Let M_0 and N_0 be the ranges of the operators $(-r^* + s^*)$ and $r^* + s^*$ respectively. Relation (3.15) allows us to define the unitary operator $U : M_0 \to N_0$, such that

$$U(-r^* + s^*) = r^* + s^*. \tag{3.16}$$

Now let us rewrite condition (3.12) for the pair $[u(\zeta), v(\zeta)]$ in terms of the operator U. Note that (3.12) is equivalent to the equality

$$[u(\zeta), v(\zeta)] J \begin{bmatrix} r^* \\ s^* \end{bmatrix} = [u(\zeta), v(\zeta)] \sigma^* \widetilde{J} \sigma \begin{bmatrix} r^* \\ s^* \end{bmatrix}$$

$$= \frac{1}{2} [-u(\zeta) + v(\zeta), u(\zeta) + v(\zeta)] \widetilde{J} \begin{bmatrix} -r^* + s^* \\ r^* + s^* \end{bmatrix}$$

$$= \frac{1}{2} \left(-(-u(\zeta) + v(\zeta))(-r^* + s^*) + (u(\zeta) + v(\zeta))(r^* + s^*) \right) = 0, \quad \zeta \in \mathbb{D}.$$

Taking into account (3.16), we can rewrite this condition in the form

$$\left(-(-u(\zeta) + v(\zeta)) + (u(\zeta) + v(\zeta)) U \right) (-r^* + s^*) = 0, \quad \zeta \in \mathbb{D}. \tag{3.17}$$

From (3.8), (3.9) we obtain

$$(u(\zeta) + v(\zeta))(u^*(\zeta) + v^*(\zeta))$$

$$= (u(\zeta)v^*(\zeta) + v(\zeta)u^*(\zeta)) + (u(\zeta)u^*(\zeta) + v(\zeta)v^*(\zeta)) > 0, \quad \zeta \in \mathbb{D}.$$

Hence for all $\zeta \in \mathbb{D}$ the matrix $u(\zeta) + v(\zeta)$ is invertible. Then (3.17) is equivalent to the condition

$$(v(\zeta) + u(\zeta))^{-1}(v(\zeta) - u(\zeta))|_{M_0} = U, \quad \zeta \in \mathbb{D}. \tag{3.18}$$

Hence (3.12) is also equivalent to this condition. Analogously, we obtain that (3.12′) is equivalent to the equality

$$(\widehat{v}^*(\zeta) + \widehat{u}^*(\zeta))^{-1}(\widehat{v}^*(\zeta) - \widehat{u}^*(\zeta))|_{N_0} = U^*, \quad \zeta \in \mathbb{D}. \tag{3.18′}$$

Thus, the following statements are proved.

Theorem 3.1. *Let $\{c_k\}_{k=0}^n \subset \mathbb{C}^{q\times q}$, the matrix C_n have the form (1.2) and let for the matrix $A_n = C_n + C_n^*$ condition (1.3) hold. Then the general solution $\mathcal{F}(\zeta)$ of basic matricial inequality (1.4) is represented in the form of the linear fractional transformation:*

$$F(\zeta) = [u(\zeta)b(\zeta) + v(\zeta)d(\zeta)]^{-1}[u(\zeta)a(\zeta) + v(\zeta)c(\zeta)],$$

where the parameter $[u(\zeta), v(\zeta)]$ is a nonsingular J-nonexpansive pair of analytical matrix-functions $[u(\zeta), v(\zeta)]$ in $\mathbb{D}$ and satisfies the condition

$$(v(\zeta) + u(\zeta))^{-1}(v(\zeta) - u(\zeta))|_{M_0} = U, \quad \zeta \in \mathbb{D}.$$

Here U is determined by the problem data from the equality

$$U(\Lambda_{q,n}(1)C_nP_0 + \Lambda_{q,n}(1)P_0) = -\Lambda_{q,n}(1)C_nP_0 + \Lambda_{q,n}(1)P_0,$$

where P_0 is the orthoprojector onto $\operatorname{Ker} A_n$, $\Lambda_{q,n}(\zeta)$ has the form (2.5). Moreover, U is a unitary mapping of M_0 to N_0, where M_0 is the range of the operator $\Lambda_{q,n}(1)C_nP_0 + \Lambda_{q,n}(1)P_0$ and N_0 is the range of the operator $(-\Lambda_{q,n}(1)C_nP_0 + \Lambda_{q,n}(1)P_0)$. The J-elementary multiple factor

$$\mathcal{B}(\zeta) = \begin{bmatrix} a(\zeta) & b(\zeta) \\ c(\zeta) & d(\zeta) \end{bmatrix}$$

of the form (2.16) with the pole of multiplicity $n+1$ at the point $\zeta = 0$ is the matrix of coefficients of the linear fractional transformation. In (2.16) P is the orthoprojector onto one of subspaces of the type $\mathcal{K}$ and $C = PC_n$.

Theorem 3.2. *Let $\{c_k\}_{k=0}^n \subset \mathbb{C}^{q\times q}$, the matrix C_n have the form (1.2) and let for the matrix $A_n = C_n + C_n^*$ condition (1.3) hold. Then the general solution $\mathcal{F}(\zeta)$ of dual matricial inequality* (1.4′) *is represented in the form of the linear fractional transformation:*

$$\mathcal{F}(\zeta) = [\widehat{a}(\zeta)\widehat{u}(\zeta) + \widehat{b}(\zeta)\widehat{v}(\zeta)][\widehat{c}(\zeta)\widehat{u}(\zeta) + \widehat{d}(\zeta)\widehat{v}(\zeta)]^{-1},$$

where the parameter $\begin{bmatrix} \widehat{u}(\zeta) \\ \widehat{v}(\zeta) \end{bmatrix}$ *is a nonsingular J-nonexpansive pair of analytical matrix-functions in* $\mathbb{D}$ *and satisfies the condition*

$$(\widehat{v}^*(\zeta) + \widehat{u}^*(\zeta))^{-1}(\widehat{v}^*(\zeta) - \widehat{u}^*(\zeta))|_{N_0} = U^*, \quad \zeta \in \mathbb{D}.$$

Here the operator U is determined as in Theorem 3.1. The J-elementary multiple factor

$$\widehat{\mathcal{B}}(\zeta) = \begin{bmatrix} \widehat{a}(\zeta) & \widehat{b}(\zeta) \\ \widehat{c}(\zeta) & \widehat{d}(\zeta) \end{bmatrix}$$

of the form (2.30) with the pole of multiplicity $n+1$ *at the point* $\zeta = \infty$ *is the matrix of coefficients of the linear fractional transformation. In (2.30)* P *is the orthoprojector onto one of subspaces of the type* $\mathcal{K}$ *and* $C = PC_n$.

R e m a r k 3.1. Assume that there exists a point $\zeta_0 \in \mathbb{D}$ such that matrix $v(\zeta_0)$ is invertible. Then analyticity of $v(\zeta)$ in $\mathbb{D}$ implies invertibility of the matrix $v(\zeta)$ everywhere in $\mathbb{D}$ excepting, may be, some set G of isolated in $\mathbb{D}$ points.

Let $\omega(\zeta) = v^{-1}(\zeta)u(\zeta)$, $\zeta \in \mathbb{D}\backslash G$. From (3.8) it follows that

$$\operatorname{Re}\omega(\zeta) = \frac{1}{2}(v^{-1}(\zeta)u(\zeta) + u^*(\zeta)(v^{-1}(\zeta))^*)$$

$$= \frac{1}{2}v^{-1}(\zeta)(u(\zeta)v^*(\zeta) + v(\zeta)u^*(\zeta))(v^{-1}(\zeta))^* \geq 0, \quad \zeta \in \mathbb{D}\backslash G.$$

Therefore the matrix $I + \omega(\zeta)$ is invertible for all $\zeta \in \mathbb{D}\backslash G$ and the function

$$s(\zeta) = (I + \omega(\zeta))^{-1}(I - \omega(\zeta)) \tag{3.19}$$

is analytical in $\mathbb{D}\backslash G$ and satisfies to the condition (see, e.g., [5, point 1.3])

$$\|s(\zeta)\| \leq 1, \quad \zeta \in \mathbb{D}\backslash G. \tag{3.20}$$

Relation (3.19) is equivalent to

$$\omega(\zeta)(I + s(\zeta)) = I - s(\zeta), \quad \zeta \in \mathbb{D}\backslash G.$$

Hence the matrix $I + s(\zeta)$ is invertible for all $\zeta \in \mathbb{D}\backslash G$ and the representation

$$\omega(\zeta) = (I - s(\zeta))(I + s(\zeta))^{-1}, \quad \zeta \in \mathbb{D}\backslash G, \tag{3.21}$$

is valid.

Denote by $\mathcal{S}_q$ the set of matrix-valued functions $S(\zeta)$ analytical in $\mathbb{D}$ with values in $\mathbb{C}^{q\times q}$ and satisfying the inequality $\|S(\zeta)\| \leq 1$ for all $\zeta \in \mathbb{D}$. From (3.20) we conclude, that all points of set G are removable singular points for the matrix-function $s(\zeta)$. Extending the $s(\zeta)$ to the points of G by continuity, we obtain function $S(\zeta) \in \mathcal{S}_q$. From (3.21) it follows that $\Omega(\zeta) = (I - S(\zeta))(I + S(\zeta))^{-1}$ belongs to the class $\mathcal{C}_q$ and it is the extension of the matrix-function $\omega(\zeta)$, $\zeta \in \mathbb{D}\backslash G$, to $\mathbb{D}$.

Now (3.18) can be rewritten

$$(I + \Omega(\zeta))^{-1}(I - \Omega(\zeta))|_{M_0} = U, \quad \zeta \in \mathbb{D}. \tag{3.22}$$

Thus, if there exists a point $\zeta_0 \in \mathbb{D}$ such that the matrix $v(\zeta_0)$ is invertible, then the corresponding solution $\mathcal{F}(\zeta)$ of Carathéodory problem (see Theorem 3.1) is represented in the form of the linear fractional transformation $\mathcal{F}(\zeta) = [\Omega(\zeta)b(\zeta) + d(\zeta)]^{-1}[\Omega(\zeta)a(\zeta) + c(\zeta)]$ of the matrix-function $\Omega(\zeta) \in \mathcal{C}_q$, satisfying condition (3.22). Moreover, this condition holds if and only if $\Omega(\zeta)$ admits the representation

$$\Omega(\zeta) = (I - S(\zeta))(I + S(\zeta))^{-1},$$

where $S(\zeta) \in \mathcal{S}_q$ and $S(\zeta)|_{M_0} = U$, $\zeta \in \mathbb{D}$. Note that from invertibility of the matrix $I + S(\zeta)$, $\zeta \in \mathbb{D}$ it follows that -1 does not belong to the spectrum of the operator U.

Analogous remark can also be made in the case of Theorem 3.2.

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