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Order-Unit Spaces which are Banach Dual Spaces

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Spaces of selfadjoint elements of a C^*-algebra or a von Neumann algebra, and also JB- and JBW-algebras are examples of order-unit spaces. A von Neumann algebra and a JBW-algebra possess predual spaces, but, generally speaking, a JB-algebra and a C^*-algebra don't have this property. In this work, conditions are found for an order-unit space to possess a predual space. Moreover, a condition is obtained characterizing JBW-algebras among order-unit spaces having a predual space.

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1. Preliminaries

Let A be a real ordered linear space. We denote by A^+ the set of positive elements of A. An element $\mathbf{e} \in A^+$ is called *order unit* if for every $a \in A$ there exists a number $\lambda \in \mathbb{R}^+$ such, that $-\lambda\mathbf{e} \leq a \leq \lambda\mathbf{e}$. If the order is Archimedean then the mapping $a \to \|a\| = \inf\{\lambda > 0 : -\lambda\mathbf{e} \leq a \leq \lambda\mathbf{e}\}$ is *a norm.* If A is a Banach space with respect to this norm, we say that $(A, \mathbf{e})$ is *an order-unit space with the order unit* $\mathbf{e}$.

Let $(A, \mathbf{e})$ be an order-unit space. An element $\rho \in A^*$ is called *positive* if $\rho(a) \geq 0$ for all $a \in A^+$, in this case one writes $\rho \geq 0$. A positive linear functional is called *a state* if $\|\rho\| = 1$. This is equivalent to $\rho(\mathbf{e}) = 1$. We denote by $S(A)$ the set of all states on A and call $S(A)$ *the states space* of A. It is known that $S(A)$ is a $*$-weakly closed subset in A^*.

As we know, the pair (A, A^*) is a dual pair. Following the work by E. Alfsen, F. Shultz [1], we suppose that A and A^* are in spectral duality. In this case every element $a \in A$ has a spectral resolution with respect to *projective units.* We denote $\mathbf{P}$ and $\mathbf{U}$ a set of P-projections and projective units of A, respectively.

Generally speaking, spectral duality in [1] is defined between A and a subspace $V \subset A^*$. Further, if the opposite is not supposed, spectral duality (A, V) means the case $V \subset A^*$.

A P-projection R is called *central* if $R + R' = I$. Here R' is the quasi-complement of R. A projective unit $u = R\mathbf{e}$ is called *central* if R is a central P-projection.

An order-unit space $(A, \mathbf{e})$ is said to be *factor* if it contains no central projective units except 0 and $\mathbf{e}$.

A projective unit $u = R\mathbf{e}$ is called *Abelian* if $imR = R(A)$ is a vector lattice.

One says that an order-unit space A *has type* I if for any central P-projection R in A, the subspace imR contains an Abelian projective unit.

An element $u \in \mathbf{U}$ is called *an atom* if u is the minimal element of the lattice $\mathbf{U}$.

If A is a factor of type I and u is an atom, then there is a unique continuous linear functional $\widehat{u}$ on A corresponding to u. This functional is the extremal point in $S(A)$ with properties: $\langle u, \widehat{u} \rangle = 1$, $\|u\| = 1$. The P-projection R corresponding to u is of the form: $Ra = \langle a, \widehat{u} \rangle u$.

Spaces of selfadjoint elements of a C^*-algebra, a von Neumann algebra, and JB- and JBW-algebras are the examples of order-unit spaces.

Let K be a compact convex subset of a local convex Hausdorff space V. We denote by $A(K)$ the space of all continuous affine functions, and by $A^b(K)$ the space of all bounded affine functions on K. Then $A(K)$ and $A^b(K)$ are order-unit spaces. The role of unit plays the affine function identically equal to 1 on K.

It is known that a von Neumann algebra, a JBW-algebra and the space $A^b(K)$ possess predual spaces, but this is not true for JB-algebras, C^*-algebras and $A(K)$ [2].

A state ρ on A is called *normal* if $\rho(a_\nu) \to 0$ for any net $\{a_\nu\} \subset A$ monotonically decreasing to zero $(a_\nu \downarrow 0)$.

Theorem (F. Shultz [2, 3]). *JB-algebra $\mathbf{A}$ has a predual space, i.e. it is a JBW-algebra if and only if it has a separating space of normal states.*

As it turns out, a similar result is valid for order-unit spaces, too.This work is devoted to this result. Moreover, in the end of the paper, we prove a theorem characterizing JBW-algebras among order-unit spaces possessing a predual space.

2. Main Results

2.1. Existence of a Predual Space

We start by studying one example of an order-unit space from [4] and prove an analog of the Shultz theorem in this case. Spaces considered in [4] and called there generalized spin-factors are constructed by the following way.

Let X and Y be real Banach spaces in separating duality [5]. Then $A = \mathbb{R} \oplus X$ and $V = \mathbb{R} \oplus Y$ form a dual pair with respect to duality:

$$\langle a, \rho \rangle = \alpha\beta + \langle x, y \rangle,$$

for $a = \alpha + x \in A$ and $\rho = \beta + y \in V$, where $\langle x, y \rangle$ is the duality between X and Y.

The order and norm on A (on V) are defined as:

$$a = \alpha + x \geq 0 \overset{def}{\Leftrightarrow} \alpha \geq \|x\| \quad \left(\rho = \beta + y \geq 0 \overset{def}{\Leftrightarrow} \beta \geq \|y\|\right),$$

$$\|a\| = |\alpha| + \|x\| \quad \Bigg(\|\rho\| = \max\left(|\beta|, \|y\|\right)\Bigg).$$

Let A have a predual space. Then $X = Y^*$ and any functional $\rho \in V$ is normal.

Indeed, let $a_\nu \downarrow 0$, then $\alpha_\nu \downarrow 0$ and $\|x_\nu\| \to 0$ since $a_\nu = \alpha_\nu + x_\nu$. Let $\rho = \beta + y \in V$, then $|\rho(a_\nu)| = |\alpha_\nu\beta + \langle x_\nu, y \rangle| \leq \alpha_\nu|\beta| + \|x_\nu\| \cdot \|y\| \to 0$. Therefore $\rho(a_\nu) \to 0$. Since (A, V) is a dual pair, then V separates points of A. Hence, V is a separating space of normal functionals for A.

Conversely, let A have a separating space of normal functionals of V, i.e. $a_\nu \downarrow 0$ follows $\rho(a_\nu) \to 0$ for any $\rho \in V$ and there exists $\rho \in V$ for any $a \neq 0$ such, that $\rho(a) \neq 0$. Since $V \subset A^*$, then an arbitrary element $\rho \in V$ is of the form $\rho = \beta + y$, where $\beta \in \mathbb{R}$, $y \in Y \subset X^*$. Since A and V are a dual pair, then X and Y are a dual pair. As it is proved in [5, Th. 1, §3, III] $Y^* = X$. Hence, generalized spin-factors possess a predual space when they have a separating space of normal states.

Let us consider the general case. Let A be an order-unit space in spectral duality, and $S(A)$ the space of normal states on A. We denote $V = lin(S(A))$ the linear hull of the normal states space. It is obvious, that $V \subset A^*$. Let $J = V^0$ be the polar of V in A^{**}.

Theorem 1. *There exists a central P-projection R in A^{**} such, that $J = R'(A^{**})$, where R' is a quasicomlement of R and the mapping $a \mapsto Ra$ is an isomorphism of A onto $R(A^{**})$.*

P r o o f. Let H be an arbitrary P-projection in A. Then $H^*(V) \subset V$. Indeed, let $a_\alpha \uparrow a$ in A and $\rho \in S(A)$. Then $H^*\rho(x) = \rho(Hx)$ for all $x \in A$. Since the P-projection H is positive and normal, $\rho(Ha_\alpha) \to \rho(Ha)$. Therefore $H^*\rho \in V$. Now it follows that if H is a P-projection in A and $x \in J$, then $H^{**}(x)(\rho) = x(H^*\rho) = 0$. Hence, $H^{**}(J) \subset J$ for any P-projection H in $A \subset A^{**}$. This means that the set J is "invariant" with respect to $\mathbf{P}$. By virtue of continuity of P-projections, we conclude that J is invariant with respect to P-projections in A^{**}. Note that $A^{**} \cong A^b(S(A))$ [2] and therefore A^{**} is an order-unit space in spectral duality.

Before continuing the proof of Th. 1, we prove the following result.

Lemma. *Let J be a weakly closed subspace invariant with respect to P-projections in A. Then there is a central P-projection H in A such, that $J = H(A)$.*

P r o o f. We denote by h the order unit J. By the condition of Lemma, J is invariant with respect to P-projections in A, then $Rh \in J$ for any $R \in \mathbf{P}$. Since h is the unit in J then $Rh \leq h$. By Proposition 5.1 in [1], we conclude that R is compatible with h. Since R is arbitrary it follows that h is a central element. Thus there is a central P-projection H such, that $h = H\mathbf{e}$. Therefore $J = H(A)$. Lemma is proved.

Return to the proof of Th. 1. By lemma, there exists a central P-projection H such, that $J = H(A^{**})$.

Let $u = \mathbf{e} - h$. Then u is a central projective unit in A^{**}. Hence, R is homomorphism of A^{**} into itself where $R\mathbf{e} = u$. Since $id = R + H$, then the kernel of R is J. Further, since the space of normal states of A is separating we have that $A \cap J = A \cap V^0 = \{0\}$. Hence, R is a one-to-one mapping of A into $R(A^{**})$. Theorem 1 is proved.

Theorem 2. *Weakly $*$-continuous extensions of states from A onto A^{**} are normal.*

P r o o f. By Proposition 1.2.11 [2], A^{**} is monotone complete and order isomorphic to $A^b(S(A))$. It is known from Cor. 1.1.22 in [2] that an arbitrary state ρ on A can be uniquely extended to a state $\overline{\rho}$ on A^{**}. Let $\{a_\alpha\}$ be a bounded increasing net in A^{**} with the least upper bound a. Since $a_\alpha \uparrow a$ implies $a_\alpha|_{S(A)} \to a|_{S(A)}$ pointwise by virtue of $A^{**} \cong A^b(S(A))$, so we have $\overline{\rho}(a_\alpha) = a_\alpha(\rho) \to a(\rho) = \overline{\rho}(a)$. Hence, $\overline{\rho}$ is a normal state on A^{**}. Theorem 2 is proved.

Theorem 3. *If A has a predual space V ($V^* \cong A$), then elements of V are normal functionals on A.*

P r o o f. If $\rho \in V$, then it is obvious that $\rho \in A^*$, and its extension is a normal functional on A^{**} by Th. 2. Hence, ρ is also a normal functional on $R(A^{**})$, where R is a P-projection from Th. 1. Since $a \mapsto Ra$ is an isomorphism of A onto $R(A^{**})$ and $\rho(a) = \rho(Ra)$ for all $a \in A$, then ρ is normal on $A = R(A^{**})$. Theorem 3 is proved.

Theorem 4. *Let A be a monotone complete order-unit space in spectral duality. Then A has a predual space if and only if it has separating space of normal states. In this case the predual space is unique and coincides with a space of normal linear functionals on A.*

P r o o f. Let A have a separating space of normal states $S(A)$. Recall that $V = lin(S(A)$, $J = V^0$. By Theorem 1, there is a central P-projection R such, that $A \cong R(A^{**})$ and $J = R'(A^{**})$. In [6] G. Godefroy has proved the following fact: a Banach space E has a predual space if and only if there exists a closed linear subspace F in E^{**} such, that $i(E) \oplus F = E^{**}$ (Prop. 1 in [6]).

In our case, the role of the subspace F plays the subspace J. From this we conclude that A has a predual space.

Conversely, let A have a predual space, i.e. there exists a subspace $V \subset A^*$ such, that $A = V^*$. Then V separates the points of A, i.e. A and V are a dual pair. Further, by Th. 3 elements of V are normal functionals. Hence, A has a separating space of normal functionals.

Later, if $\rho \in V$, then $\rho \in A^*$ and it is normal on A^{**} by Th. 2. Since $a \mapsto Ra$ is an isomorphism of A onto $R(A^{**})$ and $\rho(a) = \rho(Ra)$ for all $a \in A$, then ρ is normal on A.

Conversely, if $\rho \in A^*$ is normal, then the extension $\overline{\rho}$ on A^{**} has the form $\overline{\rho} = \rho R$. Since R is an isomorphism between A and $R(A^{**})$, then $\overline{\rho}$ is normal on A^{**} and is equal to zero on $R'(A^{**})$. Thus, ρ is equal to zero on $J = V^0$, and thus it belongs to V.

This proves that a predual space to A is unique and coincides with the space of normal functionals. Theorem 4 is proved.

2.2. Characterization of JBW-Algebra among Order-Unit Spaces Having a Predual Space

Note that JB-algebras are examples of order-unit spaces. Various authors have investigated conditions under which an order-unit space becomes a JB-algebra.

For example, in [7] it is shown that if a state space $S(A)$ of a spectral order-unit space A has the Hilbert ball property then A is a JB-algebra. In [8] geometric conditions on $S(A)$ are found: a spectral order-unit space A to be a JB-algebra if and only if $S(A)$ is symmetric.

Here, it was found another condition in this circle of problems: let a spectral order-unit space A has a predual space V $(V^* = A)$. If the spaces $L_1(\tau)$ and V are order and isometrically isomorphic then A is a JBW-algebra.

A positive linear functional τ is called *a trace* on an order-unit space $(A, \mathbf{e})$ if it satisfies the following condition:

$$\tau(a) = \tau(Ra) + \tau(R'a) \quad \forall a \in A, \quad R \in \mathbf{P}.$$

Let A be an order-unit space, τ be a faithful trace on A. For $a \in A$, we put $\|a\|_1 = \tau(|a|)$, where $|a| = a_+ + a_-$ is the module of the element a. The following result is proved in [9].

Theorem 5. *The mapping* $\|\cdot\|_1 : A \to \mathbb{R}$ *is a norm on* A.

A mapping $\|\cdot\|_1 : A \to \mathbb{R}$ is said to be L_1-*norm* on A. We denote $L_1(\tau)$ the completion of A by L_1-norm.

Let A be an order-unit space of type I having a predual space, i.e. there is a space V such, that $V^* = A$.

Consider relation between $L_1(\tau)$ and V.

Theorem 6. *The spaces* $L_1(\tau)$ *and* V *are order and isometrically isomorphic if and only if* A *is a* JBW*-algebra.*

P r o o f. It is known [10], if A is a JBW-algebra with a trace τ, then spaces $L_1(\tau)$ and V are order and isometrically isomorphic.

Conversely, suppose that $L_1(\tau)$ and V are isometrically isomorphic.

By Lemma 7.1 in [7], any order-unit space of type I can be reduced to factors of type I. Therefore we shall prove the theorem for factors of type I.

For an atom $u \in U$, $u = Re$, where $R \in \mathbf{P}$, we assume

$$\varphi_u(x) = \tau(Rx) = R^*\tau(x).$$

It is obvious, that φ_u is a positive functional on A, i.e it is an element of V. Functionals of the form $R^*\tau$ were called in [11] projective traces. If $v = Qe$ is another atom orthogonal to u, then the element $h = u + v$ corresponds to a P-projection $H = R \vee Q = R + Q$ and the functional $\varphi_h = H^*\tau = R^*\tau + Q^*\tau$. It is natural, that to their linear combination $a = \alpha u + \beta v$ corresponds the functional $\varphi_a = \alpha R^*\tau + \beta Q^*\tau$. This process can be done for an arbitrary finite number of orthogonal atoms. Since A is a spectral order-unit space, then by assumption of theorem, an arbitrary element of $L_1(\tau)$ can be approximated by finite linear combinations of functionals of type $R^*\tau$.

From the above, one can determine the following order and isometrical isomorphism between spaces $L_1(\tau)$ and V :

If $\{u_i\}$ is a family of orthogonal atoms then for $a = \sum \alpha_i u_i \in L_1(\tau)$, we define

$$\varphi_a(x) = \sum \alpha_i \tau(R_i x), \tag{1}$$

where $u_i = R_i e$.

Let $b = \sum \beta_j v_j$ be an element of $L_1(\tau)$. We define for b by formula (1) the functional $\varphi_b(x) = \sum \beta_j \tau(Q_j x)$, where $v_j = Q_j e$ are atoms.

Then

$$\varphi_a(b) = \sum \alpha_i \tau(R_i b) = \sum \sum \alpha_i \beta_j \tau(R_i v_j) = \sum \sum \alpha_i \beta_j \tau(R_i Q_j e),$$

$$\varphi_b(a) = \sum \beta_j \tau(Q_j a) = \sum \sum \beta_j \alpha_i \tau(Q_j u_i) = \sum \sum \alpha_i \beta_j \tau(Q_j R_i e).$$

In order to functionals be well defined by formula (1), the values of $\varphi_a(b)$ and $\varphi_b(a)$ have to be equal.That's why we have

$$\tau(RQe) = \tau(QRe)$$

for all atoms $u = Re$ and $v = Qe$.

The last equality means that $\tau(Rv) = \tau(Qu)$, i.e.

$$\tau(\langle v, \widehat{u} \rangle u) = \tau(\langle u, \widehat{v} \rangle v).$$

Since the trace on factors of type I takes equal values on atoms, we have

$$\langle v, \widehat{u} \rangle = \langle u, \widehat{v} \rangle$$

for all atoms u and v. But this is the Hilbert ball property. By Proposition 6.14 from [7], we conclude that A is a JBW-factor. Theorem 5 is proved.

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