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Inverse Scattering Problem on the Axis for the Schrödinger Operator with Triangular 2×2 Matrix Potential. II. Addition of the Discrete Spectrum

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The theorem of the necessary and sufficient conditions for the solvability of ISP under consideration is proved. The method of addition of the discrete spectrum to the considered matrix not self-adjoint case is developed.

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This work constitutes Part II of [15]. Notations, definitions, numerations of the statements, formulas, etc., extend those of [15]. The formulas (1)–(49) are contained in [15]. References [1–5] repeat those of [15], but references [6–14] of [15] are omitted here. Correction to Part I see at the end of the present Part II.

Theorem 1. *If a set of values* $\{R^+(k),\ k \in \mathrm{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = \overline{1,p}\}$ *forms the right SD for the scattering problem on the axis for the Schrödinger operator (1) with an upper triangular* 2×2 *matrix potential having the second moment ((2) with* $m = 2$*) and with the real diagonal and without virtual level, conditions 1)–6) should be satisfied:*

1) $R^+(k)$ *is continuous in* $k \in \mathrm{R}:$ $\overline{r_{ll}^+(k)} = r_{ll}^+(-k),$ $|r_{ll}^+(k)| \le 1 - \frac{C_l k^2}{1+k^2},$ $l = 1, 2,$ $R^+(0) = -I;$ $I - R^+(-k)R^+(k) = O(k^2)$ *as* $k \to 0$ *and* $R^+(k) = O(k^{-1})$ *with* $k \to \pm\infty$ *(note that replacing the last condition by* $R^+(k) = o(k^{-1})$ *we obtain a necessary condition too).*

2) The function

$$F_R^+(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} R^+(k)e^{ikx}dk \tag{50}$$

is absolutely continuous, and with $a > -\infty$ *one has* $\int\limits_a^{+\infty}\left(1+x^2\right)\left|\frac{d}{dx}F_R^+(x)\right|dx < \infty$.

3) The functions $za_{ll}(z)$, $l = 1, 2$, *given by (48), are continuously differentiable in the closed upper half-plane.*

4) The function

$$F_R^-(x) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} C(k)^{-1}R^+(-k)C(-k)e^{-ikx}dk \tag{51}$$

is absolutely continuous, and with $a < +\infty$ *one has* $\int\limits_{-\infty}^{a}\left(1+x^2\right)\left|\frac{d}{dx}F_R^-(x)\right|dx < \infty$. *Here* $c_{12}(z)$ *is given by (49),* $c_{ll}(z) \equiv a_{ll}(z)$ *is determined by (48) (one can show** *that condition 4 is also necessary in version 4a, namely, if* $c_{12}(z)$ *is constructed as in (42), (43), and* $c_{ll}(z) \equiv a_{ll}(z)$ *is constructed as in (39), which correspond to the absence of discrete spectrum).*

5) $\deg Z_j^+(t) \le \sum\limits_{l=1}^{2} \operatorname{sign} z_{ll}^{[j]+} - 1$, $j = \overline{1,p}$, $z_{ll}^{[j]+} \ge 0$ *and* $z_{ll}^{[j]+}$ *are constant.*

6) $\operatorname{rg} Z_j^+(t) = \operatorname{rg}\operatorname{diag} Z_j^+(t) = \operatorname{rg}\operatorname{diag} Z_j^+(0)$, $\qquad j = \overline{1,p}$.

The necessary conditions 1)–6) listed above (with condition 4 being replaced by its version 4a) become sufficient together with the following assumption:

H) The function $ka_{11}(-k)a_{22}(k)\{r_{11}^+(-k)r_{12}^+(k) + r_{12}^+(-k)r_{22}^+(k)\}$ *satisfies the Hölder condition in the finite points as well as at infinity.*

(*The claims of the theorem related solely to the diagonal matrix elements, are direct consequences of [1, 2].*)

Remark 3. *In the case when the discrete spectrum is absent, conditions 5 and 6 of Th. 1 become inapplicable, and conditions 4 and 4a become the same.*

P r o o f of Theorem 1. **The necessity**. Similarly to [1], under condition (2) one has that $R^+(k)$ is a continuous function of $k \in \mathbf{R}$. In this context, since the upper triangular potential of the scattering problem (1) has its principal diagonal formed by real functions, the following relations hold: $\overline{r_{ll}^+(k)} = r_{ll}^+(-k)$ and $\left|r_{ll}^+(k)\right| \le 1 - \frac{C_l k^2}{1+k^2}$, $l = 1, 2$ (see [1]).

*E.g., using the procedure of subsequent eliminating eigenvalues or the properties of the Fourier transform, which we omit here.

Furthermore, (9) and (11) imply with $k \in \mathbf{R}$ that there exist the limits

$$kE_-(x,k) = \{E_+(x,k)[R^+(k) + I] + E_+(x,-k) - E_+(x,k)\}kC(k). \qquad (52)$$

Since the scattering problem in (1), (2) is assumed to have no virtual level, the definition (10) implies the existence of the limits

$$\begin{gathered} \lim_{k\to 0} kA(k) = C_1; \ \lim_{k\to 0} kC(k) = C_2; \\ \det C_1 \neq 0; \quad \det C_2 \neq 0. \end{gathered} \qquad (53)$$

Thus, passage to a limit as $k \to 0$ in (52) yields $0 = \lim\limits_{k\to 0}\{E_+(x,k)[R^+(k) + I]\}kC(k) = E_+(x,0)\lim\limits_{k\to 0}[R^+(k) + I]C_2$. By a continuity of $R^+(k)$ one has $R^+(0) = -I$. Also by (53), we deduce from (14) that $I - R^+(-k)R^+(k) = O(k^2)$ as $k \to 0$.

Lemma 6. *The coefficients $A(k)$ and $B(k)$ given by (10), admit representations as follows:*

$$A(k) = I - \frac{1}{2ik}\left\{\int\limits_{-\infty}^{\infty} V(x)dx + \int\limits_{-\infty}^{0} A_1(t)e^{-ikt}dt\right\},$$

$$B(k) = \frac{1}{2ik}\int\limits_{-\infty}^{\infty} B_1(t)e^{-ikt}dt,$$

with $A_1(t)$ being a summable matrix function whose first moment with $m = 2$ is on $(-\infty; 0]$;
$B_1(t)$ is a summable matrix function whose first moment with $m = 2$ is on $(-\infty; \infty)$.

P r o o f of Lemma 6 coincides with that of the lemma by V.A. Marchenko [1, Lemma 3.5.1.], if one takes into account that under condition (2) the kernel $K(x,t)$ of the transformation operator is a summable function, which has (when $m = 2$) the first moment with respect to $t \in [x; \infty)$.

Lemma 6 and the definition of reflection coefficient (11), (24) imply $R^+(k) = o(\frac{1}{k})$ as $k \to \pm\infty$. Condition 1) of the theorem is proved completely.

Condition 2) of the theorem follows from the arguments, with the help of which the Marchenko equation is derived for the given right SD (see [1, 4, 5]).

The fact that $za_{ll}(z)$ (48) as well as the same function $zc_{ll}(z)$ are continuous in the closed upper half-plane is proved in [1] by an application of Lemma 3.5.1 from [1]. The continuous differentiability of these functions in the closed upper half-plane is a direct consequence of Lemma 6.

Use Remark 2 to Lemma 5 [15], the relation (29) between the left and the right reflection coefficients, and the argument that derives the Marchenko equation by a contour integration for the given left SD (see [1], the text that starts at (3.5.14) and ends at (3.5.19′)) to prove that condition 4) holds.

The necessity of conditions 5) and 6) of the theorem follows from claims a) and b) of Lemma 1 [15].

The necessity of assumptions of Theorem 1 is proved.

Prove a **sufficiency** of assumptions of the theorem.

1. The Case of the Absence of Discrete Spectrum

First, reconstruct the problem (1), (2) with $m = 2$, given $R^+(k)$, without eigenvalues and normalizing polynomials. In this case, use (29) and the formulas (39), (40) of Lemma 5 [15], to construct the function $R_0^-(k)$ as follows:

$$R_0^-(k) = -C_0(k)^{-1}R^+(-k)C_0(-k),\ C_0(k) = \begin{pmatrix} a_{11}^0(k) & c_{12}^0(k) \\ 0 & a_{22}^0(k) \end{pmatrix}, \tag{54}$$

where zero indices indicate the absence of eigenvalues.

Prove that $R^+(k)$ and $R_0^-(k)$ are the right and the left reflection coefficients of the same differential equation (1), whose potential is triangular, summable, and has the second moment on the real axis. Since $R^+(k)$ and $R_0^-(k)$ are upper triangular and the diagonal elements $r_{ll}^+(k)$ and $r_{ll}^{[0]-}(k)$, $l = 1, 2$, satisfy the assumptions of the Marchenko lemma [1, Lemma 3.5.3], one deduces that the Marchenko equations constructed respectively to $R^+(k)$ and $R_0^-(k)$ have unique solutions $K_+^0(x,y)$ and $K_-^0(x,y)$, and analogously $\widetilde{K}_+^0(x,y)$ and $\widetilde{K}_-^0(x,y)$. (In fact, the equations for diagonal elements are solvable unambiguously by [1], and the equations for k_{12+}^0 and k_{12-}^0 differ from those ones for the diagonal elements only by a free term). By the same Lemma 3.5.3 of [1], the functions $E_\pm^0(x,k) = e^{\pm ikx}I \pm \int\limits_x^{\pm\infty} K_\pm^0(x,t)e^{\pm ikt}dt$ are the Jost solutions of the Schrödinger equations on the entire axis, in which the potentials $V_0^\pm(x)$ possess the property (2) with $m = 2$ (i.e., have the second moment), and similarly $\widetilde{E}_\pm^0(x,k) = e^{\pm ikx}I \pm \int\limits_x^{\pm\infty} \widetilde{K}_\pm^0(x,t)e^{\pm ikt}dt$ are the Jost tilde-solutions.

To prove that $R^+(k)$ and $R_0^-(k)$ are the right and the left reflection coefficients of the same equation, it suffices to demonstrate that

$$\begin{gathered} E_-^0(x,k)C_0(k)^{-1} = E_+^0(x,-k) + E_+^0(x,k)R^+(k); \\ E_+^0(x,k)A_0(k)^{-1} = E_-^0(x,-k) + E_-^0(x,k)R_0^-(k),\ k \in \mathbf{R}. \end{gathered} \tag{55}$$

We follow the ideas of [1, 3] to prove (55).

Define a function

$$\Phi_+(x,y) := F_R^+(x+y) + \int_x^\infty K_+^0(x,t)F_R^+(t+y)dt,$$

with F_R^+ given by (50). It is quite plausible from the above that at every fixed x, the function $\Phi_+(x,y)$ is in $L(-\infty,\infty)$ since $F_R^+(y) \in L(-\infty,\infty)$. Furthermore, by virtue of (50) one has

$$\int_{-\infty}^{\infty} \Phi_+(x,y)e^{-iky}dy = E_+^0(x,k)R^+(k).$$

By the Marchenko equation $\Phi_+(x,y) = -K_+^0(x,y)$ with $x < y < \infty$.

Since $\int_x^\infty K_+^0(x,y)e^{-iky}dy = E_+^0(x,-k) - e^{-ikx}I$, one has $\int_{-\infty}^{\infty} \Phi_+(x,y)e^{-iky}dy$ $= \int_{-\infty}^{x} \Phi_+(x,y)e^{-iky}dy + e^{-ikx}I - E_+^0(x,-k)$. Thus,

$$E_+^0(x,k)R^+(k) + E_+^0(x,-k) = H_-(x,k)C_0(k)^{-1}, \tag{56}$$

where

$$H_-(k) = e^{-ikx}\left\{I + \lim_{N\to\infty}\int_{-N}^{x} \Phi_+(x,y)e^{-ik(y-x)}dy\right\}C_0(k).$$

It suffices to show that

$$H_-(x,k) = E_-^0(x,k) \tag{57}$$

and this proves (55). In fact, consider the system

$$E_+^0(x,k)R^+(k) + E_+^0(x,-k) = H_-(x,k)C_0(k)^{-1},$$

$$E_+^0(x,k) + E_+^0(x,-k)R^+(-k) = H_-(x,-k)C_0(-k)^{-1}$$

with respect to $E_+^0(x,\pm k)$ to deduce from (14) that

$$H_-(x,k)R_0^-(k) + H_-(x,-k) = E_+^0(x,k)A_0(k)^{-1}, \tag{58}$$

which by virtue of (57) yields (55).

Similarly to the proof of Theorem 6.5.1 of [3] it is possible to establish the following three properties of the function $H_-(x,k)$:

1. $H_-(x,k)$ admits an analytic continuation into the upper half-plane, and for large z one has the estimate $|H_-(x,z) - e^{-ixz}I| = O\left(\frac{e^{x\operatorname{Im} z}}{|z|}I\right)$.

2. $zH_-(x,z)$ is continuous in the closed upper half-plane, and $zH_-(x,z) = o(I)$ as $z \to 0$ (uniformly in x).

3. $H_-(x,k) - e^{-ikx}I \in L_2(-\infty,\infty)$ in k.

Use the above properties of $H_-(x,k)$ to prove (57). For $x < y$, consider an analytic in the upper half-plane function $[H_-(x,z) - e^{-ixz}I]e^{iyz}$. Use the method of contour integration to obtain, in view of properties 1–3,

$$\lim_{R\to\infty} \int\limits_{-R}^{R} \left[H_-(x,k) - e^{-ixk}I\right] e^{iyk} dk = 0, \qquad x < y.$$

Hence,

$$H_-(x,k) = e^{-ikx}I + \int\limits_{-\infty}^{x} G_-(x,y)e^{-iky}dy, \tag{59}$$

for some $G_-(x,y) \in L_2(-\infty,x)$.

From (58) and (59) one has $E^0_+(x,k)A_0(k)^{-1} - e^{ikx}I = \int\limits_{-\infty}^{x} G_-(x,y)e^{iky}dy$ $+e^{-ikx}R^-_0(k) + \int\limits_{-\infty}^{x} G_-(x,y)e^{-iky}dyR^-_0(k)$.

By a construction, $A_0(z)$ and $A_0^{-1}(z)$ are regular in the open upper half-plane, hence with $t < x$ one has

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} (E^0_+(x,k)A_0(k)^{-1} - e^{ikx}I)e^{-ikt}dt \\ &= G_-(x,t) + F^-_{R_0}(x+t) + \int\limits_{-\infty}^{x} G_-(x,y)F^-_{R_0}(t+y)dt. \end{aligned}$$

That is, $G_-(x,y)$ satisfies the Marchenko equation. It follows from the unambiguous solvability of the Marchenko equation that $K^0_-(x,t) \equiv G_-(x,t)$, whence one deduces (57) in view of (59).

Thus $R^+(k)$ and $R^-_0(k)$ are the right and the left reflection coefficients for the problem (1), (2) with $m = 2$ under the absence of discrete spectrum, so in this special case Theorem 1 is proved.

2. The Addition of Discrete Spectrum

Now consider a general case when the problem might have a finite number p of different eigenvalues (for the considered 2×2 triangular matrix potential they can be either simple or of multiplicity two, and respectively the ranks of normalizing polynomials $Z^+_j(t)$ will be either 1 or 2). We proceed by induction (cf., for example, [3] in the scalar case). Suppose that for the data

$$\{R^+(k);\ k_1^2,\ \dots\ k_p^2;\ Z^+_1(t),\ \dots\ Z^+_p(t)\} \tag{60}$$

the inverse problem on the axis is solved, that is those values from the right SD for a problem of the form (1), (2) with $m = 2$ and a potential $V_0(x)$ (do not confuse index 0 with the previous notation of Sect. 1!). We are about to show how in this case to obtain a solution of the inverse problem with the $p+1$ different eigenvalues and the normalizing polynomials, that is, with the right SD of the form

$$\{R^+(k);\ k_1^2,\ \dots\ k_p^2,\ k_{p+1}^2;\ Z_1^+(t),\ \dots\ Z_p^+(t),\ Z_{p+1}^+(t)\}. \tag{61}$$

Denote by $E_+^0(x,k)$ and $\widetilde{E}_+^0(x,k)$ the Jost solutions for the equations, respectively,

$$-Y'' + V_0(x)Y = k^2 Y,\ -\infty < x < +\infty, \tag{62}$$

$$-\widetilde{Z}'' + \widetilde{Z}V_0(x) = k^2\widetilde{Z},\ -\infty < x < +\infty, \tag{63}$$

with asymptotics $E_+^0(x,k) \sim e^{ikx}I$, $\widetilde{E}_+^0(x,k) \sim e^{ikx}I$ as $x \to +\infty$, $\operatorname{Im} k \ge 0$. Since k_{p+1}^2 is not an eigenvalue of the equations (62) and (63), one has that $E_+^0(x,k_{p+1})$ and $\widetilde{E}_+^0(x,k_{p+1})$ decay exponentially as $x \to +\infty$, and increase exponentially as $x \to -\infty$.

We generalize the procedure of attaching the discrete spectrum expounded in [3, Ch. VI, § 6] to the considered matrix not self-adjoint case.

Set

$$F(x,y) =$$
$$E_+^0(x,k_{p+1})Z_{p+1}^+(0)\widetilde{E}_+^0(y,k_{p+1}) - i\frac{d}{dk}\left\{E_+^0(x,k)(Z_{p+1}^+)'(0)\widetilde{E}_+^0(y,k)\right\}_{k_{p+1}},$$

and consider the degenerate integral equation about $B(x,y)$

$$B(x,y) + F(x,y) + \int_x^\infty B(x,t)F(t,y)dt = 0, \quad x < y. \tag{64}$$

Solve it to obtain

$$B(x,y) = -E_+^0(x,k_{p+1})Z_{p+1}^+(0)\Big[I + \int_x^\infty \widetilde{E}_+^0(t,k_{p+1})E_+^0(t,k_{p+1})dt Z_{p+1}^+(0)\Big]^{-1}$$
$$\times\widetilde{E}_+^0(y,k_{p+1}) + ib(x,y)(Z_{p+1}^+)'(0), \tag{65}$$

with

$$b(x,y) = \frac{\dot e_{11+}^0(x,k_{p+1})e_{22+}^0(y,k_{p+1})}{1+z_{22}^{[p+1]+}\int_x^\infty e_{22+}^0(t,k_{p+1})^2dt} + \frac{e_{11+}^0(x,k_{p+1})\dot e_{22+}^0(y,k_{p+1})}{1+z_{11}^{[p+1]+}\int_x^\infty e_{11+}^0(t,k_{p+1})^2dt}$$
$$-\frac{e_{11+}^0(x,k_{p+1})e_{22+}^0(y,k_{p+1})z_{11}^{[p+1]+}\int_x^\infty \dot e_{11+}^0(t,k_{p+1})e_{11+}^0(t,k_{p+1})dt}{(1+z_{11}^{[p+1]+}\int_x^\infty e_{11+}^0(t,k_{p+1})^2dt)(1+z_{22}^{[p+1]+}\int_x^\infty e_{22+}^0(t,k_{p+1})^2dt)}$$
$$-\frac{e_{11+}^0(x,k_{p+1})e_{22+}^0(y,k_{p+1})z_{22}^{[p+1]+}\int_x^\infty \dot e_{22+}^0(t,k_{p+1})e_{22+}^0(t,k_{p+1})dt}{(1+z_{11}^{[p+1]+}\int_x^\infty e_{11+}^0(t,k_{p+1})^2dt)(1+z_{22}^{[p+1]+}\int_x^\infty e_{22+}^0(t,k_{p+1})^2dt)}. \tag{66}$$

Set

$$\triangle V(x) = -2\frac{d}{dx}B(x,x),\ V(x) = V_0(x) + \triangle V(x). \tag{67}$$

Lemma 7. *The matrix function $\triangle V(x)$ given by (67) possesses the property*

$$\int_{-\infty}^{+\infty} (1+|x|^m)|\triangle V(x)|dx < \infty, \tag{68}$$

if $V_0(x)$ satisfies condition (2) with $m \geq 1$.

P r o o f. Since $E_+^0(x,k_{p+1})$ and $\widetilde{E}_+^0(x,k_{p+1})$ decay exponentially as $x \to +\infty$, it follows from (65)–(67) that $\triangle V(x)$ also decays exponentially as $x \to +\infty$.

Thus it remains to show that $\triangle V(x)$ has m moments at $-\infty$ if $V_0(x)$ satisfies (2). For this we introduce the notations

$$\begin{aligned} \Phi(x,k_{p+1}) &= e^{-ik_{p+1}x}E_+^0(x,k_{p+1}); \\ \widetilde{\Phi}(x,k_{p+1}) &= e^{-ik_{p+1}x}\widetilde{E}_+^0(x,k_{p+1}); \\ \Gamma(x,k_{p+1}) &= e^{-2ik_{p+1}x}\int_x^{\infty}\widetilde{E}_+^0(t,k_{p+1})E_+^0(t,k_{p+1})dt \end{aligned} \tag{69}$$

and generalize the techniques of [3] to obtain the following three Lemmas 8–10 which are similar to Lemmas 6.6.1–6.6.3 of [3].

Lemma 8. *The matrix functions $\Phi(x,k_{p+1})$, $\widetilde{\Phi}(x,k_{p+1})$, given by (69), and $\dot{\varphi}_{ll}(x,k_{p+1}) = \frac{d}{dk}\left(e^{-ikx}e_{ll+}^0(x,k)\right)_{k_{p+1}}$ (where φ_{ll} are the diagonal elements of Φ, $\widetilde{\Phi}$, $l=1,2$), are bounded in the neighborhood of $x=-\infty$.*

A p r o o f results immediately from the representations (5) for the Jost solutions and the inequalities for transformation operators (see [4]):

$$|K_+^0(x,t)| \leq C\int_{\frac{x+t}{2}}^{\infty}|V_0(s)|ds; \quad |\widetilde{K}_+^0(x,t)| \leq C\int_{\frac{x+t}{2}}^{\infty}|V_0(s)|ds, \tag{70}$$

with some constant C and also with the use of exponentially rising solutions as $x \to -\infty$ with asymptotics of (7).

Lemma 9. *One has the following inequalities ($m \geq 1$) for the matrix functions $\Phi(x,k_{p+1})$, $\widetilde{\Phi}(x,k_{p+1})$ (69) and $\dot{\varphi}_{ll}(x,k_{p+1})$:*

$$\begin{gathered} \int_{-\infty}^{0}(1+|t|^m)|\tfrac{d}{dt}\Phi(t,k_{p+1})|dt < \infty; \quad \int_{-\infty}^{0}(1+|t|^m)|\tfrac{d}{dt}\widetilde{\Phi}(t,k_{p+1})|dt < \infty; \\ \int_{-\infty}^{0}(1+|t|^m)|\tfrac{d}{dt}\dot{\varphi}_{ll}(t,k_{p+1})|dt < \infty,\ l=1,2. \end{gathered} \tag{71}$$

A p r o o f is similar to that of Lemma 6.6.2 [3] in view of the fact that $V_0(x)$ satisfies (2) with $m \geq 1$.

Lemma 10. *The following statements are valid for the matrix function* $\Gamma(x, k_{p+1})$ *(69):*

a) $|\Gamma(x, k_{p+1})|$ *and* $|\Gamma^{-1}(x, k_{p+1})|$ *are bounded as* $x \to -\infty$*;*

b)

$$\int\limits_{-\infty}^{0} (1 + |t|^m) |\frac{d}{dt} \Gamma(t, k_{p+1})| dt < \infty, \; m \geq 1; \tag{72}$$

c) $|\dot{\gamma}_{ll}(x, k_{p+1})|$ *are bounded as* $x \to -\infty, \; l = 1, 2,$ *and*

$$\int\limits_{-\infty}^{0} (1 + |t|^m) |\frac{d}{dt} \dot{\gamma}_{ll}(t, k_{p+1})| dt < \infty, \; m \geq 1; \; l = 1, 2, \tag{73}$$

with γ_{ll}*,* $l = 1, 2$*, being the diagonal elements of the matrix function* $\Gamma(x, k_{p+1})$*.*

P r o o f s of propositions a) and b) of Lemma 10 are similar to that of Lemma 6.6.3 [3] taking into account that $V_0(x)$ satisfies (2) with $m \geq 1$. Also note that the boundedness of $|\Gamma^{-1}(x, k_{p+1})|$ as $x \to -\infty$ follows from the fact that $|\Gamma(x, k_{p+1})|$ is bounded and $|\gamma_{ll}(x, k_{p+1})| \geq a_l > 0$, $l = 1, 2$, as $x \to -\infty$.

Prove the proposition c) of Lemma 10.

From (69) one has

$$\begin{aligned} \dot{\gamma}_{ll}(x, k_{p+1}) &= 2 \int\limits_{x}^{\infty} \dot{\varphi}_{ll}(t, k_{p+1}) \varphi_{ll}(t, k_{p+1}) e^{-2ik_{p+1}(x-t)} dt \\ &\quad -2i \int\limits_{x}^{\infty} \varphi_{ll}^2(t, k_{p+1})(x - t) e^{-2ik_{p+1}(x-t)} dt \\ &= 2 \int\limits_{-\infty}^{0} \dot{\varphi}_{ll}(x - z, k_{p+1}) \varphi_{ll}(x - z, k_{p+1}) e^{-2ik_{p+1}z} dz \\ &\quad -2i \int\limits_{-\infty}^{0} \varphi_{ll}^2(x - z, k_{p+1}) z e^{-2ik_{p+1}z} dz. \end{aligned} \tag{74}$$

Thus, in view of Lemma 8, we obtain that

$$|\dot{\gamma}_{ll}(x, k_{p+1})| \leq C_{ll} (\int\limits_{-\infty}^{0} e^{-2ik_{p+1}z} dz + \int\limits_{-\infty}^{0} |z| e^{-2ik_{p+1}z} dz) < \infty, \; l = 1, 2,$$

as $x \to -\infty$.

Since $\int\limits_0^\infty e^0_{ll+}(x,k_{p+1})dx<\infty$, with the notation

$$\gamma^0_{ll}(x,k_{p+1})=e^{-2ik_{p+1}x}\int\limits_x^0 e^0_{ll+}(t,k_{p+1})^2dt,\quad \alpha_{ll}(k_{p+1})=\int\limits_0^\infty e^0_{ll+}(t,k_{p+1})^2dt,$$

we obtain the following representation:

$$\gamma_{ll}(x,k_{p+1})=\alpha_{ll}(k_{p+1})e^{-2ik_{p+1}x}+\gamma^0_{ll}(x,k_{p+1}),$$

and hence

$$\dot\gamma_{ll}(x,k_{p+1})=-2ixe^{-2ik_{p+1}x}\alpha_{ll}(k_{p+1})+e^{-2ik_{p+1}x}\dot\alpha_{ll}(k_{p+1})+\dot\gamma^0_{ll}(x,k_{p+1}).$$

Thus it suffices to prove (73) for the function $\frac{d}{dt}\dot\gamma^0_{ll}(t,k_{p+1})$.
So,

$$\begin{aligned}\dot\gamma^0_{ll}(x,k_{p+1})&=-2ix\int\limits_x^0\varphi_{ll}(t,k_{p+1})^2e^{-2ik_{p+1}(x-t)}dt+2\int\limits_x^0[\dot\varphi_{ll}(t,k_{p+1})\\&\quad+it\varphi_{ll}(t,k_{p+1})]\varphi_{ll}(t,k_{p+1})e^{-2ik_{p+1}(x-t)}dt\\&=2\int\limits_x^0\dot\varphi_{ll}(x-z,k_{p+1})\varphi_{ll}(x-z,k_{p+1})e^{-2ik_{p+1}z}dz\\&\quad-2i\int\limits_x^0\varphi_{ll}(x-z,k_{p+1})^2ze^{-2ik_{p+1}z}dz.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\dot\gamma^0_{ll}(t,k_{p+1})&=-2\dot\varphi_{ll}(0,k_{p+1})\varphi_{ll}(0,k_{p+1})e^{-2ik_{p+1}t}\\&+2i\varphi_{ll}(0,k_{p+1})^2te^{-2ik_{p+1}t}+2\int\limits_t^0\left[\frac{d}{dt}\dot\varphi_{ll}(t-z,k_{p+1})\varphi_{ll}(t-z,k_{p+1})\right.\\&\left.+\dot\varphi_{ll}(t-z,k_{p+1})\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right]e^{-2ik_{p+1}z}dz\\&-4i\int\limits_t^0\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\varphi_{ll}(t-z,k_{p+1})ze^{-2ik_{p+1}z}dz.\end{aligned}$$

Thus, in view of Lemmas 8 and 9, one has

$$\int_{-\infty}^{0} |t|^m \left|\frac{d}{dt}\dot{\gamma}^0_{ll}(t,k_{p+1})\right| dt \le C_0 + C_1 \int_{-\infty}^{0} \left(\int_{t}^{0} \left|\frac{d}{dt}\dot{\varphi}_{ll}(t-z,k_{p+1})\right| e^{-2ik_{p+1}z} dz \right.$$

$$\left. + \int_{t}^{0} \left|\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right| e^{-2ik_{p+1}z} dz \right) |t|^m dt$$

$$+ C_2 \int_{-\infty}^{0} |t|^m \int_{t}^{0} \left|\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right| |z| e^{-2ik_{p+1}z} dz dt$$

$$= C_0 + C_1 \int_{-\infty}^{0} e^{-2ik_{p+1}z} dz \int_{-\infty}^{z} |t|^m \left\{ \left|\frac{d}{dt}\dot{\varphi}_{ll}(t-z,k_{p+1})\right| + \left|\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right| \right\} dt$$

$$+ C_2 \int_{-\infty}^{0} |z| e^{-2ik_{p+1}z} dz \int_{-\infty}^{z} |t|^m \left|\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right| dt < \infty,$$

and Lemma 10 is proved.

We return to the proof of Lemma 7. Use the notation of (69) to rewrite $\triangle V(x)$:

$$\triangle V(x) = -2\frac{d}{dx}B(x,x)$$

$$= -2\frac{d}{dx}[-\Phi(x,k_{p+1})Z^+_{p+1}(0)\{Ie^{-2ik_{p+1}x} + \Gamma(x,k_{p+1})Z^+_{p+1}(0)\}^{-1}\widetilde{\Phi}(x,k_{p+1})$$

$$+ i\left\{ \frac{e^{-2ik_{p+1}x}\frac{d}{dk}(\varphi_{11}(x,k)\varphi_{22}(x,k))_{k_{p+1}} + 2ix\varphi_{11}(x,k_{p+1})\varphi_{22}(x,k_{p+1})e^{-2ik_{p+1}x}}{(e^{-2ik_{p+1}x}+z^{[p+1]+}_{11}\gamma_{11}(x,k_{p+1}))(e^{-2ik_{p+1}x}+z^{[p+1]+}_{22}\gamma_{22}(x,k_{p+1}))} \right.$$

$$+ \frac{\dot{\varphi}_{11}(x,k_{p+1})\varphi_{22}(x,k_{p+1})z^{[p+1]+}_{11}\gamma_{11}(x,k_{p+1}) + \varphi_{11}(x,k_{p+1})\dot{\varphi}_{22}(x,k_{p+1})z^{[p+1]+}_{22}\gamma_{22}(x,k_{p+1})}{(e^{-2ik_{p+1}x}+z^{[p+1]+}_{11}\gamma_{11}(x,k_{p+1}))(e^{-2ik_{p+1}x}+z^{[p+1]+}_{22}\gamma_{22}(x,k_{p+1}))}$$

$$\left. - \frac{\frac{\varphi_{11}(x,k_{p+1})\varphi_{22}(x,k_{p+1})}{2}(z^{[p+1]+}_{11}\dot{\gamma}_{11}(x,k_{p+1}) + z^{[p+1]+}_{22}\dot{\gamma}_{22}(x,k_{p+1}))}{(e^{-2ik_{p+1}x}+z^{[p+1]+}_{11}\gamma_{11}(x,k_{p+1}))(e^{-2ik_{p+1}x}+z^{[p+1]+}_{22}\gamma_{22}(x,k_{p+1}))} \right\} (Z^+_{p+1})'(0)].$$

Now consider the possible (with the assumption 6 of the theorem being taken into account) Cases I–III:

I) $z^{[p+1]+}_{11} = 0$. In view of the assumption 5) of the theorem one has $(Z^+_{p+1})'(t) \equiv 0$, $Z^+_{p+1}(t) \equiv Z^+_{p+1}$, that is

$$\triangle V(x) = 2\frac{1}{e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}}$$
$$\times\{\Phi'(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}(x,k_{p+1})+\Phi(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}'(x,k_{p+1})\} \tag{75}$$
$$-2\frac{-2ik_{p+1}e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}'}{(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})^2}\Phi(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}(x,k_{p+1}).$$

II) $z_{22}^{[p+1]+}=0$. In view of the assumption 5) of the theorem one has $(Z_{p+1}^{+})'(t) \equiv 0$, $Z_{p+1}^{+}(t)\equiv Z_{p+1}^{+}$, that is

$$\triangle V(x) = 2\frac{1}{e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}}$$
$$\times\{\Phi'(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}(x,k_{p+1})+\Phi(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}'(x,k_{p+1})\} \tag{76}$$
$$-2\frac{-2ik_{p+1}e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}'}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})^2}\Phi(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}(x,k_{p+1}).$$

III) $z_{ll}^{[p+1]+}>0$, $l=1,2$, then

$$\triangle V(x) = 2\Phi'(x,k_{p+1})\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\}^{-1}\widetilde{\Phi}(x,k_{p+1})$$
$$+2\Phi(x,k_{p+1})\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\}^{-1}\widetilde{\Phi}'(x,k_{p+1})$$
$$-2\Phi(x,k_{p+1})\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\}^{-1}[-2ik_{p+1}e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}$$
$$+\Gamma'(x,k_{p+1})]\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\}^{-1}\widetilde{\Phi}(x,k_{p+1})$$
$$-2i\Bigg[\frac{-2ik_{p+1}e^{-2ik_{p+1}x}[\frac{d}{dk}(\varphi_{11}\varphi_{22})+2ix\varphi_{11}\varphi_{22}]}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})}$$
$$+\frac{e^{-2ik_{p+1}x}[\frac{d^2}{dkdx}(\varphi_{11}\varphi_{22})+2i\varphi_{11}\varphi_{22}+2ix(\varphi_{11}\varphi_{22})']}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})}$$
$$+\frac{z_{11}^{[p+1]+}[(\dot\varphi_{11}\varphi_{22})'\gamma_{11}+\dot\varphi_{11}\varphi_{22}\gamma_{11}'-\frac{(\varphi_{11}\varphi_{22})'}{2}\dot\gamma_{11}-\frac{\varphi_{11}\varphi_{22}}{2}\dot\gamma_{11}']}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})}$$
$$+\frac{z_{22}^{[p+1]+}[(\varphi_{11}\dot\varphi_{22})'\gamma_{22}+\varphi_{11}\dot\varphi_{22}\gamma_{22}'-\frac{(\varphi_{11}\varphi_{22})'}{2}\dot\gamma_{22}-\frac{\varphi_{11}\varphi_{22}}{2}\dot\gamma_{22}']}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})}$$
$$-\Bigg[\frac{-4ik_{p+1}e^{-4ik_{p+1}x}-2ik_{p+1}e^{-2ik_{p+1}x}(z_{11}^{[p+1]+}\gamma_{11}+z_{22}^{[p+1]+}\gamma_{22})}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})^2(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})^2}$$
$$+\frac{e^{-2ik_{p+1}x}(z_{11}^{[p+1]+}\gamma_{11}'+z_{22}^{[p+1]+}\gamma_{22}')+z_{11}^{[p+1]+}z_{22}^{[p+1]+}(\gamma_{11}\gamma_{22})'}{(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11})^2(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22})^2}\Bigg]$$
$$\times\Big\{e^{-2ik_{p+1}x}\Big[\frac{d}{dk}(\varphi_{11}\varphi_{22})+2ix\varphi_{11}\varphi_{22}\Big]+\dot\varphi_{11}\varphi_{22}z_{11}^{[p+1]+}\gamma_{11}$$
$$+\varphi_{11}\dot\varphi_{22}z_{22}^{[p+1]+}\gamma_{22}-\frac{\varphi_{11}\varphi_{22}}{2}(z_{11}^{[p+1]+}\dot\gamma_{11}+z_{22}^{[p+1]+}\dot\gamma_{22})\Big\}\Bigg](Z_{p+1})'(0).$$

It follows from Lemmas 8–10 that in each of three cases the matrix function $\triangle V(x)$ satisfies (68) if $V_0(x)$ satisfies (2) with $m \geq 1$. Lemma 7 is proved.

It is possible to deduce from (64) the following differential equation for $B(x, y)$ (compare, for instance, [3]):

$$\frac{\partial^2 B(x,y)}{\partial x^2} - V(x)B(x,y) = \frac{\partial^2 B(x,y)}{\partial y^2} - B(x,y)V_0(y), \tag{77}$$

with $B(x,x) = \frac{1}{2}\int\limits_x^\infty \triangle V(t)dt$.

It follows from (77), in view of the fact that $B(x, y)$ tends to zero as $y \to +\infty$ (see (65)), that the function

$$E_+(x,k) = E_+^0(x,k) + \int\limits_x^\infty B(x,y)E_+^0(y,k)dy, \quad \mathbf{Im}\, k \geq 0, \tag{78}$$

is the Jost solution for the equation

$$-Y'' + V(x)Y = k^2 Y, \quad -\infty < x < +\infty, \tag{79}$$

with the asymptotics $E_+(x,k) \sim e^{ikx}I$ as $x \to +\infty$.

Lemma 11. *The right SD of the problems (79), (2) with* $m \geq 1$ *coincide with the values given in (61).*

A p r o o f of the lemma is based on the computation of the coefficients $A(k)$ and $B(k)$ (10) for the equation (79).

In view of the assumptions 5) and 6) of the theorem one has three possible cases again:

Case I). $Z_{p+1}^+(t) \equiv Z_{p+1}^+ = \begin{pmatrix} 0 & z_{12}^{[p+1]+} \\ 0 & z_{22}^{[p+1]+} \end{pmatrix}$: $z_{22}^{[p+1]+} > 0$, then we obtain from (65) and (78), in view of (9) and (27) as $x \to -\infty$

$$\begin{aligned} E_+(x,k) = E_+^0(x,k) - E_+^0(x,k_{p+1})Z_{p+1}^+[I + \int\limits_x^\infty \widetilde{E}_+^0(t,k_{p+1})E_+^0(t,k_{p+1})dtZ_{p+1}^+]^{-1} \\ \times \int\limits_x^\infty \widetilde{E}_+^0(y,k_{p+1})E_+^0(y,k)dy \\ \sim e^{ikx}\left\{I - \tfrac{2k_{p+1}}{(k+k_{p+1})z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}A_0(k_{p+1})Z_{p+1}^+\right\}A_0(k) \\ +e^{-ikx}\left\{I + \tfrac{2k_{p+1}}{(k-k_{p+1})z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}A_0(k_{p+1})Z_{p+1}^+\right\}B_0(k), \end{aligned}$$

hence

$$A(k) = \alpha(k)A_0(k),\ B(k) = \alpha(-k)B_0(k), \quad k \in \mathbf{R}, \tag{80}$$

with

$$\alpha(k) = I - \frac{2k_{p+1}}{(k+k_{p+1})z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}A_0(k_{p+1})Z_{p+1}^+$$
$$= \begin{pmatrix} 1 & -\frac{2k_{p+1}(a_{11}^0(k_{p+1})z_{12}^{[p+1]+}+a_{12}^0(k_{p+1})z_{22}^{[p+1]+})}{(k+k_{p+1})z_{22}^{[p+1]+}a_{22}^0(k_{p+1})} \\ 0 & \frac{k-k_{p+1}}{k+k_{p+1}} \end{pmatrix}. \tag{81}$$

It is clear from the first relation in (80) that the equation (79) has the same eigenvalues $k_1^2, k_2^2, \ldots, k_p^2$ as the initial equation (62), and one more eigenvalue k_{p+1}^2.

To compute the right reflection coefficient $R_1^+(k)$ and $C(k)$ that correspond to the constructed equation (79), we use (24) and (14):

$$R_1^+(k) = -A(k)^{-1}B(-k) = -A_0(k)^{-1}\alpha^{-1}(k)\alpha(k)B_0(-k) = R^+(k), \quad k \in \mathbf{R}, \tag{82}$$

and

$$C(k) = (I - R_1^+(-k)R_1^+(k))^{-1}A(-k)^{-1}$$
$$= (I - R^+(-k)R^+(k))^{-1}A_0(-k)^{-1}\alpha^{-1}(-k) = C_0(k)\alpha^{-1}(-k), \quad k \in \mathbf{R}. \tag{83}$$

It is clear from (81) that $\alpha^{-1}(-k) = \alpha(k)$, hence $C(k) = C_0(k)\alpha(k)$.

Now prove that the initial p normalizing polynomials of the problem (79), (2) with $m \geq 1$ coincide with the normalizing polynomials of problem (62), (2) with $m \geq 1$.

It follows from the definition (15) of the normalizing polynomials $Z_{j<1>}^-(t)$ for the equation (79) and the relation (80) that

$$Z_{j<1>}^-(t) = -i\{\dot{W}^-(k_j)A_{-2}^{0<k_j>}\alpha(-k_j) - W^-(k_j)A_{-2}^{0<k_j>}\dot{\alpha}(-k_j)$$
$$+W^-(k_j)A_{-1}^{0<k_j>}\alpha(-k_j)\} - tW^-(k_j)A_{-2}^{0<k_j>}\alpha(-k_j), \quad j = \overline{1,p}, \tag{84}$$

with $A_{-2}^{0<k_j>} = (A_0(k)^{-1}(k-k_j)^2)_{k_j}$; $A_{-1}^{0<k_j>} = \frac{d}{dk}(A_0(k)^{-1}(k-k_j)^2)_{k_j}$ being the Laurent coefficients.

One can show, using (27) and (32) with $x \to +\infty$ and noting that k_j is an eigenvalue of the equation (62) with $k_j < k_{p+1}$, $j = \overline{1,p}$, that

$$W^-(k_j)A_{-2}^{0<k_j>} = -\alpha(-k_j)(Z_j^-)'(0);$$
$$\dot{W}^-(k_j)A_{-2}^{0<k_j>} + W^-(k_j)A_{-1}^{0<k_j>} = i\alpha(-k_j)(Z_j^-)(0) + \dot{\alpha}(-k_j)(Z_j^-)'(0),$$

hence (84) can be rewritten as follows:

$$Z^{-}_{j<1>}(t) = \alpha(-k_j) Z^{-}_{j}(t)\alpha(-k_j) - i\frac{d}{dk}(\alpha(k)(Z^{-}_{j})'(0)\alpha(k))_{k=-k_j}.$$

Next, use the relation (30) with $\tau = t$ for the right and the left normalizing polynomials $Z^{\pm}_{j<1>}(t)$ of the problem (79):

$$\begin{aligned} Z^{+}_{j<1>}(t) &= -A^{<k_j>}_{-1}(Z^{-}_{j<1>}(t)+Q_j)^{-1}C^{<k_j>}_{-1} \\ &= -[A^{0<k_j>}_{-1}\alpha(-k_j) - A^{0<k_j>}_{-2}\dot\alpha(-k_j)]\alpha(k_j) \\ &\times\{Z^{-}_{j}(t) - i\alpha(k_j)\tfrac{d}{dk}(\alpha(k)(Z^{-}_{j})'(0)\alpha(k))_{k=-k_j}\alpha(k_j) + \alpha(k_j)Q_j\alpha(k_j)\}^{-1} \\ &\times\alpha(k_j)[\alpha(-k_j)C^{0<k_j>}_{-1} - \dot\alpha(-k_j)C^{0<k_j>}_{-2}], \end{aligned}$$

with $C^{0<k_j>}_{-1} = \frac{d}{dk}(C_0(k)^{-1}(k-k_j)^2)_{k_j}$; $C^{0<k_j>}_{-2} = (C_0(k)^{-1}(k-k_j)^2)_{k_j}$ being the Laurent coefficients.

In view of (81) one has

$$\begin{aligned} &\{Z^{-}_{j}(t) + \alpha(k_j)Q_j\alpha(k_j) - i\alpha(k_j)\tfrac{d}{dk}(\alpha(k)(Z^{-}_{j})'(0)\alpha(k))_{k=-k_j}\alpha(k_j)\}^{-1} \\ &= (Z^{-}_{j}(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1} + \frac{2ik_{p+1}}{(k_j^2-k_{p+1}^2)(z^{[j]-}_{11}+q^{[j]}_{11})(z^{[j]-}_{22}+(\frac{k_j-k_{p+1}}{k_j+k_{p+1}})^2 q^{[j]}_{22})}(Z^{-}_{j})'(0), \end{aligned}$$

hence

$$\begin{aligned} Z^{+}_{j<1>}(t) &= -A^{0<k_j>}_{-1}(Z^{-}_{j}(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}C^{0<k_j>}_{-1} \\ &+A^{0<k_j>}_{-1}(Z^{-}_{j}(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}\alpha(k_j)\dot\alpha(-k_j)C^{0<k_j>}_{-2} \\ &+A^{0<k_j>}_{-2}\dot\alpha(-k_j)\alpha(k_j)(Z^{-}_{j}(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}C^{0<k_j>}_{-1} \\ &-\frac{2ik_{p+1}}{(k_j^2-k_{p+1}^2)(z^{[j]-}_{11}+q^{[j]}_{11})(z^{[j]-}_{22}+(\frac{k_j-k_{p+1}}{k_j+k_{p+1}})^2 q^{[j]}_{22})}A^{0<k_j>}_{-1}(Z^{-}_{j})'(0)C^{0<k_j>}_{-1}. \end{aligned}$$

If $z^{[j]-}_{11} > 0$ and $z^{[j]-}_{22} > 0$, then $q^{[j]}_{11} = q^{[j]}_{22} = 0$ with the definitions of Q_j of Lemma 3 [15], being taken into account;

$$A^{0<k_j>}_{-2} = \begin{pmatrix} 0 & \frac{-a^0_{12}(k_j)}{\dot a^0_{11}(k_j)\dot a^0_{22}(k_j)} \\ 0 & 0 \end{pmatrix},$$

hence

$$\begin{aligned} Z^{+}_{j<1>}(t) &= Z^{+}_{j}(t) \\ &+\frac{2k_{p+1}}{(k_j^2-k_{p+1}^2)z^{[j]-}_{22}\dot a^0_{22}(k_j)\dot a^0_{11}(k_j)}\begin{pmatrix} 0 & \frac{-a^0_{12}(k_j)}{\dot a^0_{22}(k_j)} - i\frac{(z^{[j]-}_{12})'(0)}{z^{[j]-}_{11}} \\ 0 & 0 \end{pmatrix} = Z^{+}_{j}(t), \quad j=\overline{1,p}, \end{aligned}$$

in view of (19) and (21).

On the other hand, if $z_{11}^{[j]-}=0$ or $z_{22}^{[j]-}=0$, then by the assumption 5) of Theorem 1 $(Z_j^-)'(0)=0=A_{-2}^{0<k_j>}$, hence $Z^+_{j<1>}(t)=Z_j^+(t),\ \ j=\overline{1,p}$.

Thus in any variant of Case I, the initial p normalizing polynomials of the equation (79) turn out to be the same as those of the equation (62).

Now prove that Z^+_{p+1} is a normalizing polynomial of (79).

Similarly to (84), use (80), (81) to obtain

$$\begin{aligned}
Z^-_{p+1<1>}(t)&=-i\tfrac{d}{dk}(W^-(k)A_0^{-1}(k)\alpha(-k)(k-k_{p+1})^2)_{k_{p+1}}\\
&\quad -t(W^-(k)A_0^{-1}(k)\alpha(-k)(k-k_{p+1})^2)_{k_{p+1}}\\
&=-iW^-(k_{p+1})A_0^{-1}(k_{p+1})\left[\tfrac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}A_0(k_{p+1})Z^+_{p+1}\right]
\end{aligned}$$

$$=-iW^-(k_{p+1})Z^+_{p+1}\tfrac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}=W\Big\{\widetilde{E}^\wedge_-(x,k_{p+1});\ E^0_+(x,k_{p+1})$$

$$-\frac{\int_x^\infty e_{22}^{0+}(t,k_{p+1})^2dt}{1+z_{22}^{[p+1]+}\int_x^\infty e_{22}^{0+}(t,k_{p+1})^2dt}E^0_+(x,k_{p+1})Z^+_{p+1}\Big\}\frac{1}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}Z^+_{p+1}$$

$$=\frac{1}{a_{22}^0(k_{p+1})}W\Big\{\widetilde{E}^\wedge_-(x,k_{p+1});\ E^0_+(x,k_{p+1})\Big[I-\frac{\int_x^\infty e_{22}^{0+}(t,k_{p+1})^2dt}{1+z_{22}^{[p+1]+}\int_x^\infty e_{22}^{0+}(t,k_{p+1})^2dt}Z^+_{p+1}\Big]\Big\}.$$

Assume $x\to-\infty$ and apply the asymptotics $E^0_+(x,k_{p+1})\sim e^{ik_{p+1}x}A_0(k_{p+1})$, that is, as $x\to-\infty$

$$Z^-_{p+1<1>}(t)\equiv Z^-_{p+1<1>}=\frac{1}{a_{22}^0(k_{p+1})}W\{e^{ik_{p+1}x}I;\ e^{ik_{p+1}x}A_0(k_{p+1})$$

$$\times\begin{pmatrix}1 & -\frac{z_{12}^{[p+1]+}}{z_{22}^{[p+1]+}}\\ 0 & -\frac{2ik_{p+1}e^{-2ik_{p+1}x}}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})^2}\end{pmatrix}\}=\frac{1}{a_{22}^0(k_{p+1})}A_0(k_{p+1})\cdot\begin{pmatrix}0 & 0\\ 0 & \frac{(2ik_{p+1})^2}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})^2}\end{pmatrix}$$

$$=\frac{(2ik_{p+1})^2}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})^3}A_0(k_{p+1})\begin{pmatrix}0&0\\0&1\end{pmatrix}=\frac{-4k_{p+1}^2}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})^3}\begin{pmatrix}0 & a_{12}^0(k_{p+1})\\ 0 & a_{22}^0(k_{p+1})\end{pmatrix}.$$

It follows from (30) with $\tau=t$ and (83) that

$$Z^+_{p+1<1>}=-A_{-1}^{<k_{p+1}>}(Z^-_{p+1<1>}+Q_{p+1})^{-1}C_{-1}^{<k_{p+1}>}$$

$$=-A_0^{-1}(k_{p+1})\{\tfrac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}A_0(k_{p+1})Z^+_{p+1}\}\tfrac{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})^3}{-4k_{p+1}^2a_{22}^0(k_{p+1})}\{\tfrac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})}$$

$$\times A_0(k_{p+1})Z^+_{p+1}\}C_0^{-1}(k_{p+1})=Z^+_{p+1}\tfrac{a_{22}^0(k_{p+1})}{z_{22}^{[p+1]+}}z_{22}^{[p+1]+}\tfrac{1}{a_{22}^0(k_{p+1})}=Z^+_{p+1},$$

and Lemma 11 in the Case I is proved.

Case II). $Z^+_{p+1}(t) \equiv Z^+_{p+1} = \begin{pmatrix} z_{11}^{[p+1]+} & z_{12}^{[p+1]+} \\ 0 & 0 \end{pmatrix} : z_{11}^{[p+1]+} > 0$, then similarly to the Case I we obtain

$$A(k) = \beta(k)A_0(k), \quad B(k) = \beta(-k)B_0(k), \; k \in \mathbf{R}, \tag{85}$$

with

$$\beta(k) = I - \frac{2ik_{p+1}}{(k+k_{p+1})z_{11}^{[p+1]+}a_{11}^0(k_{p+1})} Z^+_{p+1} C_0(k_{p+1})$$
$$= \begin{pmatrix} \frac{k-k_{p+1}}{k+k_{p+1}} & -\frac{2ik_{p+1}(z_{11}^{[p+1]+}c_{12}^0(k_{p+1})+z_{12}^{[p+1]+}a_{22}^0(k_{p+1}))}{(k+k_{p+1})z_{11}^{[p+1]+}a_{11}^0(k_{p+1})} \\ 0 & 1 \end{pmatrix}. \tag{86}$$

It is clear from the first relation in (85) that the equation (79) has the same eigenvalues $k_1^2, k_2^2, \ldots, k_p^2$ as the original equation (62), and one more eigenvalue k_{p+1}^2.

Just as in (83), we deduce from (24) and (14) that

$$C(k) = C_0(k)\beta^{-1}(-k) = C_0(k)\beta(k), \tag{87}$$

and the reflection coefficient of the equation (79) coincides with that of (62): $R_1^+(k) = R^+(k)$.

The coincidence of the p initial normalizing polynomials of the equation (79) with those of (62) can be proved just as in Case I with the substitution of $\beta(k)$ (86) for $\alpha(k)$ (81).

Now prove that Z^+_{p+1} is a normalizing polynomial of (79). Similarly to (84), use (85) and (86) to obtain

$$Z^-_{p+1<1>}(t) \equiv Z^-_{p+1<1>} = -iW^-(k_{p+1})A_0^{-1}(k_{p+1})\frac{2k_{p+1}}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})}Z^+_{p+1}$$
$$\times C_0(k_{p+1}) = \frac{1}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})^2} W\Big\{ \widetilde{E}^\wedge_-(x, k_{p+1});\; E^0_+(x, k_{p+1})$$
$$- \frac{e_{11}^{0+}(x,k_{p+1})}{1+z_{11}^{[p+1]+}\int_x^\infty (e_{11}^{0+}(t,k_{p+1}))^2 dt} Z^+_{p+1}\int_x^\infty \widetilde{E}^0_+(y,k_{p+1})E^0_+(y,k_{p+1})dy\Big\} Z^+_{p+1}C_0(k_{p+1})$$
$$= \frac{W\Big\{e_{11}^\wedge(x,k_{p+1}); \frac{e_{11}^{0+}(x,k_{p+1})}{z_{11}^{[p+1]+}\int_x^\infty (e_{11}^{0+}(t,k_{p+1}))^2 dt}\Big\}}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})} Z^+_{p+1}C_0(k_{p+1}).$$

Supposing $x \to -\infty$ and using the asymptotics $e_{11}^{0+}(x, k_{p+1}) \sim e^{ik_{p+1}x}a_{11}^0(k_{p+1})$, that is with $x \to -\infty$, one has

$$Z^-_{p+1<1>} = \frac{W\Big\{e^{ik_{p+1}x}; e^{-ik_{p+1}x}a_{11}^0(k_{p+1})\frac{2ik_{p+1}}{-z_{11}^{[p+1]+}a_{11}^0(k_{p+1})^2}\Big\}}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})^2} Z^+_{p+1}C_0(k_{p+1})$$
$$= \frac{-4k_{p+1}^2}{(z_{11}^{[p+1]+})^2 a_{11}^0(k_{p+1})^3} Z^+_{p+1}C_0(k_{p+1}).$$

It follows from the relation (30) with $\tau = t$ and (87) that

$$Z^+_{p+1<1>} = -A^{<k_{p+1}>}_{-1}(Z^-_{p+1<1>} + Q_{p+1})^{-1}C^{<k_{p+1}>}_{-1} = -\tfrac{2k_{p+1}}{z^{[p+1]+}_{11}a^0_{11}(k_{p+1})}A^{-1}_0(k_{p+1})$$
$$\times Z^+_{p+1}C_0(k_{p+1})\tfrac{(z^{[p+1]+}_{11})^2a^0_{11}(k_{p+1})^3}{-4k^2_{p+1}}\left(Z^+_{p+1}C_0(k_{p+1}) + Q_{p+1}\tfrac{z^{[p+1]+}_{11}a^0_{11}(k_{p+1})^3}{-4k^2_{p+1}}\right)^{-1}$$
$$\times\tfrac{2k_{p+1}}{z^{[p+1]+}_{11}a^0_{11}(k_{p+1})}Z^+_{p+1} = \tfrac{1}{a^0_{11}(k_{p+1})^2}a^0_{11}(k_{p+1})(z^{[p+1]+}_{11})^2a^0_{11}(k_{p+1})^3\tfrac{1}{z^{[p+1]+}_{11}a^0_{11}(k_{p+1})}$$
$$\times\tfrac{1}{z^{[p+1]+}_{11}a^0_{11}(k_{p+1})}Z^+_{p+1} = Z^+_{p+1},$$

and in Case II Lemma 11 is proved too.

Case III). $Z^+_{p+1}(t) = \begin{pmatrix} z^{[p+1]+}_{11} & z^{[p+1]+}_{12}(t) \\ 0 & z^{[p+1]+}_{22} \end{pmatrix}$: $z^{[p+1]+}_{11} > 0$, $z^{[p+1]+}_{22} > 0$, then, similarly to Case I we obtain

$$A(k) = \gamma(k)A_0(k), \quad B(k) = \gamma(-k)B_0(k), \; k \in \mathbf{R}, \tag{88}$$

with

$$\gamma(k) = \tfrac{k-k_{p+1}}{k+k_{p+1}}I + i\Big\{\tfrac{a^0_{11}(k_{p+1})}{(k+k_{p+1})z^{[p+1]+}_{22}a^0_{22}(k_{p+1})} + \tfrac{a^0_{22}(k_{p+1})}{(k+k_{p+1})z^{[p+1]+}_{11}a^0_{11}(k_{p+1})}$$
$$-\tfrac{2k_{p+1}a^0_{11}(k_{p+1})}{(k+k_{p+1})^2z^{[p+1]+}_{11}a^0_{11}(k_{p+1})}\Big\}(Z^+_{p+1})'(0). \tag{89}$$

The eigenvalues k^2_j, $j = \overline{1,p}$, of the equations (79) and (62) coincide, and (79) has one more eigenvalue k^2_{p+1} by (88) and (89).

(24) and (14) imply that the reflection coefficients for the equations (79) and (62) coincide $R^+_1(k) = R^+(k)$, and

$$C(k) = C_0(k)\gamma^{-1}(-k). \tag{90}$$

The coincidence of the initial p normalizing polynomials of (79) and (62) can be deduced similarly to Case I.

In this context,

$$Z^-_{j<1>}(t) = -i\{\dot{W}^-(k_j)A^{0<k_j>}_{-2}\gamma^{-1}(k_j) + W^-(k_j)A^{0<k_j>}_{-2}\frac{d}{dk}(\gamma^{-1}(k))_{k_j}$$
$$+W^-(k_j)A^{0<k_j>}_{-1}\gamma^{-1}(k_j)\} - tW^-(k_j)A^{0<k_j>}_{-2}\gamma^{-1}(k_j). \tag{91}$$

Using $k_j < k_{p+1}$, $j = \overline{1,p}$, as $x \to -\infty$, we get

$$Z^-_{j<1>}(t) = \gamma(-k_j)Z^-_j(t)\gamma^{-1}(k_j) - \frac{4ik_{p+1}(k_j + k_{p+1})}{(k_j - k_{p+1})^3}(Z^-_j)'(0).$$

Thus, (30) with $\tau = t$, (88) and (89) imply

$$Z^+_{j<1>}(t) = -[A^{0<k_j>}_{-1}\gamma^{-1}(k_j) + A^{0<k_j>}_{-2}\tfrac{d}{dk}(\gamma^{-1}(k))_{k_j}]\gamma(k_j)\{\big(Z^-_j(t) + \gamma(k_j)Q_j$$
$$\times\gamma^{-1}(-k_j)\big)^{-1} + \tfrac{4ik_{p+1}(k_j+k_{p+1})}{(k_j-k_{p+1})^3(z^{[j]-}_{11}+(\frac{k_j-k_{p+1}}{k_j+k_{p+1}})^2q^{[j]}_{11})(z^{[j]-}_{22}+(\frac{k_j-k_{p+1}}{k_j+k_{p+1}})^2q^{[j]}_{22})}\big(\tfrac{k_j-k_{p+1}}{k_j+k_{p+1}}\big)^2$$
$$\times(Z^-_j)'(0)\}\gamma^{-1}(-k_j)[\gamma(-k_j)C^{0<k_j>}_{-1} - \dot\gamma(-k_j)C^{0<k_j>}_{-2}]$$
$$= Z^+_j(t) + A^{0<k_j>}_{-1}(Z^-_j(t) + \gamma(k_j)Q_j\gamma^{-1}(-k_j))^{-1}\gamma^{-1}(-k_j)\dot\gamma(-k_j)C^{0<k_j>}_{-2}$$
$$-A^{0<k_j>}_{-2}\tfrac{d}{dk}(\gamma^{-1}(k))_{k_j}\gamma(k_j)(Z^-_j(t) + \gamma(k_j)Q_j\gamma^{-1}(-k_j))^{-1}C^{0<k_j>}_{-1}$$
$$-\tfrac{4ik_{p+1}}{(k_j^2-k^2_{p+1})(z^{[j]-}_{11}+(\frac{k_j-k_{p+1}}{k_j+k_{p+1}})^2q^{[j]}_{11})(z^{[j]-}_{22}+(\frac{k_j-k_{p+1}}{k_j+k_{p+1}})^2q^{[j]}_{22})}A^{0<k_j>}_{-1}(Z^-_j)'(0)C^{0<k_j>}_{-1} = Z^+_j(t)$$

in view of (19) and (21) and since $Q_j = 0$, $j = \overline{1,p}$, with the definitions of Q_j of Lemma 3 [15], being taken into account, as $z^{[j]-}_{11} > 0$ and $z^{[j]-}_{22} > 0$. On the other hand, if $z^{[j]-}_{11} = 0$ or $z^{[j]-}_{22} = 0$, then by assumption 5) of the theorem $(Z^-_j)'(0) = 0$, $A^{0<k_j>}_{-2} = (A_0(k)^{-1}(k-k_j)^2)_{k_j} = 0$; $C^{0<k_j>}_{-2} = (C_0(k)^{-1}(k-k_j)^2)_{k_j} = 0$, hence $Z^+_{j<1>}(t) = Z^+_j(t)$, $j = \overline{1,p}$.

Prove that $Z^+_{p+1}(t)$ is a normalizing polynomial of the problem (79), (2) with $m \geq 1$.

Using (88) and (89), similarly to (91), one has

$$Z^-_{p+1<1>}(t) = -2ik_{p+1}W^-(k_{p+1})A^-_0(k_{p+1}) - \tfrac{2k_{p+1}}{z^{[p+1]+}_{22}a^0_{22}(k_{p+1})}\dot W^-(k_{p+1})(Z^+_{p+1})'(0)$$
$$-\Big\{\tfrac{1}{z^{[p+1]+}_{22}a^0_{22}(k_{p+1})} + \tfrac{a_{22}(k_{p+1})}{z^{[p+1]+}_{11}a^0_{11}(k_{p+1})^2} - \tfrac{2k_{p+1}\dot a^0_{11}(k_{p+1})}{a^0_{11}(k_{p+1})z^{[p+1]+}_{22}a^0_{22}(k_{p+1})}\Big\}W^-(k_{p+1})(Z^+_{p+1})'(0)$$
$$+\tfrac{2ik_{p+1}t}{z^{[p+1]+}_{22}a^0_{22}(k_{p+1})}W^-(k_{p+1})(Z^+_{p+1})'(0).$$

It is easy to show using the asymptotics as $x \to -\infty$

$$E^0_+(x,k_{p+1}) \sim e^{ik_{p+1}x}A_0(k_{p+1});\ \widetilde E^0_+(x,k_{p+1}) \sim e^{ik_{p+1}x}C_0(k_{p+1}),$$

that with $x \to -\infty$

$$Z^-_{p+1<1>}(t) = (2ik_{p+1})^2[A_0(k_{p+1})Z^+_{p+1}(t)C_0(k_{p+1})]^{-1}$$
$$+2ik_{p+1}\Big[\tfrac{1}{z^{[p+1]+}_{11}z^{[p+1]+}_{22}a^0_{11}(k_{p+1})a^0_{22}(k_{p+1})}\Big(2 - 2k_{p+1}\Big(\tfrac{\dot a^0_{22}(k_{p+1})}{a^0_{22}(k_{p+1})} + \tfrac{\dot a^0_{11}(k_{p+1})}{a^0_{11}(k_{p+1})}\Big)\Big)$$
$$+\tfrac{a^0_{11}(k_{p+1})}{(z^{[p+1]+}_{22})^2a^0_{22}(k_{p+1})^3} + \tfrac{a^0_{22}(k_{p+1})}{(z^{[p+1]+}_{11})^2a^0_{11}(k_{p+1})^3}\Big](Z^+_{p+1})'(0). \quad (92)$$

(30) with $\tau = t$ implies

$$Z^+_{p+1<1>}(t) = -A^{<k_{p+1}>}_{-1}(Z^-_{p+1<1>}(t))^{-1}C^{<k_{p+1}>}_{-1}. \quad (93)$$

Use (88)–(90) and the definition to get

$$A_{-1}^{<k_{p+1}>} = \tfrac{d}{dk}(A^{-1}(k)(k-k_{p+1})^2)_{k_{p+1}} = 2k_{p+1}A_0^{-1}(k_{p+1})$$

$$-i\Big[\tfrac{1}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})} + \tfrac{a_{22}(k_{p+1})}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})^2} - \tfrac{2k_{p+1}\dot{a}_{11}^0(k_{p+1})}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})a_{11}^0(k_{p+1})}\Big](Z_{p+1}^+)'(0), \quad (94)$$

and in a similar way,

$$C_{-1}^{<k_{p+1}>} = 2k_{p+1}C_0^{-1}(k_{p+1}) - i\Big[\tfrac{1}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})}$$

$$+\tfrac{a_{11}(k_{p+1})}{z_{22}^{[p+1]+}a_{22}^0(k_{p+1})^2} - \tfrac{2k_{p+1}\dot{a}_{11}^0(k_{p+1})}{z_{11}^{[p+1]+}a_{11}^0(k_{p+1})a_{22}^0(k_{p+1})}\Big](Z_{p+1}^+)'(0). \quad (95)$$

Use (92), (94), and (95) to deduce from (93), after some obvious computations, that $Z_{p+1<1>}^+(t) = Z_{p+1}^+(t)$, and Lemma 11 is proved in Case III too and therefore it is proved completely.

It remains to notice that the scattering problems constructed above do not have virtual levels, since $R^+(0) = R^-(0) = -I$ for them.

Theorem 1 is proved completely.

Correction to Part I. In Remark 2 [15] the functions $\psi^{\pm}(z)$ and $\psi^+(0)$ must be determined by modified formulas (43), that is by (43) with additional multiplier $\prod_{j=1}^{p}(\frac{k-k_j}{k+k_j})^{s_j^1}$ in the integrand.

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