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Solution to the Homogeneous Boundary Value Problems of Free Vibrations of a Finite String

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In the paper solutions to the homogeneous boundary value problems of free vibrations of a finite string are obtained in the d'Alembert form. The initial boundary value problems to vibrations of a string with free ends as well as with one end fixed and one free are solved.

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1. Introduction

The procedures for solving a one-dimensional wave equation are detailed in [1] and they are used to study a great number of oscillatory motions. One of the simplest procedures is the method of continuation. But its application is restricted because of complexity of the analysis of the obtained solution; we do not have a unified formula solution in many important cases. In the paper [2] such a solution is given to the problem of free vibrations of a finite string with fixed ends. At the beginning we introduce the function

$$stc(x, w) = \frac{w}{2\pi} \arccos \cos \frac{2\pi x}{w} \equiv \left| x - w \left[\frac{x}{w} + \frac{1}{2} \right] \right|, \ w > 0, \tag{1}$$

where square brackets $[z]$ denote the greatest integer that is less than or equal to z. The expression $stc(x, w)$ represents the continuous even piecewise-linear periodic function of the period w. It takes the values $stc(x, 2L) = x - 2kL$ on the intervals $2kL \le x \le (2k + 1)L$ $(k \in \mathbb{Z})$ and the values $stc(x, 2L) = 2(k + 1)L - x$ on the intervals $(2k + 1)L \le x \le 2(k + 1)L$. In the paper [2] we proved two propositions.

Proposition 1.1. *Let $f(x)$ be a function defined on the interval $[0, L]$. Perform its even extension to the interval $[-L, 0]$ and then periodic extension from the interval $[-L, L]$ to the whole axis. The formula of even periodic extension looks like the following:*

$$F_e(x) = f(stc(x, 2L)). \tag{2}$$

Proposition 1.2. *Let $f(x)$ be a function defined on the interval $[0, L]$ such that $f(0) = f(L) = 0$. The odd periodic extension of the function from the interval $[0, L]$ to the whole axis can be constructed by the following formula:*

$$F_o(x) = (-1)^{[x/L]} f(stc(x, 2L)). \tag{3}$$

The proof of these propositions is performed by verification of the fact that the functions $F_e(x)$, $F_o(x)$ coincide with the function $f(x)$ on the interval $[0, L]$; the function $F_e(x)$ coincides with $f(|x|)$ and $F_o(x)$ with $-f(|x|)$ on the interval $[-L, 0]$. Both functions have a period $2L$. As a result they are even and odd periodic extensions of the function $f(x)$ from the interval $[0, L]$ to the whole real axis. Also, it is easy to see that if the function $f(x)$ is continuous on $0 \leq x \leq L$, the constructed functions $F_e(x)$ and $F_o(x)$ are continuous on $\mathbb{R}$.

We consider the first boundary value problem for the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \tag{4}$$

on a finite interval $0 < x < L$ with the general initial conditions

$$u(x, 0) = \varphi(x), \;\; u'_t(x, 0) = \psi(x), \quad 0 \leq x \leq L, \tag{5}$$

and homogeneous boundary conditions $u(0, t) = 0, \;\; u(L, t) = 0 \;\; (t \geq 0)$.

The main idea for the construction of solution to the problem in the paper [2] was proposed as follows. In accordance with the concept of the continuation method we extend the initial functions to be odd periodic functions. For this purpose we can use formula (3)

$$\Gamma(x) = (-1)^{[x/L]} \varphi(stc(x, 2L)), \quad \Psi(x) = (-1)^{[x/L]} \psi(stc(x, 2L)). \tag{6}$$

The following lemma is needed for the sequel [1].

Lemma 1.1. *If initial data in the problem on the propagation of vibrations along an infinite string are odd functions with respect to some point x_0, the corresponding solution at this point is equal to* 0.

The functions $\Gamma(x)$ and $\Psi(x)$ are odd with respect to points $x = 0$ and $x = L$. Therefore, using them in the formula of d'Alembert

$$u(x,t) = \frac{\Gamma(x+at) + \Gamma(x-at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \Psi(\alpha)d\alpha \qquad (7)$$

we get a solution to the equation of vibration of a finite string with fixed ends. Substituting Eqs. (6) into Eq. (7) and using the equality

$$\int\limits_{x-at}^{x+at} (-1)^{[\alpha/L]}\psi(stc(\alpha,2L))d\alpha = \int\limits_{stc(x-at,2L)}^{stc(x+at,2L)} \psi(\alpha)d\alpha \ ,$$

which was proved in [2], we obtain the following formula:

$$u(x,t) = \frac{1}{2}\Big((-1)^{[(x+at)/L]}\varphi(stc(x+at,2L)) + (-1)^{[(x-at)/L]}\varphi(stc(x-at,2L))\Big) + \frac{1}{2a}\int\limits_{stc(x-at,2L)}^{stc(x+at,2L)} \psi(\alpha)d\alpha. \qquad (8)$$

At any x and t we have $0 \le stc(x \pm at, 2L) \le L$. So the argument of the function $\varphi(stc(x \pm at, 2L))$ is contained in the limits of the interval $[0, L]$ on which the function $\varphi(x)$ is given. The integration in Eq. (8) is always performed over an interior section of the interval $[0, L]$ on which the function $\psi(x)$ is known too. Therefore, formula (8) gives the explicit solution to the problem of free vibrations of the finite string with fixed ends.

If the initial functions $\varphi(x) \in C^2([0, L])$ and $\psi(x) \in C^1([0, L])$ satisfy the compatibility conditions $\varphi(0) = \varphi(L) = 0$, $\psi(0) = \psi(L) = 0$, $\varphi''(0) = \varphi''(L) = 0$, the solution given by d'Alembert's formula will have continuous derivatives of the first and second orders [1, 3]. But Eq. (8) is the transformed formula of d'Alembert considered on the interval $0 \le x \le L$. Therefore, if the compatibility conditions are satisfied, then Eq. (8) gives a representation of classical solution of the problem on the interval $[0, L]$.

In the present paper a similar approach is developed to construct a solution to the initial boundary value problem for the equation of vibration of a finite string with free ends (formula (13)) and a string with one end fixed and one free (formula (19)).

2. Free Boundary Conditions

Consider a problem of vibrations of a finite string with free ends. For this purpose we have to solve the wave equation (4) with the general initial conditions (5) and boundary conditions $u'(0, t) = 0,\ u'(L, t) = 0$.

As we know [1], the functions $\Gamma(x)$ and $\Psi(x)$ from the d'Alembert solution (7) must be even periodic extensions of the initial functions $\varphi(x)$ and $\psi(x)$. Therefore we can use formula (2)

$$\Gamma(x) = \varphi(stc(x, 2L)), \quad \Psi(x) = \psi(stc(x, 2L)). \tag{9}$$

Indeed, the function $stc(x, 2L)$ coincides with $|x|$ on the interval $[-L, L]$ and hence $\Gamma(x) = \Gamma(-x)$, $\Psi(x) = \Psi(-x)$ for $-L \le x \le L$. As the function $stc(x, 2L)$ is periodic of the period $2L$, the functions $\Gamma(x)$, $\Psi(x)$ are periodic of the same period, too. Consequently, they will be even periodic extensions of initial functions from the interval $[0, L]$ to the whole real axis.

Then we employ the following lemma [1]

Lemma 2.1. *If initial data in the problem on the propagation of vibrations along an infinite string are even functions with respect to some point x_0, then the derivative in x of the corresponding solution at this point is equal to* 0.

The functions $\Gamma(x)$ and $\Psi(x)$ are even with respect to the points $x = 0$ and $x = L$. Therefore, using them in the formula of d'Alembert (7), which should be considered only on the interval $[0, L]$, we get a solution to the equation of vibrations of a finite string with free ends. Substituting (9) in (7), we obtain

$$u(x,t) = \frac{\varphi(stc(x+at, 2L)) + \varphi(stc(x-at, 2L))}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(stc(\alpha, 2L))d\alpha. \tag{10}$$

The first item of Eq. (10) is convenient for calculations. Now, let us study the integral $\Omega(x,t) = \int_{x-at}^{x+at} \psi(stc(\alpha, 2L))d\alpha$ on the right-hand side of Eq. (10). For this we will need the following lemma.

Lemma 2.2. *Let $f(x)$ be a piecewise continuous function defined on the interval $[0, L]$. Then the following equality is valid*

$$\int_0^x f(stc(\tau, 2L))d\tau = (-1)^{[x/L]} \int_0^{stc(x,2L)} f(\tau)d\tau + 2\Big[\frac{x+L}{2L}\Big] \int_0^L f(\tau)d\tau. \tag{11}$$

P r o o f. As values of the function $stc(x, 2L)$ at any x are contained in the limits of the interval $[0, L]$, the integrals on both sides of Eq. (11) exist. To prove the formula, we need to verify that the right-hand and the left-hand members of Eq. (11) coincide at any x.

Denote the right-hand side of Eq. (11) as $I(x)$ and define $\Phi(x) \equiv \int_0^x f(\tau)d\tau + C$, where C is any constant. As $f(x)$ is a piecewise continuous function on the interval $[0, L]$, then $\Phi(x)$ is a continuous function on the same interval.

Let $2kL \le x < (2k+1)L,\ \ k = 0, 1, 2, \ldots$. Then

$$\int_0^x f(stc(\tau, 2L))d\tau = \int_0^L f(\tau)d\tau + \int_L^{2L} f(2L-\tau)d\tau + \ldots$$

$$+ \int_{2kL}^x f(\tau - 2kL)d\tau = \underbrace{(\Phi(L) - \Phi(0)) + \ldots + (\Phi(L) - \Phi(0))}_{2k}$$

$$+\Phi(\tau - 2kL)\Big|_{2kL}^x = 2k(\Phi(L) - \Phi(0)) + (\Phi(x - 2kL) - \Phi(0)).$$

But in this case we have $stc(x, 2L) = x - 2kL$, $[x/L] = 2k$, and $[(x+L)/2L] = k$. Hence $I(x) = \Phi(x - 2kL) - \Phi(0) + 2k(\Phi(L) - \Phi(0))$ and we obtain $I(x) = \int_0^x f(stc(\tau, 2L))d\tau$.

Let $(2k+1)L \le x < 2(k+1)L$, $k = 0, 1, 2, \ldots$. Then

$$\int_0^x f(stc(\tau, 2L))d\tau = \int_0^L f(\tau)d\tau + \int_L^{2L} f(2L-\tau)d\tau + \ldots + \int_{(2k+1)L}^x f(2(k+1)L - \tau)d\tau$$

$$= \underbrace{(\Phi(L) - \Phi(0)) + \ldots + (\Phi(L) - \Phi(0))}_{2k+1} - \Phi(2(k+1)L - \tau)\Big|_{(2k+1)L}^x$$

$$= (2k+1)(\Phi(L) - \Phi(0)) - (\Phi(2(k+1)L - x) - \Phi(L)).$$

In this case we have $[x/L] = 2k+1$, $[(x+L)/2L] = k+1$ and $stc(x, 2L) = 2(k+1)L - x$. Hence

$$I(x) = -\left(\Phi(stc(x, 2L)) - \Phi(0)\right) + 2(k+1)\left(\Phi(L) - \Phi(0)\right)$$

$$= -\left(\Phi(2(k+1)L - x) - \Phi(L)\right) + (2k+1)\left(\Phi(L) - \Phi(0)\right).$$

Thus in this case we have $I(x) = \int_0^x f(stc(\tau, 2L))d\tau$, too. Therefore Eq. (11) is satisfied for all $x \ge 0$.

Show that Eq. (11) is valid for $x < 0$. Here by virtue of evenness of the integrand we have $\int_0^x f(stc(\tau, 2L))d\tau = -\int_0^{|x|} f(stc(\tau, 2L))d\tau$. Formula (11) already is valid for the last integral. Thus, for $x < 0$ we have

$$\int_0^x f(stc(\tau, 2L))d\tau = -(-1)^{[|x|/L]} \int_0^{stc(x,2L)} f(\tau)d\tau - 2\Big[\frac{|x| + L}{2L}\Big] \int_0^L f(\tau)d\tau. \qquad (12)$$

For a nonintegral z the equality occurs $[-z] = -[z] - 1$. By setting $z = x/L$ $(x \ne kL)$, this equality results $[-x/L] = -[x/L] - 1$, and for $x < 0$ we obtain

$[|x|/L] = -[x/L] - 1$. Then $-(-1)^{[|x|/L]} = (-1)^{-[x/L]} = (-1)^{[x/L]}$. By setting $z = (x-L)/(2L)$, we obtain $[-(x-L)/(2L)] = -[(x-L)/(2L)] - 1$. Therefore, for $x < 0$ we have $[(|x|+L)/(2L)] = -[(x-L)/(2L)+1] = -[(x+L)/(2L)]$. If we change the factors in Eq. (12), which contain $|x|$ to their values without one, we again obtain Eq. (11).

It remains only to verify that Eq. (11) is valid for $x = kL$, $k = -1, -2, \ldots$. This can be seen by direct verification. Thus the left-hand and the right-hand members of Eq. (11) coincide at all x. The lemma is proved. ■

Let us return to the integral $\Omega(x,t)$ on the right-hand side of Eq. (10). Taking into account (11), we get

$$\Omega(x,t) = \int_0^{x+at} \psi(stc(\alpha,2L))d\alpha - \int_0^{x-at} \psi(stc(\alpha,2L))d\alpha = (-1)^{\left[\frac{x+at}{L}\right]} \int_0^{stc(x+at,2L)} \psi(\alpha)d\alpha$$
$$-(-1)^{\left[\frac{x-at}{L}\right]} \int_0^{stc(x-at,2L)} \psi(\alpha)d\alpha + 2\Big(\Big[\frac{x+at+L}{2L}\Big] - \Big[\frac{x-at+L}{2L}\Big]\Big)\int_0^L \psi(\alpha)d\alpha.$$

Substituting this equality in (10), we finally obtain

$$u(x,t) = \frac{\varphi(stc(x+at,2L)) + \varphi(stc(x-at,2L))}{2}$$
$$+ \frac{1}{2a}\Big((-1)^{\left[\frac{x+at}{L}\right]} \int_0^{stc(x+at,2L)} \psi(\alpha)d\alpha - (-1)^{\left[\frac{x-at}{L}\right]} \int_0^{stc(x-at,2L)} \psi(\alpha)d\alpha$$
$$+ 2\Big(\Big[\frac{x+at+L}{2L}\Big] - \Big[\frac{x-at+L}{2L}\Big]\Big)\int_0^L \psi(\alpha)d\alpha\Big). \tag{13}$$

The argument of the function $\varphi(stc(x \pm at, 2L))$ is contained in the interval $[0, L]$ and integration is also performed over $[0, L]$. Therefore, formula (13) gives an explicit solution to the problem of vibrations of the finite string with free ends.

It is known that the formula of d'Alembert (7) gives doubly continuously differentiable solution to the wave equation provided $\Gamma(x) \in C^2(\mathbb{R})$ and $\Psi(x) \in C^1(\mathbb{R})$. But Eq. (13) actually is d'Alembert's solution with the functions $\Gamma(x) = \varphi(stc(x, 2L))$ and $\Psi(x) = \psi(stc(x, 2L))$.

The function $\Gamma(x)$ will be of class $C^2(\mathbb{R})$ provided $\varphi(x) \in C^2([0, L])$ and $\varphi'(0) = \varphi'(L) = 0$. Indeed, derivative discontinuities can be only at points

$x = kL$, $k \in \mathbb{Z}$. At points $x = 2kL$ we have

$$\Gamma'_+(2kL) = \lim_{x\to 2kL+0} \frac{\Gamma(x) - \Gamma(2kL)}{x - 2kL} = \lim_{x\to 2kL+0} \frac{\varphi(x - 2kL) - \varphi(0)}{x - 2kL} = \varphi'0),$$
$$\Gamma'_-(2kL) = \lim_{x\to 2kL-0} \frac{\Gamma(x) - \Gamma(2kL)}{x - 2kL} = \lim_{x\to 2kL-0} \frac{\varphi(2kL - x) - \varphi(0)}{x - 2kL} = -\varphi'(0).$$

If $\varphi'(0) = 0$, then the left-hand $\Gamma'_-(2kL)$ and the right-hand $\Gamma'_+(2kL)$ derivatives coincide. So $\Gamma'(2kL)$ exists. In addition, we have

$$\lim_{x\to 2kL+0} \frac{d\Gamma}{dx} = \lim_{x\to 2kL+0} \frac{d}{dx}\varphi(x - 2kL) = \varphi'(0),$$
$$\lim_{x\to 2kL-0} \frac{d\Gamma}{dx} = \lim_{x\to 2kL-0} \frac{d}{dx}\varphi(2kL - x) = -\varphi'(0).$$

As a result, the derivative $\Gamma'(x)$ will be continuous at points $x = 2kL$ provided $\varphi'(0) = 0$.

Repeating in the same way as above, we can establish that the derivative $\Gamma'(x)$ exists and it is continuous at points $x = (2k+1)L$, $k \in \mathbb{Z}$ provided $\varphi'(L) = 0$.

The same proof applies for $\Gamma''(x)$ also. In this case the second derivative exists and it is continuous at any x.

In the same way we can establish that $\Psi(x) = \psi(stc(x, 2L)) \in C^1(\mathbb{R})$ provided $\psi(x) \in C^1([0, L])$ and $\psi'(0) = \psi'(L) = 0$.

We have obtained the known result [3]. If initial functions $\varphi(x) \in C^2([0, L])$, $\psi(x) \in C^1([0, L])$ satisfy the compatibility conditions $\varphi'(0) = \varphi'(L) = 0$, $\psi'(0) = \psi'(L) = 0$, the d'Alembert solution (7) will have continuous derivatives of the first and second orders. Then solution (13), which is the transformed d'Alembert formula considered on the interval $0 \le x \le L$, gives a representation of classical solution to the problem.

R e m a r k. If $\int_0^L \psi(\tau)d\tau \ne 0$, then solution (13) grows in time. To understand this, we analyze the last item in Eq. (13), which we denote as $G(x,t) = 2A \cdot \int_0^L \psi(\alpha)d\alpha$, where $A = [(x + at + L)/(2L)] - [(x - at + L)/(2L)]$.

Taking into account $(at)/(2L) = [(at)/(2L)] + \varepsilon$, where $0 \le \varepsilon(t) < 1$, we can write

$$A = \Big[\frac{x+L}{2L} + \Big[\frac{at}{2L}\Big] + \varepsilon\Big] - \Big[\frac{x+L}{2L} - \Big[\frac{at}{2L}\Big] - \varepsilon\Big]$$
$$= \Big[\frac{x+L}{2L} + \varepsilon\Big] - \Big[\frac{x+L}{2L} - \varepsilon\Big] + 2\Big[\frac{at}{2L}\Big].$$

Since $0 \le x \le L$, then $1/2 \le (x + L)/(2L) \le 1$. Therefore the expression $[(x + L)/(2L) + \varepsilon]$ can be equal to 0 or 1. The expression $[(x + L)/(2L) - \varepsilon]$ can be equal to -1 or 0; also it can be equal to 1 in the case $x = L$ and $\varepsilon = 0$. In any case, as it is easy to see, we have $[(x + L)/(2L) + \varepsilon] - [(x + L)/(2L) - \varepsilon] \ge 0$

because this expression may be equal only to 0, 1 or 2. Therefore $A \geq 2\,[(at)/(2L)]$ and $|G(x,t)| \geq 4\,[(at)/(2L)] \left|\int_0^L \psi(\tau)d\tau\right|$. Obviously, the other items of Eq. (13) are bounded. Therefore the solution $u(x,t)$ grows in time.

R e m a r k. Solution (13) is bounded if $\int_0^L \psi(\tau)d\tau = 0$. Also, it is easy to see that in this case at a given x the function $u(x,t)$ will be $2L/a$ periodic in t.

3. Homogeneous Mixed Boundary Conditions

Consider a problem of vibrations of a finite string which is fastened at the left end and is free at the right end. In this case we have to solve the wave equation (4) with the general initial conditions (5) and boundary conditions $u(0,t) = 0$, $u'(L,t) = 0$.

Consider a function $f(x)$ that is given on the interval $[0, L]$ such that $f(0) = 0$. Construct its even extension to the interval $[L, 2L]$, i.e., construct the function $f_1(x)$, $x \in [0, 2L]$ so that $f_1(x) = f(x)$ for $x \in [0, L]$ and $f_1(x) = f(2L - x)$ for $x \in [L, 2L]$. Extend the function $f_1(x)$ to be odd on the interval $[-2L, 2L]$, i.e., construct the function $f_2(x)$, $x \in [-2L, 2L]$ such that $f_2(x) = f_1(x)$ for $x \in [0, 2L]$ and $f_2(x) = -f_1(-x)$ for $x \in [-2L, 0]$. After that, extend $f_2(x)$ to be a periodic function of the period $4L$. As a result, the obtained function $F(x)$ coincides with $f(x)$ on the interval $x \in [0, L]$, with $f_1(x)$ on the interval $x \in [0, 2L]$ and with $f_2(x)$ on the interval $x \in [-2L, 2L]$.

For the function $F(x)$ the following properties hold:

(i) $F(x) = F(x + 4kL)$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$;

(ii) $F(x) = -F(-x)$, $x \in \mathbb{R}$.

This property is satisfied by the construction for $x \in [-2L, 2L]$. Let $x \notin [-2L, 2L]$. It is always possible to choose $x_0 \in [-2L, 2L]$ and $k \in \mathbb{Z}$ such that $x = x_0 + 4kL$. Then we have

$$F(x) = F(x_0 + 4kL) = F(x_0) = -F(-x_0) = -F(-x_0 - 4kL) = -F(-x);$$

(iii) $F(L - z) = F(L + z)$, $\forall z \in \mathbb{R}$.

This property expresses the evenness with respect to the point $x = L$. We have the property $F(x) = F(2L - x)$ for $x \in [0, 2L]$ by construction. Show that it is satisfied for all x. Indeed, let $x \notin [-2L, 2L]$. It is always possible to choose $x_0 \in [-2L, 2L]$ and $k \in \mathbb{Z}$ such that $x = x_0 + 4kL$.
If $x_0 \in [0, 2L]$, we have

$$F(x) = F(x_0 + 4kL) = F(x_0) = F(2L - x_0) = F(2L - x_0 - 4kL) = F(2L - x).$$

If $x_0 \in [-2L, 0]$, we have

$$\begin{aligned} F(x) &= F(x_0 + 4kL) = F(x_0) = -F(-x_0) = -F(|x_0|) = -F(2L - |x_0|) \\ &= -F(2L + x_0) = F(-x_0 - 2L) = F(-x_0 - 2L - 4(k-1)L) \\ &= F(2L - (x_0 + 4kL)) = F(2L - x). \end{aligned}$$

Therefore the condition $F(x) = F(2L - x)$ is satisfied far all x. Replacing x by $L - z$, we obtain $F(L - z) = F(L + z)$.

We note that the condition $f(0) = 0$ is necessary to avoid the ambiguities of function $F(x)$ at points $x = 2kL$, $k \in \mathbb{Z}$. Moreover, if the function $f(x)$ is continuous on the interval $[0, L]$ and $f(0) = 0$, the obtained function $F(x)$ is continuous on the whole real axis.

We will call the described above extension of the function as an even-odd periodic continuation.

Let us return to the boundary value problem. As it is required that the string should be fastened at $x = 0$, the initial functions must be subjected to the restrictions $\varphi(0) = 0$, $\psi(0) = 0$. Extend the initial functions $\varphi(x)$ and $\psi(x)$, which are given on the segment $[0, L]$, to be even-odd periodic functions of the period $4L$. Denote the obtained functions as $\Gamma(x)$, $\Psi(x)$ and substitute them into the d'Alembert solution (7).

Lemma 3.1. *If the initial data $\Gamma(x)$ and $\Psi(x)$ in the problem on the propagation of vibrations along an infinite string are even-odd periodic continuations of the initial functions $\varphi(x)$ and $\psi(x)$, then the formula of d'Alembert (7), which is actually considered on the interval $[0, L]$, represents a solution of the considered initial boundary value problem.*

P r o o f. Indeed, we have constructed the initial functions $\Gamma(x)$, $\Psi(x)$ for the infinite string so that they are odd functions with respect to the point $x = 0$. Therefore, by Lemma 1.1 the d'Alembert solution at this point will be equal to 0. Moreover, the functions $\Gamma(x)$, $\Psi(x)$ have been constructed so that they are even functions with respect to the point $x = L$. Then, by Lemma 2.1, the derivative in x of d'Alembert's solution will be equal to 0 at this point, i.e., $u'_x(L, t) = 0$. Thus both boundary conditions $u(0, t) = 0$, $u'(L, t) = 0$ are satisfied.

From (7) we have $u(x, 0) = \frac{\Gamma(x)+\Gamma(x)}{2} + \frac{1}{2a}\int_x^x \Psi(\alpha)d\alpha = \Gamma(x)$. But we have $\Gamma(x) = \varphi(x)$ for $0 \le x \le L$ and therefore $u(x, 0) = \varphi(x)$ on this interval. Next, we have $u'_t(x, 0) = \frac{1}{2}(a\Gamma'(x) - a\Gamma'(x)) + \frac{1}{2a}(a\Psi(x) - (-a)\Psi(x)) = \Psi(x)$. But we have $\Psi(x) = \psi(x)$ for $0 \le x \le L$ and hence $u'_t(x, 0) = \psi(x)$. So the initial conditions are satisfied also. The lemma is proved. ■

Thus, when it is required that the string should be fastened at the left end and free at the right end, the initial functions $\varphi(x)$, $\psi(x)$ must be even-odd periodic extended; then the obtained functions $\Gamma(x)$, $\Psi(x)$ must be substituted into d'Alembert's solution (7).

Lemma 3.2. *Let $f(x)$ be a function defined on the interval $0 \le x \le L$ such that $f(0) = 0$. The formula of even-odd periodic continuation of the function $f(x)$ to the whole real axis looks like the following:*

$$F_{eo}(x) = (-1)^{[x/(2L)]} f(stc(x, 2L)). \tag{14}$$

P r o o f. Let a function $f(x)$ is given on the interval $[0, L]$ and $f(0) = 0$. Perform its even extension from the interval $[0, L]$ to the interval $[L, 2L]$. This can be done by formula $f_1(x) = f(L - |L - x|)$. In fact, we have $f_1(z) = f(z)$ for $0 \le z \le L$ and $f_1(L+z) = f(L-|L-(L+z)|) = f(L-z) = f_1(L-z)$. Therefore $f_1(x)$ is an even function on the interval $[0, 2L]$ with respect to the point $x = L$.

After that we perform the odd extension of the function $f_1(x)$ from the interval $[0, 2L]$ to the interval $[-2L, 0]$. Then the obtained function is extended periodically from the interval $[-2L, 2L]$ to entire line $-\infty < x < \infty$. This can be done with the help of formula (3), where it is necessary to use $2L$ as half-period. So we have

$$F_{eo}(x) = (-1)^{[x/(2L)]} f_1(stc(x, 4L)) = (-1)^{[x/(2L)]} f(L - |L - stc(x, 4L)|).$$

Let us show that $L - |L - stc(x, 4L)| \equiv stc(x, 2L)$. It is evident that the expression $I(x) = L - |L - stc(x, 4L)|$ has a period $4L$, and the function $stc(x, 2L)$ has a period $2L$. Therefore it suffices to verify their coincidence on the interval $[0, 4L]$.

For $0 \le x \le L$ we have $stc(x, 4L) = x$ and $stc(x, 2L) = x$. Therefore $I(x) = L - |L - x| = L - (L - x) = x = stc(x, 2L)$.

For $L \le x \le 2L$ we have $stc(x, 4L) = x$ and $stc(x, 2L) = 2L - x$. Hence $I(x) = L - |L - x| = L - (x - L) = 2L - x = stc(x, 2L)$.

For $2L \le x \le 3L$ we have $stc(x, 4L) = 4L - x$ and $stc(x, 2L) = x - 2L$. Thus $I(x) = L - |L - (4L - x)| = L - |x - 3L| = L - (3L - x) = x - 2L = stc(x, 2L)$.

For $3L \le x \le 4L$ we have $stc(x, 4L) = 4L - x$ and $stc(x, 2L) = 4L - x$. Hence $I(x) = L - |L - (4L - x)| = L - |x - 3L| = L - (x - 3L) = 4L - x = stc(x, 2L)$.

Thus the functions $I(x)$ and $stc(x, 2L)$ coincide on the interval $[0, 4L]$, and by virtue of periodicity they coincide on the whole axis. Therefore we have

$$F_{eo}(x) = (-1)^{[x/(2L)]} f(L - |L - stc(x, 4L)|) = (-1)^{[x/(2L)]} f(stc(x, 2L)).$$

This proves the lemma. ∎

In view of (14), the even-odd continuation of the initial functions $\varphi(x)$, $\psi(x)$ can be written

$$\Gamma(x) = (-1)^{[x/(2L)]}\varphi(stc(x,2L)), \quad \Psi(x) = (-1)^{[x/(2L)]}\psi(stc(x,4L)). \qquad (15)$$

Then the solution of the boundary value problem, represented by d'Alembert's formula (7) becomes after the substitution of (15)

$$u(x,t) = \frac{(-1)^{\left[\frac{x+at}{2L}\right]}\varphi(stc(x+at,2L)) + (-1)^{\left[\frac{x-at}{2L}\right]}\varphi(stc(x-at,2L))}{2} + \frac{1}{2a}\int_{x-at}^{x+at}(-1)^{\left[\frac{\alpha}{2L}\right]}\psi(stc(\alpha,2L))d\alpha. \qquad (16)$$

The first item of Eq. (16) is convenient for calculations. Let us now consider the integral $\Omega(x,t) = \int_{x-at}^{x+at}\Psi(\alpha)d\alpha$, where the integrand $\Psi(x) = (-1)^{\left[\frac{\alpha}{2L}\right]}\psi(stc(\alpha,2L))$ is an odd periodic function with the period $4L$.

Lemma 3.3. *Let $f(x)$ be a piecewise continuous odd periodic function. Then the function $F(x)$ defined by $F(x) = \int_0^x f(\xi)d\xi$ will be an even periodic continuous function with the same period.*

P r o o f. It is obvious that $F(x)$ is a continuous function. Let T be a period of an odd function $f(x)$. Since $\int_{x_0}^{x_0+T} f(\xi)d\xi = 0$ for any x_0, it follows that $F(x+T) = \int_0^{x+T} f(\xi)d\xi = \int_0^T f(\xi)d\xi + \int_T^{x+T} f(\xi)d\xi = \int_T^{x+T} f(\xi)d\xi$. By changing variables $\xi = \tau + T$ and using periodicity $f(\tau+T) = f(\tau)$, we obtain

$$F(x+T) = \int_0^x f(\tau+T)d\tau = \int_0^x f(\tau)d\tau = F(x.)$$

Using the oddness of the function $f(\alpha) = -f(-\alpha)$, we have

$$F(-x) = \int_0^{-x} f(\xi)d\xi = -\int_0^x f(-\alpha)d\alpha = \int_0^x f(\alpha)d\alpha = F(x).$$

This proves the lemma. ■

Lemma 3.4. *Let $f(x)$ be a piecewise continuous function defined on the interval $0 \le x \le L$. Then the following holds*

$$\int_0^x(-1)^{\left[\frac{\alpha}{2L}\right]}f(stc(\alpha,2L))d\alpha = \int_0^L f(\xi)d\xi - (-1)^{\left[\frac{x+L}{2L}\right]}\int_{stc(x,2L)}^L f(\xi)d\xi. \qquad (17)$$

P r o o f. As we have $0 \le stc(x, 2L) \le L$, and $f(x)$ is a piecewise continuous function for $0 \le x \le L$, the integrals on both sides of Eq. (17) exist. The integrand $H(x) = (-1)^{[x/(2L)]} f(stc(x, 2L))$ is an odd periodic function with the period $4L$. We can apply Lemma 3.3 to the function $F(x) = \int_0^x H(\xi)d\xi$ that will be an even periodic function with the same period. If we evaluate the integral $\tilde{F}(x) = \int_0^x H(\xi)d\xi$ only for $0 \le x \le 2L$, then we can build the whole function $F(x)$ by even periodic continuation of expression $\tilde{F}(x)$ from interval $[0, 2L]$ to entire line $-\infty < x < \infty$.

Let us denote $\Phi(x) = \int_0^x f(\alpha)d\alpha + C$, where $C = \Phi(0)$ is any constant. As $f(x)$ is a piecewise continuous function on the interval $[0, L]$, then $\Phi(x)$ is a continuous function on the same interval.

For $0 \le x < L$ we have

$$\tilde{F}(x) = \int_0^x (-1)^{[\alpha/(2L)]} f(stc(\alpha, 2L))d\alpha = \int_0^x f(\alpha)d\alpha = \Phi(x) - \Phi(0).$$

For $L \le x < 2L$ we have

$$\begin{aligned}\tilde{F}(x) &= \int_0^x H(\alpha)d\alpha \\ &= \int_0^L f(\alpha)d\alpha + \int_L^x f(2L-\alpha)d\alpha = \Phi(L) - \Phi(0) - \Phi(2L-\alpha)\Big|_L^x \\ &= \Phi(L) - \Phi(0) - (\Phi(2L-x) - \Phi(L)) = 2\Phi(L) - \Phi(2L-x) - \Phi(0).\end{aligned}$$

As $\tilde{F}(x)$ is a continuous function for $0 \le x \le 2L$, then the last equality is valid for $x = 2L$. Hence, for the expression $\tilde{F}(x)$ we obtain

$$\tilde{F}(x) = \begin{cases} \Phi(x) - \Phi(0), & 0 \le x < L, \\ 2\Phi(L) - \Phi(2L-x) - \Phi(0), & L \le x \le 2L. \end{cases}$$

Show that the even periodic continuation of the expression $\tilde{F}(x)$ with a period $4L$ from the interval $[0, 2L]$ to the whole axis is given by

$$F(x) = (-1)^{\left[\frac{x+L}{2L}\right]} (\Phi(stc(x, 2L)) - \Phi(L)) + \Phi(L) - \Phi(0). \tag{18}$$

As the factor $(-1)^{[(x+L)/(2L)]}$ is a periodic function with the period $4L$, the right-hand member of Eq. (18) has a period $4L$ instead of $2L$. Therefore, it suffices to show that $F(x) = \tilde{F}(x)$ for $x \in [0, 2L]$ and $F(x) = \tilde{F}(-x)$ for $x \in (-2L, 0)$. For $0 \le x < L$ we have $stc(x, 2L) = x$ and hence

$$F(x) = (\Phi(x) - \Phi(L)) + \Phi(L) - \Phi(0) = \Phi(x) - \Phi(0) = \tilde{F}(x).$$

For $L \le x \le 2L$ we have $stc(x, 2L) = 2L - x$ and hence

$$F(x) = -\left(\Phi(2L - x) - \Phi(L)\right) + \Phi(L) - \Phi(0) = 2\Phi(L) - \Phi(2L - x) - \Phi(0) = \tilde{F}(x).$$

For $-L \le x < 0$ we have $stc(x, 2L) = -x$ and hence

$$F(x) = \left(\Phi(-x) - \Phi(L)\right) + \Phi(L) - \Phi(0) = \Phi(-x) - \Phi(0) = \tilde{F}(-x).$$

For $-2L < x < -L$ we have $stc(x, 2L) = x + 2L$ and hence

$$F(x) = -\left(\Phi(2L + x) - \Phi(L)\right) + \Phi(L) - \Phi(0) = 2\Phi(L) - \Phi(2L + x) - \Phi(0) = \tilde{F}(-x).$$

As a result, the function $F(x)$ coincides with $\tilde{F}(x)$ for $0 \le x \le 2L$ and it coincides with $\tilde{F}(-x)$ for $-2L < x < 0$. Therefore $F(x)$ represents an even periodic continuation of the expression $\tilde{F}(x)$ from the segment $[0, 2L]$ to the whole axis. Thus equality (18) holds. When the value of $\Phi(x)$ is substituted in Eq. (18), we obtain the desired form (17). The lemma is proved. ■

Let us now consider the integral $\Omega(x,t) = \int_{x-at}^{x+at} \Psi(\alpha) d\alpha$ standing in Eq. (16). Taking into account (17), we have

$$\Omega(x,t) = \int_{x-at}^{x+at} (-1)^{\left[\frac{\alpha}{2L}\right]} \psi(stc(\alpha, 2L)) d\alpha = \int_{0}^{x+at} - \int_{0}^{x-at}$$
$$= (-1)^{\left[\frac{x-at+L}{2L}\right]} \int_{stc(x-at,2L)}^{L} \psi(\xi) d\xi - (-1)^{\left[\frac{x+at+L}{2L}\right]} \int_{stc(x+at,2L)}^{L} \psi(\xi) d\xi.$$

Substituting the value of $\Omega(x,t)$ in (16), we finally obtain

$$u(x,t) = \frac{(-1)^{\left[\frac{x+at}{2L}\right]} \varphi(stc(x+at, 2L)) + (-1)^{\left[\frac{x-at}{2L}\right]} \varphi(stc(x-at, 2L))}{2}$$
$$+ \frac{1}{2a}\Big((-1)^{\left[\frac{x-at+L}{2L}\right]} \int_{stc(x-at,2L)}^{L} \psi(\xi) d\xi - (-1)^{\left[\frac{x+at+L}{2L}\right]} \int_{stc(x+at,2L)}^{L} \psi(\xi) d\xi \Big). \qquad (19)$$

The argument of function $\varphi(stc(x \pm at, 2L))$ is contained in the limits of interval $[0, L]$ and the integration is also performed in the limits of $[0, L]$. Therefore, formula (19) gives the explicit solution to the problem of vibrations of the finite string with one end fixed and one end free.

R e m a r k. It is easy to see that the solution (19) will be $4L/a$ periodic in t at given x.

As we know, d'Alembert's formula (7) for the infinite string gives doubly continuously differentiable solution to the wave equation provided $\Gamma(x) \in C^2(\mathbb{R})$ and $\Psi(x) \in C^1(\mathbb{R})$. But Eq. (19) actually is the d'Alembert solution with the functions $\Gamma(x) = (-1)^{[x/(2L)]}\varphi(stc(x, 2L))$ and $\Psi(x) = (-1)^{[x/(2L)]}\psi(stc(x, 4L))$.

The function $\Gamma(x)$ will be of class $C^2(\mathbb{R})$ if $\varphi(x) \in C^2([0, L])$ and satisfy the additional conditions of existence and continuity of the 1-st and 2-nd derivatives at points $x = kL$, $k \in \mathbb{Z}$. It follows immediately from the condition $\varphi(0) = 0$ that $\Gamma(x)$ is continuous at points $x = 2kL$, $k \in \mathbb{Z}$. The continuity at points $x = (2k+1)L$ follows from the evenness of function $\Gamma(x)$ with respect to these points.

Now, since $\varphi(0) = 0$, we have

$$\Gamma'_+(2kL) = \lim_{x\to 2kL+0} \frac{\Gamma(x) - \Gamma(2kL)}{x - 2kL}$$

$$= \lim_{x\to 2kL+0} \frac{(-1)^k\varphi(x - 2kL) - (-1)^k\varphi(0)}{x - 2kL} = (-1)^k\varphi'(0),$$

$$\Gamma'_-(2kL) = \lim_{x\to 2kL-0} \frac{\Gamma(x) - \Gamma(2kL)}{x - 2kL}$$

$$= \lim_{x\to 2kL-0} \frac{(-1)^{k-1}\varphi(2kL - x) - (-1)^k\varphi(0)}{x - 2kL} = (-1)^k\varphi'(0).$$

The left-hand Γ'_- and the right-hand Γ'_+ derivatives coincide at points $x = 2kL$ provided $\varphi'(0)$ exists and $\varphi(0) = 0$. Then $\Gamma'(2kL)$ exists. Because of

$$\lim_{x\to 2kL+0} \Gamma'(x) = (-1)^k\varphi'(0) = \lim_{x\to 2kL-0} \Gamma'(x),$$

the derivative $\Gamma'(x)$ is continuous at points $x = 2kL$.

At points $x = (2k+1)L$, $k \in \mathbb{Z}$ we have

$$\Gamma'_+((2k+1)L) = \lim_{x\to(2k+1)L+0} \frac{(-1)^k\varphi(2(k+1)L - x) - (-1)^k\varphi(L)}{x - (2k+1)L} = -(-1)^k\varphi'(L),$$

$$\Gamma'_-((2k+1)L) = \lim_{x\to(2k+1)L-0} \frac{(-1)^k\varphi(x - 2kL) - (-1)^k\varphi(L)}{x - (2k+1)L} = (-1)^k\varphi'(L).$$

The left-hand Γ'_- and the right-hand Γ'_+ derivatives coincide at points $x = (2k+1)L$ provided $\varphi'(L) = 0$. Then $\Gamma'(x)$ exists at these points. It is easy to see that $\Gamma'(x)$ also is continuous at these points.

The requirement of the existence and continuity of the second derivative $\Gamma''(x)$ at points $x = 2kL$ results in $\varphi''(0) = 0$. For the existence and continuity of $\Gamma''(x)$

at points $x = (2k+1)L$ no additional conditions are required; the existence of $\varphi''(L)$ is necessary only.

Repeating the same procedure, we can establish that $\Psi(x) \in C^1(\mathbb{R})$ provided $\psi(x) \in C^1([0,L])$ and $\psi(0) = 0$, $\psi'(L) = 0$. The requirement $\psi(0) = 0$ is used at verification of differentiability at points $x = 2kL$ as well as $\psi'(L) = 0$ is used at points $x = (2k+1)L$.

Thus, if initial functions $\varphi(x) \in C^2([0,L])$, $\psi(x) \in C^1([0,L])$ satisfy the compatibility conditions $\varphi(0) = 0$, $\varphi''(0) = 0$, $\psi(0) = 0$ and $\varphi'(L) = 0$, $\psi'(L) = 0$, d'Alembert's solution will have continuous derivatives in t and x of the first and second orders [3]. Then the function (19), which is the transformed d'Alembert solution, has the same properties. So it will be a classical solution to the considered initial boundary value problem.

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