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Solving of Partial Differential Equations under Minimal Conditions

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It is proved that a differentiable with respect to each variable function $f : \mathbb{R}^2 \to \mathbb{R}$ is a solution of the equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ if and only if there exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = \varphi(x - y)$. This gives a positive answer to a question by R. Baire. Besides, this result is used to solve analogous partial differential equations in abstract spaces and partial differential equations of higher-order.

Key words: separately differentiable functions, partial differential equations.

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1. Introduction

Let X, Y, Z be arbitrary sets and $f : X \times Y \to Z$. For any $x \in X$ and $y \in Y$ we define the mappings $f^x : Y \to Z$ and $f_y : X \to Z$ by the following equalities: $f^x(y) = f_y(x) = f(x, y)$. We say that *a mapping f separately has P* for some property P of mappings (continuity, differentiability, etc.) if for any $x \in X$ and $y \in Y$ the mappings f^x and f_y have P.

R. Baire in the fifth section of his PhD thesis [1] raised a problem of solving differential equations with partial derivatives under minimal requirements, that is, a problem of solving some differential equation in the class of functions satisfying strongly necessary conditions for the existence of expressions contained in this equation. Besides, considering the equation

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0, \tag{1}$$

he proved, using rather laborious arguments, that a jointly continuous separately differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is a solution of (1) if and only if there exists a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = \varphi(x - y)$ for any $x, y \in \mathbb{R}$. Taking into account the solution of this equation in the class of differentiable functions f (which can be obtained by introducing new variables $t = x - y$ and $s = x + y$), the given result means that every jointly continuous separately differentiable solution of (1) is differentiable. It is clear that the continuity condition on f is not necessary for the existence of partial derivatives of f. Hence R. Baire naturally raised the following question.

Question 1.1 (R. Baire [1, p. 118]). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately differentiable solution of (1). Does there exist a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = \varphi(x - y)$ for any $x, y \in \mathbb{R}$?*

Note that the result analogous to Baire's was independently obtained in [2] where Question 1.1 was formulated too. Notice that the method used in [2] is based essentially on the joint continuity of f; it is very nice and simpler than the method from [1]. But, in fact, in [1] R. Baire solved (1) for separately differentiable functions f which are continuous on every line $y = x + c$ (see Th. 4.1).

Besides, by the end of the XX century there were known some results concerning solutions of the following equation:

$$\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} = 0. \tag{2}$$

So, in [3] it was proved that every continuously differentiable solution $f : \mathbb{R}^2 \to \mathbb{R}$ of the equation (2) depends only on one variable. This result was carried over the mappings $f : X \times Y \to Z$ with locally convex range space Z. Also, there was shown the essentiality of local convexity of space Z. An analogous result for separately differentiable functions was obtained in [4]. Moreover, using rather delicate topological arguments, it was proved that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a separately continuous function and for every point $p \in \mathbb{R}^2$ there exists at least one of the partial derivatives $\frac{\partial f}{\partial x}(p)$ and $\frac{\partial f}{\partial y}(p)$, and it is equal to zero, then f depends only on one variable. This result from [4] was generalized in [5] to the case of the so-called separately L-differentiable mappings $f : X \times Y \to Z$, where X, Y, Z are real vector spaces and L is a subspace of the space of all linear functionals on Z which separates points from Z.

In this paper we firstly develop a technique from [1] and study the properties of separately differentiable vector-valued functions of two real variables (Sect. 2). Further, in Sect. 3 we establish necessary and sufficient conditions under which the metric-valued functions defined on an interval are constant. Also we obtain the following property of separately pointwise Lipschitz (in particular, separately differentiable) functions: the restriction of the function of this type on an arbitrary set has nowhere dense discontinuity point set. This property makes it possible to give a positive answer to Question 1.1. In the two last sections we generalize this result to the case of mappings defined on the square of a vector space and then we use it in solving partial differential equations of higher-orders.

2. Auxiliary Baire Function and Separately Differentiable Functions on $\mathbb{R}^2$

In this section we introduce an auxiliary function connected with the difference relation analogously as real functions in [1], study its properties and use it for studying separately differentiable functions.

For arbitrary $a, b \in \mathbb{R}$ with $a < b$, by $[a; b]$, $[a; b)$, $(a; b]$ and $(a; b)$ we denote the corresponding intervals on $\mathbb{R}$.

Let Z be a vector space and $f : \mathbb{R} \to Z$ be a function. For the arbitrary $x, y \in \mathbb{R}$, $x \neq y$, and $B \subseteq Z$ put $r_f(x, y) = \frac{f(x)-f(y)}{x-y}$ and $\Delta(B, f, x) = \{\delta \in (0; 1] : (\forall p', p'' \in (x-\delta; x) \times (x; x+\delta))\ (r_f(p') - r_f(p'') \in B)\}$.

Define a function $\lambda(B, f) : \mathbb{R} \to \mathbb{R}$ by the following: $\lambda(B, f)(x) = \sup\Delta(B, f, x)$ if $\Delta(B, f, x) \neq \emptyset$ and $\lambda(B, f)(x) = 0$ if $\Delta(B, f, x) = \emptyset$.

Let Z be a Hausdorff topological vector space. A mapping $f : \mathbb{R} \to Z$ is called *differentiable at a point* $x_0 \in \mathbb{R}$ if there exists $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$. Note that for a topological vector space Z, a differentiable at x_0 function $f : \mathbb{R} \to Z$ and an arbitrary neighborhood W of zero in Z we have $\lambda(W, f)(x_0) > 0$. Moreover, putting $r_f(x_0, x_0) = f'(x_0)$ we obtain that $\Delta(W, f, x_0) = \{\delta \in (0; 1] : (\forall p', p'' \in (x_0-\delta; x_0] \times [x_0; x_0+\delta))\ (r_f(p') - r_f(p'') \in W)\}$ for any closed neighborhood W of zero in Z.

Theorem 2.1. *Let Z be a Hausdorff topological vector space, $f : \mathbb{R}^2 \to Z$ be a differentiable in the first variable and continuous in the second variable function and W be a closed neighborhood of zero in Z. Then the function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = \lambda(W, f_y)(x)$, is a jointly upper semicontinuous function.*

P r o o f. Let $x_0, y_0 \in \mathbb{R}$, $\gamma = g(x_0, y_0)$ and $\varepsilon > 0$. If $\gamma + \varepsilon > 1$, then $g(x, y) \leq 1 < \gamma + \varepsilon$ for every $x, y \in \mathbb{R}$.

Now, let $\gamma + \varepsilon \leq 1$. Then $\delta_0 = \gamma + \frac{\varepsilon}{3} \leq 1$. Since $g(x_0, y_0) < \delta_0$, $\delta_0 \notin A(W, f_{y_0}, x_0)$. Therefore, there exist $x_1, x_1' \in (x_0-\delta_0; x_0)$ and $x_2, x_2' \in (x_0; x_0+\delta_0)$ such that

$$\frac{f(x_2, y_0) - f(x_1, y_0)}{x_2 - x_1} - \frac{f(x_2', y_0) - f(x_1', y_0)}{x_2' - x_1'} \notin W.$$

The continuity of f in the second variable and the closedness of W imply the existence of a neighborhood V of y_0 in $\mathbb{R}$ such that

$$\frac{f(x_2, y) - f(x_1, y)}{x_2 - x_1} - \frac{f(x_2', y) - f(x_1', y)}{x_2' - x_1'} \notin W$$

for every $y \in V$. Put $s = \min\{x_0 - x_1, x_0 - x_1', x_2 - x_0, x_2' - x_0, \frac{\varepsilon}{3}\}$, $U = (x_0 - s; x_0 + s)$ and $\delta_1 = \gamma + \frac{2\varepsilon}{3}$. Then $x_1, x_1' \in (x - \delta_1; x)$ and $x_2, x_2' \in (x; x + \delta_1)$ for every $x \in U$. Therefore $\delta_1 \notin A(W, f_y, x)$ and $g(x,y) \le \delta_1 < \gamma + \varepsilon$ for every $x \in U$ and $y \in V$.

Thus g is a jointly upper semicontinuous at (x_0, y_0) function. ∎

Let $q, p \in \mathbb{R}^2$. The Euclid distance in $\mathbb{R}^2$ between q and p we denote by $d(q,p)$. If $q \neq p$, then by $\alpha(q,p)$ we denote the angle between the vector $\overrightarrow{pq}$ and the positive direction of abscissa.

The following theorem shows that using the function λ one can obtain some properties of separately differentiable functions.

Theorem 2.2. *Let Z be a topological vector space, $f : \mathbb{R}^2 \to Z$ be a separately differentiable function, $E \subseteq \mathbb{R}^2$ be a nonempty set and W be an arbitrary neighborhood of zero in Z. Then for any open in E nonempty set G there exists a point $p_0 \in G$ and its neighborhood O in E such that for any distinct points $p, q \in O$ the following inclusion holds:*

$$\frac{f(q) - f(p)}{d(q,p)} - \big(f_x'(p_0)\cos\alpha(q,p) + f_y'(p_0)\sin\alpha(q,p)\big) \in W.$$

P r o o f. Note that it is sufficient to consider the case of closed set E.

Let $G \subseteq E$ be an arbitrary nonempty open in E set and W_1 be such closed radial neighborhood of zero in Z that $W_1 + W_1 + W_1 + W_1 + W_1 + W_1 \subseteq W$. Consider the functions $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_1(x,y) = \lambda(W_1, f_y)(x)$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$, $g_2(x,y) = \lambda(W_1, f^x)(y)$. According to Th. 2.1, g_1 and g_2 are jointly upper semicontinuous. For every $n \in \mathbb{N}$ put $E_n = \{(x,y) \in E : g_1(x,y) \ge \frac{1}{n}, g_2(x,y) \ge \frac{1}{n}\}$. Evidently, all the sets E_n are closed in a Baire space E. Since $g_1(x,y) > 0$ and $g_2(x,y) > 0$ for any $(x,y) \in \mathbb{R}^2$, $E = \bigcup\limits_{n=1}^{\infty} E_n$. Then there exists an open in G nonempty set $H \subseteq G$ and $n_0 \in \mathbb{N}$ such that $H \subseteq E_{n_0}$.

Fix an arbitrary point $p_0 = (x_0, y_0) \in H$. Denote $x_1' = x_0 - \frac{1}{n_0}$, $x_2' = x_0 + \frac{1}{n_0}$, $y_1' = y_0 - \frac{1}{n_0}$ and $y_2' = y_0 + \frac{1}{n_0}$. The separate continuity of f implies that there exists $\delta < \frac{1}{2n_0}$ such that

$$r_{f^{x_0}}(y_2', y_1') - r_{f^x}(y_2', y_1') \in W_1 \quad \text{and} \quad r_{f_{y_0}}(x_2', x_1') - r_{f_y}(x_2', x_1') \in W_1$$

for any $x \in U = (x_0 - \delta; x_0 + \delta)$ and $y \in V = (y_0 - \delta; y_0 + \delta)$. Put $O = (U \times V) \cap H$.

Let $p = (x_1, y_1)$, $q = (x_2, y_2)$ be distinct points from the set O and $\alpha = \alpha(q,p)$. If $x_1 \neq x_2$ and $y_1 \neq y_2$, then

$$\frac{f(q) - f(p)}{d(q,p)} = \frac{f(q) - f(x_1, y_2)}{x_2 - x_1} \cdot \frac{x_2 - x_1}{d(q,p)}$$

$$+\frac{f(x_1,y_2)-f(p)}{y_2-y_1}\cdot\frac{y_2-y_1}{d(q,p)}=r_{f_{y_2}}(x_2,x_1)\cos\alpha+r_{f^{x_1}}(y_2,y_1)\sin\alpha.$$

If $x_1=x_2$ or $y_1=y_2$, then $\cos\alpha=0$ or $\sin\alpha=0$, and therefore

$$\frac{f(q)-f(p)}{d(q,p)}=r_{f_{y_2}}(x_2,x_1)\cos\alpha+r_{f^{x_1}}(y_2,y_1)\sin\alpha.$$

Since $p_0\in E_{n_0}$, $g_1(p_0)\geq\frac{1}{n_0}$, there exists $\delta_1>\frac{1}{2n_0}$ such that $\delta_1\in\Delta(W_1,f_{y_0},x_0)$. Hence

$$r_{f_{y_0}}(x_2',x_1')-r_{f_{y_0}}(x_0,x_0)\in W_1,$$

provided $x_1'\in(x_0-\delta_1;x_0]$, $x_2'\in[x_0;x_0+\delta_1)$ and $f_x'(p_0)=r_{f_{y_0}}(x_0,x_0)$.

Note also that $q\in E_{n_0}$, besides, $g_1(q)\geq\frac{1}{n_0}$. Since $\frac{1}{2n_0}+\delta<\frac{1}{n_0}$, there exists $\delta_2\geq\frac{1}{2n_0}+\delta>2\delta$ such that $\delta_2\in\Delta(W_1,f_{y_2},x_2)$. Then $x_1'=x_0-\frac{1}{2n_0}<x_0-\delta<x_2$, $x_2-x_1'<x_0+\delta-x_0+\frac{1}{2n_0}\leq\delta_2$, $x_2'=x_0+\frac{1}{2n_0}>x_0+\delta>x_2$ and $x_2'-x_2<x_0+\frac{1}{2n_0}-x_0+\delta\leq\delta_2$. Thus $x_1'\in(x_2-\delta_2;x_2]$ and $x_2'\in[x_2;x_2+\delta_2)$. The inequalities $|x_1-x_2|<2\delta<\delta_2$ imply

$$r_{f_{y_2}}(x_2,x_1)-r_{f_{y_2}}(x_2',x_1')\in W_1.$$

Since $y_2\in V$,

$$r_{f_{y_2}}(x_2',x_1')-r_{f_{y_0}}(x_2',x_1')\in W_1.$$

Now we have

$$r_{f_{y_2}}(x_2,x_1)-f_x'(p_0)=\Big(r_{f_{y_2}}(x_2,x_1)-r_{f_{y_2}}(x_2',x_1')\Big)$$

$$+\Big(r_{f_{y_2}}(x_2',x_1')-r_{f_{y_0}}(x_2',x_1')\Big)+\Big(r_{f_{y_0}}(x_2',x_1')-f_x'(p_0)\Big)\in W_1+W_1+W_1.$$

Analogously,

$$r_{f^{x_1}}(y_2,y_1)-f_y'(p_0)\in W_1+W_1+W_1.$$

Then

$$\frac{f(q)-f(p)}{d(q,p)}-(f_x'(p_0)\cos\alpha+f_y'(p_0)\sin\alpha)$$

$$=\cos\alpha\Big(r_{f_{y_2}}(x_2,x_1)-f_x'(p_0)\Big)+\sin\alpha\left(r_{f^{x_1}}(y_2,y_1)-f_y'(p_0)\right)$$

$$\in\cos\alpha(W_1+W_1+W_1)+\sin\alpha(W_1+W_1+W_1)\subseteq W.$$

This completes the proof. ■

3. Separately Pointwise Lipschitz Functions and Pointwise Changeable Functions

Firstly recall some definitions.

Let $(X, |\cdot - \cdot|_X)$ and $(Y, |\cdot - \cdot|_Y)$ be metric spaces. A mapping $f : X \to Y$ *satisfies the Lipschitz condition with a constant* $C > 0$ if $|f(x) - f(y)|_Y \le C|x - y|_X$ for any $x, y \in X$. A mapping $f : X \to Y$ is called *pointwise Lipschitz* if for any point $x_0 \in X$ there exists a neighborhood U of point x_0 in X and $C > 0$ such that $|f(x_0) - f(x)|_Y \le C|x_0 - x|_X$ for any $x \in U$. A mapping $f : X \to Y$ is called *pointwise changeable*, if for every $\varepsilon > 0$ the union G_ε of the system $\mathcal{G}_\varepsilon$ of all open nonempty sets $G \subseteq X$ such that $f|_G$ satisfies the Lipschitz property with the constant ε, is an everywhere dense set.

The following property of separately pointwise Lipschitz mappings plays an important role in obtaining the positive answer to Question 1.1.

Theorem 3.1. *Let $(X, |\cdot - \cdot|_X)$ and $(Y, |\cdot - \cdot|_Y)$ be metric spaces such that the space $X \times Y$ is a hereditarily Baire space, $(Z, |\cdot - \cdot|_Z)$ be a metric space and $f : X \times Y \to Z$ be a separately pointwise Lipschitz mapping. Then for any nonempty set $E \subseteq X \times Y$ the discontinuity point set $D(f|_E)$ of mapping $f|_E$ is nowhere dense in E.*

P r o o f. Note that it is sufficient to prove the theorem for the closed set E.

Let $E \subseteq X \times Y$ be a closed nonempty set and $G \subseteq X \times Y$ be an open set such that $W_0 = G \cap E \ne \emptyset$. For any $n, m \in \mathbb{N}$, by E_{nm} denote the set of all points $(x, y) \in W_0$ such that

$$|f(x', y) - f(x, y)|_Z \le n|x' - x|_X \ \textit{and}\ |f(x, y') - f(x, y)|_Z \le n|y' - y|_Y$$

for any $x' \in X$ with $|x' - x|_X < \frac{1}{m}$ and $y' \in Y$ with $|y' - y|_Y < \frac{1}{m}$. Since f is a separately pointwise Lipschitz function, $W_0 = \bigcup\limits_{n,m=1}^{\infty} E_{nm}$. We obtain that there exist $n_0, m_0 \in \mathbb{N}$ and an open in E nonempty set $W \subseteq W_0$ such that $E_{n_0 m_0}$ is dense in W, provided W_0 is an open set in a Baire space E.

Choose the open balls U_1 and V_1 with radius $\frac{1}{2m_0}$ in the spaces X and Y, respectively, such that $W_1 = (U_1 \times V_1) \cap W \ne \emptyset$. Let us show that the function f satisfies a Lipschitz condition on the set W_1 with the constant $2n_0$ with respect to the maximum metric $|\cdot - \cdot|_{X \times Y}$ on $X \times Y$.

Let $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in W_1$. Fix arbitrary $\varepsilon, \delta > 0$. Since f is continuous in the first variable at points p_1 and p_2 and the set $E_{n_0 m_0}$ is dense in W_1, there exist $(\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2) \in W_1 \cap E_{n_0 m_0}$ such that

$$|x_1 - \tilde{x}_1|_X < \delta,\ |y_1 - \tilde{y}_1|_Y < \delta,\ |x_2 - \tilde{x}_2|_X < \delta,\ |y_2 - \tilde{y}_2|_Y < \delta,$$

$$|f(x_1, y_1) - f(\tilde{x}_1, y_1)|_Z < \varepsilon \ \textit{and}\ |f(x_2, y_2) - f(\tilde{x}_2, y_2)|_Z < \varepsilon.$$

Then

$$|f(p_1) - f(p_2)|_Z \le |f(x_1, y_1) - f(\tilde{x}_1, y_1)|_Z + |f(\tilde{x}_1, y_1) - f(\tilde{x}_1, \tilde{y_1})|_Z$$
$$+|f(\tilde{x}_1, \tilde{y}_1) - f(\tilde{x}_1, \tilde{y}_2)|_Z + |f(\tilde{x}_1, \tilde{y}_2) - f(\tilde{x}_2, \tilde{y}_2)|_Z + |f(\tilde{x}_2, \tilde{y}_2) - f(\tilde{x}_2, y_2)|_Z$$
$$+|f(\tilde{x}_2, y_2) - f(x_2, y_2)|_Z \le \varepsilon + n_0|y_1 - \tilde{y}_1|_Y + n_0|\tilde{y}_1 - \tilde{y}_2|_Y + n_0|\tilde{x}_1 - \tilde{x}_2|_X$$
$$+n_0|\tilde{y}_2 - y_2|_Y + \varepsilon \le 2\varepsilon + 2\delta n_0 + n_0(|y_1 - y_2|_Y + 2\delta) + n_0(|x_1 - x_2|_X + 2\delta)$$
$$= 2\varepsilon + 6\delta n_0 + n_0(|x_1 - x_2|_X + |y_1 - y_2|_Y) \le 2\varepsilon + 6\delta n_0 + 2n_0|p_1 - p_2|_{X\times Y}.$$

Tending ε and δ to zero, we obtain

$$|f(p_1) - f(p_2)|_Z \le 2n_0|p_1 - p_2|_{X\times Y}.$$

Hence, $f|_E$ is continuous on the set W_1. ∎

Note that the obtained property of separately pointwise Lipschitz mappings is new, but for the real-valued separately differentiable functions of two variables this property can be obtained from the analog of Th. 2.2, which was presented in [1]. Besides, in [6] it was proved that the discontinuity point set of function of two real variables, which is differentiable in the first variable and continuous in the second one, is nowhere dense. This result was generalized in [7].

For a topological space X and a set $A \subseteq X$, by $\overline{A}$ we denote the closure of A in X.

The following characterization was obtained in [1] for the real-valued functions of one real variable.

Theorem 3.2. *Let $X \subseteq \mathbb{R}$ be a nonempty interval, $(Y, |\cdot - \cdot|_Y)$ be a metric space, $f : X \to Y$ be a continuous pointwise changeable on every closed set mapping. Then f is constant.*

P r o o f. For any $x_1, x_2 \in X$, $x_1 \ne x_2$ put $r(x_1, x_2) = \frac{|f(x_2) - f(x_1)|_Y}{|x_2 - x_1|}$ and for every $x \in X$ put $g(x) = \inf\limits_{\delta > 0} \sup\{r(x_1, x_2) : x - \delta < x_1 < x_2 < x + \delta\}$.

Let us show that for any $a, b \in X$, $a < b$, there exists a point $c \in [a; b]$ such that $g(c) \ge r(a, b)$.

Let $a \le x < y < z \le b$. Then $r(x, z) \le \frac{y - x}{z - x} r(x, y) + \frac{z - y}{z - x} r(y, z)$. Hence, $r(x, z) \le r(x, y)$ or $r(x, z) \le r(y, z)$. Now it is easy to construct the sequence $(I_n)_{n=1}^{\infty}$ of segments $I_n = [a_n; b_n] \subseteq [a; b]$ such that $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, $I_{n+1} \subseteq I_n$ and $r(a_n, b_n) \ge r(a, b)$ for every $n \in \mathbb{N}$. Then, for point $c \in \bigcap\limits_{n=1}^{\infty} I_n$ we have $g(c) \ge r(a, b)$.

Therefore, if $a, b \in X$, $\varepsilon > 0$ and $g(x) \le \varepsilon$ for any $x \in (a, b) \subseteq X$, then $r(x, y) \le \varepsilon$ for any $x, y \in (a; b)$, what implies $r(a, b) \le \varepsilon$, provided f is continuous.

Assume that f is not constant. Then there exists $\varepsilon > 0$ such that $E = \{x \in X : g(x) > \varepsilon\} \neq \emptyset$. Since f is pointwise changeable on the set $F = \overline{E}$, there exist $x_0 \in E$ and $\delta > 0$ such that $r(x, y) < \varepsilon$ for any distinct $x, y \in F \cap U$, where $U = (x_0 - \delta; x_0 + \delta)$.

Let $x, y \in U$ be arbitrary distinct points. Let us show that $r(x, y) \leq \varepsilon$. We assume that $x < y$. First consider the case of $x, y \notin F$. If $(x; y) \cap F = \emptyset$, then $(x; y) \cap E = \emptyset$ and $r(x, y) \leq \varepsilon$. Let $(x; y) \cap F \neq \emptyset$. Choose the points $u, v \in F$ such that $x < u \leq v < y$, $(x; u) \cap F = \emptyset$ and $(v; y) \cap F = \emptyset$. Then, as above, $r(x, u) \leq \varepsilon$ and $r(v, y) \leq \varepsilon$. If $u < v$, then

$$r(x, y) = \frac{u - x}{y - x} r(x, u) + \frac{v - u}{y - x} r(u, v) + \frac{y - v}{y - x} r(v, y),$$

therefore $r(x, y) \leq \varepsilon$. When $u = v$, we use the equality

$$r(x, y) = \frac{u - x}{y - x} r(x, u) + \frac{y - u}{y - x} r(u, y).$$

In the case of $x \in F$ or $y \in F$ we use analogous reasons.

Thus, $\sup\{r(x, y) : x_0 - \delta < x < y < x_0 + \delta\} \leq \varepsilon$. Then $g(x_0) \leq \varepsilon$, what contradicts to $x_0 \in E$. ∎

Corollary 3.3. *Let X be an arbitrary normed space, $(Y, |\cdot - \cdot|_Y)$ be a metric space, $f : X \to Y$ be a continuous pointwise changeable on every closed set mapping. Then f is constant.*

P r o o f. It is enough to prove that $f(x) = f(0)$ for any $x \in X$.

Let $x_0 \in X$, $x_0 \neq 0$, be an arbitrary point. Consider the function $g : \mathbb{R} \to Y$, $g(\alpha) = f(\alpha x_0)$. Since $|\alpha - \beta| = \frac{1}{\|x_0\|} \|\alpha x_0 - \beta x_0\|$ and f is a continuous pointwise changeable on every closed set mapping, g satisfies the conditions of Th. 3.2. Therefore, g is constant and $f(x_0) = g(1) = g(0) = f(0)$. ∎

The following two examples demonstrate that there is no analogous property for the mappings defined on an arbitrary metric space and, on the other hand, this property does not have any equivalent formulation in topological terms.

E x a m p l e 3.4. Let $(X, |\cdot - \cdot|_X)$ be a metric space with the discrete metric, i.e., $|x_1 - x_2|_X = 1$ when $x_1 \neq x_2$, and $(Y, |\cdot - \cdot|_Y)$ be an arbitrary metric space. Then every mapping $f : X \to Y$ is continuous and pointwise changeable on every closed set.

E x a m p l e 3.5. Let $0 < p < 1$ and $\mathbb{R}_p$ be a real line with the metric $|x - y|_p = |x - y|^p$. Then the identical map $f : \mathbb{R}_p \to \mathbb{R}$, $f(x) = x$, is a homeomorphism of pointwise changeable on every closed set.

4. The Equation $f'_x + f'_y = 0$.

In this section we give a positive answer to Question 1.1.

Actually, the following theorem was proved in [1], but R. Baire instead of continuity of function f on respective lines put on f a stronger condition of joint continuity.

Theorem 4.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately differentiable function and $c \in \mathbb{R}$ such that the restriction of function f to the set $A = \{(x, y) \in \mathbb{R}^2 : y-x = c\}$ is continuous and $f'_x(p)+f'_y(p) = 0$ for every $p \in A$. Then the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x, c + x)$, is constant.*

P r o o f. Since $\cos\alpha(q, p) = \sin\alpha(q, p)$ for any distinct points $p, q \in A$, Th. 2.2 implies that the continuous function g is pointwise changeable on every closed set. It remains to apply Th. 3.2. ∎

In the proof of the main result we will use the following auxiliary fact.

Lemma 4.2. *Let $I = (a; b) \subseteq \mathbb{R}$ be an arbitrary nonempty interval, $c \in \mathbb{R}$, $\delta > 0$, $W = \{(x, y) \in \mathbb{R}^2 : x \in I, |y - x - c| \le \delta\}$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be such separately continuous functions that $f(x, y) = g(x, y)$ for any $(x, y) \in W$. Then $f(x, y) = g(x, y)$ for any $(x, y) \in \overline{W}$.*

P r o o f. Let $x_0 = a$ and $|y_0 - x_0 - c| < \delta$. Then $f(x_0, y_0) = \lim\limits_{x\to a+0} f(x, y_0) = \lim\limits_{x\to a+0} g(x, y_0) = g(x_0, y_0)$. Analogously, if $x_0 = b$ and $|y_0 - x_0 - c| < \delta$, then $f(x_0, y_0) = g(x_0, y_0)$.

Now let $x_0 = a$ and $y_0 - x_0 - c = \delta$. Then $f(x_0, y_0) = \lim\limits_{y\to y_0-0} f(x_0, y) = \lim\limits_{y\to y_0-0} g(x_0, y) = g(x_0, y_0)$. We use analogous reasons in the case of $x_0 = a$ and $y_0 - x_0 - c = -\delta$, or $x_0 = b$ and $y_0 - x_0 - c = \pm\delta$. ∎

Let X be a topological space, $x_0 \in X$, $\mathcal{U}$ be a system of all neighborhoods of point x_0 in X, $(Y, |\cdot - \cdot|_Y)$ be a metric space and $f : X \to Y$. Recall that a real $\omega_f(x_0) = \inf\limits_{U\in\mathcal{U}} \sup\limits_{x',\, x''\in U} |f(x') - f(x'')|_Y$ is called *the oscillation of mapping f at x_0*.

Now let us prove our main result.

Theorem 4.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately differentiable function such that $f'_x(p) + f'_y(p) = 0$ for every $p \in \mathbb{R}^2$. Then for any $c \in \mathbb{R}$ the function f is constant on the set $A = \{(x, y) \in \mathbb{R}^2 : y - x = c\}$.*

P r o o f. According to Th. 4.1 it is enough to prove that f is continuous.

Assume that the discontinuity point set E of function f is nonempty. Theorem 3.1 implies that there exists a point $p_0 = (x_0, y_0) \in E$, in which function $f|_E$

is continuous. Denote $\varepsilon = \omega_f(p_0)$, $c_0 = y_0 - x_0$ and choose $\delta_1, \delta_2 > 0$ such that for any point $p \in E \bigcap W$, where $W = \{(x,y) \in \mathbb{R}^2 : \ |x - x_0| < \delta_1, \ |y - x - c_0| < \delta_2\}$, the inequality $|f(p) - f(p_0)| \le \frac{\varepsilon}{3}$ holds. Note that for any point $q \in W$ with $|f(q) - f(p_0)| > \frac{\varepsilon}{3}$ the function f is continuous at q.

Consider the continuous function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = f(x_0, x_0 + y - x)$ and show that $f(p) = g(p)$ for any point $p \in W$ with $|f(p) - f(p_0)| > \frac{\varepsilon}{3}$.

Let $p_1 = (x_1, y_1) \in W$, besides, $|f(p_1) - f(p_0)| > \frac{\varepsilon}{3}$. Choose $\delta > 0$ such that $|f(x_1, y) - f(x_0, y_0)| > \frac{\varepsilon}{3}$ and $(x_1, y) \in W$ for any $y \in [y_1 - \delta; y_1 + \delta]$.

Denote by $\mathcal{I}$ the system of all nonempty open intervals $I \subseteq (x_0 - \delta_1; x_0 + \delta_1)$ such that $x_1 \in I$ and $|f(x,y) - f(x_0,y_0)| > \frac{\varepsilon}{3}$ for any $x \in I$ and $y \in \mathbb{R}$ with $|y - x - c_1| \le \delta$, where $c_1 = y_1 - x_1$. Note that f is continuous at every point of compact set $K = \{(x_1, y) : y \in [y_1 - \delta; y_1 + \delta]\}$. Therefore the system $\mathcal{I}$ is nonempty.

Put $I_0 = (a;b) = \bigcup\limits_{I \in \mathcal{I}} I$ and $W_1 = \{(x,y) \in \mathbb{R}^2 : x \in I_0, |y - x - c_1| \le \delta\}$. Since $|f(p) - f(p_0)| > \frac{\varepsilon}{3}$ for every $p \in W_1 \subseteq W$, the function f is continuous at every point from W_1. According to Th. 2.2, the function $\varphi(x) = f(x, c + x)$ is pointwise changeable on I, and therefore, according to Th. 3.2, φ is constant on I for every $c \in [c_1 - \delta; c_1 + \delta]$, i.e., $f(x,y) = f(x, x + y - x) = f(x_1, x_1 + y - x)$ for any $(x,y) \in W_1$.

Let us show that $I_0 = (x_0 - \delta_1; x_0 + \delta_1)$. Assume that $a > x_0 - \delta_1$. Then Lem. 4.2 implies $f(x,y) = f(x_1, x_1 + y - x)$ for any $(x,y) \in \overline{W_1}$, besides, $f(a,y) = f(x_1, x_1 + y - a)$ for any $y \in [a + c_1 - \delta; a + c_1 + \delta]$. Note that $(x_1; x_1 + y - a) \in K$ if $y \in [a + c_1 - \delta; a + c_1 + \delta]$, then $|f(p) - f(p_0)| > \frac{\varepsilon}{3}$ for every $p \in K_1$, where $K_1 = \{(a,y) : |y - a - c_1| \le \delta\}$. Since $K_1 \subseteq W$, the function f is continuous at every point from the set K_1. Hence, there exists a nonempty interval $I_1 \subseteq (x_0 - \delta_1; x_0 + \delta_1)$ such that $a \in I_1$ and $|f(x,y) - f(x_0,y_0)| > \frac{\varepsilon}{3}$ for any $x \in I_1$ and $y \in \mathbb{R}$ with $|y - x - c_1| \le \delta$. Then $I_0 \bigcup I_1 \in \mathcal{I}$, what is contrary to the definition of set I_0. We use analogous reasons if $b < x_0 + \delta$.

Thus, $I_0 = (x_0 - \delta_1; x_0 + \delta_1)$. Then $(x_0, x_0 + c_1) \in W_1$, $f(x_0, x_0 + c_1) = f(x_1, x_1 + c_1) = f(x_1, y_1)$, and $g(x_1, y_1) = f(x_0, x_0 + c_1) = f(x_1, y_1)$.

Since $\omega_f(p_0) = \varepsilon$, then there exists a sequence $(q_n)_{n=1}^{\infty}$ of points $q_n = (u_n, v_n) \in W$ such that $|f(q_n) - f(p_0)| > \frac{\varepsilon}{3}$ and $\lim\limits_{n \to \infty} q_n = p_0$. Then, using the continuity of g, we obtain $\lim\limits_{n \to \infty} f(q_n) = \lim\limits_{n \to \infty} g(q_n) = g(p_0) = f(p_0)$. But the last equalities contradict to the choice of $(q_n)_{n=1}^{\infty}$. ■

Corollary 4.4. *Let $k \in \mathbb{R}$, $k \ne 0$, $f : \mathbb{R}^2 \to \mathbb{R}$ be such a separately differentiable function that $f'_x(p) + k f'_y(p) = 0$ for every $p \in \mathbb{R}^2$. Then there exists a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = \varphi(kx - y)$ for any $x, y \in \mathbb{R}$.*

5. Equations for Separately L-Differentiable Functions

In this section we apply Th. 4.3 for solving differential equations in abstract spaces.

Let X be a vector space, Z be a set and L be a system of functions $l : Z \to \mathbb{R}$. We say that a mapping $f : X \to Z$ is *L-differentiable at $x_0 \in X$* if for arbitrary $h \in X$ and $l \in L$ the function $g : \mathbb{R} \to \mathbb{R}$, $g(t) = l(f(x_0 + th))$, is differentiable at $t_0 = 0$, i.e., there exists $A(h,l) = \lim\limits_{t\to 0} \frac{l(f(x_0+th))-l(f(x_0))}{t}$. The mapping $A : X \times L \to \mathbb{R}$ is called *L-derivative of f at x_0* and is denoted by $Df(x_0)$. Besides, we denote $Df(x_0)(h,l)$ by $Df(x_0,h,l)$.

A mapping $f : X \to Z$ is called *L-differentiable* if f is L-differentiable at every point $x \in X$.

Recall that a system L of functions defined on a set Z *separates points from Z* if for arbitrary distinct points $z_1, z_2 \in Z$ there exists $l \in L$ such that $l(z_1) \neq l(z_2)$.

Theorem 5.1. *Let X be a vector space, Z be a set, L be a system of functions defined on Z which separates points from Z, and $f : X^2 \to Z$ be a separately L-differentiable mappings such that*

$$Df^x(y) + Df_y(x) = 0$$

for every $x, y \in X$. Then there exists an L-differentiable mapping $\varphi : X \to Z$ such that $f(x,y) = \varphi(x-y)$ for every $x, y \in X$.

P r o o f. Firstly show that $f(x,y) = \varphi(x-y)$ for some mapping $\varphi : X \to Z$. It is enough to prove that $f(x_1,y_1) = f(x_2,y_2)$ if $x_1 - y_1 = x_2 - y_2$.

Suppose that there exist $x_1, y_1, x_2, y_2 \in X$ such that $x_1 - y_1 = x_2 - y_2$ and $f(x_1,y_1) = z_1 \neq f(x_2,y_2) = z_2$. Since the system L separates points from Z, there exists $l \in L$ such that $l(z_1) \neq l(z_2)$. Put $h = x_2 - x_1 = y_2 - y_1$ and consider the function $u : \mathbb{R}^2 \to \mathbb{R}$, $u(s,t) = l(f(x_1 + sh, y_1 + th))$.

Show that u is a separately differentiable function with $u'_s + u'_t = 0$.

Let $s_0, t_0 \in \mathbb{R}$, $x_0 = x_1 + s_0 h$ and $y_0 = y_1 + t_0 h$. Then

$$\begin{aligned} u'_s(s_0,t_0) &= \lim_{s\to s_0} \frac{l(f(x_1+sh, y_1+t_0h)) - l(f(x_1+s_0h, y_1+t_0h))}{s-s_0} \\ &= \lim_{s\to s_0} \frac{l(f(x_0+(s-s_0)h, y_0)) - l(f(x_0,y_0))}{s-s_0} \\ &= \lim_{s\to s_0} \frac{l(f_{y_0}(x_0+(s-s_0)h)) - l(f_{y_0}(x_0))}{s-s_0} = Df_{y_0}(x_0,h,l). \end{aligned}$$

Analogously, $u'_t(s_0,t_0) = Df^{x_0}(y_0,h,l)$. Since $Df^{x_0}(y_0) + Df_{y_0}(x_0) = 0$, $u'_s(s_0,t_0) + u'_t(s_0,t_0) = 0$.

Thus u satisfies the conditions of Th. 4.3, therefore $l(z_1) = u(0,0) = u(1,1) = l(z_2)$ what is contrary to our assumption.

The L-differentiability of φ follows from $\varphi(x) = f(x,0)$ and L-differentiability of f_y if $y = 0$. ■

Let X, Z be topological vector spaces. A mapping $f : X \to Z$ is called the *Gateaux differentiable at point* $x_0 \in X$ if there exists a linear continuous operator $A : X \to Z$ such that

$$\lim_{t\to 0} \frac{f(x_0 + th) - f(x_0)}{t} = (Ax_0)(h)$$

for every $h \in X$. The operator A is called the *Gateaux derivative of mapping* f *at point* x_0.

Note that for a Hausdorff topological vector space Z the Gateaux derivative is unique. A mapping $f : X \to Z$, which is the Gateaux differentiable at every point $x \in X$, is called the *Gateaux differentiable.* A mapping D, which assigns to every Gateaux differentiable mapping $f : X \to Z$ the Gateaux derivative mapping, i.e., $Df(x)$ is the Gateaux derivative of f at point $x \in X$, we call the *Gateaux differentiation operator.*

Corollary 5.2. *Let X be a topological vector space, Z be a topological vector space such that the conjugate space Z^* separates points from Z, and $f : X^2 \to Z$ be a mapping such that*

$$Df^x(y) + Df_y(x) = 0$$

for every $x, y \in X$, where D is a Gateaux differentiation operator for the mapping acting from X to Z. Then there exists a Gateaux differentiable mapping $\varphi : X \to Z$ such that $f(x,y) = \varphi(x - y)$ for every x, $y \in X$.

P r o o f. Since Z^* separates points from Z, Z is a Hausdorff space and the definition of D is correct. Besides, for arbitrary $x, y, h \in X$ and $z^* \in Z^*$ we have

$$\lim_{t\to 0} \frac{z^*(f_y(x+th)) - z^*(f_y(x))}{t} = \lim_{t\to 0} z^* \left(\frac{f_y(x+th) - f_y(x)}{t} \right) = z^*(Df_y(x)(h))$$

and

$$\lim_{t\to 0} \frac{z^*(f^x(y+th)) - z^*(f^x(y))}{t} = z^*(Df^x(y)(h)) = -z^*(Df_y(x)(h)).$$

Therefore f is a separately Z^*-differentiable mapping and $\tilde{D}f^x(y) + \tilde{D}f_y(x) = 0$, where $\tilde{D}$ is the Z^*-differentiation operator. Theorem 5.1 implies that there exists a mapping $\varphi : X \to Z$ such that $f(x,y) = \varphi(x - y)$ for every $x, y \in X$. Since $\varphi(x) = f(x,0)$, φ is a Gateaux differentiable mapping. ■

6. Higher-Order Equations

Finally, we give the applications of Th. 4.3 for solving higher-order partial differential equations.

Let $n \in \mathbb{N}$ and a function $f : \mathbb{R}^2 \to \mathbb{R}$ has all n-order partial derivatives. We denote the sum of all n-order partial derivatives of f by $D_n f$. Clearly, $D_{n+k} f = D_n(D_k f)$ for every $n, k \in \mathbb{N}$ and any function $f : \mathbb{R}^2 \to \mathbb{R}$ which has all $(n+k)$-order partial derivatives.

Theorem 6.1. *Let $n \in \mathbb{N}$, a function $f : \mathbb{R}^2 \to \mathbb{R}$ have all n-order partial derivatives and $D_n f(p) = 0$ for every $p \in \mathbb{R}^2$. Then there exist differentiable functions $\varphi_1, \dots, \varphi_n : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x,y) = \varphi_1(x-y) + (x+y)\varphi_2(x-y) + \dots + (x+y)^{n-1}\varphi_n(x-y)$$

for every $x, y \in \mathbb{R}$.

P r o o f. The proof is by induction in n.

For $n = 1$ it follows from Th. 4.3.

Assume that our assertion is true for some $n = k$ and prove it for $n = k+1$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function which has all $(k+1)$-order partial derivatives and $D_{k+1} f(p) = 0$ for every $p \in \mathbb{R}^2$.

Put $g = D_1 f$. Since $D_k g = D_{k+1} f = 0$, the assumption implies that there exist differentiable functions $\psi_1, \dots, \psi_k : \mathbb{R} \to \mathbb{R}$ such that

$$g(x,y) = \psi_1(x-y) + (x+y)\psi_2(x-y) + \dots + (x+y)^{k-1}\psi_k(x-y).$$

Denote $\varphi_{i+1} = \frac{1}{2i}\psi_i$ if $1 \le i \le k$, and put

$$u(x,y) = (x+y)\varphi_2(x-y) + (x+y)^2\varphi_3(x-y) + \dots + (x+y)^k\varphi_{k+1}(x-y).$$

Then

$$D_1 u(x,y) = 2\varphi_2(x-y) + 4(x+y)\varphi_3(x-y) + \dots + 2k(x+y)^{k-1}\varphi_{k+1}(x-y)$$

$$= \psi_1(x-y) + (x+y)\psi_2(x-y) + \dots + (x+y)^{k-1}\psi_k(x-y) = g(x,y) = D_1 f(x,y).$$

Thus $D_1(f-u) = 0$ and Th. 4.3 implies that there exists a differentiable function $\varphi_1 : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = \varphi_1(x,y) + u(x,y)$. ■

Theorem 6.2. *Let a function $f : \mathbb{R}^2 \to \mathbb{R}$ have all second-order partial derivatives and*

$$f''_{xx}(p) = f''_{yy}(p) \quad \text{and} \quad f''_{xy}(p) = f''_{yx}(p)$$

for every $p \in \mathbb{R}^2$. Then there exist twice differentiable functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = \varphi(x+y) + \psi(x-y)$$

for every $x, y \in \mathbb{R}$.

P r o o f. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(p) = f'_x(p) - f'_y(p)$. Then $g'_x(p) + g'_y(p) = f''_{xx}(p) - f''_{yx}(p) + f''_{xy}(p) - f''_{yy}(p) = 0$. Theorem 4.3 implies that there exists a differentiable function $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ such that $g(x, y) = \tilde{\psi}(x - y)$.

Choose some twice differentiable function $\psi : \mathbb{R} \to \mathbb{R}$ such that $2\psi' = \tilde{\psi}$ and consider the function $\tilde{f}(x, y) = f(x, y) - \psi(x - y)$. Then

$$\tilde{f}'_x(x, y) - \tilde{f}'_y(x, y) = f'_x(x, y) - f'_y(x, y) - 2\psi'(x - y) = g(x, y) - g(x, y) = 0.$$

Therefore, Cor. 4.4 implies that there exists a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}(x, y) = \varphi(x+y)$, i.e., $f(x, y) = \varphi(x+y)+\psi(x-y)$ for every x, $y \in \mathbb{R}$. Since $\varphi(x) = f(x, 0) - \psi(x)$, φ is a twice differentiable function. ∎

R e m a r k s. The existence of f''_{xx} and f''_{yy} for a function $f : \mathbb{R}^2 \to \mathbb{R}$ does not imply the existence of f''_{xy} and f''_{yx}. For example, the Schwartz function

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if} \quad x^2 + y^2 \neq 0; \\ 0, & \text{if} \quad x = y = 0, \end{cases}$$

is a separately infinite differentiable function, but $f''_{xy}(0, 0)$ and $f''_{yx}(0, 0)$ do not exist.

On the other hand, the existence of all second-order partial derivatives of function $f : \mathbb{R}^2 \to \mathbb{R}$ and the equality $f''_{xy} = f''_{yx}$ do not imply the joint continuity of f. Really, the function

$$f(x, y) = \begin{cases} \frac{2x^3y^3}{x^6+y^6}, & \text{if} \quad x^6 + y^6 \neq 0; \\ 0, & \text{if} \quad x = y = 0, \end{cases}$$

has all second-order partial derivatives, besides, $f''_{xy} = f''_{yx}$ on $\mathbb{R}^2$ and f is jointly discontinuous at $(0, 0)$.

In this connection the following question arises naturally.

Question 6.3. *Let a function $f : \mathbb{R}^2 \to \mathbb{R}$ have partial derivatives f''_{xx} and f''_{yy}, and*

$$f''_{xx}(p) = f''_{yy}(p)$$

for every $p \in \mathbb{R}^2$. Do there exist twice differentiable functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ such that

$$f(x, y) = \varphi(x + y) + \psi(x - y)$$

for every $x, y \in \mathbb{R}$?

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