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An Invariant Form of the Euler–Lagrange Operator

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We define a class of almost $S(M)$-multilinear maps. The Euler–Lagrange operator is given by means of the trace of an almost $S(M)$-bilinear map.

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1. Introduction

The aim of the paper is to give a geometric interpretation of the Euler–Lagrange operator. The problem considered here will be described in terms of fibre bundle theory. First, let us consider a functional of the form

$$I[y] = \int\limits_{\Omega} F(x, y(x), \ldots, y^{(\alpha)}(x), \ldots) d^k x, \tag{1}$$

where Ω is a domain in $\mathbf{R}^k$, $y : \Omega \longrightarrow \mathbf{R}^m$ is a smooth function, $\alpha = (i_1, \ldots, i_k)$ denotes a multiindex, and $y^{(\alpha)} = \frac{D^{|\alpha|}y}{Dx^\alpha}$. We assume here that the functional (1) has a finite rank (say N), which means that F depends only on partial derivatives $y^{(\alpha)}$ of the rank less or equal n. It is well known [1, 2] that the Euler–Lagrange equations

$$\sum_{\alpha} (-1)^{|\alpha|} \frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\alpha)}} = 0 \ \text{ for } \ i = 1, 2, \ldots, m, \tag{2}$$

are necessary conditions for the extremum of the functional (1) in a suitable class of functions satisfying the well-known boundary conditions. These equations

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imply that for any vector field $h : \Omega \longrightarrow \mathbf{R}^m$ the form

$$\delta_h F = \sum_i h_i \sum_\alpha (-1)^{|\alpha|} \frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\alpha)}} \tag{3}$$

is identically equal to zero on the extreme of the functional. The operator δ_h is called the Euler–Lagrange operator. This operator can be expressed in the form

$$\delta_h F = \sum_{i,\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} \left\{ \frac{D^{|\alpha|}}{Dx^\alpha} \left[\sum_\beta \left(\frac{D^{|\beta|}}{Dx^\beta}((x - x_0)^\alpha h_i(x)) \right)_{x_0 = x} \frac{\partial F}{\partial y_i^{(\beta)}} \right] \right\}. \tag{4}$$

To show the equality of the formulas (3) and (4), it is enough to use the identity $\sum_k (-1)^k \binom{n}{k}\binom{k}{l} = (-1)^n \delta_{n,l}$ and the Leibnitz rule. Using the following differential operators

$$V_h F = \sum_{i,\beta} h_i^{(\beta)} \frac{\partial F}{\partial y_i^{(\beta)}} \tag{5}$$

and $(V_h^{\{\alpha\}} F)_x = (V_{h^\alpha_{x_0}} F)_{x_0 = x}$, where $h^\alpha_{x_0}(x) = (x - x_0)^\alpha h(x)$, we have

$$\delta_h F = \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \frac{D^{|\alpha|}}{Dx^\alpha} (V_h^{\{\alpha\}} F). \tag{6}$$

W.M. Tulczyjew [2] considered the analogous to (6) expressions for jet bundles in a local system of coordinates. The aim of the paper is to give a geometrical formulation of the problem. For the functionals of rank one, the invariant formula for the Euler–Lagrange equation is known [3, 4]. In [5] the authors considered the Euler–Lagrange equation invariant with respect to a Lie group. In this paper we will generalize (6) to the functionals defined on a fibre bundle, and the result will be obtained without using the coordinate system. Here we deal with the functionals of arbitrary but finite rank. The paper is organized as follows. In Section 2 we introduce some notation concerning fibre bundles, jets and differential operators. In Section 3 we define what an almost $S(M)$–multilinear map between the sections of vector bundles is ($S(M)$ denotes here the ring of smooth real functions on M). By a smooth function on a C_r manifold we understand any C_r function. The trace defined here can be considered as a generalization of the trace of matrices or linear operators. In Section 4 we deal with the adjoint of differential operators. In Section 5 we consider the jet vertical derivative on fibre bundles. Further, using the adjoint of differential operators, we give an invariant expression for the Euler–Lagrange operator by means of the trace of a suitable almost bilinear map (61) which is the main result of the paper.

2. Notation

All the manifolds considered in this paper have finite dimensions and we deal only with differential bundles. Let us consider the following notation for the homomorphism of vector bundles. Let $\mathcal{K} = \pi : K \longrightarrow X$, $\mathcal{E} = p : E \longrightarrow Y$ be given vector bundles, and $f : X \longrightarrow Y$ be a map. A map $\alpha : K \longrightarrow E$ is called a homomorphism of the vector bundles $\mathcal{K}$ and $\mathcal{E}$ over f, if $p \circ \alpha = f \circ \pi$, and the restriction $\alpha|_{K_x}^{E_{f(x)}} : K_x \longrightarrow E_{f(x)}$ is a linear homomorphism for any $x \in X$. We denote the set of all homomorphisms of these bundles over f by $\mathrm{HOM}_f(\mathcal{K}, \mathcal{E})$. In the case $X = Y$, $f = \mathrm{id}_X$ we write $\mathrm{HOM}(\mathcal{K}, \mathcal{E})$ instead of $\mathrm{HOM}_{\mathrm{id}_X}(\mathcal{K}, \mathcal{E})$.

For a bundle $\mathcal{E} = \pi : E \longrightarrow M$ by $V\mathcal{E}$ we denote the vertical bundle of $\mathcal{E}$ and by $\mathcal{J}^n\mathcal{E}$ – the n–th jet extension of $\mathcal{E}$ [6, 7]. We denote the total space of $\mathcal{J}^n\mathcal{E}$ also by $\mathcal{J}^n\mathcal{E}$ and so on. The maps

$$\pi^n : \mathcal{J}^n\mathcal{E} \longrightarrow M \ , \ \ \pi^n_k : \mathcal{J}^n\mathcal{E} \longrightarrow \mathcal{J}^k\mathcal{E} \ \text{ for } \ k \leq n \tag{7}$$

denote suitable projections. For a local section s of the bundle $\mathcal{E}$, $[s]^n$ denote the lifting of s to $\mathcal{J}^n\mathcal{E}$. If the bundle $\mathcal{E}$ is a vector bundle, then $\mathcal{J}^n\mathcal{E}$ is also a vector bundle.

A linear differential operator D of the rank less or equal n from a vector bundle $\mathcal{E}_1 = \pi_1 : E_1 \longrightarrow M$ to a vector bundle $\mathcal{E}_2 = \pi_2 : E_2 \longrightarrow M$ can be treated as a homomorphism of the vector bundles $\mathcal{J}^n\mathcal{E}_1$ and $\mathcal{E}_2$ over id_M. The bundle of these operators is denoted by $\mathrm{Ldiff}^n(\mathcal{E}_1, \mathcal{E}_2)$. In particular, $\mathrm{Ldiff}^0(\mathcal{E}_1, \mathcal{E}_2)$ is isomorphic to $\mathrm{HOM}(\mathcal{E}_1, \mathcal{E}_2)$. In the case $\mathcal{E}_1 = \mathcal{E}_2$ we write $\mathrm{Ldiff}^n(\mathcal{E}_1)$ instead of $\mathrm{Ldiff}^n(\mathcal{E}_1, \mathcal{E}_1)$. For any $f \in S(M)$ by $m_f \in \Gamma(\mathrm{Ldiff}^0(\mathcal{E}))$ we understand the differential operator of the rank 0 of multiplication by the function f.

So, we regard the vector bundle $\mathrm{Ldiff}^n(M, \mathbf{R})$ of the linear differential operators acting on $S(M)$ of the rank less or equal n as a dual one to the n–th jet bundle $\mathcal{J}^n(M, \mathbf{R})$. There exist two kinds of multiplication of functions and sections of the n–th jet bundle $\mathcal{J}^n(M, \mathbf{R})$:

$$(f\eta)_x = f(x)\eta_x \ \text{ and } \ (f * \eta)_x = [fg]^n_x \ , \tag{8}$$

where $x \in M$, $f \in S(M)$, $\eta \in \Gamma(\mathcal{J}^n(M, \mathbf{R}))$, and $g \in S(M)$ is a representative of η_x. For any $D \in \Gamma(\mathrm{Ldiff}^n(M, \mathbf{R}))$ the following equalities

$$D(f\eta) = fD(\eta) \ \text{ and } \ (D(f * \eta))(x) = (D(fg))(x) \tag{9}$$

are fulfilled. The consequence of the second formula in (9) is

$$(Dm_f)(\eta) = D(f * \eta). \tag{10}$$

We use the following notation. A bundle $\pi : P \longrightarrow B$ is denoted as P_B if the projection is obvious.

Now, let the map $\mathcal{E} = \pi : E \longrightarrow M$ be a bundle, not necessarily a vector bundle. Using this notation, we can write the equality

$$(\mathcal{J}^n((V\mathcal{E})_M))_{\mathcal{J}^n\mathcal{E}} = (V(\mathcal{J}^n\mathcal{E}))_{\mathcal{J}^n\mathcal{E}}, \tag{11}$$

which is a very well-known canonical isomorphism [3, 6] of the bundles over $\mathcal{J}^n\mathcal{E}$. The following equality

$$\mathcal{J}^n((V\mathcal{E})_M) = (V(\mathcal{J}^n\mathcal{E}))_M \tag{12}$$

is an analogue of (11) for the bundles over M.

Further, let

$$\mathcal{E} = \pi : E \longrightarrow M\ ,\ \ \mathcal{K}_E = p : K \longrightarrow E\ ,\ \ \mathcal{K}_M = \pi \circ p : K \longrightarrow M \tag{13}$$

be given bundles. Then we put

$$(\mathcal{J}^n\mathcal{K}_M)_{\mathcal{J}^n\mathcal{E}} = \mathcal{J}^n(p) : \mathcal{J}^n\mathcal{K}_M \longrightarrow \mathcal{J}^n\mathcal{E}\ , \tag{14}$$

where the projection is given by the formula

$$\mathcal{J}^n(p)([\sigma]^n_x) = [p \circ \sigma]^n_x, \tag{15}$$

and σ is a local section of $\mathcal{K}_M$ in a neighborhood of $x \in M$.

If the bundle $\mathcal{K}_E$ in (13) is a vector bundle, then there exists a pointwise multiplication of functions on M and sections of $\mathcal{K}_M$. For any $f \in S(M)$ and X being a local section of $\mathcal{K}_M$ in a neighborhood $x \in M$, we can put

$$(fX)_x = f(x)X_x, \tag{16}$$

because X_x is an element of the total space of the bundle $\mathcal{K}_E$. Moreover, either the composition $p \circ X$ or $p \circ (fX)$ is the same section of $\mathcal{E}$. In a similar way one can define the addition of local sections X, Y of $\mathcal{K}_M$ for the case $p \circ X = p \circ Y$.

One can also define two kinds of multiplication of functions and local sections of $\mathcal{J}^n(\mathcal{K}_M)$ in the following way:

$$(fh)_x = f(x)h_x \ \text{ and } \ (f * h)_x = [fX]^n_x\ , \tag{17}$$

where $x \in M$, and h is a local section of $\mathcal{J}^n(\mathcal{K}_M)$ such that $h_x = [X]^n_x$. Moreover, we put

$$(\eta h)_x = (f * h)_x \ \text{ for } \ \eta \in \Gamma(\mathcal{J}^n(M, \mathbf{R})),\ \eta_x = [f]^n_x. \tag{18}$$

3. Almost Multilinearity and Trace

Let $\mathcal{E}_i = p_i : E_i \longrightarrow M$, $i = 1, \dots, n$, and $\mathcal{P} = \pi : P \longrightarrow M$ be given vector bundles. An $\mathbf{R}$–multilinear map

$$A : \Gamma(\mathcal{E}_1) \times \Gamma(\mathcal{E}_2) \times \ldots \times \Gamma(\mathcal{E}_n) \longrightarrow \Gamma(\mathcal{P}) \tag{19}$$

will be called almost $S(M)$–multilinear if the equalities

$$A(f\alpha_1, \alpha_2, \dots, \alpha_n) = A(\alpha_1, f\alpha_2, \dots, \alpha_n) = \ldots = A(\alpha_1, \alpha_2, \dots, f\alpha_n) \tag{20}$$

are satisfied for any sections $\alpha_i \in \Gamma(\mathcal{E}_i)$ and a smooth function $f \in S(M)$.

Any almost $S(M)$–multilinear map A can be treated as an $\mathbf{R}$–linear map

$$A : \Gamma(\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \cdots \otimes \mathcal{E}_n) \longrightarrow \Gamma(\mathcal{P}) \tag{21}$$

given by

$$A(\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n) = A(\alpha_1, \alpha_2, \dots, \alpha_n). \tag{22}$$

Example I.

Any $S(M)$-multilinear map is an almost $S(M)$-multilinear map.

Example II.

A Lie bracket $[,] : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ of vector fields is not an almost $S(M)$–bilinear map.

Example III.

The map $A : \Gamma(TM) \times \Gamma(\bigwedge^{|M|} M) \longrightarrow \Gamma(\bigwedge^{|M|} M)$ given by formula $A(X, \omega) = L_X \omega$ is an almost $S(M)$–bilinear map, where L_X is the Lie derivation and $|M|$ denotes a dimension of M. Certainly,

$$A(fX, \omega) = L_{fX}\omega = (\partial_X f)\omega + fL_X\omega = L_X(f\omega) = A(X, f\omega).$$

Let us notice that for $\omega \in \Gamma(\bigwedge^k M)$ the expression $L_X\omega$ is not an almost $S(M)$-bilinear one for $k < |M|$.

For the case when $n = 2$, $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \mathcal{E}^*$, where $\mathcal{E} : E \longrightarrow M$ is a vector bundle, we can define the trace of map A by the following formula:

$$\mathrm{Tr}A = A(\mathrm{id}_{\Gamma(\mathcal{E})}), \tag{23}$$

where we treat the bundle $\mathcal{E} \otimes \mathcal{E}^*$ as $\mathrm{End}(\mathcal{E})$ using the identification

$$(u \otimes v^*)(w) = v^*(w)u. \tag{24}$$

For two sections $T_1, T_2 \in \Gamma(\mathrm{End}(\mathcal{E}))$ we put

$$l_{T_1}T_2 = T_1T_2 \ \ , \ \ r_{T_1}T_2 = T_2T_1. \tag{25}$$

Then, from (23) we get

$$\mathrm{Tr}(A \circ l_T) = \mathrm{Tr}(A \circ r_T) = A(T) \tag{26}$$

for any $T \in \Gamma(\mathrm{End}(\mathcal{E}))$.

Example IV.

The trace of a linear endomorphism.

Let V be a real k-dimensional space. Let A be an endomorphism of V and $[A^i_j]$ be a representation of A in a basis $(e_i)_{i=1,\dots,k}$ in V,

$$A^i_j = e^i(Ae_j), \quad i, j = 1, \dots, k, \tag{27}$$

where $(e^i)_{i=1,\dots,k}$ is the dual basis to e_i, it means that

$$e^i(e_j) = \delta^i_j. \tag{28}$$

Further, using the Eq.(24), we can write the decomposition of id_V in the way:

$$\mathrm{id}_V = \sum_i e_i \otimes e^i. \tag{29}$$

Certainly, for any basis vector e_i we have

$$(\sum_i e_i \otimes e^i)e_j = \sum_i e^i(e_j)e_i = \sum_i \delta^i_j e_i = e_j. \tag{30}$$

So, the right side of the Eq.(29) is the id_V. Moreover,

$$A(\mathrm{id}_V) = A(\sum_i e_i \otimes e^i) = \sum_i A^i_i = \mathrm{Tr}A. \tag{31}$$

This example shows that our definition of the trace is a generalization of the trace of a linear endomorphism.

Example V. (The Taylor expansion in terms of jets.)

Lemma. *Let $f \in S(M)$ be a smooth function and the $S(M)$-bilinear map*

$$A_f : \Gamma(\mathcal{J}^n(M, \mathbf{R})) \times \Gamma(\mathrm{Ldiff}^n(M, \mathbf{R})) \longrightarrow \Gamma(\mathcal{J}^n(M, \mathbf{R})) \tag{32}$$

be given by the formula

$$A_f(\eta, D) = (Df)\eta. \tag{33}$$

Then for any point $x \in M$ the equality

$$(\mathrm{Tr}A_f)_x = [f]^n_x \tag{34}$$

is fulfilled.

P r o o f. The proof follows immediately from the definition of the trace (23).

R e m a r k. One can treat the equality (34) as the Taylor formula in the Peano form. The basis of $\mathcal{J}^n_{x_0}(M,\mathbf{R})$ given by $e_\alpha = [f_\alpha]^n_{x_0}$, where $f_\alpha(x) = (\widetilde{x} - \widetilde{x}_0)^\alpha$, is a dual one to the basis $e^\alpha = \frac{1}{\alpha!}\frac{D^{|\alpha|}}{D\widetilde{x}^\alpha}$ of differential operators, for any choice of a chart φ at a neighborhood of the point x_0, where $\widetilde{x} = \varphi(x)$ denotes coordinates of x. Certainly,

$$e^\alpha e_\beta = \frac{1}{\alpha!}\frac{D^{|\alpha|}}{D\widetilde{x}^\alpha}(\widetilde{x} - \widetilde{x}_0)^\beta|_{x=x_0} = \delta^\alpha_\beta, \tag{35}$$

and $\sum_\alpha e^\alpha e_\alpha$ is the identity operator in the space of jets $\mathcal{J}^n_{x_0}(M,\mathbf{R})$.

4. Adjoint of Differential Operators

Let us define the adjoint of differential operators

$$^* : \Gamma(\mathrm{Ldiff}(M,\mathbf{R})) \longrightarrow \Gamma(\mathrm{Ldiff}(\bigwedge\nolimits^{|M|} M)). \tag{36}$$

For any function $f \in S(M)$ and any vector field X on M we put

$$(m_f)^* = m_f, \;\; (\partial_X)^* = -L_X. \tag{37}$$

We also put

$$(D_1 D_2)^* = D_2^* D_1^* \tag{38}$$

for $D_1, D_2 \in \Gamma(\mathrm{Ldiff}(M,\mathbf{R}))$.

By a domain in M we understand here an $|M|$-dimensional oriented compact connected submanifold Ω of M with boundary $\partial\Omega$. The operation of taking the adjoint satisfies the following integral formula:

$$\int\limits_\Omega (Df)\omega = \int\limits_\Omega f(D^*\omega) \tag{39}$$

for any differential operator $D \in \Gamma(\mathrm{Ldiff}^n(M,\mathbf{R})$, with Ω being a domain in M, $f \in S(M)$ and $\omega \in \Gamma(\bigwedge^{|M|} M)$ such that $[f\omega]^{n-1}|_{\partial\Omega} \equiv 0$. The Equation (39) is a consequence of integration by parts the formula

$$\int\limits_\Omega (\partial_X f)\omega = \int\limits_{\partial\Omega} f i_X\omega - \int\limits_\Omega f L_X\omega, \tag{40}$$

where $X \in \Gamma(TM)$.

Now we can formulate the following theorem.

Theorem 1. *Let*

$$\mathcal{E} = \;\pi : E \longrightarrow M \;,\;\; \mathcal{K}_E = \; p : K \longrightarrow E, \;\; \mathcal{K}_M = \; \pi \circ p : K \longrightarrow M \tag{41}$$

be given bundles, where $\mathcal{K}_E$ is a vector bundle. Let Ω be a domain in M, and

$$\omega \in \ \mathrm{HOM}_{\pi^n}(\mathcal{J}^n\mathcal{K}_M)_{\mathcal{J}^n\mathcal{E}}|_{\widetilde{\Omega}}, \bigwedge^{|M|}\Omega), \ \ h \in \Gamma(\mathcal{J}^n(\mathcal{K}_M)|_{\Omega}), \tag{42}$$

where $\widetilde{\Omega} = (\pi^n)^{-1}(\Omega)$. *Let the map*

$$A_h^\omega : \Gamma(\mathcal{J}^n(\Omega, \mathbf{R})) \times \Gamma(\mathrm{Ldiff}^n(\Omega, \mathbf{R})) \longrightarrow \Gamma(\bigwedge^{|M|}\Omega) \tag{43}$$

be given by the formula

$$A_h^\omega(\eta, D) = D^*(\omega(\eta h)). \tag{44}$$

Then A_h^ω is an almost $S(M)$-bilinear map, and the equality

$$\mathrm{Tr} A_{f*h}^\omega = f \mathrm{Tr} A_h^\omega \tag{45}$$

holds for any smooth function f.

First, let us prove the following:

Lemma. *The map A_h^ω satisfies the equalities:*

$$A_{f*h}^\omega(\eta, D) = A_h^\omega(f * \eta, D) \tag{46}$$

and

$$A_h^\omega(\eta, D \circ m_f) = f A_h^\omega(\eta, D). \tag{47}$$

T h e p r o o f o f t h e l e m m a.

We can write

$$A_{f*h}^\omega(\eta, D) = D^*(\omega(\eta(f * h))) = D^*(\omega((f * \eta)h)) = A_h^\omega(f * \eta, D) \ , \tag{48}$$

because $\eta(f * h) = (f * \eta)h$. The formula (47) is a consequence of the fact $(Dm_f)^* = fD$. ∎

T h e p r o o f o f T h e o r e m 1. Using the identification (22), we can write the equality (46) as

$$A_{f*h}^\omega = A_h^\omega \circ l_{f^*}. \tag{49}$$

Further, from (10, 47) we get

$$A_h^\omega \circ r_{f^*} = f A_h^\omega. \tag{50}$$

So, from Eq.(26) we obtain Eq.(45). ∎

5. Euler–Lagrange Operator

Let $\mathcal{E} = p : E \longrightarrow M$ be a fibre bundle (not necessarily a vector bundle), and

$$F : \mathcal{J}^n\mathcal{E} \longrightarrow \bigwedge^{|M|} M \tag{51}$$

be a fibre map (over id_M). The map (51) induces the functional of the form

$$I[s] = \int\limits_{\Omega} F([s]^n) \ , \tag{52}$$

where s denotes a section of $\mathcal{E}$ over the domain $\Omega \subset M$.

Let X be a local section of $(V\mathcal{E})_M$ over a neighborhood of $x \in M$ and let σ_t, $t \in (-\epsilon, \epsilon)$ be a one-parameter family of local sections of $\mathcal{E}$ such that

$$X = \frac{d}{dt}\sigma_t|_{t=0}. \tag{53}$$

The expression

$$(V_X F)_x = \frac{d}{dt} F([\sigma_t]^n_x)|_{t=0} \tag{54}$$

is a vertical jet derivative of the map F at point x in the direction of field X. It is well known that $(V_X F)_x$ depends on the n-th jet of X at x. So, we can put

$$V_h F = (V_X F)_x \tag{55}$$

for $h = [X]^n_x$.

R e m a r k. The derivative operator (5) is an analogue of the operator (54). The field X in (54) corresponds to the field h in (5).

Now let h be a local section of $\mathcal{J}^n((V\mathcal{E})_M) = (V(\mathcal{J}^n\mathcal{E}))_M$. The vertical jet derivative satisfies the following equalities:

$$V_{fh}F = fV_hF, \ \ (V_{fX}F)_x = (V_{f*h}F)_x, \tag{56}$$

where $x \in M$ and X is a local section of $(V\mathcal{E})_M|_\Omega$ such that $[X]^n_x = h_x$.

The Euler–Lagrange operator δ_X for the functional (52) is uniquely determined by the following two conditions:

$$\delta_{fX}F = f\delta_X F \tag{57}$$

(it means that the operator δ is $S(M)$-linear with respect to the field X), and

$$\int\limits_{\Omega} \delta_X F = \int\limits_{\Omega} V_X F \tag{58}$$

for any X over Ω such that $[X]^{n-1}|_{\partial\Omega} \equiv 0$.

The following theorem expresses the Euler–Lagrange operator by means of the trace of an $S(M)$-bilinear map.

Theorem 2. *Let* $F : \mathcal{J}^n\mathcal{E} \longrightarrow \bigwedge^{|M|}M$ *be a fibre map over* id_M, Ω *be a domain in* M, *and*

$$A_h^F : \Gamma(\mathcal{J}^n(\Omega,\mathbf{R})) \times \Gamma(\mathrm{Ldiff}^n(\Omega,\mathbf{R})) \longrightarrow \Gamma(\bigwedge^{|M|}\Omega) \tag{59}$$

be given by the formula

$$A_h^F(\eta, D) = D^*(V_{\eta h}F) \tag{60}$$

for $h \in \Gamma(\mathcal{J}^n((V\mathcal{E})_M)|_\Omega)$. *Then* A_h^F *is an almost* $S(M)$*–bilinear map, and the trace* $\mathrm{Tr}A^F_{[X]^n}$ *is the Euler–Lagrange operator*

$$\delta_X F = \mathrm{Tr}A^F_{[X]^n} \tag{61}$$

for any $X \in \Gamma((V\mathcal{E})_M)|_\Omega)$.

P r o o f. Let $\Delta_h F := \mathrm{Tr}A_h^F$. Theorem 1 shows that the equality $\Delta_{f*h}F = f\Delta_h F$ is fulfilled for any smooth function $f \in S(M)$. Now let us consider an operator of the form $D \circ \partial_Y$ for $D \in \Gamma(\mathrm{Ldiff}^{k-1}(M,\mathbf{R}))$, where $k \leq n$ and $Y \in \Gamma(TM)$. Let $\eta \in \Gamma(\mathcal{J}^n(M,\mathbf{R}))$ be a section of the class of infinitesimal jets of rank k. It means that $D'(\eta) \equiv 0$ for any operator D' of the rank lower than k. So

$$\int\limits_\Omega (D \circ \partial_Y)^*(V_{\eta[X]^n}F) = \int\limits_{\partial\Omega} i_Y(D^*(V_{\eta[X]^n}F)) = 0 \tag{62}$$

for $[X]^{(n-1)}|_{\partial\Omega} \equiv 0$. The bundle $\mathrm{Ldiff}^n(M,\mathbf{R})$ can be decomposed into the direct sum

$$\mathrm{Ldiff}^n(M,\mathbf{R}) = \mathrm{Ldiff}^0(M,\mathbf{R}) \oplus \mathrm{Ldiff}^{n,1}(M,\mathbf{R}), \tag{63}$$

where $\mathrm{Ldiff}^0(M,\mathbf{R})$ is the bundle of vanishing on constant functions operators of rank 0 (multiplication by functions) and $\mathrm{Ldiff}^{n,1}(M,\mathbf{R})$ is the bundle of operators of the rank not higher than n.

The decomposition of $\mathcal{J}^n(M,\mathbf{R})$, dual to (63), is

$$\mathcal{J}^n(M,\mathbf{R}) = \mathcal{J}^{n,0}(M,\mathbf{R}) \oplus \mathcal{J}^{n,1}(M,\mathbf{R}), \tag{64}$$

where $\mathcal{J}^{n,0}(M,\mathbf{R})$ is the bundle of jets represented by locally constant functions and $\mathcal{J}^{n,1}(M,\mathbf{R})$ is the bundle of infinitesimal jets. We can decompose $\Delta_h F$

$$\Delta_h F = \mathrm{Tr}^{n,0}A_h^F + \mathrm{Tr}^{n,1}A_h^F \tag{65}$$

in accordance with (63, 64). So

$$\mathrm{Tr}^{n,0}A_h^F = A_h^F([1], m_1) = V_h F,$$

and from (62) we obtain
$$\int\limits_{\Omega} \mathrm{Tr}^{n,1} A^F_{[X]^n} = 0$$
for $[X]^{n-1}|_{\partial\Omega} \equiv 0$. ■

R em a r k. Our formalism gives a geometric approach to the theory without using local coordinates. W.M. Tulczyjew considered the bundles of jets of infinite rank $\mathcal{J}^\infty(\mathcal{E})$. In the paper [2] there was introduced a family of operators θ_β acting on differential forms on $\mathcal{J}^\infty(\mathcal{E})$. Using these operators, W.M. Tulczyjew derived an expression analogous to the formula (6). We have given here a definition of the Euler–Lagrange operator by means of the trace of an almost $S(M)$–bilinear map.

In a local coordinate the Taylor formula and the Euler–Lagrange operator are given by means of differential operators of the form $\frac{D^{|\alpha|}}{Dx^\alpha}$, which commute each other, that is a consequence of the Schwarz theorem. Our approach is global and it can be applied to a noncommutative basis of differential operators. The way is natural for to study functionals on the noncommutative Lie groups. The first order operators can be globally defined as the left-invariant vector fields and they form Lie algebra of the group. Any left-invariant differential operator is a linear combination of compositions of the first order operators. Using a suitable basis in the space of the left-invariant differential operators we can study globally the Euler–Lagrange operators on Lie groups.

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