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Ruled Surfaces in E^4 with Constant Ratio of the Gaussian Curvature and Gaussian Torsion

O.A. Goncharova

B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv, 61103, Ukraine
E-mail:goncharova@ilt.kharkov.ua

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Local and global existence theorems on the ruled surfaces with a constant ratio of the Gaussian curvature and Gaussian torsion are proved.

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For 2-dimensional surfaces in E^4 the Gaussian torsion is the invariant of normal connection similar to the invariant of tangent connection, that is to the Gaussian curvature. Two-dimensional surfaces in E^4 for which the Gaussian torsion κ_Γ coincides with the Gaussian curvature K were considered in [1]. For example, if the position vector of a surface is $x, y, u(x,y), v(x,y)$ and $u = \Phi_x$, $v = \Phi_y$, where $\Phi = \Phi(x,y)$ is some function, then $K = \kappa_\Gamma$. In general case on the surface with $K = \kappa_\Gamma$ the functions u and v satisfy some differential equation of the first order. By using a method of characteristics one can integrate this equation to the system of Hamilton equations. Thus, in [1] the method of constructing the surfaces mentioned above is proposed and concrete examples are considered.

It is natural to consider a wider class of surfaces for which the ratio of the Gaussian torsion and Gaussian curvature is a constant number. In this paper a class of the ruled surfaces in E^4 is considered. It is assumed that the ratio of the Gaussian torsion and curvature is distinct from zero, as the ruled surfaces with zero Gaussian torsion were considered in [2]. In this paper from [2] the formulas for K and κ_Γ of the ruled surfaces in E^4 are used.

We formulate the basic results.

Theorem 1. *For any C^2-regular curve $\gamma \subset E^4$, with the first curvature $k_1 \neq 0$, and for any number $c_1 \neq 0$ in a neighborhood of each point $p_0 \in \gamma$, there exists a regular ruled surface with the base curve γ for which the ratio of the Gaussian torsion and curvature is constant and equal to c_1.*

Following Prop. A and B state global resolvability.

Analogously to the slope curve in E^3, denote the slope curve in E^4 for which the curvatures $k_1 \neq 0$, k_2 and k_3 satisfy the relations $k_2 = \alpha k_1$ and $k_3 = \beta k_1$, where α, β are constants, and $\alpha \neq 0$.

A generalized standard ruled surface is denoted as the two-dimensional ruled surface F^2 in Euclidean space E^n when the directing (base) curve γ is a curve with the distinct from zero curvatures $k_1, \dots, k_{n-1}$, and the generatrix in each point are directed as one of the basis vectors from the natural frame $\xi_1, \dots, \xi_n$. If the directing curve γ is a curve with constant and distinct from zero curvatures, then the generalized standard ruled surface corresponds to the standard ruled surface given in [4].

Proposition A. *If γ in E^4 is a slope curve, then there exists a regular generalized standard ruled surface along the whole base curve γ for which the ratio $\frac{\kappa_\Gamma}{K}$ is constant and equal to $\frac{\beta}{\alpha^2+\beta^2}$.*

In particular, the standard ruled surface Φ_3 from [4] is of this kind, what follows from the expressions for K and κ_Γ calculated in [4].

Proposition B. *If γ is a flat curve with $(k_1 \neq 0)$, then in E^4 there exists a regular ruled surface along the whole base curve γ for which $\frac{\kappa_\Gamma}{K} = c_1 = const$.*

If the curve γ is a closed flat curve with the length ℓ, and $c_1 = \frac{1}{2\pi}\int\limits_0^\ell k_1(t)dt$, then the ruled surface is homeomorphic to a cylinder. As a special case of the closed flat curve can be taken a circle. If $c_1 = \frac{p}{q}$ is a rational number and the curve γ is a circle taken p times, then this surface is homeomorphic to a cylinder. If c_1 is an irrational number and the curve γ is a circle taken infinite number of times, then this surface is homeomorphic to a plane.

Some definitions should be recalled. Let γ be some $C^k(k \geq 2)$-regular curve in E^4 with the position vector $\rho(t)$, where t is an arc length. Let along the curve γ the $C^k(k \geq 2)$-regular unit vector field of generatrix $a(t)$ be given. Then the position vector of the ruled surfaces F^2 has the following form:

$$r(t,u) = \rho(t) + u a(t).$$

Denote the curve γ as a base curve and also assume that a base curve is orthogonal to generatrix.

Take the unit vector field $a(t)$ as a linear combination of vectors of Frene's natural basis ξ_2, ξ_3, ξ_4, that is $a(t) = \sum\limits_{i=2}^4 a^i \xi_i$. The expression for the first derivative of vector a has the form $a' = \sum\limits_{i=1}^4 T^i \xi_i$, where the coefficients T^i are:

$$T^1 = -k_1 a^2, \quad T^2 = \frac{da^2}{dt} - k_2 a^3,$$

$$T^3 = \frac{da^3}{dt} + k_2 a^2 - k_3 a^4, \quad T^4 = \frac{da^4}{dt} + k_3 a^3.$$

From [2] the expression for the ratio of the Gaussian torsion and Gaussian curvature of the ruled surface in E^4 is taken

$$\frac{\kappa_\Gamma}{K} = \frac{c(t) + b(t)u}{d(t)},$$

where

$$c(t) = k_1 \left[\left(a^4 \frac{da^3}{dt} - a^3 \frac{da^4}{dt} \right) - k_3[(a^3)^2 + (a^4)^2] + k_2 a^2 a^4 \right],$$

$$b(t) = \left(T^4 \frac{dT^3}{dt} - T^3 \frac{dT^4}{dt} \right) a^2 + T^2 \left(\frac{dT^4}{dt} a^3 - \frac{dT^3}{dt} a^4 \right)$$
$$+ T^2 T^4 (k_2 a^2 + k_3 a^4) - (T^2)^2 k_2 a^4 + T^2 T^3 k_3 a^3 - k_3 a^2 ((T^3)^2 + (T^4)^2)$$
$$+ (T^3 a^4 - T^4 a^3) \left(\frac{dT^2}{dt} + k_1 T^1 - k_2 T^3 \right),$$

$$d(t) = -(a')^2 + (\xi, a')^2 = -[(T^2)^2 + (T^3)^2 + (T^4)^2].$$

Let us prove Th. 1. For this we consider the case $\frac{\kappa_\Gamma}{K} = c_1 = const$. To find the surfaces possessing this property it is necessary to solve the system

$$b(t) = 0, \tag{1}$$

$$c(t) = c_1 d(t). \tag{2}$$

First, consider the equation (1). By substituting the expressions for T^i, one gets

$$\frac{d^2 a^2}{dt^2} \left(a^4 \frac{da^3}{dt} - a^3 \frac{da^4}{dt} + k_2 a^2 a^4 - k_3 ((a^3)^2 + (a^4)^2) \right)$$
$$+ \frac{d^2 a^3}{dt^2} \left(a^2 \frac{da^4}{dt} - a^4 \frac{da^2}{dt} + k_3 a^2 a^3 + k_2 a^3 a^4 \right)$$
$$+ \frac{d^2 a^4}{dt^2} \left(a^3 \frac{da^2}{dt} - a^2 \frac{da^3}{dt} + k_3 a^2 a^4 - k_2 ((a^2)^2 + (a^3)^2) \right) + \ldots = 0, \tag{3}$$

hereinafter the dots designate the terms not containing the second derivative from a^i.

Rewrite (2) in the following form:

$$k_1 (T^3 a^4 - T^4 a^3) + c_1 ((T^2)^2 + (T^3)^2 + (T^4)^2) = 0. \tag{4}$$

By differentiating both parts of (4), one obtains

$$k_1 \left(\frac{dT^3}{dt} a^4 - \frac{dT^4}{dt} a^3 + \ldots \right) + 2c_1 \left(T^2 \frac{dT^2}{dt} + T^3 \frac{dT^3}{dt} + T^4 \frac{dT^4}{dt} \right) = 0.$$

By substituting the expressions for T^i, one gets

$$\frac{d^2a^2}{dt^2}\left(2c_1\frac{da^2}{dt}-2c_0k_2a^3\right)+\frac{d^2a^3}{dt^2}\left(2c_1\frac{da^3}{dt}+k_1a^4+2c_1k_2a^2-2c_1k_3a^4\right)$$

$$+\frac{d^2a^4}{dt^2}\left(2c_1\frac{da^4}{dt}-k_1a^3+2c_0k_3a^3\right)+...=0. \tag{5}$$

As a vector field $a(t)$ is unit, then $(a^2)^2+(a^3)^2+(a^4)^2=1$. Differentiate this condition twice

$$\frac{d^2a^2}{dt^2}(2a^2)+\frac{d^2a^3}{dt^2}(2a^3)+\frac{d^2a^4}{dt^2}(2a^4)+...=0. \tag{6}$$

The system of three differential equations (3), (5), (6) of the second order relatively to the coefficients a^i with additional conditions

$$c(t)-c_1d(t)=0, \tag{7}$$

$$(a^2)^2+(a^3)^2+(a^4)^2=1, \tag{8}$$

$$a^2\frac{da^2}{dt}+a^3\frac{da^3}{dt}+a^4\frac{da^4}{dt}=0 \tag{9}$$

is obtained.

If the determinant Δ of the matrix consists of the coefficients of $\frac{d^2a^i}{dt^2}$ in (3), (5), (6) and is distinct from zero, then the system in its normal form can be written as follows:

$$\frac{d^2a^k}{dt^2}=F_k\left(t,a^i,\frac{da^j}{dt}\right),\quad k=2,3,4. \tag{10}$$

Calculate the determinant of the matrix. In spite rather, of the expressions being rather complex, one gets a very simple result

$$\Delta=-k_1\left(a^4\frac{da^3}{dt}-a^3\frac{da^4}{dt}-k_3((a^3)^2+(a^4)^2)+k_2a^2a^4\right)-2c_1\left(\frac{da^2}{dt}-k_2a^3\right)^2$$

$$-2c_1\left(\frac{da^3}{dt}-k_3a^4+k_2a^2\right)^2-2c_1\left(\frac{da^4}{dt}+k_3a^3\right)^2+2c_1\left(a^2\frac{da^2}{dt}+a^3\frac{da^3}{dt}+a^4\frac{da^4}{dt}\right)$$

$$=-k_1(T^3a^4-T^4a^3)-2c_1[(T^2)^2+(T^3)^2+(T^4)^2].$$

Using the equation (4), one finally obtains

$$\Delta=-c_1[(T^2)^2+(T^3)^2+(T^4)^2].$$

The system (10) of the second orders, when using a usual method, can be reduced to the system of equations of the first order. Let y be a vector with the components $y_2, \ldots, y_7$. Put $y_2 = a^2$, $y_3 = a^4$, $y_4 = a^6$, $y_5 = \frac{da^2}{dt}$, $y_6 = \frac{da^3}{dt}$, $y_7 = \frac{da^4}{dt}$. Then the system can be written in the following form:

$$\frac{dy}{dt} = F(y, t).$$

Choose the initial conditions of the system as follows:

$$y_0^2 = a^2(t_0) = 0, y_0^3 = a^3(t_0) = 1, y_0^4 = a^4(t_0) = 0, y_0^6 = \frac{da^3}{dt}(t_0) = 0.$$

Then additional conditions (7) and (8) are executed automatically. From (9) one gets the equation

$$\left(\frac{da^2}{dt}(t_0) - k_2\right)^2 = \left(\frac{da^4}{dt} + k_3\right)\left(\frac{k_1}{c_1} - \left(\frac{da^4}{dt} + k_3\right)\right).$$

First, consider the case when the number c_1 is positive. Let us assume

$$y_0^7 = \frac{da^4}{dt}(t_0) = -k_3 + \varepsilon,$$

$$y_0^5 = \frac{da^2}{dt}(t_0) = k_2(t_0) + \sqrt{\varepsilon\left(\frac{k_1(t_0)}{c_1} - \varepsilon\right)}.$$

It is possible to choose a positive small enough ε, so that the additional condition (9) will take place. For $c_1 > 0$ at point (t_0, y_0), the determinant is distinct from zero

$$\Delta(y_0) = -c_1\left[\left(\frac{da^2}{dt}(t_0) - k_2\right)^2 + \left(\frac{da^4}{dt}(t_0) + k_3\right)^2\right]$$
$$= -c_1\left(\varepsilon\left(\frac{k_1}{c_1} - \varepsilon\right) + \varepsilon^2\right) = -\varepsilon k_1 \neq 0.$$

Now consider the case $c_1 < 0$. Let us assume

$$y_0^7 = \frac{da^4}{dt}(t_0) = -k_3 - \varepsilon,$$

$$y_0^5 = \frac{da^2}{dt}(t_0) = k_2(t_0) + \sqrt{-\varepsilon\left(\frac{k_1(t_0)}{c_1} + \varepsilon\right)},$$

then it is possible to choose a positive small enough ε, so that the additional condition (9) will take place. For $c_1 < 0$ at point (t_0, y_0), the determinant is also distinct from zero

$$\Delta(t_0) = -c_1\left(-\varepsilon\left(\frac{k_1}{c_1} + \varepsilon\right) + \varepsilon^2\right) = \varepsilon k_1 \neq 0.$$

Let us consider some neighborhood of point (t_0, y_0). The determinant Δ is a function of $t, y_2, y_3, \dots, y_7$: $\Delta = \Delta(t, y_2, \dots, y_7)$ which is the algebraic function of $t, y_2, \dots, y_7$ and, consequently, is continuous. Thus, the existence of the ruled surfaces for which the ratio of the Gaussian torsion and curvature is constant follows from the existence theorem of the theory of ordinary differential equations.

Consider the question of regularity of the constructed surfaces. The coefficients of their metric have the following form (see [2]):

$$g_{11} = 1 + 2u(\xi_1, a') + u^2(a')^2,$$

$$g_{12} = 0, \quad g_{22} = 1.$$

Then get the condition of regularity

$$g_{11}g_{22} = (1 - uk_1a^2) + u^2((T^2)^2 + (T^3)^2 + (T^4)^2) \neq 0.$$

In point (t_0, y_0), the inequality $\sum\limits_{i=2}^{4}(T^i)^2 \geq \varepsilon^2$ holds. From the above it follows that at all values of the parameter u in some neighborhood of point (t_0, y_0), the inequality $g_{11}g_{22} > 0$ is valid. The surface is regular for the whole strip limited by extreme generatrix. Thus Theorem 1 is proved.

Now let us turn to some examples of the ruled surfaces with a constant ratio of the Gaussian torsion and Gaussian curvature with some base curves.

It is known that the tangent vector of a curve with constant ratio of torsion and curvature in E^3 makes a constant angle with the fixed direction (see [3]). Denote these curves as slope curves in E^3. Recall that at the beginning of the paper there was given the definition of slope curves in E^4. For these curves

$$k_2 = \alpha k_1, \quad k_3 = \beta k_1 \tag{11}$$

hold, where α, β are constants, and $\alpha \neq 0$. Frene's equations for this type of the curves, can be written in the form:

$$\begin{aligned}
\frac{d\xi_1}{ds} &= k_1\xi_2,\\
\frac{d\xi_2}{ds} &= k_1(-\xi_1 + \alpha\xi_3),\\
\frac{d\xi_3}{ds} &= k_1(-\alpha\xi_2 + \beta\xi_4),\\
\frac{d\xi_4}{ds} &= -k_1\beta\xi_3.
\end{aligned}$$

Introduce a new parameter $t = \int k_1 ds$ on the slope curve in E^4. By differentiating several times $\frac{d\xi_1}{dt}$ and using Frene's equations, get the differential equation on ξ_1 with constant coefficients

$$\frac{d^4\xi_1}{dt^4} + (1 + \alpha^2 + \beta^2)\frac{d^2\xi_1}{dt^2} + \beta^2\xi_1 = 0.$$

From the theory of differential equations it follows that the common solution of the equation above is

$$\xi_1 = A_1 \cos\nu_1 t + B_1 \sin\nu_1 t + A_2 \cos\nu_2 t + B_2 \sin\nu_2 t,$$

where ν_1, ν_2 are some functions from α, β, and A_1, B_1, A_2, B_2 are constant vectors satisfying condition $|\xi_1| = 1$. By integrating ξ_1, it is possible to find a position vector of the slope curve in E^4.

Now prove Prop. A. Let us construct a generalized standard ruled surface along the slope curve in E^4. So, let a base curve be a slope curve in E^4 with certain conditions on the curvatures $k_1 \neq 0$, $k_2 = \alpha k_1$, $k_3 = \beta k_1$, and the generatrix be directed along ξ_3, that is $a^2 \equiv 0$, $a^3 \equiv 1$, $a^4 \equiv 0$. Then the equation $b(t) = 0$ is reduced to $k_2' k_3 - k_2 k_3' = 0$ which is executed by virtue of (11). From (2) it follows that $-k_1 k_3 = -c_1(k_2^2 + k_3^2)$, that is $c_1 = \frac{k_1 k_3}{k_2^2 + k_3^2} = \frac{\beta}{\alpha^2+\beta^2}$.

It appears that the ratio of the Gaussian torsion and curvature at the generalized standard ruled surfaces along the slope curve in E^4 is constant. From the expression $g_{11}g_{22}$ it is easy to see that the surface is regular. Proposition A is proved.

If the curvatures of the curve are constant, then this case may be referred to the standard ruled surface Φ_3 in E^4 described in [4].

To prove Prop. B, construct an example of the ruled surfaces with a constant ratio of the Gaussian torsion and Gaussian curvature along a flat curve. In this case $k_2 = 0$, $k_3 = 0$, and the functions $b(t)$, $c(t)$, $d(t)$ can be written in the following form:

$$b(t) = \left(\frac{da^4}{dt}\frac{d^2a^3}{dt^2} - \frac{da^3}{dt}\frac{d^2a^4}{dt}\right)a^2 + \frac{da^2}{dt}\left(\frac{d^2a^4}{dt^2}a^3 - \frac{d^2a^3}{dt^2}a^4\right)$$
$$+ \left(\frac{da^3}{dt}a^4 - \frac{da^4}{dt}a^3\right)\left(\frac{d^2a^2}{dt^2} - k_1^2 a^2\right),$$

$$c(t) = k_1\left(\frac{da^3}{dt}a^4 - \frac{da^4}{dt}a^3\right), \tag{12}$$

$$d(t) = -\left[\left(\frac{da^2}{dt}\right)^2 + \left(\frac{da^3}{dt}\right)^2 + \left(\frac{da^4}{dt}\right)^2\right]. \tag{13}$$

If $a_2 \equiv 0$, then the equation $b(t) = 0$ is executed identically. We put the coefficients

$$a^3 = \sin\varphi(t), \quad a^4 = \cos\varphi(t) \tag{14}$$

and substitute them into expressions for $c(t)$ and $d(t)$. Then the equation (2) by virtue of (12) - (14) has the following form:

$$k_1 \frac{d\varphi}{dt} + c_1 \left(\frac{d\varphi}{dt}\right)^2 = 0.$$

In the case when $\frac{d\varphi}{dt} = 0$, the surface is a cylinder for which $K = 0$. In this paper the surfaces of this type are not considered. Hence,

$$\frac{d\varphi}{dt} = -\frac{k_1(t)}{c_1}.$$

The solution of the differential equation is as follows:

$$\varphi(t) = -\frac{1}{c_1}\int\limits_0^t k_1(t)dt + \varphi_0.$$

As $(T^3)^2 + (T^4)^2 = \left(\frac{d\varphi}{dt}\right)^2 \neq 0$, then the surface is regular. Proposition B is proved.

Choose a closed flat curve with the length ℓ. If any point on this curve passes the whole curve, then the increment $\Delta\varphi$ of the angle φ is equal to $\Delta\varphi(t) = -\frac{1}{c_1}\int\limits_0^\ell k_1(t)dt$. When

$$c_1 = \frac{1}{2\pi}\int\limits_0^\ell k_1(t)dt,$$

then $\Delta\varphi = -2\pi$, and in this case the vector $a(t)$ will return to the initial position. Thus we obtain the surface that is homeomorphic to the cylinder with the base flat curve.

A special case of the flat closed curve is a circle, i.e., $k_1 = const$. If any point on the circle passes the whole circle, then the angle φ gets the increment $\Delta\varphi = -\frac{k_1}{c_1}\ell$, where ℓ is the length of the circle, that is $\Delta\varphi = -\frac{2\pi}{c_1}$. If $c_1 = \frac{p}{q}$ is a rational number, and the point of the curve passes it p times, then $\Delta\varphi$ is multiple 2π. In this case the vector $a(t)$ will return to the initial position, and we obtain the surface that is homeomorphic to the cylinder with the circle taken p times as a base curve. If c_1 is an irrational number, then the vector $a(t)$ will not return to the initial position. Thus, if the point of the circle passes it infinite number of times, we get the surface homeomorphic to a plane.

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