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Geometric Constructions in the Class of Busemann Nonpositively Curved Spaces

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The procedures of gluing and the Hausdorff limit in the class of metric spaces nonpositively curved in the sense of Busemann are studied in the paper. Conditions under which the resulting spaces belong to the same class are found.

Key words: Busemann nonpositive curvature, gluing, Hausdorff limit.

Mathematics Subject Classification 2000: 53C23.

1. Introduction

We study some constructions in the class of geodesic metric spaces with Busemann nonpositive curvature. Here we use not a classical definition of nonpositively curved spaces ([2]), but the one introduced by B. Bowditch in [1] where the class of spaces considered is called Busemann. This allows to include the consideration of all $CAT(0)$-spaces, i.e., the complete simply connected spaces of nonpositive curvature in the sense of A.D. Alexandrov and all strictly convex normed spaces. The main question of the paper is: what are additional conditions for gluing and limiting operations in the class of Busemann spaces to keep the resulting space in the same class?

Similar problems for Alexandrov spaces were studied sufficiently deeply in [3, 4], etc. Some operations in the class of Busemann spaces were studied in [5] and [6]. The condition of nonpositivity of curvature in the sense of Busemann is weaker than in the sense of Alexandrov. By this reason many results that are true to Alexandrov spaces do not have direct generalization for Busemann spaces. When applying Alexandrov spaces theory one should set additional requirements in a number of situations.

The paper is organized as follows. In Section 2 we recall some necessary definitions and facts from Busemann spaces theory. In Section 3 we prove the gluing theorem which generalizes Reshetnyak's gluing theorem known for Alexandrov spaces (cf. [3, Th. 9.1.21]). When we speak about Busemann spaces, the gluing theorem has the following formulation.

Theorem 3.1. *Let (X_1, d_1), (X_2, d_2) and (X_3, d_3) be three Busemann spaces represented as unions of closed convex subsets $X_i := A_i \cup B_i$. Let $g_1 : B_2 \to A_3$, $g_2 : B_3 \to A_1$ and $g_3 : B_1 \to A_2$ be three isometries such that $g_2 \circ g_1 \circ g_3 = \mathrm{Id}|_{A_1 \cap B_1}$. Then the space X obtained as a factorspace $X := (X_1 \cup X_2 \cup X_3)/\{g_1, g_2, g_3\}$ with the metric d that coincides with d_i in each X_i, is a Busemann space.*

In Section 4 we study the Hausdorff limits of Busemann spaces. The class of all Busemann spaces is not closed under Hausdorff limit: the sequence of strictly convex normed spaces can converge to the normed space with nonstrictly convex norm. B. Kleiner introduced the notion of often convex space in [6]. The class of often convex spaces is closed under limits and contains a subclass of Busemann spaces. We study the Hausdorff limits of Busemann spaces under additional requirement of unimodular convexity. The main result of the section is the following theorem.

Theorem 4.3. *Let the complete metric space (X, o, d_X) with basepoint o be a Hausdorff limit of unimodularly convex sequence (X_n, o_n, d_n) of pointed Busemann spaces. Then X is also a Busemann space and its convexity modulus $\delta_x(\epsilon, r)$ for all $x \in X$ is bounded from below by the common low boundary of convexity modules of spaces X_n.*

2. Preliminaries

The general theory of spaces with intrinsic metric can be found in [3, 4] and [7]. Here we recall some basic facts related to Busemann nonpositively curved spaces.

Definition 2.1. *Let (X, d) be a geodesic space. We use the notation $|xy|$ for the distance $d(x, y)$ between its points. A segment connecting the points $x, y \in X$ is denoted $[xy]$. We say that X is a Busemann nonpositively curved space (shortly Busemann space) if its metric is convex: if $c : [a, b] \to X$ and $d : [a', b'] \to X$ are affine parameterizations of two segments, then the function $D : [a, b] \times [a', b'] \to \mathbb{R}_+$*

$$D(s, t) = |c(s)d(t)|$$

is convex. Equivalently, the space X is Busemann nonpositively curved if for any three points $x, y, z \in X$, for the arbitrary midpoint m between x and y and for *arbitrary midpoint n between x and z the inequality*

$$|mn| \le \frac{1}{2}|yz| \tag{2.1}$$

holds.

The following properties of the considered spaces are simple corollaries from Def. 2.1. Each Busemann nonpositively curved space X is contractible, any its two points are connected by the unique segment.

The class of Busemann nonpositively curved spaces contains all $CAT(0)$-spaces and strictly convex Minkowski spaces.

The fact that some non-Minkowskian Finsler manifolds have nonpositive curvature in sense of Busemann is less trivial. Finsler metrics having nonpositive curvature in the sense of Busemann were studied in [8]. It is shown that every Finsler manifold with Berwald metric and nonpositive flag curvatures is a generalized Busemann space (geodesic space with the Busemann property of curvature nonpositivity but without symmetry condition on the metric). The Finsler metric $F(x, dx)$ on the manifold M^n is a *Berwald metric* if there is a special coordinate system, where its geodesics satisfy the system of differential equations

$$\ddot{\sigma}^i + 2G^i(\sigma, \dot{\sigma}) = 0.$$

Here $G^i := G^i(x, y)$ are positive functions homogeneous of the second degree in y. If the metric F is Riemannian, then $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$, where Γ^i_{jk} are Levi–Chivita connection coefficients.

The Berwald condition is essential here. By Kelly–Straus theorem (cf. [9]), if the Finsler space with Hilbert metric (of constant negative flag curvature) is a Busemann space, then it is a Lobachevsky space.

In connection with convexity, the spaces with nonpositive curvature are sometimes called *convex spaces* (cf. [10]).

Definition 2.2. *The metric space X is called locally convex if every its point has a neighborhood that is the Busemann nonpositively curved space in the metric of X.*

Several strengthenings of the convexity property were introduced in [11].

Definition 2.3. *The Busemann nonpositively curved space X is called strictly convex if there is the strong inequality*

$$|x_0 m| < \max\{|x_0 y|, |x_0 z|\}$$

for every triple of points $x_0, y, z \in X$*, where* m *is a midpoint between* y *and* z*. The strictly convex space* X *is called weakly uniformly convex if for any point* $x_0 \in X$ *the modulus of convexity function*

$$\delta_{x_0}(\mu, r) := \inf\{r - |x_0 m| \mid y, \quad z \in X,$$
$$|x_0 y| \leq r, |x_0 z| \leq r, |yz| \geq \mu r, |ym| = |mz| = \frac{1}{2}|yz|\}$$

is positive for any $\mu, r > 0$*. Finally, the weakly uniformly convex space* X *is called uniformly convex if*

$$\lim_{r \to +\infty} \delta_{x_0}(\mu, r) = +\infty$$

for any fixed $\mu > 0$*.*

For example, every strictly convex Minkowski space is uniformly convex, because its modulus of convexity function is homogeneous by r:

$$\delta_o(\mu, \lambda r) = \lambda \delta_o(\mu, r)$$

for all $\mu, r, \lambda > 0$.

3. Gluing

The gluing theorem known for the Alexandrov spaces in Reshetnyak's formulation (cf. [12]) is not true for Busemann spaces with nonpositive curvature. For example, the result of gluing of two normed half-planes with different norms is a plane whose metric fails to be a Busemann nonpositive curvature. We will prove the following version of the gluing theorem.

Theorem 3.1. *Let* (X_1, d_1)*,* (X_2, d_2) *and* (X_3, d_3) *be three Busemann spaces represented as unions of closed convex subsets* $X_i := A_i \cup B_i$*. Let* $g_1 : B_2 \to A_3$*,* $g_2 : B_3 \to A_1$ *and* $g_3 : B_1 \to A_2$ *be three isometries such that* $g_2 \circ g_1 \circ g_3 = \mathrm{Id}\,|_{A_1 \cap B_1}$*. Then the space* X *obtained as a factorspace* $X := (X_1 \cup X_2 \cup X_3)/\{g_1, g_2, g_3\}$ *with the metric* d *that coincides with* d_i *in each* X_i*, is a Busemann space.*

P r o f. Identifying each space X_i with the corresponding subset in X, we notice that

$$A_1 \cap B_1 = B_3 \cap A_2 \subset X_1 \cap X_2 \cap X_3.$$

As a corollary,

$$X_1 \cap X_2 \cap X_3 = A_1 \cap B_1 = A_2 \cap B_2 = A_3 \cap B_3.$$

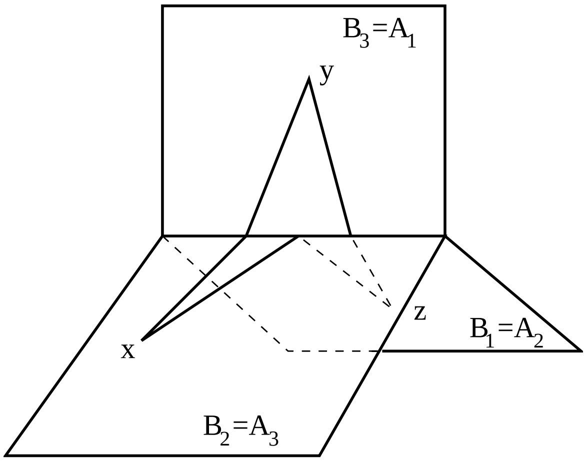

Fig. 1: The space X is a result of the gluing of spaces X_1, X_2 and X_3.

Any two points $x, y \in X$ are contained in one of X_i and connected in this X_i by a segment $[xy]$ with natural parametrization $c : [\alpha, \beta] \to X_i$. Assume for definiteness that $x, y \in X_1$. Since the distance in X between the points that belong to X_1 coincides with the distance d_1, then the parametrization c is a natural parametrization of the path c in the space X as well. Consequently, the map c represents a segment connecting x and y in X. It follows that X is a geodesic space.

Let $k \notin X_1$, that is $k \in B_2 \setminus A_2 = A_3 \setminus B_3 = (X_2 \cap X_3) \setminus X_1$, be an arbitrary point. Consider the segments $[xk]$ and $[ky]$ with natural parameterizations $p : [\alpha, \gamma] \to X$ and $q : [\delta, \beta] \to X$. Denote $s \in [\alpha, \gamma]$ the infimum of parameters σ for which $p(\sigma) \notin X_1$, and $t \in [\delta, \beta]$ the supremum of parameters τ for which $q(\tau) \notin X_1$. Since the sets A_i and B_i are closed, then $p(s), q(t) \in (B_2 = A_3) \cap X_1$. It follows

$$|xy|_X = d_1(x, y) \le d_1(x, p(s)) + d_1(p(s), q(t)) + d_1(q(t), y) < |xk|_X + |ky|_X.$$

Consequently, every segment connecting x and y passes in X_1, and the points x and y are connected by the unique segment in X. If $x, y \in X_i$, then there is a unique midpoint between x and y and it belongs to the same X_i.

Let three points $x, y, z \in X$ and the midpoints m, n of segments $[xy]$ and $[xz]$, respectively, be given. If $x, y, z \in X_i$ for some i, then also $m, n \in X_i$, and the inequality (2.1) is fulfilled automatically. Assume that $x \notin X_1$, $y \notin X_2$ and $z \notin X_3$ (as in Fig. 1). Denote p an arbitrary point of the segment $[yz]$ in the intersection $A_1 \cap B_1 = A_1 \cap A_2 \cap A_3$, and q the midpoint of the segment $[xp]$ (Fig. 2).

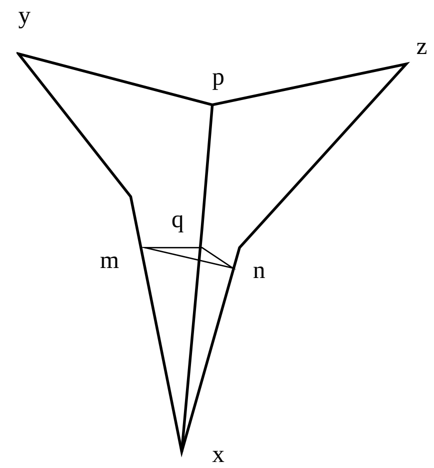

Fig. 2:

Then from $x, y, p \in X_3$ there follows the inequality

$$|mq|_X \leq \frac{1}{2}|yp|_X, \tag{3.1}$$

and from $x, p, z \in X_2$ the inequality

$$|qn|_X \leq \frac{1}{2}|pz|_X. \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$|mn|_X \leq |mq|_X + |qn|_X \leq \frac{1}{2}(|yp|_X + |pz|_X) = \frac{1}{2}|yz|_X.$$

■

4. Convergence in the Class of Busemann Nonpositively Curved Spaces

Definition 4.1. *The distortion of the map $f : X \to Y$ of the metric space (X, d_X) to the metric space (Y, d_Y) is defined by*

$$\operatorname{dis}(f) := \sup_{x,y \in X} |d_Y(f(x), f(y)) - d_X(x, y)|.$$

The uniform distance $|XY|_u$ between metric spaces (X, d_X) and (Y, d_Y) is defined by

$$|XY|_u := \inf \operatorname{dis}(f),$$

where the infimum is taken over all bijections $f : X \to Y$. *A sequence* (X_n, d_n) *of metric spaces converges uniformly to the metric space* (X, d_X) *if* $|X_n X|_u \to 0$.

For $\epsilon > 0$, ϵ*-net in the metric space* X *is a subset* $\mathcal{N} \subset X$ *such that for any* $x \in X$ *there exists* $a \in \mathcal{N}$ *with* $|xa| < \epsilon$.

Definition 4.2 [13, Part I, p. 7]. *Let* (X, d) *be a bounded metric space and* (X_i, d_i) *be a sequence of bounded metric spaces with distances* d_i. *The sequence* X_i *converges in the sense of Hausforff to the space* X *if for any* $\epsilon > 0$ *there exists* ϵ*-net* $\mathcal{N}_\epsilon \subset X$ *that is a uniform limit of* ϵ*-nets* $\mathcal{N}_{i\epsilon}$ *in* X_i.

The definition of the Hausdorff convergence in the case of nonbounded spaces is valid in the category of pointed spaces. Let (X, o, d) *be a pointed metric space with the marked point* o *and the metric* d, *and* (X_i, o_i, d_i) *be a sequence of pointed metric spaces with the marked points* o_i *and the metrics* d_i, *respectively. The sequence* X_i *converges in the sense of Hausdorff to the space* X *if for any* $r > 0$ *the sequence of balls* $B_{X_i}(o_i, r)$ *converges in the sense of Hausdorff to the ball* $B_X(o, r)$.

We say that the family of geodesic spaces $\{(X_\alpha, d_\alpha)\}$ with the metrics d_α *is unimodularly convex* if each of spaces (X_α, d_α) is weakly uniformly convex and there exists the positive function $m(\mu, r)$ defined for $\mu, r > 0$ that bounds convexity modules of all spaces X_α from below uniformly

$$\delta_x(\mu, r) \geq m(\mu, r) \tag{4.1}$$

for any $x \in X_\alpha$ and for all α.

Theorem 4.3. *Let the complete metric space* (X, o, d_X) *with basepoint* o *is a Hausdorff limit of unimodularly convex sequence* (X_n, o_n, d_n) *of pointed Busemann spaces. Then* X *is also a Busemann space and its convexity modulus* $\delta_x(\epsilon, r)$ *for all* $x \in X$ *is bounded from below by the common low boundary of convexity modules of spaces* X_n.

R e m a r k. The unimodular convexity condition is essential here.
For example, Minkowski planes with the norms

$$\|(x, y)\|_n := \sqrt[n]{|x|^n + |y|^n}$$

that are strictly convex when $n > 1$ converges in the sense of Hausdorff to the non-strictly convex Minkowski plane with maximum norm

$$\|(x, y)\|_\infty := \max\{|x|, |y|\}.$$

First, we need the following lemma.

Lemma 4.4. *Let the sequence (X_n, o_n, d_n) and the space (X, o, d_X) satisfy the conditions of Th. 4.3. Then:*

1. *X is a geodesic space;*

2. *for any two points $x, y \in X$ the midpoint m between them is unique.*

P r o f. By Claim 6.1 in [13, Part I] the metric of the space X is interior. Consequently, X is geodesic as a complete space with interior metric.

Now we prove the second statement. Assume for the contrary that for points x, $y \in X$ there exists two different midpoints m_1, $m_2 \in X$. Put $R := 2\max\{d_X(o,x), d_X(o,y)\}$. From the definition of Hausdorff convergence in unbounded spaces, the balls $B_{X_n}(o_n, R)$ converge to the ball $B_X(o, R)$.

Let the positive function $L(\mu, r)$ be defined for $\mu, r > 0$ by the equality

$$L(\mu, r) = \inf \delta_x(\mu, \alpha),$$

where the infimum is taken over all $x \in X_n$ for all natural n. By the inequality (4.1) the infimum is positive. The function $L(\mu, r)$ is nondecreasing on μ when $r > 0$ is fixed. To see this it is sufficient to observe that for all n the functions $\delta_x(\mu, r)$ have the mentioned property, where $x \in X_n$ is arbitrary. Let $\mu_2 > \mu_1 > 0$. If $d_\alpha(x, y) \le r$, $d_\alpha(x, z) \le r$ and $d_\alpha(y, z) \ge \mu_2 r$ hold for the points $x, y, z \in X_n$, then also $d_\alpha(y, z) \ge \mu_1 r$. Hence $\delta_x(\mu_1, r) \le \delta_x(\mu_2, r)$, and $\delta_\alpha(\mu_1, r) \le \delta_\alpha(\mu_2, r)$.

Consequently, for all $\mu, r > 0$ there exists $\epsilon > 0$ such that

$$\epsilon < \frac{2}{9} L\left(\mu - \frac{9\epsilon}{r}, r\right).$$

Take $\epsilon > 0$ to satisfy the conditions

$$d_X(m_1, m_2) - 3\epsilon > \epsilon M(\epsilon), \tag{4.2}$$

where $M(\epsilon) = \frac{1}{2} d_X(x, y) + 3\epsilon$, and

$$\epsilon < \frac{2}{9} L\left(2\frac{d_X(m_1, m_2) - 9\epsilon}{d_X(x, y)}, \frac{1}{2} d_X(x, y)\right). \tag{4.3}$$

Let X_ϵ be an ϵ-net in the ball $B(o, R) \subset X$ and a uniform limit of ϵ-nets $X_{\epsilon,n}$ in balls $B(o_n, R) \subset X_n$. Let the number $N \in \mathbb{N}$ be taken such that for all $n > N$ there exists a bijection $\phi_{\epsilon,n} : X_{\epsilon,n} \to X_\epsilon$ for which

$$\operatorname{dis} \phi_{\epsilon,n} < \epsilon. \tag{4.4}$$

Choose the points $x_\epsilon, y_\epsilon, m_{1,\epsilon}, m_{2,\epsilon} \in X_\epsilon$ with the conditions

$$d_X(x, x_\epsilon) < \epsilon,$$
$$d_X(y, y_\epsilon) < \epsilon,$$
$$d_X(m_1, m_{1,\epsilon}) < \epsilon,$$
$$d_X(m_2, m_{2,\epsilon}) < \epsilon.$$

For them

$$d_X(x_\epsilon, y_\epsilon) \geq d_X(x, y) - 2\epsilon,$$

$$|d_X(x_\epsilon, m_{1,\epsilon}) - \frac{1}{2} d_X(x, y)| \leq 2\epsilon,$$

$$|d_X(x_\epsilon, m_{2,\epsilon}) - \frac{1}{2} d_X(x, y)| \leq 2\epsilon,$$

$$|d_X(y_\epsilon, m_{1,\epsilon}) - \frac{1}{2} d_X(x, y)| \leq 2\epsilon,$$

$$|d_X(y_\epsilon, m_{2,\epsilon}) - \frac{1}{2} d_X(x, y)| \leq 2\epsilon$$

and

$$|d_X(m_{1,\epsilon}, m_{2,\epsilon}) - d_X(m_1, m_2)| \leq 2\epsilon.$$

For arbitrary $n > N$ we have

$$d_n(\phi_{\epsilon,n}^{-1}(x_\epsilon), \phi_{\epsilon,n}^{-1}(y_\epsilon)) \geq d_X(x, y) - 3\epsilon, \tag{4.5}$$

and also

$$|d_n(\phi_{\epsilon,n}^{-1}(x_\epsilon), \phi_{\epsilon,n}^{-1}(m_{1,\epsilon})) - \frac{1}{2} d_X(x, y)| \leq 3\epsilon, \tag{4.6}$$

$$|d_n(\phi_{\epsilon,n}^{-1}(x_\epsilon), \phi_{\epsilon,n}^{-1}(m_{2,\epsilon})) - \frac{1}{2} d_X(x, y)| \leq 3\epsilon, \tag{4.7}$$

$$|d_n(\phi_{\epsilon,n}^{-1}(y_\epsilon), \phi_{\epsilon,n}^{-1}(m_{1,\epsilon})) - \frac{1}{2} d_X(x, y)| \leq 3\epsilon,$$

$$|d_n(\phi_{\epsilon,n}^{-1}(y_\epsilon), \phi_{\epsilon,n}^{-1}(m_{2,\epsilon})) - \frac{1}{2} d_X(x, y)| \leq 3\epsilon$$

and

$$|d_n(\phi_{\epsilon,n}^{-1}(m_{1,\epsilon}), \phi_{\epsilon,n}^{-1}(m_{2,\epsilon})) - d_X(m_1, m_2)| \leq 3\epsilon.$$

Consequently, from (4.2)

$$d_n(\phi_{\epsilon,n}^{-1}(m_{1,\epsilon}), \phi_{\epsilon,n}^{-1}(m_{2,\epsilon})) \geq \epsilon M(\epsilon).$$

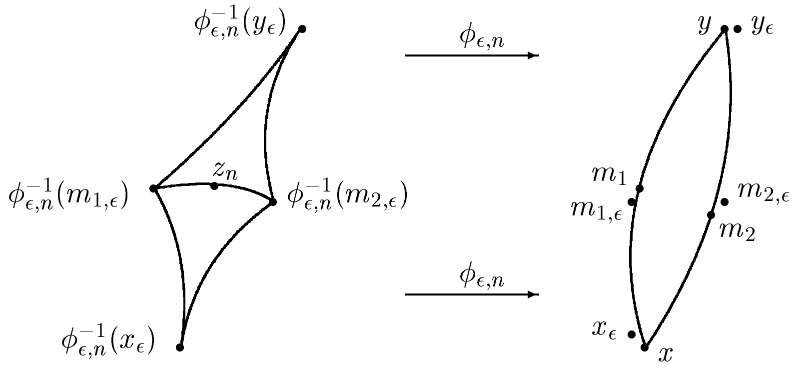

Fig. 3:

Let $z_n \in X_n$ be the midpoint between $\phi^{-1}_{\epsilon,n}(m_{1,\epsilon})$ and $\phi^{-1}_{\epsilon,n}(m_{2,\epsilon})$ (Fig. 3). Consider also the following points. The point $p_{1,n} \in X_n$ in the segment $[\phi^{-1}_{\epsilon,n}(x_\epsilon)\phi^{-1}_{\epsilon,n}(m_{1,\epsilon})]$, such that

$$d_n(\phi^{-1}_{\epsilon,n}(x_\epsilon), p_{1,n}) =$$
$$= \begin{cases} \frac{1}{2}d_X(x,y), \text{ if } d_n(\phi^{-1}_{\epsilon,n}(x_\epsilon), \phi^{-1}_{\epsilon,n}(m_{1,\epsilon})) \geq \frac{1}{2}d_X(x,y) \\ d_n(\phi^{-1}_{\epsilon,n}(x_\epsilon), \phi^{-1}_{\epsilon,n}(m_{1,\epsilon})) \text{ otherwise.} \end{cases}$$

The point $p_{1,n}$ coincides with the endpoint $\phi^{-1}_{\epsilon,n}(m_{1,\epsilon})$ of the segment if

$$d_n(\phi^{-1}_{\epsilon,n}(x_\epsilon), \phi^{-1}_{\epsilon,n}(m_{1,\epsilon})) \leq \frac{1}{2}d_X(x,y),$$

or its distance from $\phi^{-1}_{\epsilon,n}(x_\epsilon)$ is $\frac{1}{2}d_X(x,y)$ if

$$d_n(\phi^{-1}_{\epsilon,n}(x_\epsilon), \phi^{-1}_{\epsilon,n}(m_{1,\epsilon})) \geq \frac{1}{2}d_X(x,y).$$

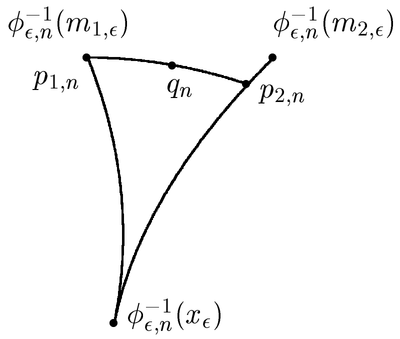

Fig. 4:

The point $p_{2,n}$ is defined analogously in the segment $[\phi_{\epsilon,n}^{-1}(x_\epsilon)\phi_{\epsilon,n}^{-1}(m_{2,\epsilon})]$. Finally, the point q_n is the midpoint of the segment $[p_{1,n}p_{2,n}]$ (Fig. 4).

Let us estimate the distance $d_n(p_{1,n}, p_{2,n})$ from below. From inequalities (4.6) and (4.7) the distances $d_n(p_i, \phi_{\epsilon,n}^{-1}(m_{i,\epsilon}))$ and $i = \overline{1,2}$ satisfy the inequality

$$d_n(p_i, \phi_{\epsilon,n}^{-1}(m_{i,\epsilon})) \le 3\epsilon.$$

Hence

$$d_n(p_{1,n}, p_{2,n}) \ge d_n(\phi_{\epsilon,n}^{-1}(m_{1,\epsilon}), \phi_{\epsilon,n}^{-1}(m_{2,\epsilon})) - 6\epsilon \ge d_X(m_1, m_2) - 9\epsilon.$$

We have

$$\begin{aligned} d_n(\phi_{\epsilon,n}^{-1}(x_\epsilon), z_n) &\le d_n(\phi_{\epsilon,n}^{-1}(x_\epsilon), q_n) + d_n(q_n, z_n) \\ &\le \frac{1}{2}d_X(x,y) - L\left(2\frac{d_X(m_1,m_2)-9\epsilon}{d_X(x,y)}, \frac{1}{2}d_X(x,y)\right) \\ &+\frac{1}{2}\left(d_n(p_{1,n}, \phi_{\epsilon,n}^{-1}(m_{1,\epsilon})) + d_n(p_{2,n}, \phi_{\epsilon,n}^{-1}(m_{2,\epsilon}))\right) \\ &\le \frac{1}{2}d_X(x,y) - L\left(2\frac{d_X(m_1,m_2)-3\epsilon}{d_X(x,y)}, \frac{1}{2}d_X(x,y)\right) + 3\epsilon \\ &< \frac{1}{2}d_X(x,y) - \frac{3}{2}\epsilon. \end{aligned}$$

Similarly,

$$d_n(\phi_{\epsilon,n}^{-1}(y_\epsilon), z_n) < \frac{1}{2}d_X(x,y) - \frac{3}{2}\epsilon.$$

Finally,

$$d_n(\phi_{\epsilon,n}^{-1}(x_\epsilon), \phi_{\epsilon,n}^{-1}(y_\epsilon)) < d_X(x,y) - 3\epsilon,$$

contradicting to the inequality (4.5). ■

Now we can complete the proof.

P r o o f of Theorem 4.3. Let the points $x, y, z \in X$ and the midpoints p and q of segments $[xy]$ and $[xz]$, respectively, be given. Denote

$$R := 2\max\{d_X(o,x), d_X(o,y), d_X(o,z)\}.$$

Fix the decreasing sequence $\epsilon_i \to 0$.

For each i, choose ϵ_i-net $X_{\epsilon_i} \subset B_X(o,R)$ which is a uniform limit of ϵ_i-nets $X_{\epsilon_i,n} \subset B_{X_n}(o,R)$. Here $B_X(o,R)$ and $B_{X_n}(o_n,R)$ are balls in the spaces X and X_n, respectively.

Let $n(i)$ be the natural number such that there exists a bijection $\phi_i : X_{\epsilon_i,n(i)} \to X_{\epsilon_i}$ with the distortion

$$\operatorname{dis}\phi_i < \epsilon_i.$$

Let the distances from the points $x_i, y_i, z_i \in X_\epsilon$ to x, y, z, respectively, be not greater than ϵ_i. Denote $\tilde{p}_i \in X_{n(i)}$ the midpoint of the segment $[\phi_i^{-1}(x_i)\phi_i^{-1}(y_i)]$, $\bar{p}_i \in X_{\epsilon_i, n(i)}$ the point on the distance not greater than ϵ_i from $\tilde{p}_i$, and $p_i = \phi_i(\bar{p}_i) \in X$ its image in the bijection ϕ_i. Since by the condition the space X is proper and the sequence p_i is bounded, one can subtract the converging subsequence. We may assume that the sequence p_i is also converging. The points p_i are $(4\epsilon_i)$-midpoints between x and y, that is

$$|d_X(x, p_i) - \frac{1}{2}d_X(x, y)| \leq 4\epsilon_i$$

and

$$|d_X(y, p_i) - \frac{1}{2}d_X(x, y)| \leq 4\epsilon_i.$$

Since $\epsilon_i \to 0$, when $i \to \infty$, the limit of the sequence p_i is the midpoint between x and y. From the uniqueness of midpoints in X, it follows that

$$\lim_{i\to\infty} p_i = p.$$

Analogously, one can construct the sequence of $(4\epsilon_i)$-midpoints q_i between x and z converging to q. We have

$$d_X(p_i, q_i) \leq d_{X_{n(i)}}(\bar{p}_i, \bar{q}_i) + \epsilon_i \leq d_{X_{n(i)}}(\tilde{p}_i, \tilde{q}_i) + 3\epsilon_i$$

$$\leq \frac{1}{2}d_{X_{n(i)}}(\phi_i^{-1}(y_i), \phi_i^{-1}(z_i)) + 3\epsilon_i \leq \frac{1}{2}d_X(y_i, z_i) + 4\epsilon_i$$

$$\leq \frac{1}{2}d_X(y, z) + 5\epsilon_i.$$

Hence

$$d_X(p, q) \leq \frac{1}{2}d_X(y, z),$$

that is X is Busemann nonpositively curved. The estimation of the convexity modulus $\delta_x(\epsilon, r)$ in X can be proven in a similar way. ■

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