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# Interaction between "Accelerating-Packing" Flows in a Low-Temperature Gas

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The Maxwellians of a special type, which correspond to inhomogeneous, nonstationary flows and describe the acceleration and packing of gas along some direction, are studied. The approximate description of interaction between these two flows for the model of hard spheres, when the temperatures are sufficiently small, is obtained in a form of bimodal distribution with various coefficient functions.

*Key words*: rarefied gas, Boltzmann equation, hard spheres, Maxwellian, bimodal distribution, approximate solution, error, low temperature.

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## 1. Introduction

In kinetic theory the state of rarefied gas is described by the distribution function $f(t,v,x)$, where $t \in R^1$ is time, $v = \left(v^1,v^2,v^3\right) \in R^3$ is the velocity of molecule, and $x = \left(x^1,x^2,x^3\right) \in R^3$ is its position in the space. This function is a solution of non-linear integro-differential Boltzmann equation [1–3]

$$D(f) = Q(f,f); \tag{1}$$

$$D(f) = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}; \tag{2}$$

$$Q(f,f) = \frac{d^2}{2}\int\limits_{R^3} dv_1 \int\limits_{\Sigma} d\alpha |(v-v_1,\alpha)|$$

$$\times[f(t,v_1^{'},x)f(t,v^{'},x) - f(t,v_1,x)f(t,v,x)], \tag{3}$$

where $\frac{\partial f}{\partial x}$ (or simply $f^{'}$) denotes the spatial gradient of distribution $f$, $d$ is the diameter of particles, $v, v_1, v^{'}, v_1^{'}$ are velocities of two molecules before and after collision, respectively; the vector $\alpha$ belongs to the unit sphere $\Sigma \subset R^3$.

The most general form of the local-equilibrium Maxwell solutions of Boltzmann equation (in short, local Maxwellians $M = M(t, v, x)$), i.e., the exact solutions of the system

$$D(M) = Q(M, M) = 0, \tag{4}$$

was studied in [2, 4, 5]. A rather full description of the solutions mentioned above and a particular analysis of their physical sense can be found, for example, in [1–3]. The geometrical structure and the physical sense of local Maxwellians were studied in detail in [6] and the complete analysis of corresponding possible motions of gas was carried out.

One of these motions was called "accelerating-packing" as it can be described by the following Maxwellian:

$$M = \rho \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(v-\widetilde{v})^2}, \tag{5}$$

where

$$\rho = \overline{\rho} \cdot e^{\beta\left(\widetilde{v}^2 + 2\overline{u}x\right)} \tag{6}$$

is the density of the flow; $\overline{\rho} = const$; $\beta = \frac{1}{2T}$ is its inverse temperature (T is the absolute temperature);

$$\widetilde{v} = \overline{v} - \overline{u}t \tag{7}$$

is its mass velocity (here $\overline{u}, \overline{v} \in R^3$ - arbitrary constant vectors). It is easy to see from (6), (7) that the vector $\overline{u}$ has a role of "mass acceleration", and the density $\rho$ changes from 0 to $+\infty$, besides for any fixed $x \in R^3$ its minimum value is reached when $t = t_0$, where

$$t_0 = \frac{1}{\overline{u}^2}(\overline{u}, \overline{v}), \tag{8}$$

but for any fixed $t \in R^1$ it increases only along the vector $\overline{u}$.

For the approximate description of interaction between two flows of "accelerating-packing" type, which have sufficiently small temperatures, let us consider, by the analogy with [6–8], the following bimodal distribution:

$$f = \varphi_1 M_1 + \varphi_2 M_2, \tag{9}$$

where the Maxwellians $M_i$, $i = 1, 2$, have the form (5)–(7) but with different hydrodynamical parameters $\rho_i$, $\beta_i$, $\overline{\rho}_i$, $\widetilde{v}_i$, $\overline{v}_i$, $\overline{u}_i$, $i = 1, 2$, and the coefficient functions

$$\varphi_i = \varphi_i(t, x), \quad i = 1, 2, \tag{10}$$

are nonnegative and smooth. The purpose is to find such a form of functions (10) and such a behaviour of parameters $\overline{v}_i, \overline{u}_i$, $i = 1, 2$, and so on that together with the "low-temperature limiting transition"

$$\beta_i \to +\infty, \ i = 1, 2, \tag{11}$$

make the error, i.e., some norm of difference between the sides of Boltzmann equation (1)–(3), arbitrary small.

In Section 2 the rigorous statement of the problem is formulated and several possible variants of its solution are presented.

## 2. Main Results

Following [6–8], consider the "mixed" or "uniform-integral" error between the values $D$ and $Q$ (see (1)–(3)):

$$\Delta = \sup_{(t,x)\in R^4} \int_{R^3} |D(f) - Q(f,f)|\, dv. \tag{12}$$

The problem is to find any possible sufficient conditions for the infinitesimality of the value (12) if the distribution $f$ has a bimodal form (9), (10) with the modes $M_i$, $i = 1, 2$, of the type (5)–(7) with the limiting restriction (11).

Now we will prove a number of theorems and corollaries which give various possibilities for solving this problem.

First, it is convenient to adopt the definition.

**Definition 1.** *Denote as $P(R^n)$ a class of nonnegative functions from $C^1(R^n)$ which have finite supports (in short, finite functions) or fast decrease at infinity.*

**Theorem 1.** *Let the functions $\varphi_i$, $i = 1, 2$, in distribution (9) have the form*

$$\varphi_i(t, x) = \frac{D_i}{(1+t^2)^{\chi_i}} C_i \left( x + \overline{u}_i \frac{(\overline{v}_i - \overline{u}_i t)^2}{2\overline{u}_i^2} \right), \ i = 1, 2, \tag{13}$$

*where the constants $\chi_i, D_i$ are as follows:*

$$D_i > 0; \quad \chi_i \geq \frac{1}{2}, \ i = 1, 2, \tag{14}$$

*and the functions $C_i$ belong to $P(R^3), i = 1, 2$. Let the conditions be fulfilled:*

$$\overline{u}_i = \frac{\overline{u}_{oi}}{\beta_i^{n_i}}, \ i = 1, 2 \tag{15}$$

$$\overline{v}_i = \frac{\overline{v}_{oi}}{\beta_i^{k_i}}, \ i = 1, 2, \tag{16}$$

*where*

$$n_i \geq 1;\; k_i \geq \frac{1}{2};\; k_i \geq \frac{1}{2}n_i, \quad i = 1, 2, \tag{17}$$

*and* $\overline{u}_{oi}, \overline{v}_{oi} \in R^3$ *are arbitrary fixed vectors.*

*Then the error* $\Delta$ *in (12) is correctly defined, and there exists such a value* $\Delta'$ *that*

$$\Delta \leq \Delta', \tag{18}$$

*besides it has the limit*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = K(\chi_1, \chi_2) \sum_{i=1}^{2} \overline{\rho}_i D_i \sup_{x \in R^3} \{\mu_i(x) C_i(x + a_i)\}, \tag{19}$$

*where* $K(\chi_1, \chi_2)$ *is some constant, the functions* $\mu_i(x)$ *are as follows:*

$$\mu_i(x) = \left[ \begin{array}{ll} 1; & n_i > 1; k_i > \frac{1}{2}, \\ \exp\{2\overline{u}_{oi} x\}; & n_i = 1; k_i > \frac{1}{2}, \\ \exp\{\overline{v}_{oi}^2 + 2\overline{u}_{oi} x\}; & n_i = 1; k_i = \frac{1}{2}, \end{array} \right. \tag{20}$$

*and the vector constants* $a_i$*,* $i = 1, 2$*, are equal to* $\frac{\overline{u}_{oi}\overline{v}_{oi}^2}{2\overline{u}_{oi}^2}$ *if* $k_i = \frac{1}{2}n_i$ *and they are equal to zero if* $k_i \neq \frac{1}{2}n_i$*.*

P r o o f. The substitution of (9) into equations (1)–(3), by taking into account (5)–(7) (with indexes $i = 1, 2$ for all values, respectively, see after (9)) and the fact that for each of $M_i$, $i = 1, 2$, the relation (4) is valid after some evident estimations, the changes of variables and transformations analogous to those done in [7,8] with the utilization of the technique developed in [6], leads to the following inequality:

$$\Delta \leq \Delta' = \sup_{(t,x) \in R^4} \sum_{i,j=1, i \neq j}^{2} \left[ \int_{R^3} \left| \frac{\partial \varphi_i}{\partial t} + \left( \frac{u}{\sqrt{\beta_i}} + \overline{v}_i - \overline{u}_i t \right) \frac{\partial \varphi_i}{\partial x} \right.\right.$$

$$\left. + \varphi_1 \varphi_2 \rho_j(t,x) \frac{d^2}{\sqrt{\pi}} \int_{R^3} F_{ij} e^{-w^2} dw \right| \overline{\rho}_i(t,x) \pi^{-3/2} e^{-u^2} du$$

$$\left. + \varphi_1 \varphi_2 \frac{\rho_1(t,x)\rho_2(t,x)}{\pi^2} d^2 \int_{R^3} e^{-w^2 - u^2} F_{ij} dw du \right], \tag{21}$$

where

$$F_{ij} = F_{ij}(u,t,w) = \left| \frac{u}{\sqrt{\beta_i}} + \overline{v}_i - \overline{v}_j + (\overline{u}_j - \overline{u}_i)t - \frac{w}{\sqrt{\beta_j}} \right|, \; i \neq j, \tag{22}$$

and

$$\rho_i(t,x) = \overline{\rho}_i \exp\{\beta_i((\overline{v}_i - \overline{u}_i t)^2 + 2\overline{u}_i x)\},\ i = 1,2. \tag{23}$$

From (21)–(23) it can be easily seen that for verification of the existence of values $\Delta$, $\Delta'$ it is sufficient to check that the products of functions (23) on the values

$$\varphi_i;\ \frac{\partial \varphi_i}{\partial t};\ \left|\frac{\partial \varphi_i}{\partial x}\right|;\ \varphi_i t;\ t\left(\overline{u}_i \frac{\partial \varphi_i}{\partial x}\right),\quad i = 1,2, \tag{24}$$

are bounded with respect to $t, x$ on $R^4$ for any fixed $\beta_i$, $i = 1,2$, if the functions $\varphi_i$, $i = 1,2$, have the form (13). Let us consider every one of the mentioned above products separately. The first one, $\varphi_i \rho_i(t,x)$, as a result of denotation

$$y = x + \overline{u}_i \frac{(\overline{v}_i - \overline{u}_i t)^2}{2\overline{u}_i^2}, \tag{25}$$

evidently, will have the form

$$\rho_i \exp\{2\beta_i \overline{u}_i y\} \cdot \frac{D_i}{(1+t^2)^{\chi_i}} \cdot C_i(y). \tag{26}$$

It follows from (26) that not only this expression itself, but also its product on $t$ is bounded with respect to $t, y$ on $R^4$ because of (14) and the properties of functions $C_i(y)$, $i = 1,2$. The analogous conclusion is also true for other three products because it follows from (13) that (once more after changing (25))

$$\frac{\partial \varphi_i}{\partial t} = -\frac{D_i}{(1+t^2)^{\chi_i}} \left[\frac{2\chi_i t}{1+t^2} C_i(y) + (\overline{u}_i, C_i') \frac{(\overline{u}_i, \overline{v}_o) - t\overline{u}_i^2}{\overline{u}_i^2}\right], \tag{27}$$

$$\frac{\partial \varphi_i}{\partial x} = \frac{D_i}{(1+t^2)^{\chi_i}} C_i'(y),\quad i = 1,2. \tag{28}$$

Further, the suppositions (15)–(17), as it can be seen from (23), guarantee that for any $(t,x) \in R^4$ there exists the limit

$$\lim_{\beta_i \to +\infty, i=1,2} \rho_i(t,x) = \overline{\rho}_i \mu_i(x),\quad i = 1,2, \tag{29}$$

with functions $\mu_i(x)$ of the form (20). Expressions (22) for every fixed $u, t, w$ (and on every compact in $R^4$, even uniformly) tend to zero

$$\lim_{\beta_i \to +\infty, i=1,2} F_{ij} = 0,\quad i \neq j. \tag{30}$$

Besides, from (25) it follows that the value $y - x$ due to (15), (16) has the form

$$\frac{\overline{u}_{oi}}{2\overline{u}_{oi}^2} \left(\frac{\overline{v}_{oi}}{\beta_i^{k_i - \frac{1}{2}n_i}} - \frac{\overline{u}_{oi} t}{\beta_i^{\frac{1}{2}n_i}}\right)^2 \tag{31}$$

and, consequently, by conditions (17) has the following finite limit:

$$\lim_{\beta_i\to+\infty, i=1,2}(y-x) = \left[\begin{array}{ll} 0, & if\ k_i > \frac{1}{2}n_i \\ \frac{\overline{u}_{oi}\overline{v}_{oi}^2}{2\overline{u}_{oi}^2}, & if\ k_i = \frac{1}{2}n_i \end{array}\right. = a_i, \quad i = 1,2. \qquad (32)$$

Thus, because of the supposition of smoothness (see Def.1), the functions $C_i(y)$ and $C_i^{'}(y)$ with $\beta_i \to +\infty$, $i = 1,2$, tend to their values at point $x + a_i, i = 1,2$. At the same time the factor near $C_i^{'}$ in (27) obviously tends to zero, therefore in (27) the second summand vanishes. Since the parentheses near $\frac{\partial\varphi_i}{\partial x}$ in (21) also tend to zero, we can, applying Lem. 1 from [7] (its conditions can be easily checked from (27)–(32)) and using the boundness and continuity of all expressions as well as good convergence of all integrals in (21), pass to the limit under the signs of supremums and integrals entering into (21). Finally, the trivial integration with respect to $w$ and $u$ yields (19), where the constant $K(\chi_1,\chi_2)$ is as follows:

$$K(\chi_1,\chi_2) = 2\max_{i=1,2}\left(\chi_i \sup_{t\in R^1}\frac{|t|}{(1+t^2)^{\chi_i+1}}\right). \qquad (33)$$

The theorem is proved.

**Corollary 1.** *Let all suppositions of Theorem 1 be fulfilled. Then*

$$\begin{gathered} \forall\varepsilon > 0,\ \exists\delta > 0,\ \forall D_1, D_2:\ 0 < D_1, D_2 < \delta; \\ \exists\beta_o > 0,\ \forall\beta_i > \beta_o, \quad i = 1,2, \\ \Delta < \varepsilon. \end{gathered} \qquad (34)$$

P r o o f is evident because of (18), (19) and the fact that $C_i(x+a_i) \in P(R^3)$, $i = 1,2$, for any $a_i$ from (32), so the products $\mu_i(x)C_i(x+a_i)$ are bounded on $R^3$.

Under the conditions of Th. 1, as it can be seen from (13), (23), the explicit dependence of the flows on the temperatures (i.e. on $\beta_i$, $i = 1,2$) is present at the densities $\rho_i(t,x)$, but not at the desired coefficient functions $\varphi_i(t,x)$ (they depend on $\beta_i$ only through (15), (16), and this dependence does not play an essential role (see (19)).

Let us now consider a result based on some other assumptions which give the possibility to compensate the increase of $\rho_i(t,x)$ with $\beta_i \to +\infty$, $i = 1,2$.

**Theorem 2.** *Let*

$$\varphi_i(t,x) = \psi_i(t,x)\exp\left\{-\beta_i((\overline{v}_i - \overline{u}_i t)^2 + 2\overline{u}_i x)\right\},\ i = 1,2, \qquad (35)$$

*where the smooth functions $\psi_i \geq 0$ are such that the values (24) with the substitution of $\psi_i$ for $\varphi_i$, $i = 1,2$, are bounded with respect to $t, x$ on $R^4$, and (15) is valid but now for*

$$n_i \geq \frac{1}{2}. \qquad (36)$$

*Then the inequality (18) holds true, besides for* $n_i > \frac{1}{2}$

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = \sum_{i=1}^{2} \overline{\rho}_i \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} + \overline{v}_i \frac{\partial \psi_i}{\partial x} + \psi_1\psi_2 \pi d^2 \overline{\rho}_j |\overline{v}_1 - \overline{v}_2| \right|$$

$$+2\pi d^2 \overline{\rho}_1 \overline{\rho}_2 |\overline{v}_1 - \overline{v}_2| \sup_{(t,x)\in R^4} (\psi_1\psi_2) = L, \tag{37}$$

*and for* $n_i = \frac{1}{2}$ *in addition to (37) a new summand arises*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = L + \frac{4}{\sqrt{\pi}} \sum_{i=1}^{2} \overline{\rho}_i |\overline{u}_{oi}| \sup_{(t,x)\in R^4} \psi_i. \tag{38}$$

P r o o f. From (35) instead of (27), (28) we have

$$\frac{\partial \varphi_i}{\partial t} = \exp\left\{-\beta_i((\overline{v}_i - \overline{u}_i t)^2 + 2\overline{u}_i x)\right\} \left\{ \frac{\partial \psi_i}{\partial t} + 2\beta_i \psi_i ((\overline{v}_i, \overline{u}_i) - t\overline{u}_i^2) \right\}, \tag{39}$$

$$\frac{\partial \varphi_i}{\partial x} = \exp\left\{-\beta_i((\overline{v}_i - \overline{u}_i t)^2 + 2\overline{u}_i x)\right\} \left\{ \frac{\partial \psi_i}{\partial x} - 2\beta_i \psi_i \overline{u}_i \right\}, \quad i = 1, 2. \tag{40}$$

The formulas (21), (22) evidently remain true. Thus the substitution of (35), (39), (40) in (21), taking into account (23), yields (18) with

$$\Delta' = \sup_{(t,x)\in R^4} \sum_{i,j=1, i\neq j}^{2} \left[ \int_{R^3} \left| \frac{\partial \psi_i}{\partial t} + 2\beta_i \psi_i ((\overline{v}_i, \overline{u}_i) - t\overline{u}_i^2) \right.\right.$$

$$+ \left( \frac{u}{\sqrt{\beta_i}} + \overline{v}_i - \overline{u}_i t \right) \left\{ \frac{\partial \psi_i}{\partial x} - 2\beta_i \psi_i \overline{u}_i \right\}$$

$$\left. + \psi_1 \psi_2 \overline{\rho}_j \frac{d^2}{\sqrt{\pi}} \int_{R^3} F_{ij} e^{-w^2} dw \right| \overline{\rho}_i \pi^{-3/2} e^{-u^2} du$$

$$\left. + \psi_1 \psi_2 \frac{d^2}{\pi^2} \overline{\rho}_1 \overline{\rho}_2 \int_{R^6} e^{-w^2 - u^2} F_{ij} dw du \right], \tag{41}$$

(the existence of the values $\Delta$ and $\Delta'$ follows from the conditions of Th. 2) or, in short, after some obvious simplifications,

$$\Delta' = \pi^{-3/2} \sup_{(t,x)\in R^4} \sum_{i=1}^{2} \overline{\rho}_i \int_{R^3} \left[ \left| \frac{\partial \psi_i}{\partial t} + A_i + B_i \right| + A_i \right] e^{-u^2} du, \tag{42}$$

where

$$A_i = A_i(u,t) = \psi_1\psi_2 \frac{d^2}{\sqrt{\pi}}\overline{\rho}_j \int\limits_{R^3} e^{-w^2} F_{ij} dw, \quad i \neq j, \tag{43}$$

$$B_i = B_i(u,t) = \frac{\partial\psi_i}{\partial x}\left(\frac{u}{\sqrt{\beta_i}} + \overline{v}_i - \overline{u}_i t\right) - 2\psi_i\sqrt{\beta_i}(u,\overline{u}_i). \tag{44}$$

The limiting transition in (42) can be done in the same way as in the proof of Th. 1, but the result will be different. Indeed it follows from (22) and (15) that

$$\lim_{\beta_i\to+\infty, i=1,2} F_{ij} = |\overline{v}_i - \overline{v}_j|, \quad i \neq j, \tag{45}$$

whence, from (43) after trivial integration with respect to $w$,

$$\lim_{\beta_i\to+\infty, i=1,2} A_i = \psi_1\psi_2\pi d^2 \overline{\rho}_j |\overline{v}_1 - \overline{v}_2|, \quad i \neq j. \tag{46}$$

The limit of $B_i$ depends on the quantity of parameter $n_i$

$$\lim_{\beta_i\to+\infty, i=1,2} B_i = \overline{v}_i \frac{\partial\psi_i}{\partial x} + 2\psi_i H_i, \quad i = 1,2, \tag{47}$$

where

$$H_i = \left[\begin{array}{ll} 0, & n_i > \frac{1}{2} \\ -(u,\overline{u}_{oi}), & n_i = \frac{1}{2}. \end{array}\right. \tag{48}$$

Passing to the limit in (42) with the use of (46)–(48), after integration with respect to $u$ (in the second case from (48) the value (42) must be bounded from above once more when one chooses a "supplementary" summand $2\psi_i|(u,\overline{u}_{oi})|$ whose integration together with the factor $e^{-u^2}$ yields the second term in (38)), we obtain (37), (38). The theorem is proved.

**Corollary 2.** *Let the requirements (35), (15), (36) be fulfilled and the functions $\psi_i$ be of the form*

$$\psi_i = D_i C_i(t), \quad i = 1,2, \tag{49}$$

*where $D_i > 0$, and smooth, nonnegative functions $C_i$ are such that the expressions $tC_i$ and $tC_i'$ are bounded on $R^1$.*

*Then:*

*a) For $C_1$, $C_2$, $\overline{v}_1$, $\overline{v}_2$ which satisfy the following conditions:*

$$supp C_1 \cap supp C_2 = \emptyset \tag{50}$$

*or*

$$\overline{v}_1 = \overline{v}_2, \tag{51}$$

*the statement (34) holds true.*

*b) For arbitrary $C_1$, $C_2$, $\overline{v}_1$, $\overline{v}_2$ the statement*

$$\begin{gathered}\forall\varepsilon>0,\ \exists\delta>0,\ \forall D_1,D_2,d:\ 0<D_1,D_2,d<\delta;\\ \exists\beta_o>0,\ \forall\beta_1,\beta_2>\beta_o\\ \Delta<\varepsilon\end{gathered}\tag{52}$$

is valid.

P r o o f. Requirement (49) under the mentioned conditions imposed on functions $C_i(t)$ ensures the fulfillment of suppositions of Th. 2. Further, by virtue of (49) and (50) or (51), or with

$$d\to 0\tag{53}$$

(the last condition is the only new fact in (52) in comparison with (34)), the nonzero terms retained in (37) or (38), will be only

$$D_i\sup_{t\in R^1}|C_i^{'}(t)|,\quad D_i\sup_{t\in R^1}|C_i(t)|,\quad i=1,2.\tag{54}$$

These two supremums due to the smoothness of functions $C_i(t)$ are finite. So, we have (34) and (52) for situations a), b) of Corollary 2, respectively. The corollary is proved.

There also exist two possible variants when the exponent in (35) contains not two summands but only one. Now we will describe these variants representing the following statements.

**Theorem 3.** *Let the conditions of Theorem 2 be valid, but now instead of (35) and (36) it is supposed that*

$$\varphi_i(t,x)=\psi_i(t,x)\exp\{-\beta_i(\overline{v}_i-\overline{u}_it)^2\},\quad i=1,2,\tag{55}$$

$$n_i\geq 1,\quad i=1,2,\tag{56}$$

*and the functions $\psi_i$, $i=1,2$, are such that the products of the values (24) (with $\psi_i$ instead of $\varphi_i$) on the factors $\exp\{2\beta_i\overline{u}_ix\}$, $i=1,2$, are bounded with respect to $t,x$ on $R^4$. Then the inequality (18) holds true, where for $n_i>1$*

$$\begin{aligned}\lim_{\beta_i\to+\infty,i=1,2}\Delta^{'}=\sum_{t=1}^{2}\overline{\rho}_i\sup_{(t,x)\in R^4}\Bigg|\mu_i(x)\left(\frac{\partial\psi_i}{\partial t}+\overline{v}_i\frac{\partial\psi_i}{\partial x}\right)\\ +\psi_1\psi_2\mu_1(x)\mu_2(x)\pi d^2\overline{\rho}_j|\overline{v}_1-\overline{v}_2|\Bigg|+2\pi d^2\overline{\rho}_1\overline{\rho}_2|\overline{v}_1-\overline{v}_2|\\ \times\sup_{(t,x)\in R^4}[\mu_1(x)\mu_2(x)\psi_1(t,x)\psi_2(t,x)]=N\end{aligned}\tag{57}$$

*with $\mu_i(x)$, $i = 1, 2$, which correspond to the first and second cases from (20), and for $n_i = 1$*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = N + 2\sum_{i=1}^{2} \overline{\rho}_i |(\overline{u}_{oi}, \overline{v}_i)| \sup_{(t,x)\in R^4} \{\mu_i(x)\psi_i(x)\}. \tag{58}$$

P r o o f. By differentiating (55), we obtain

$$\frac{\partial \varphi_i}{\partial t} = \exp\left\{-\beta_i(\overline{v}_i - \overline{u}_i t)^2\right\} \cdot \left\{\frac{\partial \psi_i}{\partial t} + 2\beta_i\psi_i\left((\overline{v}_i, \overline{u}_i) - t\overline{u}_i^2\right)\right\}, \tag{59}$$

$$\frac{\partial \varphi_i}{\partial x} = \frac{\partial \psi_i}{\partial x} \exp\left\{-\beta_i(\overline{v}_i - \overline{u}_i t)^2\right\}. \tag{60}$$

Thus, remembering (23), from (21) and the conditions of Th. 3 we will have the value $\Delta'$ for (18)

$$\Delta' = \sup_{(t,x)\in R^4} \sum_{i\cdot j=1, i\neq j}^{2} \left[ \int_{R^3} \left| \frac{\partial \psi_i}{\partial t} + 2\beta_i\psi_i\left((\overline{u}_i, \overline{v}_i) - t\overline{u}_i^2\right) \right.\right.$$

$$\left. + \left(\frac{u}{\sqrt{\beta_i}} + \overline{v}_i - \overline{u}_i t\right)\frac{\partial \psi_i}{\partial x} + \psi_1\psi_2\overline{\rho}_j e^{2\beta_j\overline{u}_j x}\frac{d^2}{\sqrt{\pi}}\int_{R^3} F_{ij}e^{-w^2}dw \right|$$

$$\left. \times \overline{\rho}_i e^{2\beta_i\overline{u}_i x}\pi^{-3/2}e^{-u^2}du + \psi_1\psi_2\left(\frac{d}{\pi}\right)^2 \overline{\rho}_1\overline{\rho}_2 e^{2x(\beta_1\overline{u}_1+\beta_2\overline{u}_2)} \cdot \int_{R^6} e^{-w^2-u^2}F_{ij}dwdu \right], \tag{61}$$

where $F_{ij}$ again has the form (22) but now (42)–(44) will be somewhat complicated:

$$\Delta' = \pi^{-3/2} \sup_{(t,x)\in R^4} \sum_{i=1}^{2} \overline{\rho}_i e^{2\beta_i\overline{u}_i x} \int_{R^3} \left[\left|\frac{\partial \psi_i}{\partial t} + A_i + B_i\right| + A_i\right] e^{-u^2}du, \tag{62}$$

$$A_i = \psi_1\psi_2\frac{d^2}{\sqrt{\pi}}\overline{\rho}_j e^{2\beta_j\overline{u}_j x}\int_{R^3} e^{-w^2}F_{ij}dw, \quad i \neq j, \tag{63}$$

$$B_i = \frac{\partial \psi_i}{\partial x}\left(\frac{u}{\sqrt{\beta_i}} + \overline{v}_i - \overline{u}_i t\right) + 2\beta_i\psi_i\left((\overline{u}_i, \overline{v}_i) - t\overline{u}_i^2\right). \tag{64}$$

It is easy to see that (45) is preserved, but in (46) the factor $\mu_j(x)$ arises and it is equal to 1 if $n_j > 1$ and to $\exp\{2\overline{u}_{oj}x\}$ if $n_j = 1$. As for (47), only the value (48) will be changed

$$H_i = \left[ \begin{array}{ll} 0, & n_i > 1 \\ (\overline{u}_{oi}, \overline{v}_i), & n_i = 1. \end{array} \right. \tag{65}$$

That is why the limiting transition in (62) leads to the expressions (57), (58). The theorem is proved.

**Corollary 3.** *Let the conditions of Theorem 3 be fulfilled and the functions* $\psi_i$ *be of the form:*

$$\psi_i = \frac{D_i}{(1+t^2)^{\chi_i}} C_i\left([x \times \overline{v}_i]\right), \ i = 1, 2, \tag{66}$$

*if*

$$(\overline{v}_i, \overline{u}_i) = 0, \ i = 1, 2, \tag{67}$$

*and*

$$\psi_i = \frac{D_i}{(1+t^2)^{\chi_i}} C_i\left(x\right), \ i = 1, 2, \tag{68}$$

*for arbitrary* $\overline{v}_1, \overline{v}_2$*, where (14) is valid and the functions* $C_i$ *belong to* $P(R^3)$*. Then the statements a), b) of Cor. 2 remain true.*

P r o o f. The expressions (68) under the indicated conditions evidently are in concord with the requirements of Th. 3. Let us check whether the same statement is true for the functions of the form (66) if (67) is fulfilled. Decompose an arbitrary vector $x \in R^3$ by the orthogonal (because of (67)) basis

$$\overline{u}_i, \overline{v}_i, [\overline{u}_i \times \overline{v}_i], \tag{69}$$

i.e.,

$$x = x_1 \overline{u}_i + x_2 \overline{v}_i + x_3 [\overline{u}_i \times \overline{v}_i], \tag{70}$$

then

$$\psi_i e^{2\beta_i \overline{u}_i x} = \frac{D_i}{(1+t^2)^{\chi_i}} C_i([x \times \overline{v}_i]) e^{2\beta_i \overline{u}_i x}$$

$$= \frac{D_i}{(1+t^2)^{\chi_i}} C_i(x_1[\overline{u}_i \times \overline{v}_i] - x_3[\overline{v}_i \times [\overline{u}_i \times \overline{v}_i]]) e^{2\beta_i x_1 \overline{u}_i^2}$$

$$= \frac{D_i}{(1+t^2)^{\chi_i}} C_i(x_1[\overline{u}_i \times \overline{v}_i] - x_3 \overline{u}_i \overline{v}_i^2) e^{2\beta_i x_1 \overline{u}_i^2}, \tag{71}$$

but with the increasing of $x_1$ when the exponent in (71) also increases, the argument of the function $C_i$ obviously increases too without any connection with the behaviour of $x_3$ (the component $x_2$ is not present in (71) at all) because of the perpendicularity of the components of this argument. Thus, the function $C_i$ either will vanish (if it is finite) or will compensate the increasing exponent in a view of supposition of its fast decrease. So, the expression (71) in whole turns out to be a bounded one, whose behaviour with $x_3 \to \infty$ is evident. The product of (71) by $t$ is also bounded because of (14). The derivative $\frac{\partial \psi_i}{\partial t}$ behaves itself completely in the same way (see (33)). Further, from (66) we will find

$$\frac{\partial \psi_i}{\partial x} = \frac{D_i}{(1+t^2)^{\chi_i}} \left[ \overline{v}_i \times C_i^{'} \right], \ i = 1, 2, \tag{72}$$

i.e., the values

$$\left| \frac{\partial \psi_i}{\partial x} \right| e^{2\beta_i \overline{u}_i x}; \quad t \left( \overline{u}_i \frac{\partial \psi_i}{\partial x} \right) e^{2\beta_i \overline{u}_i x} \tag{73}$$

are bounded by the same reasons as (71), because $C_i^{'}([x \times \overline{v}_i])$ is finite or fast-decreasing, too. Therefore, for the expressions (66), (68) all conditions of Th. 3 are fulfilled. Consequently, (57) or (58) holds true. If (67) is valid, then the second summand in (58) vanishes, i.e., it is essential only when $n_i = 1$ and (68) is fulfilled. But in the cases a), b) of Cor. 2 the only nonzero expression in (57) remains

$$\mu_i(x) \left( \frac{\partial \psi_i}{\partial t} + \overline{v}_i \frac{\partial \psi i}{\partial x} \right). \tag{74}$$

And in (58) an "additional" summand to the value $N$ may remain. However, as it can be seen from (72), under the supposition (66) we have

$$\overline{v}_i \frac{\partial \psi_i}{\partial x} = 0, \ i = 1, 2, \tag{75}$$

and for (68)

$$\overline{v}_i \frac{\partial \psi_i}{\partial x} = \frac{D_i}{(1+t^2)^{\chi_i}} \left( \overline{v}_i, C_i^{'}(x) \right), \ i = 1, 2, \tag{76}$$

that is, for all possible cases the expressions (57), (58) up to some constant factors reduce to the values of type (19), where there are either functions $C_i$ themselves or their derivatives $C_i^{'}$, $i = 1, 2$. This fact, obviously, yields to (34) and (52) in the same way as in the proofs of previous corollaries.

R e m a r k 1. The expressions (66) and (68) are similar but they do not reduce to each other. Really, $C_i([x \times \overline{v}_i])$ describes a function on $x$, which is constant along the vector $\overline{v}_i$, i.e. (because of (67)) in a direction perpendicular to the direction of acceleration and packing of $i$-th flow, $i = 1, 2$, and finite or fast-decrease particularly along the vector $\overline{u}_i$ that is, at the direction of the increasing of factor $\mu_i(x)$). However, (68) corresponds to some "clot" of a gas concentrated on a bounded in $R^3$ support, and the factors depending on $t$, which are common for (66) and (68), mean that the interaction between two flows is weakened with $t \to \pm\infty$ but not too quickly.

**Theorem 4.** *Let us suppose that instead of (35) or (55) the following equality is fulfilled:*

$$\varphi_i(t,x) = \psi_i(t,x) \exp\{-2\beta_i \overline{u}_i x\}, \tag{77}$$

*and the requirements (15), (16), (36) are valid with*

$$k_i \geq \frac{1}{2}, \ i = 1, 2, \tag{78}$$

*besides the smooth, nonnegative functions $\psi_i$ ensure boundness on $R^4$ of the same expressions as in the conditions of Th. 3, but with substitution of the factor* $\exp\{\beta_i(\overline{v}_i - \overline{u}_i t)^2\}$ *for* $\exp\{2\beta_i \overline{u}_i x\}$, $i = 1, 2$. *Then the inequality (18) holds true once more, where:*

***i.*** *If $n_i > \frac{1}{2}$; $k_i > \frac{1}{2}$, then*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = \sum_{i=1}^{2} \overline{\rho}_i \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} \right|. \tag{79}$$

***ii.*** *If $n_i > \frac{1}{2}$; $k_i = \frac{1}{2}$, then*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = \sum_{i=1}^{2} \overline{\rho}_i e^{\overline{v}_{oi}^2} \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} \right|. \tag{80}$$

***iii.*** *If $n_i = \frac{1}{2}$; $k_i > \frac{1}{2}$, then*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = \sum_{i=1}^{2} \overline{\rho}_i \sup_{(t,x)\in R^4} \left\{ e^{t^2 \overline{u}_{oi}^2} \left| \frac{\partial \psi_i}{\partial t} + 2\psi_i t \overline{u}_{oi}^2 \right| \right\}$$

$$+ \frac{2}{\sqrt{\pi}} \sum_{i=1}^{2} \overline{\rho}_i \left| \overline{u}_{oi} \right| \sup_{(t,x)\in R^4} \left\{ e^{t^2 \overline{u}_{oi}^2} \psi_i \right\}. \tag{81}$$

***iiii.*** *If $n_i = k_i = \frac{1}{2}$, then*

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' = \sum_{i=1}^{2} \overline{\rho}_i \sup_{(t,x)\in R^4} \left\{ e^{(\overline{v}_{oi} - \overline{u}_{oi} t)^2} \left| \frac{\partial \psi_i}{\partial t} + 2\psi_i t \overline{u}_{oi}^2 \right| \right\}$$

$$+2 \sum_{i=1}^{2} \overline{\rho}_i \left( \frac{|\overline{u}_{oi}|}{\sqrt{\pi}} + |(\overline{u}_{oi}, \overline{v}_{oi})| \right) \sup_{(t,x)\in R^4} \left\{ e^{(\overline{v}_{oi} - \overline{u}_{oi} t)^2} \psi_i \right\}. \tag{82}$$

P r o o f. It is evident that in our situation the analogues of formulae (59)–(61) will be the following:

$$\frac{\partial\varphi_i}{\partial t}=\frac{\partial\psi_i}{\partial t}\exp\{-2\beta_i\overline{u}_i x\}, \tag{83}$$

$$\frac{\partial\varphi_i}{\partial x}=\exp\{-2\beta_i\overline{u}_i x\}\cdot\left\{\frac{\partial\psi_i}{\partial x}-2\beta_i\psi_i\overline{u}_i\right\}, \tag{84}$$

$$\Delta'=\sup_{(t,x)\in R^4}\sum_{i\cdot j=1,i\neq j}^{2}\left[\int\limits_{R^3}\left|\frac{\partial\psi_i}{\partial t}+\left(\frac{u}{\sqrt{\beta_i}}+\overline{v}_i-\overline{u}_i t\right)\left\{\frac{\partial\psi_i}{\partial x}-2\beta_i\psi_i\overline{u}_i\right\}\right.\right.$$
$$\left.+\psi_1\psi_2\overline{\rho}_j e^{\beta_j(\overline{v}_j-\overline{u}_j t)^2}\frac{d^2}{\sqrt{\pi}}\int\limits_{R^3}F_{ij}e^{-w^2}dw\right|\cdot\overline{\rho}_i e^{\beta_i(\overline{v}_i-\overline{u}_i t)^2}\pi^{-3/2}e^{-u^2}du$$
$$\left.+\psi_1\psi_2\left(\frac{d}{\pi}\right)^2\overline{\rho}_1\overline{\rho}_2\exp\left\{\beta_1(\overline{v}_1-\overline{u}_1 t)^2+\beta_2(\overline{v}_2-\overline{u}_2 t)^2\right\}\int\limits_{R^6}e^{-w^2-u^2}F_{ij}dwdu\right]. \tag{85}$$

Thus, instead of (62)–(64) we will have:

$$\Delta'=\pi^{-3/2}\sup_{(t,x)\in R^4}\sum_{i=1}^{2}\overline{\rho}_i e^{\beta_i(\overline{v}_i-\overline{u}_i t)^2}\int\limits_{R^3}\left[\left|\frac{\partial\psi_i}{\partial t}+A_i+B_i\right|+A_i\right]e^{-u^2}du, \tag{86}$$

$$A_i=\psi_1\psi_2\frac{d^2}{\sqrt{\pi}}\overline{\rho}_j e^{\beta_j(\overline{v}_j-\overline{u}_j t)^2}\int\limits_{R^3}e^{-w^2}F_{ij}dw,\quad i\neq j, \tag{87}$$

$$B_i=\frac{\partial\psi_i}{\partial x}\left(\frac{u}{\sqrt{\beta_i}}+\overline{v}_i-\overline{u}_i t\right)+2\beta_i\psi_i\left\{-\frac{1}{\sqrt{\beta_i}}(u,\overline{u}_i)-(\overline{u}_i,\overline{v}_i)+t\overline{u}_i^2\right\}. \tag{88}$$

The limit of exponents presented in $\Delta'$ depends on the behaviour of the parameters $n_i$, $k_i$ as follows:

$$\lim_{\beta_i\to+\infty,i=1,2}e^{\beta_i(\overline{v}_i-\overline{u}_i t)^2}=\sigma_i(t)=\left[\begin{array}{ll}1, & if\ n_i>\frac{1}{2},k_i>\frac{1}{2},\\ e^{\overline{v}_{oi}^2}, & if\ n_i>\frac{1}{2},k_i=\frac{1}{2},\\ e^{t^2\overline{u}_{oi}^2}, & if\ n_i=\frac{1}{2},k_i>\frac{1}{2},\\ e^{(\overline{v}_{oi}-\overline{u}_{oi}t)^2}, & if\ n_i=k_i=\frac{1}{2}.\end{array}\right. \tag{89}$$

Therefore, with passing the limit the analogous exponent in $A_i$ (with index $j\neq i$) always has a finite limit $\sigma_j(t)$, i.e., $A_i$ in whole tends to zero because of (15), (16) which yield (30). Finally, since $\overline{v}_i\to 0, i=1,2$, in (47) there will be maintained only the second summand:

$$\lim_{\beta_i\to+\infty,i=1,2}B_i=2\psi_i H_i, \tag{90}$$

where

$$H_i = \left[ \begin{array}{ll} 0, & if\ n_i > \frac{1}{2}, k_i \geq \frac{1}{2}, \\ t\overline{u}_{oi}^2 - (u, \overline{u}_{oi}), & if\ n_i = \frac{1}{2}, k_i > \frac{1}{2}, \\ t\overline{u}_{oi}^2 - (u, \overline{u}_{oi}) - (\overline{u}_{oi}, \overline{v}_{oi}), & if\ n_i = k_i = \frac{1}{2}. \end{array} \right. \tag{91}$$

Taking into account all these facts, the equality (86) leads to (79)–(82).
The theorem is proved.

**Corollary 4.** *Let all conditions of Theorem 4 be fulfilled. Then the statement (34) holds true if the functions* $\psi_i$ *have the form*

$$\psi_i(t, x) = D_i C_i(t) E_i(x), \quad i = 1, 2, \tag{92}$$

*where* $D_i > 0$*;* $C_i(t) \in P(R^1)$ *and* $E_i(x) \geq 0$ *are smooth and bounded together with* $E_i^{'}(x)$ *functions on* $x \in R^3$*.*

P r o o f. The functions of the form of (92) by the conditions imposed here ensure the boundless of all expressions mentioned in Th. 4. Moreover, directly from (79)–(82) it can be seen that the supremums entering into these formulas are finite for any possible values of the constants $n_i$, $k_i \geq \frac{1}{2}$, $i = 1, 2$, and the presence of factors $D_i$, $i = 1, 2$, in (92) leads to (34).

R e m a r k 2. It would be of interest to try to minimize the expressions (81), (82) by solving the following differential equation (for each $i = 1, 2$):

$$\frac{\partial \psi_i}{\partial t} + 2\psi_i t \overline{u}_{oi}^2 = 0. \tag{93}$$

However, it can be easily seen that its solution

$$\psi_i = e^{-t^2 \overline{u}_{oi}^2} E_i(x), \tag{94}$$

with $E_i(x)$ being the same as in Cor. 4, in spite of ensuring the existence of both supremums in (81) and under the additional condition (67) also in (82), does not satisfy the conditions of Th. 4, because it cannot guarantee the boundness of indicated there expressions before the limiting transition $\beta_i \to +\infty$, $i = 1, 2$.

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