Journal of Mathematical Physics, Analysis, Geometry
2009, vol. 5, No. 1, pp. 54–81

On the Limit of Regular Dissipative and Self-Adjoint Boundary Value Problems with Nonseparated Boundary Conditions when an Interval Stretches to the Semiaxis

V.I. Khrabustovskyi

Ukrainian State Academy of Railway Transport
7, Feyerbakh Sq., Kharkiv, 61050, Ukraine
E-mail:v_khrabustovskyi@ukr.net

Received April 2, 2007

For the symmetric differential system of the first order that contains a spectral parameter in Nevanlinna's manner the limit of regular boundary value problems with dissipative or accumulative nonseparated boundary conditions is studied when the interval stretches to the semiaxis. When for the considered system the case of the limit point takes place in one of the complex half-planes, we obtain the condition which guarantees the non-self-adjointness of the boundary condition at zero that corresponds to the limit boundary problem. This result is illustrated on the perturbed almost periodic systems. When the boundary condition in the prelimit regular problems is periodic, we show that the limit characteristic matrix is also the characteristic matrix on the whole axis if the coefficients of the system are extended in a certain way on the negative semiaxis. In the general case we find the condition when the convergence of characteristic matrixes implies the convergence of resolvents.

Key words: characteristic matrix, nonseparated boundary conditions, resolvent convergence, almost periodic function.

Mathematics Subject Classification 2000: 34B07, 34B20, 34B40, 47E05.

1. Introduction

In a finite dimensional Hilbert space $\mathcal{H}$ we consider the system

$$\frac{i}{2}\Big(\left(Q\left(t\right)x\left(t\right)\right)' + Q\left(t\right)x'\left(t\right)\Big) - H_{\lambda}\left(t\right)x\left(t\right) = w_{\lambda}\left(t\right)f\left(t\right), \;\; t \in [0, \;\infty)\,, \qquad (1)$$

where $Q(t) = Q^*(t) \in AC_{loc}, H_\lambda(t) = H^*_{\bar\lambda}(t)$ are $n \times n$ matrixes, $\det Q(t) \neq 0, H_\lambda(t) \in L^1_{loc}$ on t and analytically depends on nonreal λ, the weight $w_\lambda(t) = \Im H_\lambda(t) / \Im\lambda \geq 0, (\Im\lambda \neq 0)$.

In the paper, the limits of characteristic matrixes of the Weyl–Titchmarsh type and the limits of resolvents of regular boundary value problems on the interval $(0, b)$ with nonseparated dissipative or accumulative boundary conditions (in particular, with self-adjoint boundary conditions that include periodic ones) are studied when $(0, b)$ stretches to the semiaxis $(0, \infty)$. The limiting transition from the regular problems to the singular ones was studied and used by many authors [1–13]. However, the question on a type of boundary condition at zero that corresponds to the limit of the boundary problems with nonseparated boundary conditions and the resolvent convergence for the system (1) have not been studied in the general case.

Let $w_\lambda(t)$ be the weight of positive type (see (4) below) and let for some nonreal λ the number of linearly independent and square integrable on the semiaxis $(0, \infty)$ with the weight $w_\lambda(t)$ solutions of homogeneous system (1) be minimal*, and therefore (see Remark 6 below) any characteristic matrix of the system (1) on the semiaxis depends only on the boundary condition at zero. Under this assumption we obtain the conditions for which the boundary condition at zero that corresponds to the limit characteristic matrix depends on λ and it is strictly dissipative or strictly accumulative as $\Im\lambda > 0$ or $\Im\lambda < 0$ ** (even if the prelimit boundary conditions are self-adjoint). It is shown that when these conditions are not fulfilled, the boundary condition at zero can be self-adjoint (even if the prelimit boundary conditions are nonself-adjoint). The classes of systems satisfying these conditions are given. In the case of periodic boundary conditions in the prelimit regular boundary value problems that are often met in Physics (see, e.g., [15, Ch. 1, § 6]) we show that under some assumptions the limit characteristic matrix of the system (1) on the semiaxis $(0, \infty)$ is the characteristic matrix of this system on the axis $(-\infty, \infty)$ if the coefficients of the system are extended in a certain way to the negative semiaxis. In some cases the limit characteristic matrix can be calculated due to the mentioned above. The proofs of our results are based on the possibility to approximate the limit characteristic matrix by means of characteristic matrixes considered in [13] belonging to special regular boundary value problems with separated and strictly dissipative or strictly accumulative boundary conditions.

Also, in the paper in the general case (i.e, without requirement that for some

*For constant $Q(t) = J = J^{-1}$ this means that the rank of one of the radiuses of the Weyl limit disc for the system (1) is minimal [14].

**And therefore, if in (1) $H_\lambda(t) = H_0(t) + \lambda H(t), H_0(t) = H_0^*(t)$, then the nonorthogonal generalized resolvent of the corresponding minimal relation corresponds to the limit boundary problem.

nonreal λ there should be the minimal number of linearly independent and square integrable on the semiaxis $(0,\infty)$ with the weight $w_\lambda(t)$ solutions of homogeneous system (1)) it is shown that the convergence of characteristic matrixes of regular dissipative or accumulative boundary value problems, in particular, self-adjoint ones, with separated or nonseparated boundary conditions implies the convergence of resolvents as $\lambda = \lambda_0$ (as $\Im\lambda\Im\lambda_0 > 0$, in the case when $H_\lambda(t)$ contains λ in the linear manner) only if the limit characteristic matrix corresponds to the self-adjoint boundary conditions as $\lambda = \lambda_0$. This fact (in another form) is announced in [8] for the limits of characteristic matrixes of the self-adjoint regular scalar differential operators of even order with real coefficients.

In the general case also a criterion of the separating of boundary conditions that correspond to the limit of regular boundary value problems with nonseparated boundary conditions is obtained.

We notice that due to the fact that in the considered system (1) the weight $w_\lambda(t)$ can be degenarated, the obtained results should be applied to the matrix differential operators and relations of arbitrary order (dissipative and, in particular, symmetric).

2. Auxiliary Results

In this section, some results of [13] are given in a convenient but not most general form. These results will be used later on. By (,) and $\|\cdot\|$ the scalar product and the norm in various spaces with special indexes, if necessary, are denoted.

Let $X_\lambda(t)$ be the matrix solution of the homogeneous system (1) satisfying the initial condition $X_\lambda(0) = \|\delta_{jk}\|_{j,k=1}^n \stackrel{def}{=} I$. Since $H_\lambda(t) = H^*_{\bar\lambda}(t)$, then

$$X^*_{\bar\lambda}(t)\,Q(t)\,X_\lambda(t) = Q(0) \stackrel{def}{=} G, \quad \Im\lambda \neq 0. \tag{2}$$

Under the condition $0 \le \alpha \le \beta < \infty$, we denote $\Delta_\lambda(\alpha,\beta) = \int_\alpha^\beta X^*_\lambda(t) w_\lambda(t) X_\lambda(t)dt$. One has

$$X^*_\lambda(\beta)\,Q(\beta)\,X_\lambda(\beta) - X^*_\lambda(\alpha)\,Q(\alpha)\,X_\lambda(\alpha) = 2\Im\lambda\Delta_\lambda(\alpha,\ \beta) \quad \Im\lambda \neq 0. \tag{3}$$

Further it is supposed that*

$$\exists\mu^0\left(\Im\mu^0 \neq 0\right), \quad \beta \in (0,\ \infty): \quad \Delta_{\mu^0}(0,\ \beta) > 0 \tag{4}$$

(by [13] the condition (4) is valid if μ^0 is changed by an arbitrary nonreal λ).

For $x(t) \in \mathcal{H}$ or $x(t) \in B(\mathcal{H})$ we denote $U[x(t)] = x^*(t)\,Q(t)\,x(t)$.

*The condition $\Im\mu^0 \neq 0$ can be weakened in the same way as in [13].

Definition 1 [13]. *An analytic $n \times n$-matrix function $M(\lambda) = M^*(\bar{\lambda})$ on nonreal λ is called a characteristic matrix (c.m.) of the system (1) on $\mathcal{I} = (0,b), (b \le \infty)$ (or simply, c.m.) if for $\Im\lambda \neq 0$ and for any vector function $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support the corresponding solution $x_\lambda(t)$ of the system (1) of the form*

$$x_\lambda(t) = R_\lambda f = \int_{\mathcal{I}} X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} sgn(s-t)(iG)^{-1} \right\} X^*_{\bar{\lambda}}(s) w_\lambda(s) f(s)\, ds \tag{5}$$

satisfies the condition

$$(\Im\lambda) \lim_{\beta \uparrow b} \left(U[x_\lambda(\beta)] - U[x_\lambda(0)] \right) \le 0, \quad \Im\lambda \neq 0. \tag{6}$$

The following remark establishes a relation between the c.m. and the regular boundary value problems for the system (1) with the boundary condition depending on spectral parameter.

Remark 1 [13]. *Let the interval $\mathcal{I} = (0, b)$ be finite. Then:*

1^0. *If the matrix-functions $\mathcal{M}_\lambda$, $\mathcal{N}_\lambda$ analytically depend on nonreal λ and*

$$\Im\lambda \left(\mathcal{N}^*_\lambda Q(b) \mathcal{N}_\lambda - \mathcal{M}^*_\lambda Q(0) \mathcal{M}_\lambda \right) \le 0, \quad \Im\lambda \neq 0, \tag{7}$$

$$\|\mathcal{M}_\lambda h\| + \|\mathcal{N}_\lambda h\| > 0, \quad 0 \neq h \in \mathcal{H}, \quad \Im\lambda \neq 0, \tag{8}$$

$$\mathcal{M}^*_{\bar{\lambda}} Q(0) \mathcal{M}_\lambda = \mathcal{N}^*_{\bar{\lambda}} Q(b) \mathcal{N}_\lambda, \quad \Im\lambda \neq 0, \tag{9}$$

then the boundary value problem, obtained by connecting to the system (1) the boundary condition

$$\exists h = h(\lambda, f) \in \mathcal{H}: \; x(0) = \mathcal{M}_\lambda h, \quad x(b) = \mathcal{N}_\lambda h, \tag{10}$$

has the unique solution as $\Im\lambda \neq 0$. The solution is equal to $x_\lambda(t)$ (5), where

$$M(\lambda) = -\frac{1}{2} \left(\mathcal{M}_\lambda + X^{-1}_\lambda(b) \mathcal{N}_\lambda \right) \left(\mathcal{M}_\lambda - X^{-1}_\lambda(b) \mathcal{N}_\lambda \right)^{-1} (iG)^{-1}, \tag{11}$$

where

$$\det \left(\mathcal{M}_\lambda - X^{-1}_\lambda(b) \mathcal{N}_\lambda \right) \neq 0, \quad \Im\lambda \neq 0.$$

The matrix-function $M(\lambda)$ (11) is a c.m. of the system (1) on $\mathcal{I}$.

2^0. *If $M(\lambda)$ is the c.m. of the system (1) on $\mathcal{I}$, then $x_\lambda(t)$ (5) is the solution of some boundary value problem from n^0 1^0.*

Definition 2 [13]. *Let $M(\lambda)$ be the c.m. of the system (1) on $\mathcal{I}$. We say that the corresponding condition (6) is separated for nonreal $\lambda = \mu_0$ if for any vector function $f(t) \in L^2_{w_{\mu_0}}(\mathcal{I})$ with compact support the following two inequalities hold simultaneously for the solution $x_{\mu_0}(t)$ (5) of the system (1):*

$$\Im\mu_0 U\left[x_{\mu_0}(0)\right] \geq 0, \quad \lim_{\beta\uparrow b} \Im\mu_0 U\left[x_{\mu_0}(\beta)\right] \leq 0. \tag{12}$$

Theorem 1 [13]. *Let $M(\lambda)$ be the c.m. of the system (1). We represent $M(\lambda)$ in the form*

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2}I\right)(iG)^{-1}. \tag{13}$$

Then the condition (6) corresponding to $M(\lambda)$ is separated for $\lambda = \mu_0$ if and only if the operator $\mathcal{P}(\mu_0)$ is a projection, i.e.,

$$\mathcal{P}(\mu_0) = \mathcal{P}^2(\mu_0). \tag{14}$$

The following remark establishes a relation between the c.m. with separated boundary condition (6) and the boundary value problems with separated boundary conditions depending on spectral parameter.

Remark 2 [13]. *Let the interval $\mathcal{I} = (0, b)$ be finite. Then:*

1^0. *If the operator-functions $\mathcal{M}_\lambda, \mathcal{N}_\lambda$ from $n^0\,1^0$ of Remark 1 are such that $\Im\lambda\mathcal{M}^*_\lambda Q(0)\mathcal{M}_\lambda \geq 0$, $\Im\lambda\mathcal{N}^*_\lambda Q(b)\mathcal{N}_\lambda \leq 0\,(\Im\lambda \neq 0)$ (i.e., the boundary condition (7)–(10) is separated), then the solution of the boundary value problem (1),(7)–(10) for any vector function $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support is equal to $x_\lambda(t)$ (5), where $M(\lambda)$ is some c.m. of the system (1) on $(0, b)$ with separated condition (6). And therefore $M(\lambda)$ admits the representation (13), where $\mathcal{P}(\lambda)$ is a projection that is equal to*

$$\mathcal{P}(\lambda) = -X^{-1}_\lambda(b)\mathcal{N}_\lambda\left(\mathcal{M}_\lambda - X^{-1}_\lambda(b)\mathcal{N}_\lambda\right)^{-1}. \tag{15}$$

2^0. *If $M(\lambda)$ (13) is the c.m. of the system (1) on $\mathcal{I}$ and, moreover, $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then $x_\lambda(t)$ (5) is a solution of some boundary value problem from $n^0\,1^0$.*

Definition 3 [13]. *If the matrix-function $M(\lambda)$ of the form (13) is the c.m. of the system (1) on $\mathcal{I}$ and, moreover, $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then $\mathcal{P}(\lambda)$ is called a characteristic projection (c.p.) of the system (1) on $\mathcal{I}$ (or simly c.p.).*

3. Main Results

In the lemma below, the matrixes $\mathcal{M}_\lambda$ and $\mathcal{N}_\lambda$ from Remark 1 can depend on b. We denote the corresponding matrix (15) by $\mathcal{P}(\lambda, b)$.

It follows from [13] (and [9] as $Q(t) = J = J^{-1}$):

Lemma 1. *Any sequence $b_n \uparrow \infty$ contains a subsequence b_{n_k} such that for any nonreal λ there exists*

$$\lim \mathcal{P}(\lambda,\ b_{n_k}) = \mathcal{P}(\lambda), \tag{16}$$

and $M(\lambda)$ (13), (16) are the c.m.'s of the system (1) on $(0,\infty)$.

Any c.m. of the system (1) on $(0,\infty)$ can be obtained in a similar way.

Remark 3. *As it is seen from Example 2 (see below), even with the constant $Q(t)$, with the periodic $H_\lambda(t)$ and with periodic boundary condition (10), the limit (16) can depend on the choice of subsequence b_{n_k}.*

By $R_\lambda(\Pi)$ we denote the operator (5), (13) corresponding to $\mathcal{P}(\lambda) = \Pi$.

The following theorem shows when the convergence of c.m.'s implies the convergence of corresponding resolvents.

Theorem 2.1^0. *Let for the c.m. $M(\lambda)$ (13), (16) of the system (1) on $(0,\infty)$ for some nonreal $\lambda = \lambda_0$ there be an equality in the condition (6)*.*

Then, for $\lambda = \lambda_0$ and for any vector function $f(t) \in L^2_{w_{\lambda_0}}(0,\ \infty)$ with compact support:

$$\lim_{b_{n_k}\to\infty} \|[R_\lambda(\mathcal{P}(\lambda)) - R_\lambda(\mathcal{P}(\lambda,\ b_{n_k}))]\, f\|_{L^2_{w_\lambda}(0,\, b_{n_k})} \to 0. \tag{17}$$

If in (1)

$$H_\lambda(t) = H_0(t) + \lambda H(t),\ H_0(t) = H_0^*(t), \tag{18}$$

then (17) holds for $\Im\lambda\Im\lambda_0 > 0$.

2^0. *If for the prelimit c.m.'s $M(\lambda, b_{n_k})$ (13), (16) there is the equality in the condition (6) for some nonreal $\lambda = \lambda_0$ (and therefore by [9, 13] it takes place for $\Im\lambda \neq 0$) and (17) is valid for $\lambda = \lambda_0$, then for $\lambda = \lambda_0$ (for $\Im\lambda\Im\lambda_0 > 0$ in the case (18)) there is the equality in (6) for the limit c.m. $M(\lambda)$ (13), (16).*

P r o o f. **1^0.** In (1) substitute

$$x(t) = T(t)\,\tilde{x}(t), \quad T(t) = \mathbf{\Gamma}(t)\, S, \tag{19}$$

where matrix $\Gamma(t)$ is the solution of the Cauchy problem

$$\frac{i}{2}\left((Q(t)\,\mathbf{\Gamma}(t))' + Q(t)\,\mathbf{\Gamma}'(t)\right) = 0, \quad \mathbf{\Gamma}(0) = I,$$

*Therefore there will also be an equality for $\lambda = \bar{\lambda}_0$ if the number of linearly independent and square-integrable on the semiaxis $(0,\infty)$ with the weight $w_\lambda(t)$ solutions of the homogeneous system (1) for some nonreal $\lambda = \mu_+$, $\lambda = \mu_-$, $\Im\mu_+\Im\mu_- < 0$ coincides. This statement can be obtained from [14], [16–18] and from Lem. 3.2 from [13].

and the matrix S is such that $S^*GS = J = J^{-1}$.

We get the system

$$iJ\tilde{x}'(t) - \tilde{H}_\lambda(t)\,\tilde{x}_\lambda(t) = \tilde{w}_\lambda(t)\,\tilde{f}(t)\,, \tag{20}$$

where $\tilde{H}_\lambda(t) = T^*(t)H_\lambda(t)T(t)$, $\tilde{w}_\lambda(t) = T^*(t)\,w_\lambda(t)\,T(t)$, $\tilde{f}(t) = T^{-1}(t)\,f(t)$.

The c.m.'s of the system (1) and (20) are in a one-to-one correspondence

$$M(\lambda) = S\tilde{M}(\lambda)\,S^*, \quad \mathcal{P}(\lambda) = S\tilde{\mathcal{P}}(\lambda)\,S^{-1},$$

and, moreover, for the corresponding resolvents we have

$$\|[R_\lambda(\mathcal{P}(\lambda)) - R_\lambda(\mathcal{P}(\lambda,\ b_{n_k}))]\,f(t)\|_{L^2_{w_\lambda}(0,\,b_{n_k})}$$
$$= \left\|\left[\tilde{R}_\lambda\left(\tilde{\mathcal{P}}(\lambda)\right) - \tilde{R}_\lambda\left(\tilde{\mathcal{P}}(\lambda,\ b_{n_k})\right)\right]\tilde{f}(t)\right\|_{L^2_{\tilde{w}_\lambda}(0,\,b_{n_k})},$$

where the analog of $R_\lambda(\Pi)$ for the system (20) is denoted by $\tilde{R}_\lambda\left(\tilde{\Pi}\right)$.

Since (4) is valid for the system (20), then the instruments of the matrix discs from [14] can be applied to (20).

Due to [9, 14] any c.m. of the system of the type of (20) on $(0,\ b)$ can be represented in the form

$$\tilde{M}(\lambda,\ b) = \frac{1}{2}i\left[R_1^2(\lambda,\ b) + R^2(\lambda,\ b) + 2R_1(\lambda,\ b)\ v_\lambda(b)\,R(\lambda,\ b)\right],\ \Im\lambda > 0, \tag{21}$$

where

$$R_1^2(\lambda, b) = [2\Im\lambda\Delta_\lambda(0,b)]^{-1},\ R^2(\lambda,\ b) = [2\Im\lambda\Delta_{\bar{\lambda}}(0,b)]^{-1},$$
$$v_\lambda(b) \in B(\mathcal{H}),\ v_\lambda^*(b)\,v_\lambda(b) \le I,$$

and any c.m. of the system of the type of (20) on $(0,\ \infty)$ can be represented in the form

$$\tilde{M}(\lambda) = \frac{1}{2}i\left[R_1^2(\lambda) + R^2(\lambda) + 2R_1(\lambda)\,v_\lambda R(\lambda)\right],\ \Im\lambda > 0, \tag{22}$$

where

$$R_1(\lambda) = \lim_{b\to\infty} R_1(\lambda, b),\ R(\lambda) = \lim_{b\to\infty} R(\lambda,\ b)\,, v_\lambda \in B(\mathcal{H}), v_\lambda^* v_\lambda \le I.$$

It can be proved (compare with [9], where $v_\lambda^{-1} = v_\lambda^*$) that the fact that the fulfilment of the condition (6) with $b = \infty$ for $\tilde{M}(\lambda)$ (22) as $\Im\lambda > 0$ or $\Im\lambda < 0$ is equivalent to the following inequality conditions, respectively:

$$C_1^*(\lambda)\,(I - P_1(\lambda))\,C_1(\lambda) + R(\lambda)\,(I - v_\lambda^* v_\lambda)\,R(\lambda) \ge 0, \quad \Im\lambda > 0, \tag{23}$$
$$C(\lambda)\,(I - P(\lambda))\,C^*(\lambda) + R_1(\lambda)\,(I - v_\lambda v_\lambda^*)\,R_1(\lambda) \ge 0, \quad \Im\lambda > 0, \tag{24}$$

where $C_1(\lambda) = R_1(\lambda) + v_\lambda R(\lambda), C(\lambda) = R(\lambda) + R_1(\lambda) v_\lambda$, $P_1(\lambda)$ and $P(\lambda)$ are orthoprojections on $R_1(\lambda)\mathcal{H}$ and $R(\lambda)\mathcal{H}$, respectively. The equality in (6) for $\tilde{M}(\lambda)$ (22) for $\Im\lambda > 0$ ($\Im\lambda < 0$) is equivalent to the equality in (23) ((24)). But the equality in (23) or (24) is equivalent to the simultaneous fulfilment of two inequalities, respectively:

$$(I - P_1(\lambda)) v_\lambda P(\lambda) = 0, \quad P(\lambda) = v_\lambda^* v_\lambda P(\lambda), \quad \Im\lambda > 0, \tag{25}$$

$$(I - P(\lambda)) v_\lambda^* P_1(\lambda) = 0, \quad P_1(\lambda) = v_\lambda v_\lambda^* P_1(\lambda), \quad \Im\lambda > 0. \tag{26}$$

Let

$$\tilde{M}(\lambda) = \lim_{b_{n_k}\to\infty} \tilde{M}(\lambda, b_{n_k}). \tag{27}$$

For definiteness, set $\Im\lambda_0 < 0$.

Since $\tilde{M}(\lambda)$ is a c.m. of the system (20) on $(0, b_{n_k})$, then $\tilde{M}(\lambda)$ can be represented in the form (21) with $v_\lambda(b_{n_k})$ being replaced by some $\tilde{v}_\lambda(b_{n_k})$.

Then

$$\left\| \left[\tilde{R}_{\lambda_0}\left(\tilde{\mathcal{P}}(\lambda_0)\right) - \tilde{R}_{\lambda_0}\left(\tilde{\mathcal{P}}_0(\lambda, b_{n_k})\right)\right] \tilde{f} \right\|^2_{L^2_{\tilde{w}_{\lambda_0}}(0, b_{n_k})} \tag{28}$$

$$= \frac{1}{2\Im\bar{\lambda}_0} \left\| \left[\tilde{v}^*_{\bar{\lambda}_0}(b_{n_k}) - v^*_{\bar{\lambda}_0}(b_{n_k}) \right] R_1\left(\bar{\lambda}_0, b_{n_k}\right) \tilde{h} \right\|^2, \tag{29}$$

where $\tilde{h} = \tilde{h}(\lambda_0, \tilde{f}) = \int\limits_0^\infty \tilde{X}^*_{\bar{\lambda}_0}(s)\, \tilde{w}_{\bar{\lambda}_0}(s)\, \tilde{f}(s)\, ds$.

We denote by $P_1(\lambda, b_{n_k})$ and $P(\lambda, b_{n_k})$ the Riesz projections corresponding to those parts of $\sigma(R_1(\lambda, b_{n_k}))$ and $\sigma(R(\lambda, b_{n_k}))$ that are separated from zero as $b_{n_k} \to \infty$. Then, the using of (21),(22),(26),(27) makes it possible to show that there exist equal limits

$$\lim \tilde{v}^*_{\bar{\lambda}_0}(b_{n_k}) P_1\left(\bar{\lambda}_0, b_{n_k}\right) = \lim v^*_{\bar{\lambda}_0}(b_{n_k}) P_1\left(\bar{\lambda}_0, b_{n_k}\right) = P\left(\bar{\lambda}_0\right) v^*_{\bar{\lambda}_0} P_1\left(\bar{\lambda}_0\right),$$

and therefore the right-hand side in (28) converges to 0 as $\lambda = \lambda_0$.

If for the c.m. $M(\lambda)$ of the system (1), (18) on $(0, \infty)$ there is the equality in (6) for $\lambda = \lambda_0$, then there is the equality in (6) for $\Im\lambda\Im\lambda_0 > 0$ in view of Lem. 3.2 [14]. Thus $\mathrm{n}^0 1^0$ is proved.

$\mathbf{2}^0$ follows from the formula (1.70) [13] and from the concluding arguments of the proof of (17) from $n^\circ 1$. The theorem is proved.

Notice that $\mathrm{n}^0 2^0$ of Th. 2 or Remark 8 shows that in $\mathrm{n}^0 1^0$ it is impossible to omit the condition of the equality in (6). Also notice that, as it is seen from the example in the proof of Remark 10, the equality in (6) is possible for the c.m. (13) of the system (1) on $(0, \infty)$, although there exists the sequence of c.m. $M(\lambda, b_{n_k})$ for which there is no equality in (6), but (16) is valid.

The remark below follows from (25), (26).

Remark 4. *Let the matrix $Q(t)$ in the system (1) be constant, moreover $Q(t) = J = J^{-1}$. Then:*

1^0. *Let us choose a basis such that $P(\lambda_0) = diag(I_m, O_{n-m})$ if $\Im\lambda_0 > 0$ and $P_1(\bar{\lambda}_0) = diag(I_{m_1}, O_{n-m_1})$ if $\Im\lambda_0 < 0$. Then the statement that for the c.m. $\tilde{M}(\lambda)$ of the system (1) on $(0, \infty)$ with nonreal $\lambda = \lambda_0$ there is the equality in the condition (6) is equivalent to the statement that all contractions v_λ in the representation $\tilde{M}(\lambda)$ (22) with $\lambda = \lambda_0$ are described by the formulae:*

$$v_{\lambda_0} = \left(\vec{v}_1^1, \ldots, \vec{v}_m^1, \vec{x}_1^1, \ldots, \vec{x}_{n-m}^1\right), \ m = \mathrm{rg}P(\lambda_0), \ \Im\lambda_0 > 0, \tag{30}$$

$$v_{\bar{\lambda}_0} = \left(\vec{v}_1, \ldots, \vec{v}_{m_1}, \vec{x}_1, \ldots, \vec{x}_{n-m_1}\right)^*, \ m_1 = \mathrm{rg}P_1(\bar{\lambda}_0), \ \Im\lambda_0 < 0, \tag{31}$$

where $\vec{v}_j^1(\vec{v}_j)$ are some fixed orthonormal eigenvectors columns of $P_1(\lambda_0)$ $(P(\bar{\lambda}_0))$ that correspond to the eigenvalue 1, $\vec{x}_k^1(\vec{x}_k)$ are arbitrary vector columns such that v_{λ_0} is the contraction.

2^0. *Let for the c.m. $\tilde{M}(\lambda)$ (22) of the system (1) on $(0, \infty)$ there be the equality in the condition (6) for $\lambda = \lambda_0$, $\Im\lambda_0 > 0$, and $\lambda = \bar{\lambda}_0$*⋆*.*

This is equivalent to the simultaneous fulfilment of the following three equalities:

$$v_{\lambda_0}P(\lambda_0) = P_1(\lambda_0)v_{\lambda_0}, P(\lambda_0) = v^*(\lambda_0)P_1(\lambda_0)v(\lambda_0),$$
$$P_1(\lambda_0) = v(\lambda_0)P(\lambda_0)v^*(\lambda_0),$$

that is, in its turn, equivalent to the fact that all contractions v_λ in the representation $\tilde{M}(\lambda)$ (22) with $\lambda = \lambda_0$ are described by the formula

$$v_{\lambda_0} = U_1^*(\lambda_0)\,diag(U, V)\,U(\lambda_0), \tag{32}$$

where U is some fixed unitary $m\times m$-matrix, V is an arbitrary $(n-m)\times(n-m)$-contraction; $U_1(\lambda), U(\lambda)$ are such unitary matrixces that

$$P_1(\lambda) = U_1^*(\lambda)\,diag(I_m, O_{n-m})\,U_1(\lambda), \ P(\lambda) = U^*(\lambda)\,diag(I_m, O_{n-m})\,U(\lambda).$$

For $V = I_{n-m}$, (32) passes into the formula obtained in [9] under the condition that v_{λ_0} is unitary.

Theorem 3. *If in the system (1) $H_\lambda(t)$ =(18), and for the c.m. $M(\lambda)$ of this system on $(0, \infty)$ in the condition (6) there is the equality for some nonreal $\lambda = \lambda_0$, then there should be the sequence of c.m.'s $M(\lambda, b_{n_k})$ for which there is the equality in (6) for $\Im\lambda \neq 0$, and (16) is valid.*

⋆As it can be obtained from [9, 14] and Lem. 3.2 from [13], $rgR_1(\lambda) = rgR(\lambda) = m$, $\Im\lambda > 0$.

P r o o f. Taking into account the beginning of the proof of Th. 2, one can see that it is sufficient to prove Th. 3 for the system (20). For definiteness, set $\Im\lambda_0 > 0$.

If for the c.m. $\tilde{M}(\lambda)$ of the system (20) on $(0, \infty)$ in (6) there is the equality for $\lambda = \lambda_0$, then in view of Remark 4 the matrix v_{λ_0} in the representation (22) has the form (30) as $\lambda = \lambda_0$ if the basis is chosen such that $P(\lambda_0) = diag(\mathcal{I}_m, \mathrm{O}_{n-m})$.

Choose the vectors $\vec{x}_1^1, \ldots, \vec{x}_{n-m}^1$ so that the matrix v_{λ_0} (30) becomes unitary.

Consider the matrix $\tilde{M}(\lambda_0, b) = \tilde{M}(\lambda_0, b, v_{\lambda_0})$ (21) with the unitary matrix v_{λ_0} and the boundary value problem (7)–(10) for the system (20) with

$$\mathcal{M}_\lambda = \tilde{\mathcal{P}}(\lambda_0, b) - I, \ \mathcal{N}_\lambda = \tilde{X}_\lambda(b)\tilde{\mathcal{P}}(\lambda_0, b), \ \tilde{\mathcal{P}}(\lambda_0, b) = \tilde{\mathcal{P}}(\lambda_0, b, v_{\lambda_0}), \quad (33)$$

where $\tilde{\mathcal{P}}(\lambda_0, b)$ and $\tilde{M}(\lambda_0, b)$ are joined with $G = J$ by (13).

From [14] it is known:

1) The c.m. $\tilde{M}(\lambda, b)$ of this problem as $\lambda = \lambda_0$ coincides with $\tilde{M}(\lambda_0, b, v_{\lambda_0})$ and, therefore, $\tilde{M}(\lambda_0, b) \xrightarrow[b\to\infty]{} \tilde{M}(\lambda_0)$.

2) Since v_{λ_0} is unitary, then in the condition (6)for the c.m. $\tilde{M}(\lambda, b)$ there is the equality as $\lambda = \lambda_0$ (and therefore as $\Im\lambda \neq 0$ by [9, 13]).

By virtue of $\mathrm{n}^0 1$ of Remark 4 in (6) for the c.m. $\tilde{\tilde{M}}(\lambda) = \lim \tilde{M}(\lambda, b_{n_k})$ there is the equality as $\lambda = \lambda_0$ and, consequently, in (6) (see the proof of $\mathrm{n}^0 1$ Th. 2) there is the equality for it as $\Im\lambda\Im\lambda_0 > 0$.

By virtue of Lem. 3.2 from [13], the operators $R_\lambda\left(\tilde{\mathcal{P}}(\lambda, b)\right)$ and $R_\lambda\left(\tilde{\mathcal{P}}(\lambda)\right)$, $R_\lambda\left(\tilde{\tilde{\mathcal{P}}}_\lambda\right)$ corresponding to $\tilde{M}(\lambda, b)$ and $\tilde{M}(\lambda)$, $\tilde{\tilde{M}}(\lambda)$, respectively, are the resolvents of maximal symmetric relations as $\Im\lambda\Im\lambda_0 > 0$ generated by the system (20) in $L^2_{\tilde{H}}(0, b)$ and $L^2_{\tilde{H}}(0, \infty)$, respectively. Therefore $\tilde{M}(\lambda) = \tilde{\tilde{M}}(\lambda)$, and from the strong convergence of the resolvents $R_{\lambda_0}\left(\tilde{\mathcal{P}}(\lambda_0, b_{n_k})\right) \to R_{\lambda_0}\left(\tilde{\tilde{\mathcal{P}}}(\lambda_0)\right)$ in $L^2_{\tilde{H}}(0, \infty)^\star$ taking place due to Th. 2, there follows their strong convergence as $\Im\lambda\Im\lambda_0 > 0$. In view of (4) we obtain $\tilde{M}(\lambda, b_{n_k}) \to \tilde{M}(\lambda)$ as $\Im\lambda\Im\lambda_0 > 0$, and it is valid for $\Im\lambda \neq 0$ since for any c.m. $M(\lambda) = M^*(\bar{\lambda})$. The theorem is proved.

The remark below follows from the proofs of Ths. 2, 3 and from Remark 4.

Remark 5. *Let for the c.m.* $\tilde{M}(\lambda)$ *(22) of the system (20) on* $(0, \infty)$ *the contraction* v_{λ_0}, $\Im\lambda_0 > 0$ *or* $v_{\bar{\lambda}_0}$, $\Im\lambda_0 < 0$ *be unitary, but not having the representation of the form (30) or (31)* ** *(such c.m. exists in view of [14]). Then for the c.m. corresponding to the boundary value problem of the type (20), (10), (33),*

*Here we set $R_\lambda\left(\tilde{\mathcal{P}}(\lambda, b)\right)\tilde{f} = 0$ if $\tilde{f} \in L^2_{\tilde{H}}(b, \infty)$

**Then, if the system (1) corresponds to the Sturm–Liouville scalar equation or if for this system the condition of Lem. 2 holds as $\Im\lambda_0\Im\mu_0 < 0$, then the boundary condition of the type (33) for the corresponding system (20) with this unitary matrix v_{λ_0} is not separated.

there is the equality in (6) as $\Im\lambda \neq 0$. *However, for the corresponding limit c.m. this is not true as* $\Im\lambda\Im\lambda_0 > 0$.

Theorems 2 (in another form), 3 and Remark 5 are announced in [8] for the limits of the c.m. of the self-adjoint regular scalar differential operators of even order with real coefficients.

Lemma 2. *Let for some nonreal* $\lambda = \mu_0$ *the number of linearly independent solutions of the homogeneous system (1) that belong to* $L^2_{w_{\mu_0}}(0,\ \infty)$ *is equal to the number of negative eigenvalues of the matrix* $\Im\mu_0 G$ *(i.e., it is minimal in view of [16]).*

Then for any nonreal λ, *for any c.m.* $M(\lambda)$ *(13) of the system (1) on the semi-axis* $(0,\ \infty)$ *in the representation (13) the operator* $\mathcal{P}(\lambda)$ *is the projection.*

P r o o f. In view of [14, 16] the conditions of the lemma hold for any λ such that $\Im\lambda\Im\mu_0 > 0$. For definiteness, set $\Im\mu_0 > 0$. From the definition of c.p. it follows (see [13]) that for $\Im\lambda > 0$ the initial condition $x(0)$ for the solution of the homogeneous system (1) that belongs to $L^2_{w_\lambda}(0,\infty)$ satisfies $x(0) \in \mathcal{P}_+(\lambda)\mathcal{H}$, where $\mathcal{P}_+(\lambda)$ is the c.p. (4.17) from [13] of this system on $(0,\ \infty)$.

We extend the coefficients of the system (1) to the left semi-axis $(-\infty,\ 0)$ in such a way that

$$Q(t) = G,\ \ H_\lambda(t) = \lambda I,\ \ t < 0. \tag{34}$$

If for $\Im\lambda > 0$ this system with $f(t) = 0$ has the solution from $L^2_{w_\lambda}(R^1)$, then, in view of (34), its initial condition satisfies $x(0) \in \mathbf{P}_+\mathcal{H} \bigcap \mathcal{P}_+(\lambda)\mathcal{H}$, where $\mathbf{P}_+$ is the Riesz projection of the operator G that corresponds to its positive spectrum. But the subspace $\mathbf{P}_+\mathcal{H}$ is G-positive, and $\mathcal{P}_+(\lambda)\mathcal{H}$ is G-negative [13]. Therefore $x(0) = 0$.

Hence in view of $n^0 6$ of Th. 1.1 [13] the c.m. $\tilde{M}(\lambda)$ of the system (1) extended by means of (34) is unique and in view of [13] equal to (13), with $\mathcal{P}(\lambda)$ being substituted by

$$\begin{array}{c} \tilde{\mathcal{P}}(\lambda) = \mathcal{P}_+(\lambda)\left(\mathcal{P}_+(\lambda) + \mathbf{P}_\pm\right)^{-1},\ \ \mathbf{P}_- = I - \mathbf{P}_+, \\ \det\left(\mathcal{P}_+(\lambda) + \mathbf{P}_\pm\right) \neq 0, \end{array} \quad \pm\Im\lambda > 0. \tag{35}$$

In $L^2_{\Im\tilde{H}_{\lambda_0}}(R^1)$, where $\Im\lambda_0 > 0$, consider the minimal closed relation L_0 that corresponds to the differential expression

$$i\left(\left(\tilde{Q}(t)\,y(t)\right)' + \tilde{Q}(t)\,y'(t)\right) - Re\tilde{H}_{\lambda_0}(t)\,y(t),$$

where $\tilde{Q}(t)$, $\tilde{H}_\lambda(t)$ are the coefficients of the system (1) extended by means of (34).

The analog of $X_\lambda(t)$ for the system (1), (34) on the axis we denote by $\tilde{X}_\lambda(t)$. In view of [17] (see also [18, 19]) and [13] the operator in $L^2_{\tilde{H}_{\lambda_0}}\left(R^1\right)$

$$\tilde{x}_{\bar{\lambda}_0}(t) = \tilde{R}_{\bar{\lambda}_0}\tilde{f}$$

$$= \int_{-\infty}^{\infty} \tilde{X}_{\bar{\lambda}_0}(t)\left\{\tilde{M}\left(\bar{\lambda}_0\right) - \frac{1}{2}sgn\,(s-t)\,(iG)^{-1}\right\}\tilde{X}^*_{\bar{\lambda}_0}(s)\,\Im\tilde{H}_{\lambda_0}(s)\,\tilde{f}(s)\,ds$$

is equal to $(L_0+i)^{-1}$. In view of Lagrange's formula $\forall\tilde{f}(s) \in L^2_{\tilde{w}_{\lambda_0}}\left(R^1\right)$

$$\lim_{(s,t)\uparrow R^1}\left(U\left[\tilde{x}_{\bar{\lambda}_0}(t)\right] - U\left[\tilde{x}_{\bar{\lambda}_0}(s)\right]\right) = 0.$$

It is obvious that for any $\tilde{f}(s) \in L^2_{\tilde{w}_{\lambda_0}}\left(R^1\right)$ with compact support

$$\lim_{s\to-\infty}\tilde{x}_{\bar{\lambda}_0}(s) = 0,$$

and therefore for any $f(s) \in L^2_{w_{\lambda_0}}(0,\ \infty)$ with compact support

$$\lim_{t\to\infty} U\left[\tilde{x}_{\bar{\lambda}_0}(t)\right] = 0,$$

and hence in view of (35), we have

$$\lim_{t\to\infty}\mathcal{P}^*_+\left(\bar{\lambda}_0\right)X^*_{\bar{\lambda}_0}(t)\,Q(t)\,X_{\bar{\lambda}_0}(t)\,\mathcal{P}_+\left(\bar{\lambda}_0\right) = 0. \tag{36}$$

Let $M(\lambda)$ be an arbitrary c.m. of the system (1) on $(0,\ \infty)$. In view of Theorem 4.4 from [13] for the matrix $\mathcal{P}(\lambda)$ corresponding to $M(\lambda)$ by (13) the following holds:

$$\mathcal{P}\left(\bar{\lambda}_0\right)\mathcal{H} \subseteq \mathcal{P}_+\left(\bar{\lambda}_0\right)\mathcal{H}, \tag{37}$$

whence

$$\mathcal{P}\left(\bar{\lambda}_0\right) = \mathcal{P}_+\left(\bar{\lambda}_0\right)\mathcal{P}\left(\bar{\lambda}_0\right), \tag{38}$$

and therefore for the matrix $\mathcal{P}(\lambda)$ that corresponds to $M(\lambda)$ there is the equality of the type (36). Thus for the c.m. $M(\lambda)$ the condition (6) is separated for $\lambda = \bar{\lambda}_0$, and in view of Th. 1 $\mathcal{P}\left(\bar{\lambda}_0\right) = \mathcal{P}^2\left(\bar{\lambda}_0\right)$. Consequently, $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$ as $\Im\lambda \neq 0$ because λ_0 is arbitrary and in view of (3.6) from [13]. Lemma 2 is proved.

Let the condition of Lemma 2 hold. Then, in view of [13] and (38), for the system (1) on the semi-axis $(0,\ \infty)$ the Weyl function $K_+(\lambda) = K^*_+\left(\bar{\lambda}\right) \in$

$B\left(P_{\mp}\mathcal{H},\ P_{\pm}\mathcal{H}\right)$, $\pm\Im\lambda > 0$ is unique, where $P_{\pm}$ are such complementary orthoprojections that $\Gamma^{*}G\Gamma = P_{+} - P_{-}$. In view of Theorem 4.4 from [13] the following remark is valid.

Remark 6. *Let the condition of Lemma 2 hold. Then the formula*

$$\mathcal{P}\left(\lambda\right) = \Gamma\left(P_{\mp} + \mathrm{K}_{+}\left(\lambda\right)P_{\mp}\right)\left(I_{\mp} - U_{-}\left(\lambda\right)\mathrm{K}_{+}\left(\lambda\right)\right)^{-1}\left(P_{\mp} - U_{-}\left(\lambda\right)P_{\pm}\right)\Gamma^{-1},$$
$$\pm\Im\lambda > 0, \tag{39}$$

where $I_{\pm}$ are identical operators in $P_{\pm}\mathcal{H}$, establishes a one-to-one correspondence between the c.p.'s $\mathcal{P}\left(\lambda\right)$ of the system (1) on semi-axis $(0,\ \infty)$ and the contractions $U_{-}\left(\lambda\right) = U_{-}^{*}\left(\bar{\lambda}\right) \in B\left(P_{\pm}\mathcal{H},\ P_{\mp}\mathcal{H}\right)$, $\pm\Im\lambda > 0$ that analytically depend on nonreal λ. Moreover, $x_{\lambda}\left(t\right)$ (5), (13), (39) for any vector function $f\left(t\right) \in L^{2}_{w_{\lambda}}\left(0,\ \infty\right)$ with compact support is the unique solution of the system (1) that belongs to $L^{2}_{w_{\lambda}}\left(0,\ \infty\right)$ and satisfies the boundary condition[⋆]

$$\exists h = h\left(\lambda,\ f\right) \in \mathcal{H}:\ x\left(0\right) = \Gamma\left(P_{\pm} + U_{-}\left(\lambda\right)P_{\pm}\right)h,\ \pm\Im\lambda > 0. \tag{40}$$

Now let us consider the boundary problem (1), (10) as $b \geq \beta$ (see (4)) with

$$\mathcal{M}_{\lambda} = I, \quad \mathcal{N}_{\lambda} = \Gamma\left(b\right)U_{\lambda}\left(b\right), \tag{41}$$

where $\Gamma\left(b\right)$ is an arbitrary matrix such that

$$\Gamma^{*}\left(b\right)Q\left(b\right)\Gamma\left(b\right) = G, \tag{42}$$

the matrix $U_{\lambda}\left(b\right)$, analytically depending on the nonreal λ, is such that[⋆⋆]

$$\left(\Im\lambda\right)\left(U_{\lambda}^{*}\left(b\right)GU_{\lambda}\left(b\right) - G\right) \leq 0, \quad U_{\bar{\lambda}}^{*}\left(b\right)GU_{\lambda}\left(b\right) = G \tag{43}$$

In view of Theorem 2.6 [13] this problem satisfies the conditions of Remark 1 and therefore its c.m. is equal to (13), where[⋆⋆⋆]

$$\mathcal{P}\left(\lambda\right) = \left(I - Z_{\lambda}\left(b\right)\right)^{-1} \overset{def}{=} \mathcal{P}\left(\lambda,\ b\right) \tag{44}$$

hence

$$I - \mathcal{P}\left(\lambda,\ b\right) = \left(Z_{\lambda}\left(b\right) - I\right)^{-1}Z_{\lambda}\left(b\right), \tag{45}$$

[⋆] This condition can be self-adjoint as $\lambda = \lambda_0$ only if the signature of the matrix $\Im\lambda_0 G$ is nonpositive. Otherwise it is accumulative or dissipative, but not necessarily strictly.

[⋆⋆] If for $\Im\lambda \neq 0$ (43) holds and for some nonreal λ and $b = b_0$ there is the equality in (43), then $U_{\lambda}\left(b_0\right)$ does not depend on λ [8] (see also [13] as $\dim\mathcal{H} = \infty$).

[⋆⋆⋆] Below we denote $\mathcal{P}\left(\lambda,\ b\right)$ (in contrast to Lem. 1 and Ths. 2, 3) only (44).

where

$$Z_\lambda(b) = U_\lambda^{-1}(b)\,\Gamma^{-1}(b)\,X_\lambda(b)\,. \tag{46}$$

In view of (3), (42), (43)

$$\frac{1}{\Im\lambda}\left[Z_\lambda^*(b)\,GZ_\lambda(b) - G\right] \geq 2\Delta_\lambda(0,\ b)\,, \tag{47}$$

moreover, if there is the equality in (43), then it is in (47), too.

It follows from (4) and (47) that the matrix $Z_\lambda(b)$ (46) is unitary dichotomic if the matrix G is indefinite*. We denote by $\mathcal{P}(Z_\lambda(b))$ the Riesz projection of the matrix $Z_\lambda(b)$ that corresponds to its spectrum lying inside the unit circle. Since in view of [20, p. 64] the subspaces $\mathcal{P}(Z_\lambda(b))\,\mathcal{H}$ and $(I - \mathcal{P}(Z_\lambda(b)))\,\mathcal{H}$ are $\Im\lambda G$-negative and $\Im\lambda G$-positive, respectively, then in view of (2), (43) $\mathcal{P}(Z_\lambda(b))$ is the c.p. of the system (1) on $(0,\ b)$.

From Lemma 1 there follows

Lemma 3. *Any sequence $b_n \uparrow \infty$ contains such subsequences $\{b_{n_k}\}, \{b_{n'_k}\}$ that for any nonreal λ the following limits exist:*

$$\lim \mathcal{P}(\lambda,\ b_{n_k}) = \mathcal{P}(\lambda)\,, \tag{48}$$

$$\lim \mathcal{P}\left(Z_\lambda\left(b_{n'_k}\right)\right) = \mathcal{Q}(\lambda) = \mathcal{Q}^2(\lambda)\,. \tag{49}$$

The matrix functions $M(\lambda)$ (13) with $\mathcal{P}(\lambda)$ (48) and $\mathcal{P}(\lambda) = \mathcal{Q}(\lambda)$ (49) are the c.m.'s of the system (1) on $(0,\ \infty)$.

The following proposition follows from Th. 1 and (44).

Proposition. *Boundary condition (6), that corresponds to the limit c.m. $M(\lambda)$ (13), (48) is separated for nonreal $\lambda = \mu_0$ if and only if*

$$\forall 0 \neq f \in \mathcal{H}:\ \|(Z_{\mu_0}(b_{n_k}) + Z_{\mu_0}^{-1}(b_{n_k}))f\| \to \infty. \tag{50}$$

Example 1. Let in the system (1) $Q(t) = G, H_{\mu_0}(t) \in L^1(0,\infty), \Im\mu_0 \neq 0$. Then in view of [20, p. 166] $\exists \lim\limits_{t\to\infty} X_{\mu_0}(t) = X$, $\det X \neq 0$, and (50) holds if in Lemma 3 $b_n = f(n), U_{\mu_0}(b_n) = X^{-n}$, where $f(n)$ is such a function that $(\|X\|^n + \|X^{-1}\|^n)\|X_{\mu_0}(f(n)) - X\| \to 0$. Here $\mathcal{P}(\lambda)$(48)$=\mathcal{P}(X)$. $\mathcal{P}(X)$ is the analog of $\mathcal{P}(Z_\lambda(b))$ for X. In this example the condition of Lem. 2 does not hold as $\lambda = \mu_0$ because $\forall h \in \mathcal{H}:\ X_{\mu_0}(t)h \in L^2_{w_{\mu_0}}(0,\infty)$.

*Further for simplicity it is considered to be true. In the case of definite G the statements and proofs are modified in a obvious way.

Lemma 4. *Let the condition of Lemma 2 hold and in Lemma 3 let one of the subsequences* $\{b_{n_k}\}$, $\{b_{n'_k}\}$ *be a subsequence of another one. Then in (48), (49)*

$$\mathcal{P}(\lambda) = \mathcal{Q}(\lambda). \tag{51}$$

P r o o f. We denote the subsequence that is a subsequence of another one by $\{\beta_n\}$. Since in view of Lem. 2 $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then

$$\|\mathcal{P}(\lambda, \beta_n)(I - \mathcal{P}(\lambda, \beta_n))\| \to 0. \tag{52}$$

Thus in view of (44), (45)

$$\left\|(I - Z_\lambda(\beta_n))^{-2} Z_\lambda(\beta_n)\right\| \to 0, \tag{53}$$

and therefore the spectrum

$$\sigma\left((I - Z_\lambda(\beta_n))^{-2} Z_\lambda(\beta_n)\right) \to 0. \tag{54}$$

From the theorem on a mapping of spectrum and with the unitary dichotomy of $Z_\lambda(b)$ being taking into account, we obtain

$$\max\{|\mu|\,|\mu \in \sigma_i(Z_\lambda(\beta_n))\} \to 0, \ \min\{|\mu|\,|\mu \in \sigma_e(Z_\lambda(\beta_n))\} \to \infty, \tag{55}$$

where $\sigma_i(Z_\lambda(\beta_n))$ and $\sigma_e(Z_\lambda(\beta_n))$ are the parts of spectrum $Z_\lambda(\beta_n)$ lying inside and outside the unit circle, respectively.

One has

$$\mathcal{P}(\lambda, \beta_n) - \mathcal{P}(Z_\lambda(\beta_n)) = A_n + B_n, \tag{56}$$

where

$$A_n = (\mathcal{P}(\lambda, \beta_n) - I)\,\mathcal{P}(Z_\lambda(\beta_n)), \quad B_n = \mathcal{P}(\lambda, \beta_n)(I - \mathcal{P}(Z_\lambda(b_n))). \tag{57}$$

In view of (52) it follows from (57) that

$$\|\mathcal{P}(\lambda, \beta_n) A_n\| \to 0.$$

Since in view of (55) $\sigma\left(\mathcal{P}(\lambda, \beta_n)\,|_{\mathcal{P}(Z_\lambda(\beta_n))\mathcal{H}}\right) \to 1$, then (for example, in view of [21, p. 42]) there could be found a constant $\delta = \delta(\lambda) > 0$ not depending on n and such that

$$\forall f \in \mathcal{H}: \quad \|\mathcal{P}(\lambda, \beta_n) A_n f\| \geq \delta \|A_n f\|,$$

from which $A_n \to 0$. In the same way one can prove that $B_n \to 0$. The lemma is proved in view of (56).

The first statement in (55) is strengthened by

Remark 7. *There exists such not depending on b constant $K = K(\lambda)$, that*

$$\| Z_\lambda (b) \mathcal{P} (Z_\lambda (b)) \| \le K. \tag{58}$$

Besides if the conditions of Lem. 2 hold, then

$$Z_\lambda (\beta_n) \mathcal{P} (Z_\lambda (\beta_n)) \to 0. \tag{59}$$

P r o o f. In view of (4),(47) and $\Im\lambda G$-nonnegativity of $\mathcal{P}(Z_\lambda (b)) \mathcal{H}$ one has $\exists \delta = \delta(\lambda) > 0 \; \forall f \in \mathcal{H}$:

$$\frac{1}{\Im\lambda} f^* \mathcal{P}^* (Z_\lambda (b)) X_\lambda^* (b) Q (b) X_\lambda (b) \mathcal{P} (Z_\lambda (b)) f \le -\delta \| Z_\lambda (b) \mathcal{P} (Z_\lambda (b)) f \|^2 . \tag{60}$$

The statement (58) is based on the fact that the left-hand side of the inequality (60) is bounded as $\|f\| = 1$ in view of Lem. 1.8, formula (1.77) and Th. 4.2 [13]. The statement (59) follows from (53), (58) and Lem. 1.8 [13] since

$$Z_\lambda (\beta_n) \mathcal{P} (Z_\lambda (\beta_n)) = ((I - Z_\lambda (\beta_n)) \mathcal{P} (Z_\lambda (\beta_n)))^2 (I - Z_\lambda (\beta_n))^{-2} Z_\lambda (\beta_n) . \tag{61}$$

The remark is proved.

From Lemma 4, Remark 7, formula (47) and the fact that $Z_\lambda (b) \mathcal{P} (\lambda,\ b) = \mathcal{P} (\lambda,\ b) - I$ there follows

Corollary 1. *Let the condition of Lemma 2 hold and let $\{\beta_n\}$ be the subsequence from the proof of Lemma 4. Then for any $f(t) \in L^2_{w_\lambda} (0,\ \infty)$ with compact support*

$$\overline{\lim_{\beta_n \to \infty}} \| [R_\lambda (\mathcal{P} (Z_\lambda (\beta_n))) - R_\lambda (\mathcal{P} (\lambda,\ \beta_n))] f(t) \|^2_{L^2_{w_\lambda}(0,\ \beta_n)} \le \frac{1}{2\Im\lambda} (G (I - \mathcal{P}(\lambda)) h,\ (I - \mathcal{P}(\lambda)) h) , \tag{62}$$

where $h = (iG)^{-1} \int\limits_0^\infty X_{\bar\lambda}^* (s) w_\lambda (s) f(s) ds$. If there is the equality in (43), then there is also the equality in (62) with lim *instead of* $\overline{\lim}$.

Remark 8. *For the system (1) with 1-periodic coefficients and with periodic condition (43) the equality (17) does not hold with $b_{n_k} = n$ for any nonreal λ.*

P r o o f. Let $Q(t+1) = Q(t)$, $H_\lambda(t+1) = H_\lambda(t)$ in the system (1) and let $\beta \le 1$ in the condition (4). Then, as shown in [22–24] (see also Th. 5 below), the system (1) satisfies the condition of Lem. 2. Let $\Gamma(n) = U_\lambda(n) = I$, $n = 1, 2\ldots$ in the boundary conditions (41)–(43). Then (48) holds with $b_{n_k} = n$ and $\mathcal{P}(Z_\lambda(n)) = \mathcal{P}(Z_\lambda(1))$ in view of Floke theorem. Therefore $\mathcal{P}(\lambda) = \mathcal{P}(Z_\lambda(1))$ by Lem. 4 with $\beta_n = n$ and, consequently, $\|R_\lambda(\mathcal{P}(Z_\lambda(n))) - R_\lambda(\mathcal{P}(\lambda))\|_{B\left(L^2_{w_\lambda}(0,n)\right)}$ $= 0$.

Therefore in view of Cor. 1 for any $f(t) \in L^2_{w_\lambda}(0,\ \infty)$ with compact support

$$\lim_{n\to\infty} \|[R_\lambda(\mathcal{P}(\lambda)) - R_\lambda(\mathcal{P}(\lambda,\, n))]\, f(t)\|^2_{L^2_{w_\lambda}(0,\,n)}$$
$$= \frac{1}{2\Im\lambda}\left(G\left[I - \mathcal{P}(Z_\lambda(1))\right]h,\ \left[I - \mathcal{P}(Z_\lambda(1))\right]h\right)e$$
$$\ge \delta(\lambda)\left\|I - \mathcal{P}(Z_\lambda(1))\,h\right\|^2,$$

where $\delta(\lambda) > 0$ as $\Im\lambda \ne 0$ in view of (4), (47), [20, p. 64]. The corollary is proved.

It follows from [9, 14, 27] that $\Im M(\lambda)/\Im\lambda > 0$ for limit c.m. (13), (48) which is obtained if in the boundary condition (41)–(43) we set $\Gamma(b) = \boldsymbol{\Gamma}(b)$, $U_\lambda(b) = \Gamma^{-1}(b)\,\tilde{T}(b)\,S$. Here $\tilde{T}(b) = T(b)U(b)$; $T(t), S$ — see (19), $U(t) \in AC_{loc}$ is the J-unitary matrix such that J-module (see, e.g., [24]) of matrix $S^{-1}\Gamma^{-1}(t)X_i(t)S$ is equal to $U^{-1}(t)S^{-1}\Gamma^{-1}X_i(t)S$. But

$$\Im M(\lambda) = G^{-1}\left(\left(I - \mathcal{P}^*(\lambda)\right)G\left(I - \mathcal{P}(\lambda)\right) - \mathcal{P}^*(\lambda)\,G\mathcal{P}(\lambda)\right)G^{-1}.$$

Let the condition of Lem. 2 hold. Then $\mathcal{P}^2(\lambda) = \mathcal{P}(\lambda)$. Since the subspace $\mathcal{P}(\lambda)\,\mathcal{H}$ is maximal $\Im\lambda G$-negative [13], then $rg(I - \mathcal{P}(\lambda))$ is equal to the number of positive eigenvalues of the matrix $\Im\lambda G$. Therefore $rg\left(I - \mathcal{P}^*(\lambda)\right)G\left(I - \mathcal{P}(\lambda)\right)$ is equal to the number of these eigenvalues because $\Im M(\lambda)/\Im\lambda > 0$. Therefore the following remark is valid.

Remark 9. *Let the condition of Lemma 2 hold. Then there exists such a sequence $\beta_n \uparrow \infty$ and such independent from λ matrixes $U_\lambda(\beta_n)$ satisfying (41)–(43) (with the equality in (43)) that the strictly dissipative or accumulative boundary condition (40) at zero corresponds to the limit c.m. $M(\lambda)$ (13), (48), $(b_{n_k} = \beta_n)$.*

Theorem 4 below gives the conditions under which any limit c.m. corresponds to strictly dissipative or accumulative boundary condition at zero (if $\mathcal{P}(\lambda) \ne I$). To prove it the following lemma is necessary.

Lemma 5. *Let for some nonreal $\lambda = \lambda_0$ the sequences $\{\beta_n\}, \{\gamma_n\}$ exist and the constants $\delta > 0$, M, $\mu > 0$ exist such that $\Gamma(\beta_n) = I$, and the following* *conditions hold:*

$$\Delta_{\lambda_0}(0,\,\beta_n) \geq \delta X^*_{\lambda_0}(\gamma_n)\, X_{\lambda_0}(\gamma_n)\,, \tag{63}$$

$$\left| \int\limits_{\gamma_n}^{\beta_n} \left\| Q^{-1}(t) \left(H_{\lambda_0}(t) - \frac{i}{2} Q'(t) \right) \right\| dt \right| \leq M, \tag{64}$$

$$\mu\left(U^*_{\lambda_0}(\beta_n)\, U_{\lambda_0}(\beta_n)\right) \geq \mu, \tag{65}$$

where $\mu(A)$ is the smallest eigenvalue of the matrix $A = A^*$.

Then for $\beta_n \geq \beta$ (see. (4))

$$\frac{1}{\Im\lambda_0}(I - \mathcal{P}^*(Z_{\lambda_0}(\beta_n)))\, G\, (I - \mathcal{P}(Z_{\lambda_0}(\beta_n)))$$
$$> 2\delta\mu e^{-M}(I - \mathcal{P}^*(Z_{\lambda_0}(\beta_n)))\,(I - \mathcal{P}(Z_{\lambda_0}(\beta_n)))\,.$$

P r o o f. We extend the coefficients $Q(t)$ and $H_\lambda(t)$ of the system (1) periodically on $[-\beta_n,\, 0)$ from $[0,\, \beta_n)$ if the period is equal to β_n.

In view of (42) $Q(t) \in AC_{loc}$ and therefore

$$X_\lambda(t) = X_\lambda(s)\, X^{-1}_\lambda(\beta_n)\,,$$

where $t \in [-\beta,\, 0]\,,\ s = t + \beta_n$. Whence in view of (63), (64) $\forall f \in \mathcal{H}$

$$f^*\Delta_{\lambda_0}(-\beta_n,\, 0)\, f = f^* X^{*-1}_{\lambda_0}(\beta_n)\, \Delta_{\lambda_0}(0,\, \beta_n)\, X^{-1}_{\lambda_0}(\beta_n)\, f$$
$$\geq \delta f^* X^{-1*}_{\lambda_0}(\beta_n)\, X^*_{\lambda_0}(\gamma_n)\, X_{\lambda_0}(\gamma_n)\, X^{-1}_{\lambda_0}(\beta_n)\, f \geq \frac{\delta\, \|f\|^2}{\left\| X_{\lambda_0}(\beta_n)\, X^{-1}_{\lambda_0}(\gamma_n) \right\|^2}$$
$$\geq \delta \exp\left\{ - \left| \int\limits_{\gamma_k}^{\beta_k} \left\| Q^{-1}(t) \left(H_{\lambda_0}(t) - \frac{i}{2} Q'(t) \right) \right\| dt \right| \right\} \|f\|^2 \geq \delta e^{-M} \|f\|^2\,,$$

where the inequality before the last one holds true in view of [20, p. 162]. As a result, in view of (47), (65):

$$\frac{1}{\Im\lambda_0}\left[G - Z^{*-1}_{\lambda_0}(\beta_n)\, G Z^{-1}_{\lambda_0}(\beta_n) \right] \geq 2U^*_{\lambda_0}(\beta_n)\Delta(-\beta_n,\, 0)\, U_{\lambda_0}(\beta_n) \geq 2\delta\mu e^{-M} I,$$

whence

$$\frac{1}{\Im\lambda_0}\left[(I - \mathcal{P}^*(Z_{\lambda_0}(\beta_n)))\, G\, (I - \mathcal{P}(Z_{\lambda_0}(\beta_n))) \right.$$
$$\left. - Z^{*-1}_{\lambda_0}(-\beta_n)\, (I - \mathcal{P}^*(Z_{\lambda_0}(\beta_n)))\, G\, (I - \mathcal{P}(Z_{\lambda_0}(\beta_n)))\, Z^{-1}_{\lambda_0}(-\beta_n) \right]$$
$$\geq 2\delta\mu e^{-M}(I - \mathcal{P}^*(Z_{\lambda_0}(\beta_n)))\,(I - \mathcal{P}(Z_{\lambda_0}(\beta_n)))\,.$$

But for $\beta_n \geq \beta$ due to (4), (47) [20, p. 61] we have

$$\exists \delta_n > 0 : \frac{1}{\Im \lambda_0} \left(I - \mathcal{P}^* \left(Z_{\lambda_0}\left(\beta_n\right)\right)\right) G \left(I - \mathcal{P}\left(Z_{\lambda_0}\left(\beta_n\right)\right)\right)$$
$$\geq \delta_n \left(I - \mathcal{P}^* \left(Z_{\lambda_0}\left(\beta_n\right)\right)\right) \left(I - \mathcal{P}\left(Z_{\lambda_0}\left(\beta_n\right)\right)\right).$$

Lemma is proved.

Theorem 4. *Let the condition of Lemma 2 hold and for the sequence $b_n \uparrow \infty$ let the following limits** *exist for all nonreal λ*

$$\lim \mathcal{P}\left(\lambda,\ b_n\right) = \lim \mathcal{P}(Z_\lambda(b_n)) = \mathcal{P}\left(\lambda\right), \tag{66}$$

that is the c.p. by Lem. 3. Let the sequence $\{b_n\}$ contain the subsequence $\{\beta_n\}$, for which the following conditions hold with some nonreal $\lambda = \lambda_0$:

1^0. For $U_{\lambda_0}(b)$ (43) a constant $\mu > 0$ exists such that (65) holds.

2^0. There exists the sequence $\{\gamma_n\}$ and the constants $\delta > 0$ and M such that either

a) $\Gamma(\beta_n) = I$ and (63),(64) hold for sufficiently large n

or

b) in (42) $\Gamma(t) \in AC_{loc}$, $\Gamma(0) = I$, and for sufficiently large n

$$\Delta_{\lambda_0}\left(0,\ \beta_n\right) \geq \delta X^*_{\lambda_0}\left(\gamma_n\right) \Gamma^{*-1}\left(\gamma_n\right) \Gamma^{-1}\left(\gamma_n\right) X_{\lambda_0}\left(\gamma_n\right), \tag{67}$$

$$\left| \int_{\gamma_n}^{\beta_n} \left\| \frac{i}{2} \left(\Gamma^{*'}(t)\, \Gamma^{*-1}(t)\, G - G\Gamma^{-1}(t)\, \Gamma'(t)\right) + \Gamma^*(t)\, H_{\lambda_0}(t)\, \Gamma(t) \right\| dt \right| \leq M. \tag{68}$$

Then for $\Im\lambda\Im\lambda_0 > 0$:

$$\exists \delta_0 = \delta_0(\lambda) > 0 : \quad \frac{1}{\Im\lambda}\left(I - \mathcal{P}^*(\lambda)\right) G \left(I - \mathcal{P}(\lambda)\right) \geq \delta_0 \left(I - \mathcal{P}^*(\lambda)\right)\left(I - \mathcal{P}(\lambda)\right). \tag{69}$$

P r o o f. The proof of (69) with $\lambda = \lambda_0$ and when 1^0, 2^0a) hold, follows from Lems. 4, 5. The proof of (69) with $\lambda = \lambda_0$ and when 1^0, 2^0b) hold, follows from from Lems. 4, 5 and the fact that the c.m.'s of the problems (1), (10), (41)–(43) and

$$iGy'(t) - \left[\frac{i}{2}\left(\Gamma^{*'}(t)\,\Gamma^{*-1}(t)\,G - G\Gamma^{-1}(t)\,\Gamma'(t)\right) + \Gamma^*(t)\,H_\lambda(t)\,\Gamma(t)\right] y(t)$$
$$= \Gamma^*(t)\, w_\lambda \Gamma(t)\, g(t) \quad \exists h = h(\lambda,\ g) : \ \ y(0) = h,\ \ y(b) = U_\lambda(b)\, h$$

*That are equal by Lem. 4.

coincide. It follows from (69), ($\lambda = \lambda_0$) that the contraction $U_{-}(\lambda)$ in (40) is strict as $\lambda = \lambda_0$. Therefore, in view of [26, p. 210], this contraction is strict for any λ such that $\Im\lambda\Im\lambda_0 > 0$. Thus (69) is valid for any of these λ in view of (40). The theorem is proved.

Remark 10. *If such sequences $\{\beta_n\}, \{\gamma_n\}$ that the conditions (63) and (64) (or (67) and (68)) hold simultaneously do not not exist, then it may happen that $(I - \mathcal{P}^*(\lambda))\, G\,(I - \mathcal{P}(\lambda)) = 0$ for self-adjoint or even nonself-adjoint boundary condition (10), (41) for a sequence of regular boundary problems not depending on b.*

P r o o f. Let in the system (1)

$$Q(t) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad H_\lambda(t) = \lambda H(t),$$

$$0 \le H(t) = \begin{cases} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, & 0 \le t < 1, \\ \begin{pmatrix} 0 & 0 \\ 0 & A_n \end{pmatrix}, & 1 \le n \le t < n+1, \end{cases}$$

where $B > 0$, $\sum\limits_{n=1}^{\infty} A_n > 0$. It can be directly verified (it also follows from [7]) that the condition of Lem. 2 holds for this equation if and only if $\sum\limits_{n=1}^{\infty} A_n = \infty$. We have for $\mathcal{N}_\lambda = \begin{pmatrix} 1 & -\lambda A_0 \\ 0 & 1 \end{pmatrix}$, where $A_0 \ge 0$:

$$I - \mathcal{P}(\lambda) = \lim_{n \to \infty} \left[I + (Z_\lambda(n+1) - I)^{-1} \right] = \begin{pmatrix} 0 & -\frac{B}{\lambda} \\ \frac{1}{\lambda \sum\limits_{j=0}^{\infty} A_j} & 0 \end{pmatrix}.$$

Thus with $b_{n_k} = n+1$ for the limit $\mathcal{P}(\lambda)$ (48) we have $(I - \mathcal{P}^*(\lambda))G(I - \mathcal{P}(\lambda)) = 0$ if and only if $\sum\limits_{n=1}^{\infty} A_n = \infty$ (for self-adjoint boundary condition (41) ($A_0 = 0$) or for nonself-adjoint boundary condition (41) (as $A_0 > 0$)). Thereby for the considered equation the simultaneous fulfilment of the conditions (63), (64) is impossible as $\sum\limits_{n=1}^{\infty} A_n = \infty$, although (63) holds when for example $\gamma_n = 0$, while (64) holds when for example A_n and $|\beta_n - \gamma_n|$ are bounded simultaneously. The remark is proved.

We will illustrate Th. 4 by using the perturbed almost periodic systems as an example.

Theorem 5. *Let in the system (1) the matrix-function $Q(t)$ be uniformly almost periodic, $\inf\limits_t |\det Q(t)| > 0$.*

Let for some nonreal λ_0

$$H_{\lambda_0}(t) = A(t) + B(t), \text{ where } \Im A(t)/\Im\lambda_0 \geq 0. \tag{70}$$

Suppose the matrix function $A(t)$ *to be uniformly almost periodic, and the normalized at zero on* I *fundamental matrix of the system*

$$\frac{i}{2}\left(Q(t)\,y(t)\right)' + Q(t)\,y'(t) = A(t)\,y(t) \tag{71}$$

to have a representation of the Floke type

$$Y(t) = Z(t)\,e^{\Lambda t}, \tag{72}$$

where the matrix function $Z(t)$ *is uniformly almost periodic,* $\inf_t |\det Z(t)| > 0$*, and*

$$\exists \beta_0 > 0: \quad \int_0^{\beta_0} Y^*(t)\left[\frac{\Im A(t)}{\Im\lambda_0}\right] Y(t)\,dt > 0. \tag{73}$$

Let

$$\int_0^{\infty} t^{2q-2}\,\|B(t)\|\,dt < \infty, \tag{74}$$

where q *is equal to the maximal order of the Jordan cells of matrix* Λ *in representation (72)***.*

Then for the system (1):

$\mathbf{1^0}$**.** *The condition of Lem. 2 holds for any* μ_0 *such that* $\Im\mu_0\Im\lambda_0 > 0$*.*

$\mathbf{2^0}$**.** *The condition (63) in* $n^0 2^0$ *of Th. 4 holds for any sequence* $\{\beta_n\} \uparrow \infty$ *if* $\{\gamma_n\} \uparrow \infty$ *is such a sequence of positive common* ε*-almost periods of matrixes* $A(t)$ *and* $Z(t)$*, that* $\beta_n - \gamma_n \geq \beta_0$ *(see (73)),* ε *is sufficiently small. For these* β_n *and* γ_n *the condition (67) in* $n^0 2^0$ *of Th. 4 also holds if the norms* $\|\Gamma(\gamma_n)\|$ *are bounded.*

Besides, if the difference $\beta_n - \gamma_n$ *is bounded, then all other conditions in* $n^0 2^0$ *of Th. 4 also hold*

in the case a) if we additionally require in (42) $\Gamma(\beta_n) = I$ *and the norms* $\|Q'(t)\|$ *to be bounded;*

in the case b) if we additionally require in (42) $\Gamma(t) \in AC_{loc}$*,* $\Gamma(0) = I$ *and the norms* $\|\Gamma(t)\|$*,* $\|\Gamma'(t)\|$ *to be bounded on the semi-axis* $(0, \infty)$*.*

---

*As it is seen from (80), obtained below in the proof of Th. 5, the fulfilment of the condition (4) for the equation (1) follows from (72)–(74).

P r o o f. It follows from (71),(72),(74) and Yakubovich's Theorem [28, p. 383] (see also [20, p. 275]) that

$$X_{\lambda_0}(t) = Z(t)(I + S(t))\, e^{\Lambda t} C, \tag{75}$$

where the matrixes $S(t) \to 0$ as $t \to \infty$.

For definiteness, set $\Im\lambda_0 > 0$.

Let us prove 1^0. In view of the equality of the type (3) for the equation (71) and the conditions (72), (73) we have

$$e^{\Lambda t} Z^*(t)\, Q(t)\, Z(t)\, e^{\Lambda t} - G > 0, \quad t \geq \beta_0. \tag{76}$$

In view of (76): $\sigma(e^{\Lambda\tau} \cap \{\lambda \in \mathbb{C} : \ |\lambda| = 1\} = \varnothing$, where $\beta_0 \leq \tau$ is the common ε-almost period of $Q(t)$ and $Z(t)$ for sufficiently small ε. Therefore $\sigma(e^{\Lambda} \cap \{\lambda \in \mathbb{C} : \ |\lambda| = 1\} = \varnothing$. Let us denote the invariant subspaces of e^{Λ} that correspond to the parts of its spectrum lying inside and outside the unit circle by $\mathcal{H}_i$ and $\mathcal{H}_e$, respectively. We denote the corresponding Riesz projections of the matrix e^{Λ} by $\mathcal{P}_i$ and $\mathcal{P}_e$. If $f \in \mathcal{P}_i\mathcal{H}$, then $e^{\Lambda t} f \to 0$ as $t \to \infty$. Therefore $(Gf, f) < 0$ for $f \neq 0$ in view of (76). Therefore $\dim\mathcal{P}_i\mathcal{H}$ does not exceed the number of negative eigenvalues of the matrix G.

On the other hand, in view of (70), (72), (75) $\forall \tau > 0$:

$$\Delta_{\lambda_0}(\alpha, \alpha + \tau) = C^* e^{\Lambda^* \alpha} \int_0^\tau e^{\Lambda^* t}(I + S^*(t + \tau)) Z^*(t + \tau)[\Im A(t + \tau)/\Im\lambda_0$$
$$+ \Im B(t + \tau)/\Im\lambda_0] Z(t + \tau)(I + S(t + \tau)) e^{\Lambda t} dt e^{e^{\Lambda\alpha}} C. \tag{77}$$

Now let $0 < \tau$ be a fixed and sufficiently large ε-almost period common for $Z(t)$ and $A(t)$, where ε is sufficiently small. In view of boundness of $A(t), Z(t), S(t)$ and in view of (74) there exists a constant δ_1, and also in view of (73) there exists a constant $\delta_2 > 0$ such that due to (77) we have

$$\delta_1 \sum_{n=0}^{\infty} \|e^{\Lambda n\tau} C f\|^2 \geq f^* \Delta_{\lambda_0}(0, \infty) f \geq \delta_2 \sum_{n=0}^{\infty} \|e^{\Lambda n\tau} C f\|^2. \tag{78}$$

It follows from (78) that

$$f^* \Delta_{\lambda_0}(0, \infty) f = \infty, \text{ if } 0 \neq f \in C^{-1}\mathcal{P}_e\mathcal{H},$$
$$f^* \Delta_{\lambda_0}(0, \infty) f < \infty, \text{ if } 0 \neq f \in C^{-1}\mathcal{P}_i\mathcal{H},$$

Therefore the number of linearly independent solutions of the homogeneous equations (1), (71)–(74), belonging to $L^2_{W_{\lambda_0}}(0, \infty)$, is equal to $\dim\mathcal{P}_i\mathcal{H}$, i.e., it does not exceed the number of negative eigenvalues of the matrix G, but in view of [16] the number of these solutions is not less than the number of these eigenvalues. The statement 1^0 is proved in view of [16].

Let us prove 2^0. We check, for example, the fulfilment of condition (67) of Theorem 4. Using (70), (74), (77) we have

$$\Delta_{\lambda_0}(0,\beta_n) \geq \int\limits_{\gamma_n}^{\beta_n} X^*_{\lambda_0}(t) w_{\lambda_0}(t) X_{\lambda_0}(t) dt$$

$$\geq C^* e^{\Lambda^*\gamma_n} \left\{ \int\limits_0^{\beta_0} e^{\Lambda^* t}(I + S^*(t+\gamma_n)) Z^*(t+\gamma_n) \left[\frac{\Im A(t+\gamma_n)}{\Im\lambda_0}\right] \right.$$

$$\left. Z(t+\gamma_n)(I+S(t+\gamma_n))e^{\Lambda t} dt + \Omega(\gamma_n) \right\} e^{\Lambda\gamma_n} C, \tag{79}$$

where the expressions under integrals in (73) differs a little from those in (79) for large γ_n uniformly on $t \in [0, \beta_0]$, $\Omega(\gamma_n) \to 0$. Therefore in view of (73), (79) there exists such constant $\delta_3 > 0$ not depending on n that

$$\Delta_{\lambda_0}(0,\beta_n) \geq \delta_3 C^* e^{\Lambda^*\gamma_n} e^{\Lambda\gamma_n} C \tag{80}$$

for sufficiently large n. Since

$$X^*_\lambda(\gamma_n)\Gamma^{*-1}(\gamma_n)\Gamma^{-1}(\gamma_n)X_\lambda(\gamma_n)$$
$$\leq \left\{ \max_n \|\Gamma^{-1}(\gamma_n) Z(\gamma_n)(I + S(\gamma_n))\|^2 \right\} C^* e^{\Lambda^*\gamma_n} e^{\Lambda\gamma_n} C,$$

then (67) holds in view of (42).

Under the assumptions of Th. 5 all other conditions of $\text{n}^0 2^0$ of Th. 4 evidently hold and Th. 5 is proved.

Remark 11. *We can set in (42):*

$$\Gamma(t) = [Q^2(t)]^{-1/4} V(t) [Q^2(0)]^{1/4} \in AC_{loc},$$

where the unitary matrix $V(t)$ is a solution to Cauchy problem

$$V' = P'(t)(2P(t) - I)V(t), V(0) = I,$$

$P(t)$ is a Riesz projection of the matrix $Q(t)$ that corresponds to its negative spectrum. Moreover, the norms $\|\Gamma(t)\|, \|\Gamma'(t)\|$ are bounded on axis if the matrix function $Q(t)$ is uniformly almost periodic, $\inf\limits_t |\det Q(t)| > 0$, $Q'(t)$ is bounded.

P r o o f is the corollary of [7].

The following theorem shows that under some assumptions in the case of periodic boundary condition (10),(41) the limit c.m. of the system (1) on the semiaxis $(0,\infty)$ is a c.m. of this system on the axis $(-\infty,\infty)$ if its coefficients are extended in a certain way to the negative semiaxis.

Theorem 6. *Let the condition of Lem. 2 hold and let for the sequence $\{\beta_n\}$ from the proof of Lemma 4 there be the subsequence $\{\beta_{n_j}\}$ such that in (41) $\Gamma(\beta_{n_j}) = U_\lambda(\beta_{n_j}) = I$ and $Q'(t+\beta_{nj})$ are locally uniformly bounded for $t<0$.*

Let the following limits exist for:

1) any $t \le 0:\ \lim Q(t+\beta_{n_j}) = Q^-(t)$, where $\det Q^-(t) \ne 0$;

2) almost all $t<0$: $\lim Q'(t+\beta_{n_j})$;

3) any nonreal λ and any $a<0$: $H_\lambda(t+\beta_{n_j}) \to H_\lambda^-(t)$ in $L^1(a,0)$, where $H_\lambda^-(t)$ depends on the nonreal λ in Nevanlinna's manner.

Then $\mathcal{P}(\lambda)$ (48) ($\{b_{n_k}\} = \{\beta_n\}$) is a c.p. of the system (1) on the axis (see [13]) with the coefficients[⋆]

$$Q(t) = \begin{cases} Q^-(t) \\ Q(t) \end{cases}, \quad H_\lambda(t) = \begin{cases} H_\lambda^-(t), & t \le 0 \\ H_\lambda(t), & t > 0. \end{cases} \tag{81}$$

P r o o f. We extend the coefficients $Q(t)$ and $H_\lambda(t)$ of the system (1) periodically on $[-\beta_{n_j},0)$ if the period equals β_{n_j}. The corresponding extension of $X_\lambda(t)$ we denote $X_\lambda^{n_j}(t)$. Let us fix $t=t_0<0$. Then for $-\beta_{n_j}<t_0$ and due to (3) we have

$$\begin{aligned} &\frac{1}{\Im\lambda}(I-\mathcal{P}^*(X_\lambda(\beta_{n_j})))X_\lambda^{n_j^*}(t_0)Q(t_0)X_\lambda^{n_j}(t_0)(I-\mathcal{P}(X_\lambda(\beta_{n_j}))) \\ \le &\frac{1}{\Im\lambda}(I-\mathcal{P}^*(X_\lambda(\beta_{n_j})))X_\lambda^{n_j^*}(-\beta_{n_j})Q(-\beta_{n_j})X_\lambda^{n_j}(-\beta_{n_j})(I-\mathcal{P}(X_\lambda(\beta_{n_j}))) \\ = &\frac{1}{\Im\lambda}X_\lambda^{*-1}(\beta_{n_j})(I-\mathcal{P}^*(X_\lambda(\beta_{n_j})))G(I-\mathcal{P}(X_\lambda(\beta_{n_j})))X_\lambda^{-1}(\beta_{n_j}) \ge 0. \end{aligned}$$

Passing to the limit in the inequality $\ldots Q(t_0)\ldots \ge 0$ as $\beta_{n_j}\to\infty$ and using the fact that for any nonreal λ $X_\lambda^{n_j}(t_0) \to X_\lambda(t_0)$ due to 1)–3) and [20, p. 160], we obtain

$$\frac{1}{\Im\lambda}(I-\mathcal{P}(\lambda))X_\lambda^*(t_0)Q^-(t_0)X_\lambda(t_0)(I-\mathcal{P}(\lambda)) \ge 0,$$

where $X_\lambda(t)$ is the matrix solution of homogeneous system (1), (81) such that $X_\lambda(0)=I$. The theorem is proved in view of [13, p. 450] and since t_0 is arbitrary.

⋆Whose properties on t and on λ are similar to those of the system (1) on the semiaxis (see introduction) due to 1)–3).

We notice that for any sequence $\{\beta_n\} \uparrow \infty$ there is a subsequence $\{\beta_{n_j}\}$ for which the conditions 1)–3) of Th. 6 hold if the matrix coefficients of the system (1) satisfy the following conditions: the coefficient $Q(t)$ is uniformly almost periodic matrix function having uniformly almost periodic derivative and $\inf_t |detQ(t)| > 0$; the coefficient $H_\lambda(t) =(18)$ and for some nonreal λ_0: $H_{\lambda_0}(t) = h(t) + \eta(t)$, where $h(t)$ is almost periodic Stepanov's matrix function, $\int_x^{x+1} \|\eta(t)\|dt \to 0$ as $x \to +\infty$.

Remark 12. *If for the system (1), (81) from Th. 6 the analog of the condition (4) on the left semi-axis $(-\infty, 0)$ holds,*[⋆] *then (69) for c.p. $\mathcal{P}(\lambda)$ from this theorem holds for any nonreal λ in view of [13, p. 459]. If additionally this c.p. is the unique c.p. of the system (1),(81) on the axis, then $\mathcal{P}(\lambda)$ can be found explicitly with the help of Th. 4.1 from [13] by taking any pair of c.p.'s of the system (1), (81) on the left and right semiaxes.*

Example 2. In the system (1) let $Q(t) = G$, $H_\lambda(t) \in C_{loc}$ on t, $H_\lambda(t+T) = H_\lambda(t), T > 0$, and therefore the condition of Lem. 2 holds. A T-periodic extension of $H_\lambda(t)$ on the whole axis is again denoted by $H_\lambda(t)$. Then the sequence $\{\beta_n\}$ from Theorem 6 contains the subsequence $\{\beta_{n_j}\}$ such that $H_\lambda(t+\beta_{n_j}) \to H_\lambda(t+\tau)$ uniformly on $t \in (-\infty, \infty)$, where $\tau \in [0, T)$ and $\tau = \lim \beta_{n_j} (\mathrm{mod} T)$.

Therefore in the considered case the conditions 1)–3) of Th. 6 hold with $\{\beta_{n_j}\}$ mentioned above, and $H_\lambda^-(t) = H_\lambda(t+\tau)$. In view of [29, p. 225] the c.m. of the system (1), (81) on axis is unique, and if $\Gamma(\beta_{n_j}) = U_\lambda(\beta_{n_j}) = I$, then the limit c.p. $\mathcal{P}(\lambda)$ from Th. 6 of the system (1) on $(0, \infty)$ equals

$$\mathcal{P}(\lambda) = \mathcal{P}(X_\lambda(T))[\mathcal{P}(X_\lambda(T)) + X_\lambda(\tau)(I - \mathcal{P}(X_\lambda(T)))X_\lambda^{-1}(\tau)]^{-1}.$$

The corresponding spectral matrix that generates Parseval's equality (when $H_\lambda(t) =(18)$) is equal [29] to $\sigma(\mu) = \sigma_{ac}(\mu) + \sigma_d(\mu)$, where $\sigma'_{ac}(\mu)$ can be found by the formula from [29] for the spectral matrix of the system (1) on axis in the case $H_\lambda(t) = H_\lambda(t \pm T_\pm), T_\pm > 0, t \in R_\pm$. In particular, it follows from this formula and [22]–[24][⋆⋆] that the absolutely continuous spectrum of the limit problem and its multiplicity are similar to those in the system (1) on axis if its coefficients are extended periodically on the left semi-axis. The component σ_d can be nonzero if, for instance, $H_\lambda^-(t) = H_\lambda(-t)$ $(t < 0)$ and $H_\lambda(t)$ (81) is not T-periodic function on the axis (it follows, e.g., from [29, p. 218]).

[⋆] For example, it takes place if the system (1) corresponds to the symmetric matrix differential operator of arbitrary order.

[⋆⋆] We notice that for the system (1), (18) with periodic coefficients the c.m. on the axis and its asymptotic in singular points have been obtained in [22] (when $H(t)$ is the weight of the positive type) and in [23, 24] (in the general case). Later an equivalent formulae for the c.m. and its asymptotic in the mentioned points for the canonical periodic systems with the weight of the positive type have been obtained by L.A. Sakhnovitch (see [12]) apparently in a more complicated way.

Example 3. All conditions of Th. 6 and Remark 12 hold if: a) the system (1) corresponds to the Sturm–Liouville equation on $[0,\infty)$ with matrix potential $\mathbb{P}(t) \in C_{loc}$ for which there exists such sequence of intervals $(a_n, b_n) \subset (0,\infty)$ that: $a_{n+1} > b_n \uparrow \infty$, $b_n - a_n \uparrow \infty$, $\Gamma(b_n) = U_\lambda(b_n) = I$, $\mathbb{P}(t) \in C^1[a_n, b_n]$, $\exists M : ||\mathbb{P}(t)|| + ||\mathbb{P}'(t)|| \leq M$, $t \in [a_n, b_n]$; b) the subsequence $\{\beta_n\}$ is chosen from $\{b_n\}$. (Here the uniqueness of c.p. for the system (1), (81) follows, for example, from Cor. 3 from [30].)

Aknowlegments. The author is grateful to Prof. F.S. Rofe-Beketov for the attention he paid to this work.

References

[1] *B.M. Levitan*, Proof of the Theorem on the Expansion in Eigenfunctions of Self-adjoint Differential Equations. — *Dokl. Akad. Nauk SSSR* **73** (1950), 651–654. (Russian)

[2] *B.M. Levitan and I.S. Sargasyan*, Introduction to spectral theory. Nauka, Moscow, 1970. (Russian). (Engl. Transl.: Transl. AMS **39** (1975), Providence, RI, xi+525.)

[3] *M.A. Naimark*, Investigation of the Spectrum and the Expansion in Eigenvalues of a Nonself-Adjoint Differential Operator Second Order on a Semi-Axis. — *Tr. Moscow. Mat. Obsch.* **3** (1954), 181–270. (Russian)

[4] *M.L. Gorbachuk*, On Spectral Functions of a Second Order Differential Operator with Operator Coefficients. — *Ukr. Mat. Zh.* **18** (1966), No. 2, 3–21. (Russian) (Engl. Transl.: AMS Transl. Ser. II. **72** (1968), 177–202.)

[5] *F.S. Rofe-Beketov*, Expansion in Eigenfunctions of Infinite Systems of Differential Equations in the Nonself-Adjoint and Self-Adjoint cases. — *Mat. Sb.* **51** (1960), No. 3, 293–342. (Russian)

[6] *F. Atkinson*, Discrete and Continuous Boundary Problems. Acad. Press, New York, London, 1964. (Russian Transl.: With Supplements to the Russian Edition by I.S. Kats and M.G. Krein: I. R-functions — Analytic Functions Mapping the Upper Half Plane into itself.) (Engl. Transl. of the Supplement I: AMS Transl. Ser. II. **103** (1974) 1–18.; II. On Spectral Functions of a String, Mir, Moscow.)

[7] *V.I. Kogan and F.S. Rofe-Beketov*, On Square Integrable Solutions of Systems of Differential Equations of Arbitrary Order. FTINT AN UkrSSR, 1973, 60. Prepr. (Russian) (Engl. Transl.: *Proc. Roy. Soc.*, Edinburgh Sect. **A 74** (1975), 5–40.)

[8] *S.A. Orlov*, The Structure of Resolvents and Spectral Functions of Unidimensional Linear Selfadjoined Singular Differential Operators of Order 2n. — *Dokl. Akad. Nauk SSSR* **111** (1956), No. 6, 1175–1177. (Russian)

[9] *S.A. Orlov*, Description of the Green Functions of Canonical Differential Systems I. II. — *Teor. Funkts., Funkts. Anal. i Prilozh.* (1956), No. 51, 78–88; No. 52, 33–39. (Russian) (Engl. Transl.: *J. Soviet Math.* **52** (1990), No. 5, 3372–3377; No. 6, 3500–3508.)

[10] *F.S. Rofe-Beketov*, Perturbations and Friedrichs Extensions of Semibounded Operators on Variable Domains. — *Dokl. Akad. Nauk SSSR* **255** (1980), No. 5, 1054–1058. (Russian) (Engl. Transl.: *Soviet Math. Dokl.* **22** (1980), No 3, 795–799.)

[11] *F.S. Rofe-Beketov and A.M. Khol'kin*, Spectral Analysis of Differential Operators. Interplay between Spectral and Oscillatory Properties. World Sci. Monogr. Ser. Math. NJ **7** (2005), +XXII, 438.

[12] *L.A. Sakhnovich*, Spectral Theory of Canonical Differential Systems. Method of Operator Identities. Birkhauser, Basel, Boston, Berlin, 1999.

[13] *V.I. Khrabustovskyi*, On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Dissipative and Accumulative Operator Systems. I. General Case. — *J. Math. Phys., Anal., Geom.* **2** (2006), No. 2, 149–175. II. Abstract Theory. — Ibid, No. 3, 299–317. III. Separated Boundary Conditions. — Ibid, No. 4, 449–473.

[14] *S.A. Orlov*, Nested Matrix Discs that Depend Analytically on a Parameter, and Theorems on the Invariance of the Ranks of the Radii of the Limit Matrix Discs. — *Izv. Akad. Nauk. USSR* **40** (1976), No. 3, 593–644. (Russian) (Engl. Transl.: *Math. USSR. Izv.* **10** (1976), 565–613.)

[15] *J.M. Ziman*, Principles of the Theory of Solids. Cambridge Univ. Press, Cambridge, 1972. (Russian Transl.: Mir, Moscow, 1974.)

[16] *F.S. Rofe-Beketov*, Square-Integrable Solutions, Selfadjoint Extensions and Spectrum of Differential Systems. Differential Equations. — In: *Proc. Intern. Conf.*, Uppsala, 1977.

[17] *V.M. Bruk*, Linear Relations in a Space of Vector Functions. — *Mat. Zametki* **24** (1978) No. 6, 499–511. (Russian) (Engl. Transl.: *Math. Notes* **24** (1979), 767–773.)

[18] *M. Lesch and M. Malamud*, On the Number of Square Integrable Solutions and Selfadjointness of Symmeteric First Order Systems of Differential Equations. Prepr. Math. Inst. der Univ. zu Koln, Jan. 2001, arXiv: math.FA/0012080 11 Dec. 2000, 1–50.

[19] *A. Dijksma and de H.S.V. Snoo*, Self-Adjoint Extensions of Symmetric Subspaces. — *Pacific J. Math.* **54** (1974), No. 1, 71–100.

[20] *Yu.L. Daletskiy and M.G. Krein*, Stability of Solutions of Differential Equations in Banach Space. Nauka, Moscow, 1970. (Russian) (Engl. Transl.: *AMS Transl.* **43** Providence, RI, 1974, vi+386.)

[21] *T. Kato*, Perturbation Theory for Linear Operators. Repr. 1980 Ed. Classics in Mathematics. Springer Verlag, Berlin, xxii+619. (Russian Transl.: Mir, Moscow, 1972.)

[22] *V.I. Khrabustovskyi*, The Spectral Matrix of a Periodic Symmetric System with a Degenerate Weight on the Axis. — *Teor. Funkts., Funkts. Anal. i Prilozh.* (1981), No. 35, 111–119. (Russian)

[23] *V.I. Khrabustovskyi*, Eigenfunction Expansions of Periodic Systems with Weight. — *Dokl. Akad. Nauk. UkrSSR.* Ser. A (1984), No. 5, 26–29.

[24] *V.I. Khrabustovskyi*, Spectral Analysis of Periodic Systems with Degenerate Weight on the Axis and Half-Axis. — *Teor. Funkts., Funkts. Anal. i Prilozh.* (1985), No. 44, 122–133. (Russian) (Engl. Transl.: *J. Soviet Math.* **48** (1990), No. 5, 345–355).

[25] *V.A. Zolotarev*, Analytic Methods of Spectral Representations of Nonself-Adjoint and Nonunitary Operators. Kharkiv National Univ., Kharkiv, 2003. (Russian)

[26] *B. Sz.-Nagy and C. Foias*, Analyse Harmonique des Operateurs de l'Espace de Hilbert. Masson et Cie, Paris; Akademiai Kiado, Budapest, (1967), xi+373. (French) (Engl. Transl.: Harmonic Analysis of Operators on Hilbert Space. Transl. from the French and Revised North-Holland Publish. Co., Amsterdam, London; Amer. Elsevier Publish. Co., Inc., New York; Akademiai Kiado, Budapest, 1970, xiii+389.) (Russian Transl.: Ju.P. Ginzburg, Ed.; a Foreword by M.G. Krein, Mir, Moscow, 1970.)

[27] *V.P. Potapov*, On the Theory of the Weyl Matrix Circles. — *Funct. Anal. and Appl. Math.* **215** (1982), 113–121.

[28] *B.F. Bylov, R.E. Vinograd, D.M. Grobman, and V.V. Nemyckii*, Theory of Ljapunov Exponents and its Application to Problems of Stability. Nauka, Moscow, 1966. (Russian)

[29] *V.I. Khrabustovskyi*, On Characteristic Matrix of Weyl–Titchmarsh Type for Differential Operator Equations which Contains the Spectral Parameter in Linear or Nevanlinna's Manner. — *Mat. fiz., anal., geom.* **2** (2003), 205–227. (Russian)

[30] *A.G. Brusentsev and F.S. Rofe-Beketov*, Self-Adjointness Conditions for Elliptic Systems of Arbitrary Order. — *Mat. Sb.* **95** (1974), No. 1, 108–129. (Russian) (Engl. Transl.: *Sb. Math.* **24** (1974), No. 1, 103–126.)