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On Convergence of Solutions of Singularly Perturbed Boundary-Value Problems

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A perturbation of the Poisson equation by a biharmonic operator with a small multiplier ε is considered. The asymptotic behavior of the solution of the Dirichlet problem for this equation as $\varepsilon \to 0$ is studied. The gradient of the solution is proved to converge to the gradient of the solution to Poisson equation in $L_1(\Omega)$ as $\varepsilon \to 0$. The difference of the gradients is also estimated.

Key words: singular perturbation, elliptical equations, the Green functions.

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1. Problem Statement and Main Result

Let Ω be a bounded domain in $\mathbb{R}^3$ with a sufficiently smooth boundary. Consider the following boundary-value problem:

$$\begin{cases} \varepsilon\Delta^2 u_\varepsilon - \Delta u_\varepsilon = F \text{ in } \Omega, \\ u_\varepsilon = 0, \ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Here ν is the outer normal to $\partial\Omega$ at the point x, $F \in L_p(\Omega)$ $\left(p > \frac{6}{5}\right)$, and $\varepsilon > 0$ is a small parameter. As known, there exists a unique solution of this problem $u_\varepsilon \in W_p^4(\Omega)$ (see, e.g., [1]). We are interested in the asymptotic behavior of solution of this problem when $\varepsilon \to 0$. Similar questions for more general equations were studied by M. Vishik and L. Lyusternik in [2], where the asymptotic expansion with respect to the powers of ε was constructed. The method proposed in the paper was widely used at that time. However, all the known results appeared to be not sufficient to our work. We use our result to construct the regularized solutions of Navier–Stokes–Vlasov–Poisson boundary value problem [3].

To formulate the main result we consider the following boundary-value problem:

$$\begin{cases} \Delta u = F \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where F is the same function as in (1.1). There exists a unique solution to this problem $u \in W_p^2(\Omega)$ (see, e.g., [1]).

The main result of the paper is the following

Theorem 1. *Let u_ε and u be the solutions of problems (1.1), (1.2), respectively. Then*

$$\lim_{\varepsilon\to 0} \int\limits_\Omega |\nabla u_\varepsilon(x) - \nabla u(x)|\, dx = 0$$

uniformly with respect to all functions F such that $\|F\|_{L_p(\Omega)} \le C$.

This theorem is proved in Sections 2 and 3.

2. Estimates of the Green Functions

Let $G_\varepsilon(x, y)$ and $G_0(x, y)$ be the Green functions of problems (1.1) and (1.2), respectively.

Lemma 1. *The following estimates for normal derivatives of the Green function $G_0(x, y)$ hold:*

$$\left|\frac{\partial G_0}{\partial \nu}(x, y)\right| \le \frac{C_1}{|x-y|^2},\ y \in \Omega,\ x \in \partial\Omega,$$

$$\left|D_\tau^k \frac{\partial G_0}{\partial \nu}(x, y)\right| \le \frac{C_2}{(d(y))^{2+|k|+\alpha}},\ y \in \Omega,\ x \in \partial\Omega,$$

where $k = (k_1, k_2)$ is a multiindex, $k_i \in \mathbb{Z}$, $k_1 + k_2 \ge 1$, $|k| = k_1 + k_2$, D_τ^k is a derivative at the point $x \in \partial\Omega$ in tangent directions to $\partial\Omega$, $d(y)$ is a distance from the point $y \in \Omega$ to $\partial\Omega$, $0 < \alpha < 1$, C_1 and C_2 are constants that depend on the minimal radius of curvature of $\partial\Omega$, k, and α only.

P r o o f. As is well known, the Green function $G_0(x,y)$ has the form

$$G_0(x,y) = \frac{1}{4\pi|x-y|} - g_0(x,y), \tag{2.1}$$

where the regular function $g_0(x,y)$ is a solution of the following boundary-value problem with respect to the variable $x \in \Omega$ ($y \in \Omega$ is a parameter):

$$\begin{cases} \Delta g_0 = 0 \text{ in } \Omega, \\ g_0 = \frac{1}{4\pi|x-y|} \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

Let us represent $g_0(x,y)$ as a simple layer potential

$$g_0(x,y) = \frac{1}{4\pi}\int\limits_{\partial\Omega} \frac{\sigma(\xi,y)}{|x-\xi|} dS_\xi. \tag{2.3}$$

The simple layer potential satisfies the Laplace equation in $\mathbb{R}^3\backslash\Omega$, tends to zero when $|x| \to \infty$, and it is a continuous function in x in $\mathbb{R}^3$. Therefore, by (2.2), it equals to the function $\frac{1}{4\pi|x-y|}$ in $\mathbb{R}^3\backslash\bar{\Omega}$. Then its normal derivative in $\mathbb{R}^3\backslash\Omega$ is given by $\left(\frac{\partial g_0}{\partial\nu}\right)_e = \frac{\partial}{\partial\nu}\frac{1}{4\pi|x-y|}$.

Hence, taking into account the properties of the simple layer potential, we obtain the integral equation for the density $\sigma(x,y)$

$$\frac{1}{2}\sigma(x,y) - \frac{1}{4\pi}\int\limits_{\partial\Omega} \frac{\cos\theta(x,\nu)}{|x-\xi|^2}\sigma(\xi,y)\,dS_\xi = -\frac{1}{4\pi}\frac{\partial}{\partial\nu}\frac{1}{|x-y|}, \tag{2.4}$$

where $\theta(x,\nu)$ is the angle between the outer normal to $\partial\Omega$ at the point $x \in \partial\Omega$ and the vector $x-\xi$.

This equation corresponds to the representation of the solution to the external Neumann boundary-value problem in the form of simple layer potential and, therefore, it has a unique solution in the class $C(\partial\Omega)$ (see, e.g., [4]). Applying the iteration method, we obtain the estimate

$$|\sigma(x,y)| \le \frac{C}{|x-y|^2}. \tag{2.5}$$

On the other hand, from (2.1) it is clear that

$$\left(\frac{\partial G_0}{\partial\nu}\right)_i = \frac{1}{4\pi}\frac{\partial}{\partial\nu_x}\frac{1}{|x-y|} - \left(\frac{\partial g_0}{\partial\nu}\right)_i$$

and according to the properties of the simple layer potential $\left(\frac{\partial g_0}{\partial\nu}\right)_i - \left(\frac{\partial g_0}{\partial\nu}\right)_e = \sigma(x,y)$. Consequently, $\frac{\partial G_0}{\partial\nu} = -\sigma(x,y)$. So, the first estimate of Lemma 1 is proved.

To establish the second estimate we use the Schauder estimates (see, e.g., [5])

$$|g_0|_{k+\alpha,\overline{\Omega}} \leq C\left(|\Delta g_0| + |g_0|_{k+\alpha,\partial\Omega}\right),$$

where $|g|_{k+\alpha,\partial\Omega}$ is the norm of g in $C^{k+\alpha}(\partial\Omega)$, $k \geq 2$, $0 < \alpha < 1$. The constant C depends on k, α, and $\partial\Omega$. The second estimate follows easily from (2.2). Lemma 1 is proved.

Lemma 2. *The following estimate holds:*

$$\int\limits_{\Omega} |\nabla G_\varepsilon(x,y) - \nabla G_0(x,y)|\, dx \leq C\left(\frac{\sqrt[4]{\varepsilon}}{d(y)^{1+\alpha}} + \frac{e^{-\frac{d(y)}{\sqrt{\varepsilon}}}}{\sqrt{\varepsilon}d(y)}\right), \tag{2.6}$$

where d(y) is the distance from the point y to $\partial\Omega$, $0 < \alpha < 1$, *and the constant* C *depends on* Ω *and* α *only.*

P r o o f. It is easy to verify that the function

$$\Gamma_\varepsilon(x,y) = \frac{1}{4\pi|x-y|}\left(1 - e^{-\frac{|x-y|}{\sqrt{\varepsilon}}}\right), \ \varepsilon > 0, \tag{2.7}$$

is a fundamental solution of the equation (1.1) in $\mathbb{R}^3$.

As is well known, the Green function $G_\varepsilon(x,y)$ can be represented in the form $G_\varepsilon(x,y) = \Gamma_\varepsilon(x,y) - g_\varepsilon(x,y)$, where $g_\varepsilon(x,y)$ is a regular function, which is a solution of the following boundary-value problem:

$$\begin{cases} \varepsilon\Delta^2 g_\varepsilon - \Delta g_\varepsilon = 0 \text{ in } \Omega, \\ g_\varepsilon = \Gamma_\varepsilon, \ \frac{\partial g_\varepsilon}{\partial\nu} = \Gamma_\varepsilon \text{ on } \partial\Omega. \end{cases} \tag{2.8}$$

According to (2.2) and (2.8),

$$G_\varepsilon(x,y) - G_0(x,y) = -\frac{e^{-\frac{|x-y|}{\sqrt{\varepsilon}}}}{4\pi|x-y|} - v_\varepsilon(x,y), \tag{2.9}$$

where the function $v_\varepsilon(x,y) = g_\varepsilon(x,y) - g_0(x,y)$ is a solution of

$$\begin{cases} \varepsilon\Delta^2 v_\varepsilon - \Delta v_\varepsilon = 0 \ \text{ in } \Omega, \\ v_\varepsilon = -\frac{e^{-\frac{|x-y|}{\sqrt{\varepsilon}}}}{4\pi|x-y|}|_{x=x(s,\tau)} = \eta_\varepsilon^0(s,\tau) \ \text{ on } \partial\Omega, \\ \frac{\partial v_\varepsilon}{\partial\nu} = \left(\frac{\partial G_0}{\partial\nu} - \frac{\partial}{\partial\nu}\left(\frac{e^{-\frac{|x-y|}{\sqrt{\varepsilon}}}}{4\pi|x-y|}\right)\right)|_{x=x(s,\tau)} = \eta_\varepsilon^1(s,\tau) \ \text{ on } \partial\Omega. \end{cases} \tag{2.10}$$

Here we use a local coordinate system such that (s,τ,t) are coordinates in a neighborhood of $x = x(s,\tau,0) \in \partial\Omega$, where t is the distance from the point x to $\partial\Omega$, and (s,τ) are coordinates on $\partial\Omega$.

Let us introduce a class of functions

$$\widetilde{W} = \left\{ w \in W_2^2(\Omega) : \ w|_{\partial\Omega} = \eta_\varepsilon^0, \ \frac{\partial w}{\partial \nu}|_{\partial\Omega} = \eta_\varepsilon^1 \right\}.$$

By [6], $v_\varepsilon(x,y)$ minimizes the functional

$$J_\varepsilon(w) = \int\limits_\Omega \left\{ \varepsilon(\Delta w)^2 + |\nabla w|^2 \right\} dx.$$

Then

$$J_\varepsilon(v_\varepsilon) \le J_\varepsilon(w_\varepsilon) \ \ \forall w_\varepsilon \in \widetilde{W}. \qquad (2.11)$$

To estimate $J_\varepsilon(v_\varepsilon)$ let us construct a representative of the class $\widetilde{W}$ in the form

$$w_\varepsilon(x) = \left(\eta_\varepsilon^0(s,\tau) + t\eta_\varepsilon^1(s,\tau)\right)\varphi\left(\frac{t}{\sqrt{\varepsilon}}\right), \qquad (2.12)$$

where $\varphi(t)$ is a smooth function such that $\varphi(t) = 1$ for $t \le 1/2$, $\varphi(t) = 0$ for $t \ge 1$, $\varphi(t) \in C^2(0,\infty)$.

Suppose that $y \in \Omega_\delta \subset \Omega$, with Ω_δ being a subdomain of Ω,

$$\Omega_\delta = \{x \in \Omega : dist(x,\partial\Omega) > \delta\}, \qquad (2.13)$$

where δ satisfies the condition $\delta > r_{\partial\Omega} \gg \sqrt{\varepsilon} > 0$, and $r_{\partial\Omega}$ is the minimal radius of curvature of the surface $\partial\Omega$.

Then, using (2.12), the explicit expressions for the functions η_ε^0, η_ε^1 (see (2.10)), and Lemma 1, we obtain the estimate

$$J_\varepsilon(w_\varepsilon) \le C\left(\frac{e^{-\frac{2d}{\sqrt{\varepsilon}}}}{\varepsilon d^2} + \frac{\sqrt{\varepsilon}}{d^{2+\alpha}}\right),$$

where $d = d(y)$ is the distance from y to $\partial\Omega$, and the constant C depends on $\partial\Omega$ and α, $0 < \alpha < 1$. Therefore, it follows from (2.11) that

$$\int\limits_\Omega |\nabla v_\varepsilon|^2 dx \le J_\varepsilon(v_\varepsilon) \le C\left(\frac{e^{-\frac{2d}{\sqrt{\varepsilon}}}}{\varepsilon d^2} + \frac{\sqrt{\varepsilon}}{d^{2+2\alpha}}\right).$$

Using this estimate and taking into account (2.9), we obtain (2.6). Lemma 2 is proved.

3. Proof of Theorem 1

We begin with the following lemma.

Lemma 3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary of the class $C^{2+\alpha}$, and φ_ε be a solution to the following boundary-value problem:*

$$\begin{cases} \varepsilon\Delta^2\varphi_\varepsilon - \Delta\varphi_\varepsilon = F \ \text{in}\ \Omega, \\ \varphi_\varepsilon = 0,\ \varepsilon\frac{\partial\varphi_\varepsilon}{\partial\nu} = 0 \quad \text{on}\ \partial\Omega, \end{cases} \tag{3.1}$$

where $\varepsilon \geq 0$, $F \in L_p(\Omega)$ $\left(p > \frac{6}{5}\right)$ with a support $S_F \subset \overline{\Omega}$. Then

$$\int_\Omega |\nabla\varphi_\varepsilon|\,dx \leq C\,\|F\|_{L_p(\Omega)}\,(mesS_F)^{\frac{5}{6}-\frac{1}{p}},$$

where C is a constant that does not depend on ε.

P r o o f. The solution of the problem (3.1) minimizes the functional

$$\hat{F}_\varepsilon(\varphi_\varepsilon) = \int_\Omega \left\{\varepsilon\,(\Delta\varphi_\varepsilon)^2 + |\nabla\varphi_\varepsilon|^2 - 2F\chi_F\varphi_\varepsilon\right\} dx$$

in the class of functions φ_ε of $\overset{\circ}{W}{}_2^1(\Omega)$ for $\varepsilon > 0$ and of $W_2^1(\Omega)$ for $\varepsilon = 0$. Here by $\chi_F = \chi_F(x)$ we denote the characteristic function of the set S_F.

Since $\hat{F}_\varepsilon(0) = 0$, then we have $\hat{F}_\varepsilon(\varphi_\varepsilon) \leq 0$.

This leads to the inequality

$$\int_\Omega \left\{\varepsilon\,(\Delta\varphi_\varepsilon)^2 + |\nabla\varphi_\varepsilon|^2\right\} dx \leq 2\int_\Omega |F(x)|\,|\chi_F(x)|\,|\varphi_\varepsilon(x)|\,dx.$$

Applying the Hölder inequality with p, $q = \frac{6p}{5p-6}$, and $r = 6$ $\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\right)$ to the right-hand side of this bound, we get

$$\int_\Omega \left\{\varepsilon\,(\Delta\varphi_\varepsilon)^2 + |\nabla\varphi_\varepsilon|^2\right\} dx \leq 2\,\|F\|_{L_p(\Omega)}\,\|mes\chi_F\|_{L_q(x)}\,\|\varphi_\varepsilon\|_{L_6(\Omega)}$$

$$\leq C\,\|F\|_{L_p(\Omega)}\,(mesS_F)^{\frac{5}{6}-\frac{1}{p}}\,\|\nabla\varphi_\varepsilon\|_{L_2(\Omega)}. \tag{3.2}$$

Here the norm of $\varphi_\varepsilon \in \overset{\circ}{W}{}_2^1$ is estimated according to the embedding of $\overset{\circ}{W}{}_2^1(\Omega)$ in $L_6(\Omega)$.

From (3.2) we conclude that

$$\|\nabla\varphi_\varepsilon\|_{L_2(\Omega)} \le C\,\|F\|_{L_p(\Omega)}\,(mesS_F)^{\frac{5}{6}-\frac{1}{p}}.$$

Now this bound and the Cauchy–Schwarz inequality

$$\int\limits_\Omega |\nabla\varphi_\varepsilon|\,dx \le \left(\int\limits_\Omega |\nabla\varphi_\varepsilon|^2\,dx\right)^{1/2} |\Omega|^{1/2}$$

yield the statement of Lemma 3.

We are now in position to complete the proof of Theorem 1.

Denote by Ω_δ a subdomain of Ω defined in (2.13).

Let us represent the function $F(x)$ as a sum of three components $F(x) = F_1(x) + F_2(x) + F_3(x)$, where

$F_1(x) \in \overline{C^1}(\Omega)$, $suppF_1(x) \subset \Omega_\delta$, $\|F_1\|_{L_p(\Omega)} \le \|F\|_{L_p(\Omega)}$;

$F_2(x) : \|F_2\|_{L_p(\Omega)} < \delta\,\|F\|_{L_p(\Omega)}$;

$F_3(x) = F(x)\,\chi_\delta(x)$, where $\chi_\delta(x)$ is a characteristic function of the set $\Omega\backslash\Omega_\delta$.

The solutions of problems (1.1) and (1.2) can be represented as $u_\varepsilon = u_{1\varepsilon} + u_{2\varepsilon} + u_{3\varepsilon}$, $u = u_1 + u_2 + u_3$, respectively. Then we have

$$\int\limits_\Omega |\nabla u_\varepsilon - \nabla u|\,dx \le \int\limits_\Omega |\nabla u_{1\varepsilon} - \nabla u_1|\,dx + \int\limits_\Omega |\nabla u_{2\varepsilon}|\,dx$$

$$+\int\limits_\Omega |\nabla u_2|\,dx + \int\limits_\Omega |\nabla u_{3\varepsilon}|\,dx + \int\limits_\Omega |\nabla u_3|\,dx. \qquad (3.3)$$

Using Lemma 2, we estimate the first integral as follows:

$$\int\limits_\Omega |\nabla u_{1\varepsilon} - \nabla u_1|\,dx \le \int\limits_\Omega |\nabla G_\varepsilon(x,y) - \nabla G_0(x,y)|\,|F_1(x)|\,dx$$

$$\le C\left(\frac{\sqrt[4]{\varepsilon}}{\delta^{1+\alpha}} + \frac{e^{-\frac{\delta}{\sqrt{\varepsilon}}}}{\sqrt{\varepsilon}\delta}\right)\|F_1\|_{L_p(\Omega)}. \qquad (3.4)$$

To estimate the remaining integrals we use Lemma 3. Thus, we have:

$$\int\limits_\Omega |\nabla u_{2\varepsilon}|\,dx \le C\,\|F_2\|_{L_p(\Omega)}\,|\Omega_\delta|^{\frac{5}{6}-\frac{1}{p}} \le C\delta\,\|F\|_{L_p(\Omega)}\,|\Omega_\delta|^{\frac{5}{6}-\frac{1}{p}}; \qquad (3.5)$$

$$\int\limits_\Omega |\nabla u_{3\varepsilon}|\,dx \le C\,\|F_3\|_{L_p(\Omega)}\,|\Omega\backslash\Omega_\delta|^{\frac{5}{6}-\frac{1}{p}} \le C\,\|F\|_{L_p(\Omega)}\,\delta^{\frac{5}{6}-\frac{1}{p}}. \qquad (3.6)$$

We set $\delta = \delta(\varepsilon) = \varepsilon^{\frac{6p}{4(11p-6+6\alpha p)}}$. Then, according to (3.3)–(3.6), we obtain

$$\int_{\Omega} |\nabla u_\varepsilon - \nabla u| \, dx \le C\varepsilon^{\gamma} \|F\|_{L_p(\Omega)},$$

where $\gamma = \frac{5p-6}{4(11p-6+6\alpha p)}$.

Since $\|F\|_{L_p(\Omega)} \le C$ and $p > \frac{6}{5}$, then

$$\int_{\Omega} |\nabla u_\varepsilon - \nabla u| \, dx \to 0$$

as $\varepsilon \to 0$.

Theorem 1 is proved.

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