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The Hardy–Littlewood Theorem and the Operator of Harmonic Conjugate in Some Classes of Simply Connected Domains with Rectifiable Boundary

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The analogue of known theorem Hardy–Littlewood about L^p-estimations of derivative analytical function through norm to the function, also are proved L^p-weight estimations the operator of harmonic conjugate in some classes of simply connected domains with rectifiable boundary for all $0 < p < +\infty$.

Key words: operator of conjugate, class BMOA, estimations of derivative analytical function.

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Let G be a simply connected domain in the complex plane C, $d(w, \partial G)$ be a distance from the point $w \in G$ to ∂G.

Denote by $L^p_\beta(G)$ the space of measurable functions f in G such that

$$\|f\|^p_{L^p_\beta(G)} = \int\limits_G |f(w)|^p d^\beta(w, \partial G) dm_2(w) < +\infty, 0 < p < +\infty, \beta > -1, \qquad (1)$$

where dm_2 is the plane Lebesque measure, and denote by $H(G)$ the set of all analytic functions in G. Also, put $A^p_\beta(G) = H(G) \bigcap L^p_\beta(G)$. Denote by $h^p_\beta(G)$ the subspace of $L^p_\beta(G)$ consisting of harmonic functions.

In this paper we generalize the Hardy–Littlewood theorem [1]: *if* $f \in H(S)$, $0 < p < +\infty$, $f(0) = 0$, $\beta > -1$, *then there exist positive constants* c_1 *and* c_2 *such* *that*

$$c_1 \int\limits_S |f(z)|^p (1-|z|)^\beta dm_2(z)$$
$$\leq \int\limits_S |f'(z)|^p (1-|z|)^{p+\beta} dm_2(z) \leq c_2 \int\limits_S |f(z)|^p (1-|z|)^\beta dm_2(z), \qquad (2)$$

where S is an open unit disk in the complex plane C.

Considerable attention was paid to this result in papers [2, 3]. The estimation (2) was carried out in [2] for simply connected domains with the boundary from the class C^1, and in [3] — for the addition of the convex bounded domains, but only at $p = 2$.

Notice that Γ is the curve of Lavrentiev class (L) if $l(w_1, w_2) \leq c|w_1 - w_2|$, where for any $w_1, w_2 \in \Gamma$, and $l(w_1, w_2)$ is the length of the shortest arc of Γ with endpoints w_1, w_2.

We prove an analogue of the left estimation of (2) for any open set and of the right estimation of (2) for simply connected domains G with the boundary from class (L).

The received estimations allow us to construct explicitly the bounded linear integral operator from $h^p_\beta(G)$ onto $A^p_\beta(G)$ for any $0 < p < +\infty$ and from $L^p_\beta(G)$ onto $A^p_\beta(G)$ for any $1 \leq p < +\infty$.

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1. Auxiliary Lemmas

In [5], M.M. Dzrbashyan proved that if $f \in A^p_\beta(S)$, $1 \leq p < +\infty$, $\beta > -1$, then the integral representation is valid

$$f(z) = \frac{\beta+1}{\pi} \int\limits_S \frac{(1-|\zeta|^2)^\beta f(\zeta)}{(1-\overline{\zeta} z)^{\beta+2}} dm_2(\zeta), z \in S. \qquad (3)$$

Let us prove (3) for $0 < p < 1$.

Lemma 1. *Suppose $f \in A^p_\beta(S)$, $0 < p < 1$, $\beta > -1$, $\eta > \dfrac{\beta+2}{p} - 1$; then $f \in A^1_\eta(S)$.*

Here and in the sequel we denote by $c, c_1, \ldots, c_n(\alpha, \beta, \ldots)$ some arbitrary positive constants depending on $\alpha, \beta, \ldots$ whose specific values are immaterial.

P r o o f. Let $K_\rho(z) = \{w : |w - z| < \rho\}$, where $\rho = \dfrac{1 - |z|}{2}$. Then, by the subharmonicity of $|f|^p$,

$$|f(z)|^p \le \frac{1}{\pi\rho^2} \int\limits_{K_\rho(z)} |f(\zeta)|^p \, dm_2(\zeta).$$

It is easy to see that for all $\zeta \in K_\rho(z)$ we have $\dfrac{1-|z|}{2} \le 1 - |\zeta| \le \dfrac{3(1-|z|)}{2}$. Hence, we get

$$|f(z)|^p \,(1-|z|)^\beta \le \frac{(1-|z|)^\beta}{\pi \left(\frac{1-|z|}{2}\right)^2} \int\limits_{K_\rho(z)} |f(\zeta)|^p \, dm_2(\zeta)$$

$$= \frac{4(1-|z|)^\beta}{\pi\,(1-|z|)^2} \int\limits_{K_\rho(z)} |f(\zeta)|^p \, dm_2(\zeta) \le \frac{4 \cdot 2^\beta}{\pi\,(1-|z|)^2} \int\limits_{K_\rho(z)} |f(\zeta)|^p \,(1-|\zeta|)^\beta dm_2(\zeta)$$

$$\le \frac{c}{(1-|z|)^2} \int\limits_S |f(\zeta)|^p \,(1-|\zeta|)^\beta dm_2(\zeta).$$

Therefore, we obtain

$$|f(z)|^p \le \frac{c}{(1-|z|)^{\beta+2}} \int\limits_S |f(\zeta)|^p \,(1-|\zeta|)^\beta dm_2(\zeta) \le \frac{c_1}{(1-|z|)^{\beta+2}}$$

and $|f(z)| \le \dfrac{c_1^{\frac{1}{p}}}{(1-|z|)^{\frac{\beta+2}{p}}}$. Thus, if $\eta > \dfrac{\beta+2}{p} - 1$, then

$$\int\limits_S |f(z)|\,(1-|z|)^\eta dm_2(z) \le c_2 \int\limits_S \frac{dm_2(z)}{(1-|z|)^{\frac{\beta+2}{p}-\eta}} \le c_3 \int\limits_0^1 \frac{dr}{(1-r)^{\frac{\beta+2}{p}-\eta}} < +\infty.$$

The lemma is proved.

If $f \in A^p_\beta(S)$, $0 < p < 1$, $\beta > -1$, $\eta > \dfrac{\beta+2}{p} - 1$, using Lemma 1 we have

$$f(z) = \frac{\eta+1}{\pi} \int\limits_S \frac{(1-|\zeta|^2)^\eta f(\zeta)}{(1-\overline{\zeta}z)^{\eta+2}} dm_2(\zeta). \tag{3'}$$

Lemma 2. *Suppose* $f \in H(S)$, $f^{(n)} \in A^p_\beta(S)$, $0 < p < +\infty$, $\beta > -1$, $f^{(k)}(z_0) = 0$, $k = 0, 1, \ldots, n-1$, $n \in N$, $z_0 \in S$; $0 < p < +\infty$, $\eta > n - 1 + \dfrac{\beta+2}{p}$. *Then*

$$f(z) = c(n,\eta) \int\limits_S \frac{(1-|\zeta|^2)^\eta f^{(n)}(\zeta) P(z,\overline{\zeta})}{(1-\overline{\zeta}z)^{\eta-n+2}} \, dm_2(\zeta), \tag{4}$$

where $P(z,\overline{\zeta})$ is some polynomial in z and $\overline{\zeta}$, $z \in S$.

P r o o f. By the condition of the lemma $f(z) = \frac{1}{(n-1)!}\int\limits_{z_0}^{z}(z-t)^{n-1}f^{(n)}(t)dt$.
Using (3) for $1 < p < +\infty$ or (3$'$) for $0 < p \le 1$, we get

$$f^{(n)}(z) = c\int\limits_S \frac{(1-|\zeta|^2)^{\eta} f^{(n)}(\zeta)}{(1-\overline{\zeta}z)^{\eta+2}}dm_2(\zeta).$$

Integrating this equality n times and taking into account $\int\limits_{z_0}^{z}\frac{(z-t)^{n-1}}{(1-\overline{\zeta}t)^{\eta+2}}dt = \frac{P(z,\overline{\zeta})}{(1-\overline{\zeta}z)^{\eta-n+2}}$, where $P(z,\overline{\zeta})$ is some polynomial in z and $\overline{\zeta}$, $z \in S$, we obtain (4).

Lemma 3 (see [6]). *Let $v(z)$ be a nonnegative subharmonic function on S. Suppose $0 < p \le 1$, $\eta > -1$; then the following is valid*:

$$\left(\int\limits_S v(z)(1-|z|)^{\eta}dm_2(z)\right)^p \le c\int\limits_S (v(z))^p\,(1-|z|)^{\eta p+2p-2}dm_2(z).$$

Let BMOA be a space of analytic functions of a bounded mean oscillation. This is the class of functions $f(z)$ analytic on the unit disc S for which

$$\sup_{|a|<1}\|f_a\|_1 < \infty,\ f_a(z) = f(\frac{z+a}{1+\overline{a}z}) - f(a),$$

where $\|\cdot\|_1$ denotes the H^1-norm.

Lemma 4 (see [7]). *Let G be a simply connected domain with boundary $\Gamma \in (L)$. Suppose $\varphi : S \to G$ conformally, $f(z) = a\ln\varphi'(z)$, and a is any positive constant. Then $f \in BMOA$.*

Lemma 5 (see [7]). *Suppose $f \in BMOA$, $|t| < 1$, and any $a \in C\backslash\{0\}$. Then there exists such $M = M(a)$ that the following inequality is valid*:

$$\frac{1}{2\pi}\int\limits_{|s|=1}\left|e^{af(s)}\right|^2\frac{(1-|t|^2)}{|1-\overline{t}s|^2}|ds| \le M\left|e^{af(t)}\right|^2.$$

Lemma 6. *Let G be a simply connected domain. Suppose $\varphi : S \to G$ conformally, $f^{(k)} \in A^p_{\beta}(G)$, $k = 0,1,\ldots,n$, $n \in N$, $0 < p < +\infty$, $\beta > -1$.*
Then $f^{(k)}(\varphi) \in A^p_{\alpha}(S)$, $k = 0,1,\ldots,n$, $n \in N$, $0 < p < +\infty$, $\alpha \ge 2(\beta+1)$.

P r o o f. By the condition of the lemma, $\int\limits_G \left|f^{(k)}(w)\right|^p d^\beta(w,\partial G)dm_2(w) = c\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p d^\beta(\varphi(z),\partial G)\left|\varphi'(z)\right|^2 dm_2(z) < +\infty$. Then, using Koebe's inequality (see [8, p. 51])

$$\frac{1}{4}\frac{d(\varphi(z),\partial G)}{1-|z|} \le \left|\varphi'(z)\right| \le 4\frac{d(\varphi(z),\partial G)}{1-|z|}, \tag{5}$$

we get
$\int\limits_G \left|f^{(k)}(w)\right|^p d^\beta(w,\partial G)dm_2(w) \ge c\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^\beta \left|\varphi'(z)\right|^{\beta+2} dm_2(z).$
The following estimate for the univalent analytic functions is well known (see [8, p. 53]):

$$\frac{1-|z|}{(1+|z|)^3} \le \left|\varphi'(z)\right| \le \frac{1+|z|}{(1-|z|)^3}. \tag{6}$$

Using it, we obtain

$$\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^\beta \left|\varphi'(z)\right|^{\beta+2} dm_2(z)$$

$$\ge c_1 \int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^\beta (1-|z|)^{\beta+2} dm_2(z)$$

$$\ge c_1 \int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^\alpha dm_2(z),$$

where $\alpha \ge 2(\beta+1)$.
Finally, since $\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^\alpha dm_2(z) \le c_2 \int\limits_G \left|f^{(k)}(w)\right|^p d^\beta(w,\partial G)dm_2(w) < +\infty$, then $f^{(k)}(\varphi) \in A^p_\alpha(S)$, $k = 0,1,\ldots,n$, $n \in N$, $0 < p < +\infty$, $\alpha \ge 2(\beta+1)$. The lemma is proved.

Lemma 7. *Suppose* $1 < p < +\infty$, $z \in S$, $\eta > 0$, $0 < \frac{\gamma}{p} < \eta$.
Then

$$\int\limits_S \frac{(1-|\zeta|^2)^\eta}{\left|1-\overline{\zeta}z\right|^{\eta+1}(1-|\zeta|)^{\frac{\gamma}{p}+1}}\,dm_2(\zeta) \le c(1-|z|)^{-\frac{\gamma}{p}}.$$

P r o o f. Suppose $z = re^{i\sigma}$, $\zeta = \rho e^{i\theta}$; then $\int\limits_S \frac{(1-|\zeta|^2)^\eta}{\left|1-\overline{\zeta}z\right|^{\eta+1}(1-|\zeta|)^{\frac{\gamma}{p}+1}}\,dm_2(\zeta)$

$$= \int\limits_0^1\int\limits_{-\pi}^{\pi} \frac{(1-\rho^2)^\eta}{\left|1-r\rho e^{i(\sigma-\theta)}\right|^{\eta+1}(1-\rho)^{\frac{\gamma}{p}+1}}d\theta d\rho = \int\limits_0^1 \frac{(1-\rho^2)^\eta}{(1-\rho)^{\frac{\gamma}{p}+1}}\int\limits_{-\pi}^{\pi}\frac{d\theta}{\left|1-r\rho e^{i(\sigma-\theta)}\right|^{\eta+1}}d\rho.$$

Since $\int\limits_{-\pi}^{\pi} \frac{d\theta}{\left|1 - r\rho e^{i(\sigma-\theta)}\right|^{\eta+1}} \leq \frac{c_1}{(1-r\rho)^{\eta}}$, then

$$\int\limits_0^1 \frac{(1-\rho^2)^{\eta}}{(1-\rho)^{\frac{\gamma}{p}+1}} \int\limits_{-\pi}^{\pi} \frac{d\theta}{\left|1 - r\rho e^{i(\sigma-\theta)}\right|^{\eta+1}} d\rho \leq c_2 \int\limits_0^1 \frac{(1-\rho^2)^{\eta}}{(1-r\rho)^{\eta}(1-\rho)^{\frac{\gamma}{p}+1}} d\rho.$$

However, if $\eta > 0$, $0 < \frac{\gamma}{p} < \eta$, then

$$\int\limits_0^1 \frac{(1-\rho^2)^{\eta}}{(1-r\rho)^{\eta}(1-\rho)^{\frac{\gamma}{p}+1}} d\rho$$

$$\leq c_3 \int\limits_0^r \frac{(1-\rho^2)^{\eta}}{(1-\rho)^{\eta}(1-\rho)^{\frac{\gamma}{p}+1}} d\rho + c_4 \int\limits_r^1 \frac{(1-\rho^2)^{\eta}}{(1-r)^{\eta}(1-\rho)^{\frac{\gamma}{p}+1}} d\rho \leq \frac{c}{(1-r)^{\frac{\gamma}{p}}}.$$

This completes the proof.

Lemma 8. *Let G be a simply connected domain with boundary $\Gamma \in (L)$. Suppose $\varphi : S \to G$ conformally, $\zeta \in S$, $\tau > -1$, $k \in Z_+$. If $1 < p$, $q < +\infty$, $\chi_\gamma(\zeta) = (1-|\zeta|)^{-\left(\frac{\gamma}{pq}\right)}$, $0 < \frac{\gamma}{q} < kp+\tau+1$, $\eta > kp+\tau+2+\frac{\gamma}{q}$, then*

$$\int\limits_S \frac{|\varphi'(z)|^{kp+\tau+2}(1-|z|)^{kp+\tau}\chi_\gamma^p(z)}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(z)$$

$$\leq \frac{c_1 |\varphi'(\zeta)|^{kp+\tau+2}(1-|\zeta|)^{kp+\tau}\chi_\gamma^p(\zeta)}{(1-|\zeta|)^{\eta-1}}. \tag{7}$$

If $0 < p \leq 1$, $\eta > k - 1 + \frac{\tau+3}{p}$, then

$$\int\limits_S \frac{|\varphi'(z)|^{kp+\tau+2}(1-|z|)^{kp+\tau}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} dm_2(z)$$

$$\leq \frac{c_2 |\varphi'(\zeta)|^{kp+\tau+2}(1-|\zeta|)^{kp+\tau}}{(1-|\zeta|)^{p(\eta+1)-2}}. \tag{7'}$$

P r o o f. Let $f(z) = \frac{kp+\tau+2}{2}\ln\varphi'(z)$, $z \in S$, $z = re^{i\sigma}$. Using Lemmas 4 and 5, we get

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \left|\varphi'(e^{i\sigma})\right|^{kp+\tau+2} \frac{(1-|t|^2)}{\left|1-\overline{t}e^{i\sigma}\right|^2} d\sigma \leq M\left|\varphi'(t)\right|^{kp+\tau+2}, \tag{8}$$

where $0 < |t| < 1$.
Suppose

$$I = \int\limits_{S} \frac{|\varphi'(z)|^{kp+\tau+2}(1-|z|)^{kp+\tau}\chi_{\gamma}^{p}(z)}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(z).$$

Since $\zeta = \rho e^{i\theta}$, we obtain

$$I = \int\limits_{0}^{1} (1-r)^{kp+\tau-\frac{\gamma}{q}} \int\limits_{-\pi}^{\pi} \left|\varphi'(re^{i\sigma})\right|^{kp+\tau+2} \frac{1}{\left|1-r\rho e^{i\sigma}e^{-i\theta}\right|^{\eta+1}} d\sigma dr$$

$$\leq c_0 \int\limits_{0}^{1} \frac{(1-r)^{kp+\tau-\frac{\gamma}{q}}}{(1-r\rho)^{\eta-1}} \int\limits_{-\pi}^{\pi} \left|\varphi'(re^{i\sigma})\right|^{kp+\tau+2} \frac{1}{\left|1-r\rho e^{i\sigma}e^{-i\theta}\right|^{2}} d\sigma dr.$$

By the construction, $\varphi'(z) \neq 0$, $z \in S$, and $(\varphi'(z))^{kp+\tau+2}$ is an analytic function in the unit disk S. The function $\Psi_{\zeta}(z) = \dfrac{1}{(1-\overline{\zeta}z)^2}$ is also analytic in S for the fixed $\zeta \in S$. Then $\Psi_{\zeta}(z)(\varphi'(z))^{kp+\tau+2}$ is an analytic function in S.

It follows that if

$$I_1(r) = \int\limits_{-\pi}^{\pi} \left|\varphi'(re^{i\sigma})\right|^{kp+\tau+2} \frac{1}{\left|1-r\rho e^{i\sigma}e^{-i\theta}\right|^{2}} d\sigma = \int\limits_{-\pi}^{\pi} \left|\varphi'(re^{i\sigma})\right|^{kp+\tau+2} \left|\Psi_{\zeta}(re^{i\sigma})\right| d\sigma,$$

then $I_1(r)$ monotonically grows on $[0,1)$. Hence we obtain

$$I_1(r) \leq \int\limits_{-\pi}^{\pi} \left|\varphi'(e^{i\sigma})\right|^{kp+\tau+2} \frac{(1-\rho^2)}{\left|1-\rho e^{i\sigma}e^{-i\theta}\right|^{2}} \frac{1}{(1-\rho^2)} d\sigma$$

$$= \frac{1}{(1-\rho^2)} \int\limits_{-\pi}^{\pi} \left|\varphi'(e^{i\sigma})\right|^{kp+\tau+2} \frac{(1-\rho^2)}{\left|1-\rho e^{i\sigma}e^{-i\theta}\right|^{2}} d\sigma.$$

With $t = \zeta$ and (8) being taken into account, we get

$$I_1(r) \leq \frac{c_1 \left|\varphi'(\rho e^{i\theta})\right|^{kp+\tau+2}}{(1-\rho^2)}.$$

Using the above, we have

$$I \leq \frac{c_2 \left|\varphi'(\rho e^{i\theta})\right|^{kp+\tau+2}}{(1-\rho^2)} \int\limits_{0}^{1} \frac{(1-r)^{kp+\tau-\frac{\gamma}{q}}}{(1-r\rho)^{\eta-1}} dr.$$

But, if $0 < \frac{\gamma}{q} < kp + \tau + 1$, $\eta > kp + \tau + 2 + \frac{\gamma}{q}$, then

$$\int_0^1 \frac{(1-r)^{kp+\tau-\frac{\gamma}{q}}}{(1-r\rho)^{\eta-1}} dr \le c_3 \int_0^\rho \frac{(1-r)^{kp+\tau-\frac{\gamma}{q}}}{(1-r)^{\eta-1}} dr + c_4 \int_\rho^1 \frac{(1-r)^{kp+\tau-\frac{\gamma}{q}}}{(1-\rho)^{\eta-1}} dr$$

$$\le \frac{c_5(1-\rho)^{kp+\tau-\frac{\gamma}{q}}}{(1-\rho)^{\eta-2}}.$$

However, we see that $I \le c_6 \left|\varphi'(\rho e^{i\theta})\right|^{kp+\tau+2} \frac{(1-\rho)^{kp+\tau-\frac{\gamma}{q}}}{(1-\rho)^{\eta-1}}$. Finally, we obtain

$$\int_S \frac{|\varphi'(z)|^{kp+\tau+2}(1-|z|)^{kp+\tau}\chi_\gamma^p(z)}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(z)$$

$$\le \frac{c\,|\varphi'(\zeta)|^{kp+\tau+2}(1-|\zeta|)^{kp+\tau}\chi_\gamma^p(\zeta)}{(1-|\zeta|)^{\eta-1}}.$$

The analogous estimate $(7')$ follows easily. The proof is finished.

2. The Formulation and the Proof of Basic Theorems

Theorem 1. *Let G be any connected open set in the complex plane C. Suppose $f \in A^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$. Then for any $n \in N$ we have*

$$\int_G \left|f^{(n)}(w)\right|^p d^{np+\beta}(w, \partial G) dm_2(w) \le c(n,\beta) \int_G |f(w)|^p d^\beta(w,\partial G) dm_2(w).$$

P r o o f. Let $G = \bigcup_k Q_k$ be the Whitney decomposition sets G, where Q_k defined is a square such that $c_1 diam(Q_k) \le dist(Q_k, {}^cG) \le c_2 diam(Q_k)$, the constants c_1, c_2 do not depend on G (see [9, p. 199]). It is possible to take $c_1 = 1, c_2 = 4$. Then

$$\int_G \left|f^{(n)}(w)\right|^p d^{np+\beta}(w,\partial G) dm_2(w) = \sum_k \int_{Q_k} \left|f^{(n)}(w)\right|^p d^{np+\beta}(w,\partial G) dm_2(w)$$

$$\le c \sum_k \max_{w \in Q_k} \left|f^{(n)}(w)\right|^p d^{np+\beta+2}(w,\partial G) \le c \sum_k \left|f^{(n)}(w_k)\right|^p d^{np+\beta+2}(w_k,\partial G),$$

where $w_k \in \partial Q_k$. Next, by Q_k^* denote the square with the same center as Q_k but stretched in $(1+\varepsilon)$ times, $0 < \varepsilon < \frac{1}{4}$. Then $Q_k \subset Q_k^*$.

Let $0 < \rho = \frac{1}{4} dist(Q_k, \partial Q_k^*)$, $C_\rho(w_k) = \{w : |w - w_k| < \rho\}$.
Since $f^{(n)}(w_k) = \frac{n!}{2\pi i} \int\limits_{\partial C_\rho} \frac{f(w)}{(w - w_k)^{n+1}} dw$, it follows that

$$\left| f^{(n)}(w_k) \right| \le n! \frac{1}{\rho^n} \max_{w \in \partial C_\rho} |f(w)| \le \frac{c}{d^n(\tilde{w}_k, \partial G)} |f(\tilde{w}_k)|,$$

where $\tilde{w}_k \in \partial C_\rho$.

Hence we get $\left| f^{(n)}(w_k) \right|^p \le \frac{c_1 |f(\tilde{w}_k)|^p}{d^{np}(\tilde{w}_k, \partial G)}$. Using the facts that $d(w_k, \partial G) \le d(\tilde{w}_k, \partial G)$, we have

$$\sum_k \left| f^{(n)}(w_k) \right|^p d^{np+\beta+2}(w_k, \partial G) \le c_1 \sum_k |f(\tilde{w}_k)|^p d^{\beta+2}(\tilde{w}_k, \partial G).$$

Next, let $0 < \rho' = \frac{1}{8} dist(Q_k, \partial Q_k^*)$ and $K_{\rho'}(\tilde{w}_k) = \{w : |w - \tilde{w}_k| < \rho'\}$. It is clear that $K_{\rho'}(\tilde{w}_k) \subset Q_k^*$. Therefore, we see that

$$|f(\tilde{w}_k)|^p \le \frac{1}{\pi \rho^2} \int\limits_{K_{\rho'}(\tilde{w}_k)} |f(w)|^p dm_2(w) \le \frac{c_2}{d^2(\tilde{w}_k, \partial G)} \int\limits_{Q_k^*} |f(w)|^p dm_2(w).$$

Thus we get $|f(\tilde{w}_k)|^p d^{\beta+2}(\tilde{w}_k, \partial G) \le c_3 \int\limits_{Q_k^*} |f(w)|^p d^\beta(w, \partial G) dm_2(w)$.

Finally, we have

$$\int\limits_G \left| f^{(n)}(w) \right|^p d^{np+\beta}(w, \partial\Omega) dm_2(w)$$

$$\le \sum_k \int\limits_{Q_k^*} |f(w)|^p d^\beta(w, \partial G) dm_2(w) \le c_4 \int\limits_G |f(w)|^p d^\beta(w, \partial G) dm_2(w).$$

The theorem is proved.

Similarly, the following theorem holds.

Theorem 2 (see [10]). *Let G be any connected open set in the complex plane C. Suppose $u \in h^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$. Then*

$$\int\limits_G |grad\, u(w)|^p d^{p+\beta}(w, \partial G) dm_2(w) \le c(\beta) \int\limits_G |u(w)|^p d^\beta(w, \partial G) dm_2(w).$$

Theorem 3. *Let G be a simply connected domain with boundary $\Gamma \in (L)$. Suppose $f \in H(G)$, $f^{(k)}(w_0) = 0$, $k = 0, 1, \ldots, n-1$, $n \in N$, $w_0 \in G$; $\tau > -1$,*

$0 < p < +\infty$. *Then the following is valid:*

$$c_1(n,\tau)\int\limits_G \left|f^{(n)}(w)\right|^p d^{np+\tau}(w,\partial G)dm_2(w)$$

$$\leq \int\limits_G |f(w)|^p d^\tau(w,\partial G)dm_2(w)$$

$$\leq c_2(n,\tau)\int\limits_G \left|f^{(n)}(w)\right|^p d^{np+\tau}(w,\partial G)dm_2(w). \tag{10}$$

P r o o f. Using Theorem 1, we see that

$$c_1(n,\tau)\int\limits_G \left|f^{(n)}(w)\right|^p d^{np+\tau}(w,\partial G)dm_2(w)$$

$$\leq \int\limits_G |f(w)|^p d^\tau(w,\partial G)dm_2(w).$$

In the proof of the right estimation the induction method is used.
For n=1, let us prove that

$$I = \int\limits_G |f(w)|^p d^\tau(w,\partial G)dm_2(w) \leq c\int\limits_G \left|f'(w)\right|^p d^{p+\tau}(w,\partial G)\, dm_2(w). \tag{11}$$

Without loss of generality, assume that the integral on the right is convergent. Suppose $\varphi : S \to G$ conformally, $\varphi(0) = w_0, \varphi'(0) > 0$, $w = \varphi(z)$; then

$$\int\limits_S |f(\varphi(z))|^p d^\tau(\varphi(z),\partial G)\left|\varphi'(z)\right|^2\, dm_2(z)$$

$$\leq c\int\limits_S \left|f'(\varphi(z))\right|^p d^{p+\tau}(\varphi(z),\partial G)\left|\varphi'(z)\right|^2\, dm_2(z).$$

Thus, using (5), we can see that

$$\int\limits_S |f(\varphi(z))|^p (1-|z|)^\tau \left|\varphi'(z)\right|^{\tau+2}\, dm_2(z)$$

$$\leq c\int\limits_S \left|f'(\varphi(z))\right|^p (1-|z|)^{p+\tau}\left|\varphi'(z)\right|^{p+\tau+2}\, dm_2(z). \tag{12}$$

Let $F(z) = f(\varphi(z))$, then $\int\limits_S \left|F'(z)\right|^p (1-|z|)^{p+\tau} \left|\varphi'(z)\right|^{\tau+2} dm_2(z) < +\infty$.
Using (6), we get $|\varphi'(z)| \geq c(1-|z|)$. Hence, we see that

$$\int\limits_S \left|F'(z)\right|^p (1-|z|)^{p+2(\tau+1)} dm_2(z) < +\infty.$$

Taking into account (2), we obtain

$$\int\limits_S \left|F'(z)\right|^p (1-|z|)^{2(\tau+1)} dm_2(z) < c \int\limits_S |F(z)|^p (1-|z|)^{p+2(\tau+1)} dm_2(z) < +\infty,$$

that is $f(\varphi) \in A^p_\alpha(S)$, $0 < p < +\infty$, $\alpha \geq 2(\tau+1)$.

Let us consider the two cases of the proof (12).

Case 1: $0 < p \leq 1$. Using $f(\varphi) \in A^p_\alpha(S)$, $0 < p < +\infty$, $\alpha \geq 2(\tau+1)$, and Lemma 2 for $\eta > -1$, we have

$$f(\varphi(z)) = \int\limits_S \frac{(1-|\zeta|^2)^\eta (f(\varphi(\zeta)))' P(z,\overline{\zeta})}{(1-\overline{\zeta}z)^{\eta+1}} dm_2(\zeta).$$

However, we see that $|f(\varphi(z))| \leq c \int\limits_S \frac{(1-|\zeta|^2)^\eta}{\left|1-\overline{\zeta}z\right|^{\eta+1}} \left|f'(\varphi(\zeta))\right| \left|\varphi'(\zeta)\right| dm_2(\zeta)$.
Applying Lemma 3, we obtain

$$|f(\varphi(z))|^p \leq c_1 \int\limits_S \frac{(1-|\zeta|^2)^{\eta p+2p-2}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} \left|f'(\varphi(\zeta))\right|^p \left|\varphi'(\zeta)\right|^p dm_2(\zeta).$$

Now we get

$$|f(\varphi(z))|^p (1-|z|)^\tau \left|\varphi'(z)\right|^{\tau+2}$$
$$\leq c_1 (1-|z|)^\tau \left|\varphi'(z)\right|^{\tau+2} \int\limits_S \frac{(1-|\zeta|^2)^{\eta p+2p-2}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} \left|f'(\varphi(\zeta))\right|^p \left|\varphi'(\zeta)\right|^p dm_2(\zeta).$$

Integrating with respect to z and changing the order of integration, we have

$$\int\limits_S |f(\varphi(z))|^p (1-|z|)^\tau \left|\varphi'(z)\right|^{\tau+2} dm_2(z)$$

$$\leq c_2 \int\limits_S \left|f'(\varphi(\zeta))\right|^p (1-|\zeta|^2)^{\eta p+2p-2} \left|\varphi'(\zeta)\right|^p \int\limits_S \frac{|\varphi'(z)|^{\tau+2} (1-|z|)^\tau}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} dm_2(z) dm_2(\zeta).$$

Using Lemma 8 for $k = 0$, $\eta > \dfrac{\tau+3}{p} - 1$, we obtain

$$\int_S \frac{|\varphi'(z)|^{\tau+2}(1-|z|)^{\tau}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} dm_2(z) \le \frac{c_3 |\varphi'(\zeta)|^{\tau+2}(1-|\zeta|)^{\tau}}{(1-|\zeta|)^{p(\eta+1)-2}}.$$

Combing this with the last inequality, we get (12) and, consequently, (10) for $n = 1$, $0 < p \le 1$.

Case 2: $1 < p < +\infty$. As above, we have

$$|f(\varphi(z))| \le c \int_S \frac{(1-|\zeta|^2)^{\eta}}{\left|1-\overline{\zeta}z\right|^{\eta+1}} \left|f'(\varphi(\zeta))\right| \left|\varphi'(\zeta)\right| dm_2(\zeta).$$

Multiplying and dividing the right-hand side of the above inequality by the function $\chi_{\gamma}(\zeta) = (1-|\zeta|)^{-(\frac{\gamma}{pq}+\frac{1}{q})}$, $0 < \dfrac{\gamma}{q} < \tau + 1$, and then using Holder's inequality with the exponent p, we get

$$|f(\varphi(z))|^p \le c_1 \int_S \frac{(1-|\zeta|^2)^{\eta}}{\left|1-\overline{\zeta}z\right|^{\eta+1}\chi_{\gamma}^{p}(\zeta)} \left|f'(\varphi(\zeta))\right|^p \left|\varphi'(\zeta)\right|^p dm_2(\zeta)$$

$$\times \left(\int_S \frac{(1-|\zeta|^2)^{\eta}\chi_{\gamma}^{q}(\zeta)}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(\zeta)\right)^{\frac{p}{q}}.$$

Using Lemma 7, we obtain $\displaystyle\int_S \frac{(1-|\zeta|^2)^{\eta}\chi_{\gamma}^{q}(\zeta)}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(\zeta) \le \frac{c_2}{(1-|z|)^{\frac{\gamma}{p}}}$.

Likewise as in the above, we have

$$\int_S |f(\varphi(z))|^p (1-|z|)^{\tau} \left|\varphi'(z)\right|^{\tau+2} dm_2(z) \le c_3 \int_S \left|f'(\varphi(\zeta))\right|^p (1-|\zeta|^2)^{\eta} \left|\varphi'(\zeta)\right|^p$$

$$\frac{1}{\chi_{\gamma}^{p}(\zeta)} \int_S \frac{|\varphi'(z)|^{\tau+2}(1-|z|)^{\tau}(1-|z|)^{-\frac{\gamma}{q}}}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(z)\, dm_2(\zeta).$$

Applying Lemma 8 for $k = 0$, $0 < \dfrac{\gamma}{q} < 1 + \tau$, $\eta > \tau + 2 + \dfrac{\gamma}{q}$, we get

$$\int_S \frac{|\varphi'(z)|^{\tau+2}(1-|z|)^{\tau}(1-|z|)^{-\frac{\gamma}{q}}}{\left|1-\overline{\zeta}z\right|^{\eta+1}} dm_2(z) \le \frac{c_4 |\varphi'(\zeta)|^{\tau+2}(1-|\zeta|)^{\tau}(1-|\zeta|)^{-\frac{\gamma}{q}}}{(1-|\zeta|)^{\eta-1}}.$$

Combing this with the last inequality, we get (12) and, consequently, (10) for $n = 1$, $1 < p < +\infty$. Now, by the induction hypothesis, the inequality

$$\int_G |f(w)|^p d^{\tau}(w, \partial G)\, dm_2(w) \le c \int_G \left|f^{(k)}(w)\right|^p d^{kp+\tau}(w, \partial G)\, dm_2(w)$$

holds and it is equivalent to

$$\int\limits_S |f(\varphi(z))|^p d^\tau(\varphi(z), \partial G) \left|\varphi'(z)\right|^2 \, dm_2(z)$$

$$\le c_1 \int\limits_S \left|f^{(k)}(\varphi(z))\right|^p d^{kp+\tau}(\varphi(z), \partial G) \left|\varphi'(z)\right|^2 \, dm_2(z).$$

Using (5), we obtain

$$\int\limits_S |f(\varphi(z))|^p (1-|z|)^\tau \left|\varphi'(z)\right|^{\tau+2} \, dm_2(z)$$

$$\le c_2 \int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2} \, dm_2(z). \quad (13)$$

Prove that

$$\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2} \, dm_2(z)$$

$$\le c_5 \int\limits_S \left|f^{(k+1)}(\varphi(z))\right|^p (1-|z|)^{(k+1)p+\tau} \left|\varphi'(z)\right|^{(k+1)p+\tau+2} \, dm_2(z). \quad (14)$$

Without loss of generality, similarly as in the above we may again assume that

$$\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2} \, dm_2(z) < +\infty.$$

Then

$$\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{2(kp+\tau+1)} \, dm_2(z) < +\infty.$$

Hence, we obtain $f^{(k)}(\varphi) \in A^p_\alpha(S)$, $0 < p < +\infty$, $\alpha > 2(kp+\tau+1)$. By Lemma 2, for $\eta > -1$ we have

$$f^{(k)}(\varphi(z)) = \int\limits_S \frac{(1-|\zeta|^2)^\eta (f^{(k)}(\varphi(\zeta)))' P(z,\overline{\zeta})}{(1-\overline{\zeta}z)^{\eta+1}} \, dm_2(\zeta).$$

Therefore, we get $\left|f^{(k)}(\varphi(z))\right| \le c \int\limits_S \frac{(1-|\zeta|^2)^\eta}{\left|1-\overline{\zeta}z\right|^{\eta+1}} \left|f^{(k+1)}(\varphi(\zeta))\right| \left|\varphi'(\zeta)\right| dm_2(\zeta).$

Let us consider the two cases of the proof (14).

Case 1: $0 < p \le 1$. Applying Lemma 3, we see that

$$\left|f^{(k)}(\varphi(z))\right|^p \le c_1 \int\limits_S \frac{(1-|\zeta|^2)^{\eta p+2p-2}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} \left|f^{(k+1)}(\varphi(\zeta))\right|^p \left|\varphi'(\zeta)\right|^p dm_2(\zeta).$$

On the other hand,

$$\left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2} \le c_2 (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2}$$

$$\times \int\limits_S \frac{(1-|\zeta|^2)^{\eta p+2p-2}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} \left|f^{(k+1)}(\varphi(\zeta))\right|^p \left|\varphi'(\zeta)\right|^p dm_2(\zeta).$$

Integrating with respect to z and changing the order of integration, we have

$$\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2} dm_2(z)$$

$$\le c_3 \int\limits_S \left|f^{(k+1)}(\varphi(\zeta))\right|^p (1-|\zeta|^2)^{\eta p+2p-2} \left|\varphi'(\zeta)\right|^p$$

$$\int\limits_S \frac{|\varphi'(z)|^{kp+\tau+2} (1-|z|)^{kp+\tau}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} dm_2(z) dm_2(\zeta).$$

Applying Lemma 8 for $\eta > k-1+\dfrac{\tau+3}{p}$, we see that

$$\int\limits_S \frac{|\varphi'(z)|^{kp+\tau+2} (1-|z|)^{kp+\tau}}{\left|1-\overline{\zeta}z\right|^{p(\eta+1)}} dm_2(z) \le \frac{c_4 \left|\varphi'(\zeta)\right|^{kp+\tau+2} (1-|\zeta|)^{kp+\tau}}{(1-|\zeta|)^{p(\eta+1)-2}}.$$

Combing this with the last inequality, we get (14) for $0 < p \le 1$.

Case 2: $1 < p < +\infty$. As above, we obtain

$$\left|f^{(k)}(\varphi(z))\right| \le c \int\limits_S \frac{(1-|\zeta|^2)^{\eta}}{\left|1-\overline{\zeta}z\right|^{\eta+1}} \left|f^{(k+1)}(\varphi(\zeta))\right| \left|\varphi'(\zeta)\right| dm_2(\zeta).$$

Let $\chi_\gamma(\zeta) = (1-|\zeta|)^{-(\frac{\gamma}{pq}+\frac{1}{q})}$, $0 \le \dfrac{\gamma}{q} < kp+\tau+1$.

Applying Holder's inequality, we conclude that

$$\left|f^{(k)}(\varphi(z))\right|^p \le c \int\limits_S \frac{(1-|\zeta|^2)^{\eta}}{\left|1-\overline{\zeta}z\right|^{\eta+1} \chi_\gamma^p(\zeta)} \left|f^{(k+1)}(\zeta)\right|^p \left|\varphi'(z)\right|^p dm_2(\zeta) \times$$

$$\left(\int\limits_S \frac{(1-|\zeta|^2)^\eta \chi_\gamma^q(\zeta)}{\left|1-\overline{\zeta}z\right|^{\eta+1}}\,dm_2(\zeta)\right)^{\frac{p}{q}}.$$

However, by Lemma 7, we obtain $\int\limits_S \frac{(1-|\zeta|^2)^\eta \chi_\gamma^q(\zeta)}{\left|1-\overline{\zeta}z\right|^{\eta+1}}\,dm_2(\zeta) \le \frac{c_2}{(1-|z|)^{\frac{\gamma}{p}}}$.

Thus, we have

$$\int\limits_S \left|f^{(k)}(\varphi(z))\right|^p (1-|z|)^{kp+\tau} \left|\varphi'(z)\right|^{kp+\tau+2}\,dm_2(z)$$

$$\le c_3 \int\limits_S \left|f^{(k+1)}(\varphi(\zeta))\right|^p (1-|\zeta|^2)^\eta \left|\varphi'(\zeta)\right|^p \frac{1}{\chi_\gamma^{\frac{p}{q}}(\zeta)}$$

$$\int\limits_S \frac{|\varphi'(z)|^{kp+\tau+2}\,(1-|z|)^{kp+\tau}(1-|z|)^{-\frac{\gamma}{q}}}{\left|1-\overline{\zeta}z\right|^{\eta+1}}\,dm_2(z)\,dm_2(\zeta).$$

Applying Lemma 8 for $0<\frac{\gamma}{q}<kp+\tau+1$, $\eta>kp+\tau+2+\frac{\gamma}{q}$, we see that

$$\int\limits_S \frac{|\varphi'(z)|^{kp+\tau+2}\,(1-|z|)^{kp+\tau}(1-|z|)^{-\frac{\gamma}{q}}}{\left|1-\overline{\zeta}z\right|^{\eta+1}}\,dm_2(z)$$

$$\le \frac{c_4\,|\varphi'(\zeta)|^{kp+\tau+2}\,(1-|\zeta|)^{kp+\tau}(1-|\zeta|)^{-\frac{\gamma}{q}}}{(1-|\zeta|)^{\eta-1}}.$$

Combing this with the last inequality, we get (14) for $1<p<+\infty$.
Also, we claim that

$$\int\limits_G |f(w)|^p d^\tau(w,\partial G)dm_2(w) \le c_3 \int\limits_G \left|f^{(k+1)}(w)\right|^p d^{(k+1)p+\tau}(w,\partial G)dm_2(w) \quad (15)$$

or

$$\int\limits_S |f(\varphi(z))|^p (1-|z|)^\tau \left|\varphi'(z)\right|^{\tau+2}\,dm_2(z)$$

$$\le c_4 \int\limits_S \left|f^{(k+1)}(\varphi(z))\right|^p (1-|z|)^{(k+1)p+\tau} \left|\varphi'(z)\right|^{(k+1)p+\tau+2}\,dm_2(z),$$

where $0<p<+\infty$. Indeed, using (13) and (14) for $0<p<+\infty$, we obtain (15).
Finally, we have proved that

$$\int\limits_G |f(w)|^p d^\tau(w,\partial G)\,dm_2(w) \le c \int\limits_G \left|f^{(n)}(z)\right|^p d^{np+\tau}(w,\partial G)\,dm_2(w)$$

for every $n \in N$, $0<p<+\infty$.

Theorem 4. *Let G be a simply connected domain with boundary $\Gamma \in (L)$. Suppose $f \in H(G)$, $f(w_0) = 0$, $w_0 \in G$; $\varphi : S \to G$ conformally, $\varphi(0) = w_0$, $\varphi'(0) > 0$, ψ is the converse function. If $f = u + iv$, $u \in h^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$, then $f \in A^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$, and the operator*

$$P_\alpha(u)(w) = \frac{\alpha+1}{\pi} \int\limits_G \frac{(1 - |\psi(\mu)|^2)^\alpha}{(1 - \overline{\psi(\mu)}\psi(w))^{\alpha+2}} u(\mu) \left|\psi'(\mu)\right|^2 dm_2(\mu) \tag{16}$$

determines a bounded linear operator $h^p_\beta(G) \to A^p_\beta(G)$ for $\alpha \geq 2(\beta+1)$.
In particular, the operator of harmonic conjugate $v = \Gamma(u)$ determines a bounded linear operator $h^p_\beta(G) \to h^p_\beta(G)$ for all $0 < p < +\infty$, $\beta > -1$.

P r o o f. We claim that if $u \in h^p_\beta(G)$, then $f \in A^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$. Indeed, using Theorem 3, we get

$$\int\limits_G |f(w)|^p d^\beta(w, \partial G) dm_2(w) \leq c \int\limits_G \left|f'(w)\right|^p d^{p+\beta}(w, \partial G) dm_2(w). \tag{17}$$

Since $|f'(w)| = |gradu(w)|$, it follows that

$$\int\limits_G |f(w)|^p d^\beta(w, \partial G) dm_2(w) \leq c \int\limits_G |gradu(w)|^p d^{p+\beta}(w, \partial G) dm_2(w).$$

Using Theorem 2, we obtain

$$\int\limits_G |gradu(w)|^p d^{p+\beta}(w, \partial G) dm_2(w) \leq c_1 \int\limits_G |u(w)|^p d^\beta(w, \partial G) dm_2(w).$$

Hence we have

$$\int\limits_G |f(w)|^p d^\beta(w, \partial G) dm_2(w) \leq c_1 \int\limits_G |u(w)|^p d^\beta(w, \partial G) dm_2(w) < +\infty. \tag{18}$$

However, we see that $f \in A^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$. By Lemma 6, for $f \in A^p_\beta(G)$, $0 < p < +\infty$, $\beta > -1$ we get $f(\varphi) \in A^p_\alpha(S)$, $\alpha \geq 2(\beta+1)$. By (3) for $f(\varphi(0)) = f(w_0) = 0$, so that

$$f(\varphi(z)) = \frac{\alpha+1}{\pi} \int\limits_S \frac{(1 - |\zeta|^2)^\alpha u(\varphi(\zeta))}{(1 - \overline{\zeta} z)^{\alpha+2}} dm_2(\zeta).$$

Substituting z for $\psi(w)$, ζ for $\psi(\mu)$, we get

$$f(w) = \frac{\alpha+1}{\pi} \int\limits_G \frac{(1 - |\psi(\mu)|^2)^\alpha u(\mu)}{(1 - \overline{\psi(\mu)}\psi(w))^{\alpha+2}} \left|\psi'(\mu)\right|^2 dm_2(\mu).$$

Combing this with (18), we get the statement of the theorem.

R e m a r k 1. For the case of the domains with smooth boundary a similar statement was carried out by the second author in [11] for $0 < p < +\infty$.

Theorem 5. *Let G be a simply connected domain with boundary $\Gamma \in (Ł)$. Suppose $f \in H(G)$, $f(w_0) = 0$, $w_0 \in G$; $\varphi : S \to G$ conformally, ψ is the converse function. Then the operator*

$$F(w) = P_\alpha(f)(w) = \frac{\alpha+1}{\pi} \int\limits_G \frac{(1-|\psi(\mu)|^2)^\alpha}{(1-\overline{\psi(\mu)}\psi(w))^{\alpha+2}} f(\mu)\left|\psi'(\mu)\right|^2 dm_2(\mu)$$

is a bounded projection from $L^p_\beta(G)$ to $A^p_\beta(G)$ for $1 \le p < +\infty$, $\alpha \ge \beta$, moreover,

$$\|F\|_{A^p_\beta(G)} \le c(\beta,p)\, \|f\|_{L^p_\beta(G)}. \tag{19}$$

P r o o f. If $f \in A^p_\beta(G)$, then $F(w) = f(w)$, $w \in G$, $\alpha \ge \beta$. We claim that if $f \in L^p_\beta(G)$, then $F \in A^p_\beta(G)$ and

$$\int\limits_S |F(\varphi(z))|^p (1-|z|)^\beta \left|\varphi'(z)\right|^{\beta+2} dm_2(z)$$

$$\le \int\limits_S |f(\varphi(z))|^p (1-|z|)^\beta \left|\varphi'(z)\right|^{\beta+2} dm_2(z). \tag{20}$$

Indeed, we get $F(\varphi(z)) = c \int\limits_S \frac{(1-|\zeta|^2)^\alpha f(\varphi(\zeta))}{(1-\overline{\zeta}z)^{\alpha+2}} dm_2(\zeta)$. And hence, we have

$$|F(\varphi(z))| \le c \int\limits_S \frac{(1-|\zeta|^2)^\alpha}{\left|1-\overline{\zeta}z\right|^{\alpha+2}} |f(\varphi(\zeta))|\, dm_2(\zeta).$$

Multiplying and dividing the right-hand side of the above inequality by the function $\chi_\gamma(\zeta) = (1-|\zeta|)^{-(\frac{\gamma}{pq})}$, $0 < \dfrac{\gamma}{q} < \beta+1$, and applying Holder's inequality with the exponent p, we get

$$|F(\varphi(z))|^p$$

$$\le c_1 \int\limits_S \frac{(1-|\zeta|^2)^\alpha}{\left|1-\overline{\zeta}z\right|^{\alpha+2} \chi^p_\gamma(\zeta)} |f(\varphi(\zeta))|^p\, dm_2(\zeta) \times \left(\int\limits_S \frac{(1-|\zeta|^2)^\alpha \chi^q_\gamma(\zeta)}{\left|1-\overline{\zeta}z\right|^{\alpha+2}} dm_2(\zeta) \right)^{\frac{p}{q}}.$$

It is easy to prove that $\int\limits_S \frac{(1-|\zeta|^2)^\alpha \chi_\gamma^q(\zeta)}{\left|1-\overline{\zeta} z\right|^{\alpha+2}}\, dm_2(\zeta) \le \frac{c_2}{(1-|z|)^{\frac{\gamma}{p}}}$.

Hence we get

$$\int\limits_S |F(\varphi(z))|^p\,(1-|z|)^\beta \left|\varphi'(z)\right|^{\beta+2} dm_2(z) \le c_3 \int\limits_S |f(\varphi(\zeta))|^p\,(1-|\zeta|^2)^\alpha \frac{1}{\chi_\gamma^p(\zeta)}$$

$$\int\limits_S \frac{|\varphi'(z)|^{\beta+2}\,(1-|z|)^\beta (1-|z|)^{-\frac{\gamma}{q}}}{\left|1-\overline{\zeta} z\right|^{\alpha+2}}\, dm_2(z)\, dm_2(\zeta). \tag{21}$$

Using Lemma 8 for $k=0$, $\tau=\beta$, $0<\frac{\gamma}{q}<1+\beta$, $\alpha>\beta+1+\frac{\gamma}{q}$, we obtain

$$\int\limits_S \frac{|\varphi'(z)|^{\beta+2}\,(1-|z|)^\beta (1-|z|)^{-\frac{\gamma}{q}}}{\left|1-\overline{\zeta} z\right|^{\alpha+2}}\, dm_2(z) \le \frac{c_4\,|\varphi'(\zeta)|^{\beta+2}\,(1-|\zeta|)^\beta (1-|\zeta|)^{-\frac{\gamma}{q}}}{(1-|\zeta|)^\alpha}.$$

Combing this with (21), we get the statement of the theorem for the case $1<p<+\infty$. Using Lemma 8 for $\alpha>\beta+2$, we obtain the statement of the theorem for the case $p=1$.

R e m a r k 2. An analogue of Theorem 5 for integral operators with Bergman kernel is proved by a different method in [12] for domains with piecewise smooth boundary, and in [13] for domains having the angle $\frac{\pi}{\vartheta}$. However, it is shown in[12, 13] that for $p \notin (\frac{2}{1+\vartheta}; \frac{2}{1-\vartheta})$, $\frac{1}{2} \le \vartheta < 1$, the operator is not bounded as the operator from $L_0^p(\Omega)$ to $A_0^p(\Omega)$. According to [4], the operator acting from $L_0^p(\Omega)$ to $A_0^p(\Omega)$ is bounded in the case of simply connected domains for $p_0<p<\frac{p_0}{p_0-1}$, $p_0 \in [\frac{4}{3}; 2)$.

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