Journal of Mathematical Physics, Analysis, Geometry
2010, vol. 6, No. 1, pp. 96–127

On the Law of Addition of Random Matrices: Covariance of Traces of Resolvent for Random Summands

V. Vasilchuk

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv, 61103, Ukraine
E-mail:vasilchuk@ilt.kharkov.ua

Received June 12, 2009

We consider the ensemble of $n \times n$ random matrices $H_n = A_n + U_n^\dagger B_n U_n$, where A_n and B_n are random Hermitian (real symmetric) matrices, having the limiting Normalized Counting Measures of eigenvalues, and U_n is unitary (orthogonal) uniformly distributed over $U(n)$ ($O(n)$). We find the leading term of the asymptotic expansion of covariance of traces of resolvent of H_n and establish the Central Limit Theorem for linear eigenvalue statistics of H_n as $n \to \infty$.

Key words: Random matrices, Central limit theorem, eigenvalue distribution, linear statistics.

Mathematics Subject Classification 2000: 15A52 (primary); 60F05, 60G50 (secondary).

1. Results and Discussions

The paper deals with the Hermitian (real symmetric) $n \times n$ random matrices

$$H_n = A_n + U_n^\dagger B_n U_n, \tag{1.1}$$

where A_n and B_n are Hermitian (real symmetric) random matrices such that if $\{\lambda_l^{A_n}\}_{l=1}^n$ and $\{\lambda_l^{B_n}\}_{l=1}^n$ are eigenvalues of A_n and B_n and N_{A_n} and N_{B_n} are their Normalized Counting Measures (NCM), defined as

$$N_{A_n}(\Delta) = \#\{\lambda_l^{A_n} \in \Delta,\ l = 1, \ldots, n\} n^{-1},$$
$$N_{B_n}(\Delta) = \#\{\lambda_l^{B_n} \in \Delta,\ l = 1, \ldots, n\} n^{-1} \tag{1.2}$$

for any interval $\Delta \subset \mathbb{R}$, then there exist nonrandom probability measures N_A and N_B and for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\{|N_{A_n}(\Delta) - N_A(\Delta)| > \varepsilon\} = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\{|N_{B_n}(\Delta) - N_B(\Delta)| > \varepsilon\} = 0, \ \forall \Delta \subset \mathbb{R}. \tag{1.3}$$

We assume further that U_n in (1.1) is the random unitary (orthogonal) matrix, whose probability law is given by the normalized to unity Haar measure on the unitary (orthogonal) group $U(n)$, and all three random matrices A_n, B_n and U_n are independent. We will confine ourselves to the technically simplest case of the Hermitian A_n and B_n and unitary U_n in (1.1).

Our goal in this paper is to study the eigenvalue distribution of H_n of (1.1), given that of A_n and B_n. The simplest but important for practically any random matrix problem is the weak convergence of the Normalized Counting Measures of eigenvalues $\{\lambda_l^{H_n}\}_{l=1}^n$ of H_n

$$N_n(\Delta) = \#\{\lambda_l^{H_n} \in \Delta, \ l = 1, \dots, n\} n^{-1} \tag{1.4}$$

to a nonrandom measure as $n \to \infty$. Following general ideas of spectral theory, we study N_n via the resolvent

$$G_n(z) = (H_n - z)^{-1}, \ \mathrm{Im}\, z \neq 0, \tag{1.5}$$

of H_n and its normalized trace

$$g_n(z) = n^{-1} \mathrm{Tr}\, G_n(z), \tag{1.6}$$

related to the Normalized Counting Measures of eigenvalues of H_n by spectral theorem

$$g_n(z) = \int \frac{N_n(d\lambda)}{\lambda - z}, \ \mathrm{Im}\, z \neq 0. \tag{1.7}$$

Here and below the integrals without limits denote integrals over $\mathbb{R}$. To study the asymptotic behavior of g_n we use an approach, based on certain differentiation formulas (matrix analogs of the integration by parts), leading to certain identities for the moments of g_n and to bounds for the variance of g_n, allowing one to convert the identities into functional equations, determining uniquely g_n, hence the limiting measure.

In [12] the following theorem was proved.

Theorem 1.1. *Consider the random matrices (1.1) and assume (1.3). Then there exists a nonrandom probability measure N such that*

$$\lim_{n\to\infty} \mathbf{P}\{|N_n(\Delta) - N(\Delta)| > \varepsilon\} = 0, \ \forall \Delta \subset \mathbb{R}. \tag{1.8}$$

Moreover, the Stieltjes transform

$$f(z)=\int\frac{N(d\lambda)}{\lambda-z},\ \operatorname{Im}z\neq 0,$$

of N is a unique solution of the system

$$\begin{cases} f(z) &= f_A(h_B(z)),\\ f(z) &= f_B(h_A(z)),\\ (f(z))^{-1} &= z-h_A(z)-h_B(z),\end{cases} \tag{1.9}$$

where

$$f_{A,B}(z)=\int\frac{N_{A,B}(d\lambda)}{\lambda-z}, \tag{1.10}$$

$f(z)$ is a Nevanlinna function, $h_{A,B}(z)$ are analytic in $\mathbb{C}\backslash\mathbb{R}$ and

$$f(z)=-z^{-1}+o(z^{-1}),\ h_{A,B}(z)=z+o(z),\ z\to\infty. \tag{1.11}$$

In [13] the following theorem was proved for the nonrandom matrices A_n and B_n.

Theorem 1.2. *Consider random matrices (1.1), assume that N_{A_n} and N_{B_n} converge weakly to the probability measures N_A and N_B, respectively, and that*

$$\sup_n\int|\lambda|^4N_{A_n,B_n}(d\lambda)\le M<\infty. \tag{1.12}$$

Then we have for g_n of (1.5)–(1.7) and n-independent $z_{1,2}\in\mathbb{C}\backslash\mathbb{R}$

$$\mathbf{Cov}\{g_n(z_1),g_n(z_2)\}=\frac{1}{n^2}S_n(z_1,z_2)+\psi_n(z_1,z_2), \tag{1.13}$$

where

$$S_n(z_1,z_2)=\frac{\partial^2}{\partial z_1\partial z_2}\log\frac{(h_{A_n}(z_1)-h_{A_n}(z_2))(h_{B_n}(z_1)-h_{B_n}(z_2))}{(z_1-z_2)(r_n(z_1)-r_n(z_2))}, \tag{1.14}$$

$$r_n(z)=-\frac{1}{\mathbf{E}\{g_n(z)\}}, \tag{1.15}$$

$$h_{A_n}(z)=z-\frac{\mathbf{E}\{n^{-1}\operatorname{Tr}G_n(z)A_n\}}{\mathbf{E}\{g_n(z)\}},\ h_{B_n}(z)=z-\frac{\mathbf{E}\{n^{-1}\operatorname{Tr}G_n(z)U_n^{\dagger}B_nU_n\}}{\mathbf{E}\{g_n(z)\}} \tag{1.16}$$

and $\psi_n(z_1,z_2)$ admits the bound

$$|\psi_n(z_1,z_2)|\le C/n^3,$$

where C is independent of n and finite if $\min\{|\operatorname{Im}z_1|,|\operatorname{Im}z_2|\}>0$.

R e m a r k 1.3. *It follows from Theorem 1.1 that for $z_{1,2} \in \mathbb{C}\backslash\mathbb{R}$ we have*

$$\lim_{n\to\infty} S_n(z_1,z_2) = S(z_1,z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{(h_A(z_1)-h_A(z_2))(h_B(z_1)-h_B(z_2))}{(z_1-z_2)(r(z_1)-r(z_2))}, \tag{1.17}$$

where $r(z) = -f^{-1}(z)$ and $S(z_1,z_2)$ is defined also for $z_1 = z_2$ as well as $S_n(z_1,z_2)$. Indeed, using (1.9) and (1.10), we obtain

$$\begin{aligned} r(z_1)-r(z_2) &= \frac{z_1-z_2}{f(z_1)f(z_2)} I_A(z_1,z_2) I_B(z_1,z_2) \qquad (1.18)\\ &\times \left(I_A(z_1,z_2) + I_B(z_1,z_2) - \frac{I_A(z_1,z_2)I_B(z_1,z_2)}{f_n(z_1)f_n(z_2)} \right)^{-1}, \\ h_A(z_1)-h_A(z_2) &= I_B^{-1}(z_1,z_2) f(z_1)f(z_2)(r(z_1)-r(z_2)), \\ h_B(z_1)-h_B(z_2) &= I_A^{-1}(z_1,z_2) f(z_1)f(z_2)(r(z_1)-r(z_2)), \end{aligned}$$

where we denote

$$\begin{aligned} I_A(z_1,z_2) &:= \int \frac{N_A(d\lambda)}{(\lambda-h_B(z_1))(\lambda-h_B(z_2))}, \\ I_B(z_1,z_2) &:= \int \frac{N_B(d\lambda)}{(\lambda-h_A(z_1))(\lambda-h_A(z_2))}. \end{aligned}$$

Note that for $z_1 = z_2$ the term in the parentheses in the r.h.s. (1.18) coincides up to a factor with with the determinant of linear system on the triple of derivatives (f', h'_A, h'_B), which is nonzero. Using (1.18) we can rewrite $S_n(z_1,z_2)$ in the form, which has no singularity at $z_1 = z_2$

$$S(z_1,z_2) = -\frac{\partial^2}{\partial z_1 \partial z_2} \log \left(I_A(z_1,z_2) + I_B(z_1,z_2) - \frac{I_A(z_1,z_2)I_B(z_1,z_2)}{f_n(z_1)f_n(z_2)} \right).$$

Moreover, because of (1.17) and Theorem 1.2, $S_n(z_1,z_2)$ is also defined at $z_1 = z_2$ for sufficiently large n.

In this paper we study the asymptotic behaviour of the covariance $\mathbf{Cov}\{g_n(z_1), g_n(z_2)\}$ of the normalized traces of resolvents (1.6) for random A_n and B_n of (1.1). An analogous problem for the normalized traces of moments of (1.1) was considered in the recent paper [17] under the condition that A_n and B_n have the second order distribution in the following sense.

Definition 1.4. *[17]. Let M_n be a Hermitian random matrix. Then, we say that it has a second order limit distribution if for any p, $q \geq 1$ the limits*

$$m_M^{(p)} := \lim_{n\to\infty} \mathbf{E}\{n^{-1} \mathrm{Tr}\, M_n^p\}$$

and

$$m_M^{(p,q)} := \lim_{n\to\infty} \mathbf{Cov}\{\mathrm{Tr}\, M_n^p, \mathrm{Tr}\, M_n^m\}$$

exist and if for all $r \geq 3$ and all $p(1), \dots, p(r) \geq 1$,

$$\lim_{n\to\infty} k_r(\mathrm{Tr}\, M_n^{p(1)}, \dots, \mathrm{Tr}\, M_n^{p(r)}) = 0$$

where k_r denotes the r^{th} classical multivariate cumulant.

It was proved in [17, 18] by a rather involved and nontrivial combinatorial analysis that under these conditions on A_n and B_n the random matrix (1.1) has also the second order limit distribution. Moreover, if

$$\begin{aligned}
&m_A^{(p)} := \lim_{n\to\infty} \mathbf{E}\{n^{-1}\mathrm{Tr}\, A_n^p\},\ m_B^{(p)} := \lim_{n\to\infty} \mathbf{E}\{n^{-1}\mathrm{Tr}\, B_n^p\},\\
&m_H^{(p)} := \lim\nolimits_{n\to\infty} \mathbf{E}\{n^{-1}\mathrm{Tr}\, H_n^p\},\ m_A^{(p,q)} := \lim_{n\to\infty} \mathbf{Cov}\{\mathrm{Tr}\, A_n^p, \mathrm{Tr}\, A_n^q\},\\
&m_B^{(p,q)} := \lim\nolimits_{n\to\infty} \mathbf{Cov}\{\mathrm{Tr}\, B_n^p, \mathrm{Tr}\, B_n^q\},\ m_H^{(p,q)} := \lim_{n\to\infty} \mathbf{Cov}\{\mathrm{Tr}\, H_n^p, \mathrm{Tr}\, H_n^q\},
\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}
f_A(z) &:= -\sum_{p=1}^{\infty} \frac{m_A^{(p)}}{z^{p+1}},\ f_B(z) := -\sum_{p=1}^{\infty} \frac{m_B^{(p)}}{z^{p+1}},\\
f(z) &:= -\sum_{p=1}^{\infty} \frac{m_H^{(p)}}{z^{p+1}},\ C_A(z_1, z_2) := \sum_{p,q=1}^{\infty} \frac{m_A^{(p,q)}}{z_1^{p+1} z_2^{q+1}},\\
C_B(z_1, z_2) &:= \sum_{p,q=1}^{\infty} \frac{m_B^{(p,q)}}{z_1^{p+1} z_2^{q+1}},\ C(z_1, z_2) := \sum_{p,q=1}^{\infty} \frac{m_H^{(p,q)}}{z_1^{p+1} z_2^{q+1}}
\end{aligned}$$

are the correspondent formal power series then the *second order R-transforms* $R_A(w_1, w_2)$, $R_B(w_1, w_2)$ and $R_H(w_1, w_2)$ defined in [17] as

$$\begin{aligned}
&C_A(z_1, z_2)\\
&= \left(R_A(f_A(z_1), f_A(z_2)) + \frac{1}{(f_A(z_1) - f_A(z_2))^2}\right) f_A'(z_1) f_A'(z_2) - \frac{1}{(z_1 - z_2)^2},\\
&C_B(z_1, z_2)\\
&= \left(R_B(f_B(z_1), f_B(z_2)) + \frac{1}{(f_B(z_1) - f_B(z_2))^2}\right) f_B'(z_1) f_B'(z_2) - \frac{1}{(z_1 - z_2)^2},\\
&C(z_1, z_2)\\
&= \left(R(f(z_1), f(z_2)) + \frac{1}{(f(z_1) - f(z_2))^2}\right) f'(z_1) f'(z_2) - \frac{1}{(z_1 - z_2)^2},
\end{aligned} \tag{1.20}$$

satisfy

$$R(w_1, w_2) = R_A(w_1, w_2) + R_B(w_1, w_2). \tag{1.21}$$

We will prove an asymptotic relation between the covariance of $g_n(z)$ for $z = z_1, z_2$ and those of $n^{-1}\mathrm{Tr}\,(A_n - z)^{-1}$, $z = z_1, z_2$ and $n^{-1}\mathrm{Tr}\,(B_n - z)^{-1}$, $z = z_1, z_2$. The relation can be viewed as a version of the second order asymptotic distribution in the terms of traces of resolvents rather than of traces of powers of the corresponding matrices as in Definition 1.4. This requires the existence (in fact boundedness in n) of expectations of traces of several first powers of corresponding random matrices rather than all moments as in (1.19). If, in addition, the limits of covariances of $\mathrm{Tr}\,(A_n - z)^{-1}$, $z = z_1, z_2$ and $\mathrm{Tr}\,(B_n - z)^{-1}$, $z = z_1, z_2$ exist, then we obtain a formula relating the limit of covariance of $ng_n(z)$ for $z = z_1, z_2$ and those of $\mathrm{Tr}\,(A_n - z)^{-1}$ and $\mathrm{Tr}\,(B_n - z)^{-1}$. The formula is a version of equality (1.21), but is valid as the equality of analytic functions rather than the formal power series.

Thus our goal is to express

$$C_n(z_1, z_2) = \mathbf{Cov}\{g_n(z_1), g_n(z_2)\} \tag{1.22}$$

via the covariances of the normalized traces of resolvent of the summands of (1.1)

$$C_{A_n}(z_1, z_2) = \mathbf{Cov}\{g_{A_n}(z_1), g_{A_n}(z_2)\},\ C_{B_n}(z_1, z_2) = \mathbf{Cov}\{g_{B_n}(z_1), g_{B_n}(z_2)\}, \tag{1.23}$$

where

$$g_{A_n}(z) = \int \frac{N_{A_n}(d\lambda)}{\lambda - z} = n^{-1}\mathrm{Tr}\ G_{A_n}(z),\ g_{B_n}(z) = \int \frac{N_{B_n}(d\lambda)}{\lambda - z} = n^{-1}\mathrm{Tr}\ G_{B_n}(z) \tag{1.24}$$

and

$$G_{A_n}(z) = (A_n - z)^{-1},\ G_{B_n}(z) = (B_n - z)^{-1},\ \mathrm{Im}\, z \neq 0. \tag{1.25}$$

Note that because of (1.3) the covariances (1.23) tend to zero as $n \to \infty$ as well as the variances

$$u_{A_n}(z) = \mathbf{Var}\{g_{A_n}(z)\},\ u_{B_n}(z) = \mathbf{Var}\{g_{B_n}(z)\}. \tag{1.26}$$

The second question is: are the rates of convergence to zero of the covariance (1.22) and the variance

$$u_n = \mathbf{Var}\{g_n(z)\}$$

the same as for u_{A_n} and u_{B_n}?

The main result of this paper is

Theorem 1.5. *Consider the random matrices of the form (1.1). Assume (1.3) and the following asymptotic relations:*

(i) for any n-independent z with $\operatorname{Im} z \neq 0$

$$\begin{array}{rclcl} \tilde{u}_{A_n} & := & \mathbf{E}\{|g_{A_n}(z) - \mathbf{E}\{g_{A_n}(z)\}|^4\} & = & o\left(u_{A_n}(z)\right), \\ \tilde{u}_{B_n} & := & \mathbf{E}\{|g_{B_n}(z) - \mathbf{E}\{g_{B_n}(z)\}|^4\} & = & o\left(u_{B_n}(z)\right) \end{array} \tag{1.27}$$

as $n \to \infty$;

(ii)

$$\sup_n \mathbf{E}\left\{\int \lambda^4 N_{A_n,B_n}(d\lambda)\right\} \le M < \infty,\ \ M \ge 1; \tag{1.28}$$

(iii) for any $z \in K$-compact, $K \subset \mathbb{C} \setminus \mathbb{R}$

$$\begin{array}{rcl} \mathbf{Var}\left\{\int |\lambda| N_{A_n}(d\lambda)\right\} & = & O\left(u_{A_n}(z)\right), \\ \mathbf{Var}\left\{\int |\lambda| N_{B_n}(d\lambda)\right\} & = & O\left(u_{B_n}(z)\right), \end{array} \tag{1.29}$$

$$\begin{array}{rcl} \mathbf{E}\left\{\left|\int |\lambda| N_{A_n}(d\lambda) - \mathbf{E}\left\{\int |\lambda| N_{A_n}(d\lambda)\right\}\right|^4\right\} & = & o\left(u_{A_n}(z)\right), \\ \mathbf{E}\left\{\left|\int |\lambda| N_{B_n}(d\lambda) - \mathbf{E}\left\{\int |\lambda| N_{B_n}(d\lambda)\right\}\right|^4\right\} & = & o\left(u_{B_n}(z)\right) \end{array} \tag{1.30}$$

as $n \to \infty$.

Then we have for any $z_{1,2} \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$

$$\begin{array}{rcl} C_n(z_1,z_2) & = & C_{A_n}(h_{B_n}(z_1), h_{B_n}(z_2)) h'_{B_n}(z_1) h'_{B_n}(z_2) \\ & + & C_{B_n}(h_{A_n}(z_1), h_{A_n}(z_2)) h'_{A_n}(z_1) h'_{A_n}(z_2) \\ & + & n^{-2} S_n(z_1,z_2) + \psi_n(z_1,z_2), \end{array} \tag{1.31}$$

where

$$\Gamma_{\alpha,\beta} = \{z \in \mathbb{C} : |\operatorname{Re} z| \le \alpha |\operatorname{Im} z|,\ \ |\operatorname{Im} z| \ge \beta\},\ \ \alpha > 0,\ \ \beta \ge (11\alpha + 15)M, \tag{1.32}$$

$$\psi_n(z_1,z_2) = o\left(\max\{n^{-2}, C_{A_n}(z_1,z_2), C_{B_n}(z_1,z_2)\}\right),\ \ n \to \infty. \tag{1.33}$$

In the proof of the theorem the techniques of [12] are used and it is given in the next section. Here we discuss the theorem and its applications.

(i) Conditions on absolute moments (1.29), (1.30) are technical. For example, they can be omitted in the case of uniformly in n bounded matrices, $\sup_n ||A_n|| < \infty$, $\sup_n ||B_n|| < \infty$. They can be replaced by the following conditions on the moments

$$\begin{array}{rcl} \mathbf{Var}\left\{\int \lambda^2 N_{A_n}(d\lambda)\right\} & = & O\left(u_{A_n}\right), \\ \mathbf{Var}\left\{\int \lambda^2 N_{B_n}(d\lambda)\right\} & = & O\left(u_{B_n}\right), \end{array} \tag{1.34}$$

and

$$\begin{array}{rcl} \mathbf{E}\left\{\left|\int \lambda^2 N_{A_n}(d\lambda) - \mathbf{E}\left\{\int \lambda^2 N_{A_n}(d\lambda)\right\}\right|^4\right\} & = & o\left(u_{A_n}\right), \\ \mathbf{E}\left\{\left|\int \lambda^2 N_{B_n}(d\lambda) - \mathbf{E}\left\{\int \lambda^2 N_{B_n}(d\lambda)\right\}\right|^4\right\} & = & o\left(u_{B_n}\right). \end{array} \tag{1.35}$$

(ii) Conditions (1.27), (1.34), (1.35) can be verified directly for Gaussian ensembles via Poincaré–Nash inequality [3, 10] (e.g., Gaussian unitary ensemble or Marchenko–Pastur ensemble with Gaussian entry). It can be also verified for the general matrix models. Indeed, it was proved in [14] that if we have probability distribution of Hermitian random matrix M_n

$$p_n(M_n)dM_n = \frac{1}{Z_n}\exp\{-n\mathrm{Tr}\,V(M_n)\}dM_n, \tag{1.36}$$

where

$$dM_n = \prod_{j=1}^{n} dM_{jj} \prod_{1\le j<k\le n}^{n} (d\mathrm{Re}\,M_{jk})(d\mathrm{Im}\,M_{jk})$$

obeying conditions

$$V(\lambda) \ge (2+\varepsilon)\ln|\lambda|,\ \varepsilon > 0,\ \lambda \to \infty,$$

and

$$|V(\lambda) - V(\mu)| \le C(L)|\lambda - \mu|^{\gamma},\ \gamma > 0,\ |\lambda|, |\mu| \le L,$$

then for any smooth bounded function $\varphi : \mathbb{R} \to \mathbb{C}$ with bounded derivative we have the bound for r-th classical cumulant of $n^{-1}\mathrm{Tr}\,\varphi(M_n)$

$$|k_r(n^{-1}\mathrm{Tr}\,\varphi(M_n))| \le \frac{C_\varphi r! a^r}{n^r},\ a > 0,\ r \ge 2, \tag{1.37}$$

where constant C_φ depends only of L^∞-norm of φ and φ'. Since we have

$$\begin{aligned}\varphi(\lambda) &= (\lambda - z)^{-1},\ |\varphi(\lambda)| \le \frac{1}{|\mathrm{Im}\,z|},\ |\varphi'(\lambda)| \le \frac{1}{|\mathrm{Im}\,z|^2},\\ k_2(a) &= \mathbf{Var}\{a\},\ \mathbf{E}\left\{|a - \mathbf{E}\{a\}|^4\right\} = k_4(a) + 3k_2^2(a),\end{aligned}$$

then condition (1.27) follows directly from (1.37). The rest of the conditions also follow from (1.37) and the fact that the support of the limiting NCM of (1.36) is compact and that NCM of (1.36) decays exponentially apart of this compact (see, e.g., [11]).

(iii) Theorem 1.5 is related with the result of recent paper [17] as follows. First, fixing in addition to the conditions of Theorem 1.5, the order of covariances of Stieltjes transforms by n^{-2} and supposing the convergence of their asymptotics we have

$$\lim_{n\to\infty} n^2 C_{A_n}(z_1, z_2) = C_A(z_1, z_2),\ \lim_{n\to\infty} n^2 C_{B_n}(z_1, z_2) = C_B(z_1, z_2),$$
$$\lim_{n\to\infty} n^2 C_n(z_1, z_2) = C(z_1, z_2).$$

Then, multiplying (1.31) by n^2 and passing to the limit $n \to \infty$, we obtain

$$\begin{aligned} C(z_1, z_2) &= C_A(h_B(z_1), h_B(z_2))h'_B(z_1)h'_B(z_2) \\ &+ C_B(h_A(z_1), h_A(z_2))h'_A(z_1)h'_A(z_2) \\ &+ S(z_1, z_2). \end{aligned} \tag{1.38}$$

Besides, (1.9) and (1.20) imply

$$\begin{aligned} C_A(h_B(z_1), h_B(z_2)) &= \left(R_A(f(z_1), f(z_2)) + \frac{1}{(f(z_1) - f(z_2))^2} \right) \\ &\times f'_A(h_B(z_1))f'_A(h_B(z_2)) - \frac{1}{(h_B(z_1) - h_B(z_2))^2}. \end{aligned}$$

Using this relation, an analogous relation for $C_B(h_A(z_1), h_A(z_2))$ and the equalities

$$h'_A(z) = \frac{f'(z)}{f'_B(h_A(z))}, \; h'_B(z) = \frac{f'(z)}{f'_A(h_B(z))},$$

we obtain from (1.38)

$$\begin{aligned} C(z_1, z_2) &= \left(R_A(f(z_1), f(z_2)) + R_B(f(z_1), f(z_2)) + \frac{1}{(f(z_1) - f(z_2))^2} \right) \\ &\times f'(z_1)f'(z_2) - \frac{1}{(z_1 - z_2)^2}. \end{aligned}$$

This leads to

$$R(f(z_1), f(z_2)) = R_A(f(z_1), f(z_2)) + R_B(f(z_1), f(z_2)).$$

Thus, because of Nevanlinnianess of $f(z)$ and one-to-one correspondence $f :, \mathbb{C}_\pm \rightarrow \mathbb{C}_\pm$ (see [4]) we have obtained for $w_{1,2} \in \mathbb{C}\backslash\mathbb{R}$ the analytic functions equality

$$R(w_1, w_2) = R_A(w_1, w_2) + R_B(w_1, w_2).$$

Moreover, supposing the existence of $k+1$-th moments of the measures $N_{A,B}$ and using $k+1$ first terms of the asymptotic expansion oin $z_{1,2}^{-1}$ of analytic functions in (1.38), we can obtain the relations for moment covariances, moments and free cumulants up to the k-th order correspondent with the relations obtained in [17].

(iv) Conditions (1.27)–(1.30) are verified for the second order distributions, having convergent resolvent asymptotic power series. Indeed, since the orders of all variances of moments and Stieltjes transforms are fixed by n^{-2}

$$\lim_{n\to\infty} n^2 u_{A_n} = C_A(z, \overline{z}), \; \lim_{n\to\infty} n^2 u_{B_n} = C_B(z, \overline{z}),$$

and due to the correspondence between moments and cumulants

$$\begin{aligned}\mathbf{E}\{a^\circ b^\circ c^\circ d^\circ\} &= k_4(a^\circ,b^\circ,c^\circ,d^\circ)+k_2(a^\circ,b^\circ)k_2(c^\circ,d^\circ)\\ &\quad +k_2(a^\circ,c^\circ)k_2(b^\circ,d^\circ)+k_2(a^\circ,d^\circ)k_2(b^\circ,c^\circ),\end{aligned}$$

where

$$\begin{aligned}&a^\circ = a-\mathbf{E}\{a\}\,,\ b^\circ = b-\mathbf{E}\{b\}\,,\\ &c^\circ = c-\mathbf{E}\{c\}\,,\ d^\circ = d-\mathbf{E}\{d\}\,,\end{aligned} \tag{1.39}$$

we obtain, in view of convergence of the correspondent series,

$$\begin{aligned}&\mathbf{E}\{|g_{A_n}(z)-\mathbf{E}\{g_{A_n}(z)\}|^4\}\\ &= n^{-4}\sum_{p_1,p_2,p_3,p_4=1}^{\infty}\frac{\mathbf{E}\{(\mathrm{Tr}\,A_n^{p_1})^\circ\,(\mathrm{Tr}\,A_n^{p_2})^\circ\,(\mathrm{Tr}\,A_n^{p_3})^\circ\,(\mathrm{Tr}\,A_n^{p_4})^\circ\}}{z^{p_1+p_2+2}\overline{z}^{p_3+p_4+2}}\\ &= n^{-4}(3C_A^2(z,\overline{z})+o(1)) = O(u_{A_n}^2) = o\,(u_{A_n})\,,\\ &\mathbf{E}\{|g_{B_n}(z)-\mathbf{E}\{g_{B_n}(z)\}|^4\} = n^{-4}(3C_B^2(z,\overline{z})+o(1)) = O(u_{B_n}^2) = o\,(u_{B_n})\,.\end{aligned}$$

Thus, the condition (1.27) is verified. The rest of conditions follow directly from the behavior in n of k_2 and k_4 and the existence of all moments of the measures $N_{A,B}$.

2. Proofs

We denote $\langle\ldots\rangle$ the conditional expectation with respect to the normalized Haar measure of $U(n)$. We are going to use often the following fact on this expectation.

Proposition 2.1. *Let $\mathcal{H}_n$ be the space of $n\times n$ Hermitian matrices, and $\Phi:\mathcal{H}_n\to\mathbb{C}$ be a continuously differentiable function. Then we have for any $X\in\mathcal{H}_n$:*

$$\left\langle \Phi'(U^\dagger MU)\cdot[X,U^\dagger MU]\right\rangle = 0,$$

where

$$[M_1,M_2] = M_1M_2 - M_1M_2.$$

The proof of the proposition is given in [12].

We will use the resolvent identity for resolvents G_1 and G_2 of two Hermitian matrices M_1 and M_2:

$$G_2(z)-G_1(z) = G_1(z)(M_1-M_2)G_2(z) = G_2(z)(M_1-M_2)G_1(z), \tag{2.1}$$

the formula for the derivative of the resolvent of a Hermitian matrix M:

$$G' \cdot X = -GXG, \ \forall X \in \mathcal{H}_n \tag{2.2}$$

and the bounds valid for any matrices M_1 and M_2 and Hermitian matrix Q:

$$|\mathrm{Tr} M_1 M_2| \leq (\mathrm{Tr} M_1 M_1^\dagger)^{1/2} (\mathrm{Tr} M_2 M_2^\dagger)^{1/2}, \tag{2.3}$$

$$|\mathrm{Tr} M_1 Q| \leq ||M_1|| \mathrm{Tr}|Q|, \ |Q| = \sqrt{Q^\dagger Q} = \sqrt{QQ^\dagger}. \tag{2.4}$$

We will also need the notion of the Nevanlinna functions (see, e.g., [1]). Namely, an analytic in $\mathbb{C}\backslash\mathbb{R}$ function f is a Nevanlinna function if

$$\overline{f(z)} = f(\overline{z}), \ \mathrm{Im}\, f(z) \mathrm{Im}\, z > 0, \ \mathrm{Im}\, z \neq 0. \tag{2.5}$$

Any Nevanlinna function admits the representation

$$f(z) = az + b + \int \frac{1 + \mu z}{\mu - z} m(d\mu), \tag{2.6}$$

where $a \geq 0, \ b \in \mathbb{R}$, m is a finite nonnegative measure and we write here and below the integrals without limits for the integrals over $\mathbb{R}$. The representation takes the form

$$f(z) = \int \frac{m(d\mu)}{\mu - z}, \tag{2.7}$$

with a finite nonnegative m if and only if $\sup_{y \geq 1} |yf(iy)| < \infty$, and in this case

$$\lim_{y \to \infty} |yf(iy)| = m(\mathbb{R}) < \infty.$$

Lemma 2.2. *Assume (1.28) and denote*

$$\begin{aligned} F(z_1, z_2) \ &: \ = \psi(g_n(z_1), g_n(z_2)), \\ F_A(z_1, z_2) \ &: \ = \psi(\delta_{A_n}(z_1), \delta_{A_n}(z_2)), \\ \delta_{A_n}(z_1) \ &: \ = n^{-1} \mathrm{Tr} G_n(z) A_n, \\ \delta_{B_n}(z_1) \ &: \ = n^{-1} \mathrm{Tr} G_n(z) U_n^\dagger B_n U_n, \end{aligned}$$

where $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a smooth enough function. Then for $z_{1,2} \in \Gamma_{\alpha,\beta}$ (1.32) we have

$$\begin{aligned} \mathbf{Cov}\{F(z_1, z_2), g_n(z_2)\} \ &= \ \mathbf{Cov}\{F(z_1, z_2), g_{A_n}(h_{B_n}(z_2))\} \\ &+ \ \frac{1}{\mathbf{E}\{g_n(z_2)\}} \left(\mathbf{Cov}\{F(z_1, z_2), g_n^\circ(z_2) k_{A_n}(z_2)\}\right. \\ &- \ \left.\mathbf{Cov}\{F(z_1, z_2), \delta_{B_n}^\circ(z_2) p_{A_n}(z_2)\}\right) \\ &+ \ \frac{1}{n^2} \gamma_{A_n B_n}(z_1, z_2), \end{aligned}$$

and

$$\begin{aligned}\mathbf{Cov}\{F_A(z_1,z_2),\delta_{A_n}(z_2)\} &= \mathbf{Cov}\{F(z_1,z_2),g_{A_n}(h_{B_n,A_n}(z_2))\}h_{B_n}(z_2)\\ &+ \frac{1}{\mathbf{E}\{g_n(z_2)\}}\Big(\mathbf{Cov}\{F_A(z_1,z_2),g_n^\circ(z_2)\tilde{k}_{A_n}(z_2)\}\\ &- \mathbf{Cov}\{F_A(z_1,z_2),\delta_{B_n}^\circ(z_2)\tilde{p}_{A_n}(z_2)\}\Big)\\ &+ \frac{1}{n^2}\widetilde{\gamma}_{A_nB_n}(z_1,z_2),\end{aligned}$$

where

$$\begin{aligned}k_{A_n}(z) &= n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))U_n^\dagger B_nU_nG_n(z),\\ \tilde{k}_{A_n}(z) &= n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))A_nU_n^\dagger B_nU_nG_n(z),\\ p_{A_n}(z) &= n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))G_n(z),\\ \tilde{p}_{A_n}(z) &= n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))A_nG_n(z),\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\gamma_{A_nB_n}(z_1,z_2)\\ =\ & \frac{\mathbf{E}\{\psi_1'(g_n(z_1),g_n(z_2))n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))[U_n^\dagger B_nU_n,G_n^2(z_1)]G_n(z_2)\}}{\mathbf{E}\{g_n(z_2)\}}\\ +\ & \frac{\mathbf{E}\{\psi_2'(g_n(z_1),g_n(z_2))n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))[U_n^\dagger B_nU_n,G_n^2(z_2)]G_n(z_2)\}}{\mathbf{E}\{g_n(z_2)\}},\end{aligned}$$

$$\begin{aligned}&\widetilde{\gamma}_{A_nB_n}(z_1,z_2)\\ =\ & \frac{\mathbf{E}\{\psi_1'(\delta_{A_n}(z_1),\delta_{A_n}(z_1))n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))[U_n^\dagger B_nU_n,G_n(z_1)A_nG_n(z_1)]G_n(z_2)\}}{\mathbf{E}\{g_n(z_2)\}}\\ +\ & \frac{\mathbf{E}\{\psi_2'(\delta_{A_n}(z_1),\delta_{A_n}(z_1))n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))[U_n^\dagger B_nU_n,G_n(z_2)A_nG_n(z_2)]G_n(z_2)\}}{\mathbf{E}\{g_n(z_2)\}}.\end{aligned}$$

P r o o f. We omit the subscript n in A_n, B_n and $G_n(z)$ in the cases where there will be no confusion.

(i) Denote $\{G_{jk}(z)\}_{j,k=1}^n$ the matrix of $G(z)$. Taking in Proposition 2.1 $\Phi = F^\circ(z_1,z_2)G_{ac}(z_2)$, $a,c=1,\dots,n$ and using (2.2), we obtain

$$\begin{aligned}&\Big\langle F^\circ(z_1,z_2)\Big(G(z_2)[X,U^\dagger BU]\Big)_{ac}\Big\rangle\\ +\ & \Big\langle \psi_1'(g_n(z_1),g_n(z_2))\Big(n^{-1}\mathrm{Tr}\,G(z_1)\Big[X,U^\dagger BU\Big]G(z_1)\Big)G_{ac}(z_2)\Big\rangle\\ +\ & \Big\langle \psi_2'(g_n(z_1),g_n(z_2))\Big(n^{-1}\mathrm{Tr}\,G(z_2)\Big[X,U^\dagger BU\Big]G(z_2)\Big)G_{ac}(z_2)\Big\rangle = 0.\end{aligned}$$

Take here $X = E^{(a,b)}$ and then apply the operation $n^{-1}\sum_{a=1}^{n}$. This yields the matrix relation

$$\begin{aligned} &\langle F^{\circ}(z_1,z_2)\delta_{B_n}(z_2)G(z_2)\rangle = \left\langle F^{\circ}(z_1,z_2)g_n(z_2)U^{\dagger}BUG(z_2)\right\rangle \\ + &\frac{1}{n^2}\left(\left\langle \psi_1'(g_n(z_1),g_n(z_2))\left[U^{\dagger}BU,G^2(z_1)\right]G(z_2)\right\rangle\right. \\ + &\left.\left\langle \psi_2'(g_n(z_1),g_n(z_2))\left[U^{\dagger}BU,G^2(z_2)\right]G(z_2)\right\rangle\right). \end{aligned}$$

Writing (cf (1.39))

$$\delta_{B_n}(z_2) = \delta^{\circ}_{B_n}(z_2) + \mathbf{E}\{\delta_{B_n}(z_2)\},\ g_n(z_2) = g^{\circ}_n(z_2) + \mathbf{E}\{g_n(z_2)\}$$

and regrouping terms, we obtain

$$\begin{aligned} &\mathbf{E}\{\delta_{B_n}(z_2)\}\langle F^{\circ}(z_1,z_2)G(z_2)\rangle - \mathbf{E}\{g_n(z_2)\}\left\langle F^{\circ}(z_1,z_2)U^{\dagger}BUG(z_2)\right\rangle \\ = &\left\langle F^{\circ}(z_1,z_2)g^{\circ}_n(z_2)U^{\dagger}BUG(z_2)\right\rangle - \langle F^{\circ}(z_1,z_2)\delta^{\circ}_{B_n}(z_2)G(z_2)\rangle \\ + &\frac{1}{n^2}\left(\left\langle \psi_1'(g_n(z_1),g_n(z_2))\left[U^{\dagger}BU,G^2(z_1)\right]G(z_2)\right\rangle\right. \\ + &\left.\left\langle \psi_2'(g_n(z_1),g_n(z_2))\left[U^{\dagger}BU,G^2(z_2)\right]G(z_2)\right\rangle\right). \end{aligned}$$

Now the resolvent identity

$$-U^{\dagger}BUG(z_2) = AG(z_2) - z_2G(z_2) - I$$

allows us to write

$$\begin{aligned} &\mathbf{E}\{g_n(z_2)\}(A - h_{B_n}(z_2))\langle F^{\circ}(z_1,z_2)G(z_2)\rangle = \mathbf{E}\{g_n(z_2)\}\langle F^{\circ}(z_1,z_2)\rangle I \\ &+\left\langle F^{\circ}(z_1,z_2)g^{\circ}_n(z_2)U^{\dagger}BUG(z_2)\right\rangle - \langle F^{\circ}(z_1,z_2)\delta^{\circ}_{B_n}(z_2)G(z_2)\rangle \\ &+\frac{1}{n^2}\left(\left\langle \psi_1'(g_n(z_1),g_n(z_2))\left[U^{\dagger}BU,G^2(z_1)\right]G(z_2)\right\rangle\right. \\ &+\left.\left\langle \psi_2'(g_n(z_1),g_n(z_2))\left[U^{\dagger}BU,G^2(z_2)\right]G(z_2)\right\rangle\right). \end{aligned} \tag{2.9}$$

Besides, in view of (1.28) and relations

$$\begin{aligned} \mathbf{E}\{g_n(z)\} &= -\frac{1}{z}(1 - \mathbf{E}\{\delta_{A_n}(z)\} - \mathbf{E}\{\delta_{B_n}(z)\}), \\ h_{A_n,B_n}(z) &= z\left(1 + \frac{\mathbf{E}\{\delta_{A_n,B_n}(z)\}}{1 - \mathbf{E}\{\delta_{A_n}(z)\} - \mathbf{E}\{\delta_{B_n}(z)\}}\right) \\ &= z\frac{1 - \mathbf{E}\{\delta_{B_n,A_n}(z)\}}{1 - \mathbf{E}\{\delta_{A_n}(z)\} - \mathbf{E}\{\delta_{B_n}(z)\}}, \end{aligned} \tag{2.10}$$

where

$$|g_n(z)| \le \frac{1}{|\mathrm{Im}\, z|},\ |\delta_{A_n,B_n}(z)| \le \frac{m^{(1)}_{A_n,B_n}}{|\mathrm{Im}\, z|},\ m^{(k)}_{A_n,B_n} = \int |\lambda|^k N_{A_n,B_n}(d\lambda), \quad (2.11)$$

we have for any $z \in \Gamma_{\alpha,\beta}$ (1.32)

$$\begin{aligned} |\mathbf{E}\{g_n(z)\}| &\ge \frac{1}{|z|}\left(1-\frac{2M}{\beta}\right) \ge \frac{13}{15|z|}, \\ |h_{A_n,B_n}(z)| &\le |z|\frac{1+\dfrac{M}{\beta}}{1-\dfrac{2M}{\beta}} \le |z|\frac{16}{13}, \\ |\mathrm{Im}\, h_{A_n,B_n}(z)| &\ge \beta\left(1-\frac{|z|\dfrac{M}{\beta}}{1-\dfrac{2M}{\beta}}\right) \ge 12M. \end{aligned} \quad (2.12)$$

Hence, the matrix $A - h_{B_n}(z_2)$ is invertible uniformly in n for any $z \in\in \Gamma_{\alpha,\beta}$ (1.32) and

$$G_{A_n}(h_{B_n}(z_2)) = (A - h_{B_n}(z_2))^{-1},\ ||G_{A_n}(h_{B_n}(z_2))|| \le \frac{1}{12M}.$$

Thus, multiplying (2.9) from the left by $G_{A_n}(h_{B_n}(z_2))$, then applying $n^{-1}\mathrm{Tr}$, and taking the expectation $\mathbf{E}\{...\}$ of the result, we obtain the first identity. The second identity can be proved analogously by using Proposition 2.1 with $\Phi = F_A^\circ(z_1, z_2)\,(G(z_2)A)_{ac}$ and taking into account that the traces of resolvents of the matrices

$$A_n + U_n^\dagger B_n U_n \text{ and } U_n A_n U_n^\dagger + B_n$$

coincide. ∎

Lemma 2.3. *Under conditions (1.28)–(1.27) we have for $z \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$ (1.32) as $n \to \infty$*

(i)

$$\begin{aligned} u_n :=& \quad \mathbf{Var}\{g_n(z)\} = O\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \\ v_n :=& \quad \mathbf{Var}\{\delta_{A_n}(z)\} = O\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \\ w_n :=& \quad \mathbf{Var}\{\delta_{B_n}(z)\} = O\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right); \end{aligned}$$

(ii)

$$\mathbf{Var}\{p_{A_n}(z)\} = O\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right);$$

(iii)

$$\mathbf{Var}\{k_{A_n}(z)\} = O\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right);$$

(iv)

$$\begin{aligned} \tilde{u}_n :=& \quad \mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^4\} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \\ \tilde{v}_n :=& \quad \mathbf{E}\{|\delta_{A_n}(z) - \mathbf{E}\{\delta_{A_n}(z)\}|^4\} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \\ \tilde{w}_n :=& \quad \mathbf{E}\{|\delta_{B_n}(z) - \mathbf{E}\{\delta_{B_n}(z)\}|^4\} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right). \end{aligned}$$

P r o o f. (i) Note that we have

$$u_n = \mathbf{Var}\{g_n(z)\} = \mathbf{Cov}\{g_n(z), g_n(\bar{z})\}.$$

On the other hand, the resolvent identity implies that

$$zG_n(z) + I = A_nG_n(z) + U_n^\dagger B_n U_n G_n(z). \tag{2.13}$$

We obtain

$$g_n(z) = \frac{1}{z}(\delta_{A_n}(z) + \delta_{B_n}(z) - 1), \tag{2.14}$$

hence

$$\mathbf{Cov}\{g_n(z), g_n(\bar{z})\} = \frac{1}{\bar{z}}\left(\mathbf{Cov}\{g_n(z), \delta_{A_n}(\bar{z})\} + \mathbf{Cov}\{g_n(z), \delta_{B_n}(\bar{z})\}\right). \tag{2.15}$$

This and the Schwarz inequalities

$$|\mathbf{Cov}\{g_n(z), \delta_{A_n}(\bar{z})\}| \le u_n^{1/2}v_n^{1/2}, \;\; |\mathbf{Cov}\{g_n(z), \delta_{B_n}(\bar{z})\}| \le u_n^{1/2}w_n^{1/2},$$

yield that for $z \in \Gamma_{\alpha,\beta}$ (1.32)

$$u_n \le \frac{1}{|z|}\left(u_n^{1/2}v_n^{1/2} + u_n^{1/2}w_n^{1/2}\right) \le \alpha_{12}u_n^{1/2}v_n^{1/2} + \alpha_{13}u_n^{1/2}w_n^{1/2}, \tag{2.16}$$

where

$$\alpha_{13} = \alpha_{12} = \frac{1}{15}. \tag{2.17}$$

Taking into account that

$$v_n = \mathbf{Var}\{\delta_{A_n}(z)\} = \mathbf{Cov}\{\delta_{A_n}(z), \delta_{A_n}(\bar{z})\}.$$

and the second identity of Lemma 2.2 with $F_A(z_1, z_2) = \delta_{A_n}(z_1)$, we obtain for $z_1 = z$, $z_2 = \bar{z}$

$$\mathbf{Var}\{\delta_{A_n}(z)\} = \mathbf{Cov}\{\delta_{A_n}(z), g_{A_n}(h_{B_n}(\bar{z}))\}h_{B_n}(\bar{z}) + \frac{I_1 - I_2}{\mathbf{E}\{g_n(\bar{z})\}} + \frac{\tilde{\gamma}_{A_nB_n}(z,\bar{z})}{n^2}, \tag{2.18}$$

where

$$I_1 = \mathbf{E}\{\delta^\circ_{A_n}(z) g^\circ_n(\bar z)\tilde k_{A_n}(\bar z)\},\ I_2 = \mathbf{E}\{\delta^\circ_{A_n}(z)\delta^\circ_{B_n}(\bar z)\tilde p_{A_n}(\bar z)\}.$$

Furthermore, (2.14) and the resolvent identity $G_{A_n}(z)A_n = zG_{A_n}(z) + I$ imply

$$g^\circ_n(\bar z) = \frac{1}{\bar z}\left(\delta^\circ_{A_n}(\bar z) + \delta^\circ_{B_n}(\bar z)\right),\ \tilde k_{A_n}(\bar z) = h_{B_n}(\bar z)k_{A_n}(\bar z) + \delta_{B_n}(\bar z),$$

hence

$$I_1 = \frac{h_{B_n}(\bar z)}{\bar z}(I_3 + I_4) + \frac{1}{\bar z}(I_5 + I_6),$$

where

$$\begin{aligned} I_3 &= \mathbf{E}\{\delta^\circ_{A_n}(z)\delta^\circ_{A_n}(\bar z)k_{A_n}(\bar z)\},\ I_4 = \mathbf{E}\{\delta^\circ_{A_n}(z)\delta^\circ_{B_n}(\bar z)k_{A_n}(\bar z)\},\\ I_5 &= \mathbf{E}\{\delta^\circ_{A_n}(z)\delta^\circ_{A_n}(\bar z)\delta_{B_n}(\bar z)\},\ I_6 = \mathbf{E}\{\delta^\circ_{A_n}(z)\delta^\circ_{B_n}(\bar z)\delta_{B_n}(\bar z)\}. \end{aligned}$$

According to (2.4), we have for $\tilde p_{A_n}$ and k_{A_n} of (2.8)

$$|\tilde p_{A_n}(\bar z)| \le \frac{m^{(1)}_{A_n}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|},\ |k_{A_n}(\bar z)| \le \frac{m^{(1)}_{B_n}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|}.$$

Thus, using the centered quantities of absolute moments (2.11)

$$\left(m^{(1)}_{A_n}\right)^\circ = m^{(1)}_{A_n} - \mathbf{E}\left\{m^{(1)}_{A_n}\right\},\ \left(m^{(1)}_{B_n}\right)^\circ = m^{(1)}_{B_n} - \mathbf{E}\left\{m^{(1)}_{B_n}\right\}$$

and (2.4), we obtain

$$\begin{aligned} |I_2| &\le \frac{\mathbf{E}\left\{m^{(1)}_{A_n}\right\}\mathbf{E}\{|\delta^\circ_{A_n}(z)||\delta^\circ_{B_n}(\bar z)|\} + \mathbf{E}\left\{|\delta^\circ_{A_n}(z)||\delta^\circ_{B_n}(\bar z)|\left(m^{(1)}_{A_n}\right)^\circ\right\}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|}\\ &\le \frac{Mv_n^{1/2}w_n^{1/2}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{\mathbf{E}\left\{|\delta_{A_n}(z)|\right\}\mathbf{E}\left\{|\delta^\circ_{B_n}(\bar z)|\left(m^{(1)}_{A_n}\right)^\circ\right\}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|}\\ &\quad + \frac{\mathbf{E}\left\{|\delta_{A_n}(z)||\delta^\circ_{B_n}(\bar z)|\left(m^{(1)}_{A_n}\right)^\circ\right\}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|}\\ &\le \frac{Mv_n^{1/2}w_n^{1/2}}{|\mathrm{Im}\, z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{2M\mathbf{E}\left\{|\delta^\circ_{B_n}(\bar z)|\left|\left(m^{(1)}_{A_n}\right)^\circ\right|\right\}}{|\mathrm{Im}\, z|^2|\mathrm{Im}\, h_{B_n}(z)|}\\ &\quad + \frac{2M\mathbf{E}\left\{|\delta^\circ_{B_n}(\bar z)|\left|\left(m^{(1)}_{A_n}\right)^\circ\right|\right\} + \mathbf{E}\left\{|\delta^\circ_{B_n}(\bar z)|\left|\left(m^{(1)}_{A_n}\right)^\circ\right|^2\right\}}{|\mathrm{Im}\, z|^2|\mathrm{Im}\, h_{B_n}(z)|} \end{aligned}$$

$$\leq \frac{Mv_n^{1/2}w_n^{1/2}}{|\mathrm{Im}\,z||\mathrm{Im}\,h_{B_n}(z)|} + \frac{2M\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}w_n^{1/2}}{|\mathrm{Im}\,z|^2|\mathrm{Im}\,h_{B_n}(z)|}$$
$$+\frac{\mathbf{E}\left\{|\delta_{B_n}(\bar{z})|\right\}\mathbf{Var}\left\{m_{A_n}^{(1)}\right\}}{|\mathrm{Im}\,z|^2|\mathrm{Im}\,h_{B_n}(z)|} + \frac{\mathbf{E}\left\{|\delta_{B_n}(\bar{z})|\left|\left(m_{A_n}^{(1)}\right)^\circ\right|^2\right\}}{|\mathrm{Im}\,z|^2|\mathrm{Im}\,h_{B_n}(z)|}$$
$$\leq \frac{Mv_n^{1/2}w_n^{1/2}}{|\mathrm{Im}\,z||\mathrm{Im}\,h_{B_n}(z)|} + \frac{2M\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}w_n^{1/2}}{|\mathrm{Im}\,z|^2|\mathrm{Im}\,h_{B_n}(z)|} + \frac{2M\mathbf{Var}\left\{m_{A_n}^{(1)}\right\}}{|\mathrm{Im}\,z|^3|\mathrm{Im}\,h_{B_n}(z)|}.$$

Analogously, we have

$$|I_3| \leq \frac{Mv_n}{|\mathrm{Im}\,z||\mathrm{Im}\,h_{B_n}(z)|} + \frac{2M\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}v_n^{1/2}}{|\mathrm{Im}\,z|^2|\mathrm{Im}\,h_{B_n}(z)|}$$
$$+ \frac{\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}\left(2M+\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}\right)}{|\mathrm{Im}\,z|^3|\mathrm{Im}\,h_{B_n}(z)|},$$

$$|I_4| \leq \frac{Mv_n^{1/2}w_n^{1/2}}{|\mathrm{Im}\,z||\mathrm{Im}\,h_{B_n}(z)|} + \frac{2M\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}w_n^{1/2}}{|\mathrm{Im}\,z|^2|\mathrm{Im}\,h_{B_n}(z)|}$$
$$+ \frac{\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}\left(2M+\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}\right)}{|\mathrm{Im}\,z|^3|\mathrm{Im}\,h_{B_n}(z)|},$$

$$|I_5| \leq \frac{Mv_n}{|\mathrm{Im}\,z|} + \frac{2M\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}v_n^{1/2}}{|\mathrm{Im}\,z|^2}$$
$$+ \frac{\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}\left(2M+\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}\right)}{|\mathrm{Im}\,z|^3},$$

$$|I_6| \leq \frac{Mv_n^{1/2}w_n^{1/2}}{|\mathrm{Im}\,z|} + \frac{2M\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}w_n^{1/2}}{|\mathrm{Im}\,z|^2}$$
$$+ \frac{\mathbf{Var}^{1/2}\left\{m_{A_n}^{(1)}\right\}\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}\left(2M+\mathbf{Var}^{1/2}\left\{m_{B_n}^{(1)}\right\}\right)}{|\mathrm{Im}\,z|^3}.$$

Substituting the above bounds into (2.18), we obtain

$$v_n \leq a(z)v_n + b(z)v_n^{1/2}w_n^{1/2} + c(z)v_n^{1/2} + d(z)w_n^{1/2} + |h_{B_n}(z)|v_n^{1/2}u_{A_n}^{1/2}(h_{B_n}(z))$$
$$+\frac{|\tilde{\gamma}_{A_nB_n}|}{n^2} + \varkappa_n,$$

and in view of (1.29), (2.11) and (2.12) we have for $z \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$ (1.32)

$$\begin{aligned}
a(z) &= \frac{M}{|\mathbf{E}\{g_n(\bar z)\}||\mathrm{Im}\, z|}\left(\frac{|h_{B_n}(\bar z)|}{|z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{1}{|\mathrm{Im}\, z|}\right) \le \frac{1}{5},\\
b(z) &= \frac{M}{|\mathbf{E}\{g_n(\bar z)\}||\mathrm{Im}\, z|}\left(\frac{2\,|h_{B_n}(\bar z)|}{|z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{1}{|\mathrm{Im}\, z|}\right) \le \frac{3}{5},\\
c(z) &= \frac{2M\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{|\mathbf{E}\{g_n(\bar z)\}||\mathrm{Im}\, z|^2}\left(\frac{|h_{B_n}(\bar z)|}{|z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{1}{|\mathrm{Im}\, z|}\right)\\
&\le \frac{\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{36} = O\left(u^{1/2}_{B_n}\right),\\
d(z) &= \frac{2M}{|\mathbf{E}\{g_n(\bar z)\}||\mathrm{Im}\, z|^2}\\
&\times \left(\frac{|h_{B_n}(\bar z)|\left(\mathbf{Var}^{1/2}\left\{m^{(1)}_{A_n}\right\} + \mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}\right)}{|z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{|\mathrm{Im}\, z|}\right)\\
&\le \frac{\mathbf{Var}^{1/2}\left\{m^{(1)}_{A_n}\right\} + \mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{12} = O\left(u^{1/2}_{B_n}\right),\\
|\tilde\gamma_{A_nB_n}| &\le \frac{2\mathbf{E}\left\{\sqrt{m^{(2)}_{B_n} m^{(2)}_{A_n}}\right\}}{|\mathrm{Im}\, z|^3\,|\mathbf{E}\{g_n(\bar z)\}||\mathrm{Im}\, h_{B_n}(z)|} \le \frac{1}{32},\ \frac{|\tilde\gamma_{A_nB_n}|}{n^2} = O(n^{-2}),
\end{aligned}$$

$$|h_{B_n}(z)| u^{1/2}_{A_n}(h_{B_n}(z)) = O\left(u^{1/2}_{A_n}\right),$$

$$\begin{aligned}
\varkappa_n &= \frac{\mathbf{Var}^{1/2}\left\{m^{(1)}_{A_n}\right\}\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{|\mathbf{E}\{g_n(\bar z)\}||\mathrm{Im}\, z|^3}\\
&\times\left(4M + \mathbf{Var}^{1/2}\left\{m^{(1)}_{A_n}\right\} + \mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}\right)\\
&\times\left(\frac{2\,|h_{B_n}(\bar z)|}{|z||\mathrm{Im}\, h_{B_n}(z)|} + \frac{1}{|\mathrm{Im}\, z|}\right)\\
&= O(\max\{u_{A_n}, u_{B_n}\}).
\end{aligned}$$

Thus, we obtained the bound (cf (2.16))

$$v_n \le \alpha_{23} v_n^{1/2} w_n^{1/2} + \beta_{22} v_n^{1/2} + \beta_{23} w_n^{1/2} + \gamma_{2n}, \tag{2.19}$$

where

$$\alpha_{23} = \frac{3}{4},\ \beta_{22} = \beta_{23} = O\left(\max\left\{u^{1/2}_{A_n}, u^{1/2}_{B_n}\right\}\right),\ \gamma_{2n} = O\left(\max\left\{n^{-2}, u_{A_n}, u_{B_n}\right\}\right). \tag{2.20}$$

Similarly, using the second identity of Lemma 2.2 with interchanged A_n and B_n and $F_B(z_1, z_2) = \delta_{B_n}(z_1)$, $z_1 = \bar{z}_2 = z$ and Schwarz inequality, we obtain an analog of (2.16) and (2.19)

$$w_n \le \alpha_{32} w_n^{1/2} v_n^{1/2} + \beta_{32} v_n^{1/2} + \beta_{33} w_n^{1/2} + \gamma_{3n}, \qquad (2.21)$$

where

$$\alpha_{32} = \frac{3}{4},\ \beta_{32} = \beta_{33} = O\left(u_{A_n}^{1/2}\right),\ \gamma_{3n} = O\left(\max\left\{n^{-2}, u_{A_n}, u_{B_n}\right\}\right). \qquad (2.22)$$

Now, introducing new variables

$$s_1 = u_n^{1/2},\ s_2 = v_n^{1/2},\ s_3 = w_n^{1/2}$$

and the quantities

$$\begin{aligned} \beta_n &= \max\{\beta_{22}, \beta_{23}, \beta_{32}, \beta_{33}\}, \\ \gamma_n &= \max\{\gamma_{2n}, \gamma_{3n}\}, \end{aligned} \qquad (2.23)$$

we rewrite (2.16), (2.19) and (2.21) as the system of quadratic inequalities

$$s_i^2 \le \sum_{j=1, j\neq i}^{3} \alpha_{ij} s_i s_j + \beta_n(s_2 + s_3) + \gamma_n,\ i = 1, 2, 3.$$

Let i_0 be defined as $s_{i_0} = \max_{i=1,2,3} u_i$. Then we have for $\bar{s} = s_{i_0}$

$$\bar{s}^2 \le \bar{s}^2 \sum_{j=1, j\neq i}^{3} \alpha_{ij} + 2\beta_n \bar{s} + \gamma_n \le \frac{3}{4}\bar{s}^2 + 2\beta_n \bar{s} + \gamma_n, \qquad (2.24)$$

where we took into account that $\sum_{j=1, j\neq i}^{3} \alpha_{ij} \le 3/4$ (see (2.17), (2.20) and (2.22)). This implies that $\bar{s} = O(\beta_n)$, and, in view of (2.17), (2.20), (2.22) and (2.23), assertion (i) of the lemma.

(ii) Note that we have

$$\mathbf{Var}\{p_{A_n}(z)\} = \mathbf{Cov}\{p_{A_n}(z), p_{A_n}(\bar{z})\}.$$

Taking in Proposition 2.1 $\Phi = p_{A_n}^{\circ}(z) G_{ac}(\bar{z})$ and using (2.2), we obtain for any $a.c = 1, \ldots, n$:

$$\begin{aligned} &\left\langle p_{A_n}^{\circ}(z) \left(G(\bar{z}) \left[X, U^{\dagger} B U\right] G(\bar{z})\right)_{ac} \right\rangle \\ &+ \left\langle \left(n^{-1} \mathrm{Tr}\, G(z) \left[X, U^{\dagger} B U\right] G(z) G_{A_n}(h_{B_n}(z))\right) G_{ac}(\bar{z}) \right\rangle = 0. \end{aligned}$$

We take $X = E^{(a,b)}$ and apply the operation $n^{-1}\sum\limits_{a=1}^{n}$ to the result. This yields the matrix relation

$$\left\langle p^{\circ}_{A_n}(z)\delta_{B_n}(\bar z)G(\bar z)\right\rangle = \left\langle p^{\circ}_{A_n}(z)g_n(\bar z)U^{\dagger}BUG(\bar z)\right\rangle$$
$$+\frac{1}{n^2}\left\langle \left[U^{\dagger}BU, G(z)G_{A_n}(h_{B_n}(z))G(z)\right]G(\bar z)\right\rangle.$$

Then, applying the same procedure as in the proof of Lemma 2.2, i.e., regrouping the terms, using the centered quantities $g^{\circ}_n(z)$ and $\delta^{\circ}_{B_n}(z)$, multiplying from the left by the $G^2_{A_n}(h_{B_n}(\bar z))$, then applying $n^{-1}\mathrm{Tr}$ and taking the expectation $\mathbf{E}\{...\}$, we obtain

$$\begin{aligned}\mathbf{Var}\{p_{A_n}(z)\} &= \mathbf{Cov}\{p_{A_n}(z), g'_{A_n}(h_{B_n}(\bar z))\} \\ &+\frac{1}{\mathbf{E}\{g_n(\bar z)\}}\Big(\mathbf{E}\{p^{\circ}_{A_n}(z)g^{\circ}_n(\bar z)\widehat{k}_{A_n}(\bar z)\} \\ &- \mathbf{E}\{p^{\circ}_{A_n}(z)\delta^{\circ}_{B_n}(\bar z)\widehat{p}_{A_n}(\bar z)\}\Big) + \frac{\widehat{\gamma}_n}{n^2},\end{aligned} \tag{2.25}$$

where

$$\begin{aligned} g'_{A_n}(h_{B_n}(\bar z)) &= n^{-1}\mathrm{Tr}G^2_{A_n}(h_{B_n}(\bar z)) \\ \widehat{k}_{A_n}(\bar z) &= n^{-1}\mathrm{Tr}G^2_{A_n}(h_{B_n}(\bar z))U^{\dagger}_nB_nU_nG_n(\bar z), \\ \widehat{p}_{A_n}(\bar z) &= n^{-1}\mathrm{Tr}G^2_{A_n}(h_{B_n}(\bar z))G_n(\bar z), \\ \widehat{\gamma}_n &= \frac{\mathbf{E}\{n^{-1}\mathrm{Tr}\,G^2_{A_n}(h_{B_n}(\bar z))\left[U^{\dagger}BU, G(z)G_{A_n}(h_{B_n}(z))G(z)\right]G(\bar z)\}}{\mathbf{E}\{g_n(\bar z)\}}.\end{aligned}$$

Besides, we have for $z \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$ (1.32)

$$\begin{aligned}\frac{|\mathbf{E}\{p^{\circ}_{A_n}(z)g^{\circ}_n(\bar z)\widehat{k}_{A_n}(\bar z)\}|}{|\mathbf{E}\{g_n(\bar z)\}|} &\le \frac{\mathbf{E}\{m^{(1)}_{B_n}\}\mathbf{Var}^{1/2}\{p_{A_n}(z)\}u_n^{1/2}}{|\mathrm{Im}\,h_{B_n}(z)|^2|\mathrm{Im}\,z||\mathbf{E}\{g_n(\bar z)\}|} \\ &+\frac{2\mathbf{Var}^{1/2}\{p_{A_n}(z)\}\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{|\mathrm{Im}\,h_{B_n}(z)|^2|\mathrm{Im}\,z|^2|\mathbf{E}\{g_n(\bar z)\}|} \\ &\le \frac{\mathbf{Var}^{1/2}\{p_{A_n}(z)\}u_n^{1/2}}{10} \\ &+\frac{\mathbf{Var}^{1/2}\{p_{A_n}(z)\}\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}}{10},\end{aligned}$$

$$\begin{aligned}\frac{|\mathbf{E}\{p^\circ_{A_n}(z)\delta^\circ_{B_n}(\bar z)\widehat{p}_{A_n}(\bar z)\}|}{|\mathbf{E}\{g_n(\bar z)\}|} &\le \frac{\mathbf{Var}^{1/2}\{p_{A_n}(z)\}w_n^{1/2}}{|\mathrm{Im}\,h_{B_n}(z)|^2|\mathrm{Im}\,z||\mathbf{E}\{g_n(\bar z)\}|}\\ &\le \frac{\mathbf{Var}^{1/2}\{p_{A_n}(z)\}w_n^{1/2}}{10},\\ |\widehat{\gamma}_n| &\le \frac{2\mathbf{E}\{m^{(1)}_{B_n}\}}{|\mathrm{Im}\,h_{B_n}(z)|^3|\mathrm{Im}\,z|^3|\mathbf{E}\{g_n(\bar z)\}|}\le\frac{1}{10}\end{aligned}$$

and then the analyticity of $g_{A_n}(z)$ for $z\in\mathbb{C}\backslash\mathbb{R}$ and Cauchy theorem imply that

$$\mathbf{Var}\{g'_{A_n}(h_{B_n}(\bar z))\}=O(u_{A_n}(h_{B_n}(\bar z))),\ n\to\infty. \tag{2.26}$$

Thus, we obtain from (2.25) by using Schwarz inequality

$$\begin{aligned}\mathbf{Var}\{p_{A_n}(z)\} \le\ & \mathbf{Var}^{1/2}\{p_{A_n}(z)\}\Big(\mathbf{Var}^{1/2}\{g'_{A_n}(h_{B_n}(z))\}\\ &+0,1\left(u_n^{1/2}+w_n^{1/2}+\mathbf{Var}^{1/2}\left\{m^{(1)}_{B_n}\right\}\right)\Big)+0,1n^{-2}.\end{aligned}$$

This, assertion (i) and (2.26) yield (ii).

(iii) It follows from (2.13) and $zG_{A_n}(z)+I=G_{A_n}(z)A_n$ that

$$\begin{aligned}G_{A_n}(h_{B_n}(z))U_n^\dagger B_nU_nG_n(z) &= G_{A_n}(h_{B_n}(z))-G_n(z)\\ &\quad+(z-h_{B_n}(z))G_{A_n}(h_{B_n}(z))G_n(z),\end{aligned}$$

hence

$$k_{A_n}(z)=g_{A_n}(h_{B_n}(z))-g_n(z)+(z-h_{B_n}(z))p_{A_n}(z).$$

Using this relation and the Schwarz inequality, we obtain

$$\begin{aligned}\mathbf{Var}\{k_{A_n}(z)\}\le\ &\mathbf{Var}\{g_{A_n}(h_{B_n}(z))\}+u_n+|z-h_{B_n}(z)|^2\mathbf{Var}\{p_{A_n}(z)\}\\ &+2|z-h_{B_n}(z)|\mathbf{Var}^{1/2}\{p_{A_n}(z)\}\left(\mathbf{Var}^{1/2}\{g_{A_n}(h_{B_n}(z))\}+u_n^{1/2}\right)\\ &+2\mathbf{Var}^{1/2}\{g_{A_n}(h_{B_n}(z))\}u_n^{1/2}.\end{aligned}$$

This and the assertions (i) and (ii) yield (iii).

(iv) Note that in view of assertion (i) and (1.29) we already have for $z\in K$-compact, $K\subset\Gamma_{\alpha,\beta}$ (1.32)

$$\tilde u_n,\tilde v_n,\tilde w_n=O\left(\max\{n^{-2},u_{A_n},u_{B_n}\}\right). \tag{2.27}$$

To prove (iv) we apply the procedure analogous to that of the proof of assertion (i) and obtain the system of inequalities, now not a quadratic one, but of the degree four. We have

$$\tilde u_n=\mathbf{Cov}\{(g^\circ_n(z))^2\,g^\circ_n(\bar z),g(\bar z)\}.$$

Using (2.14), we get

$$\begin{aligned}\mathbf{Cov}\{(g_n^\circ(z))^2\, g_n^\circ(\bar z), g(\bar z)\} &= \frac{1}{\bar z}\mathbf{Cov}\{(g_n^\circ(z))^2\, g_n^\circ(\bar z), \delta_{B_n}(\bar z)\} \\ &+ \frac{1}{\bar z}\mathbf{Cov}\{(g_n^\circ(z))^2\, g_n^\circ(\bar z), \delta_{A_n}(\bar z)\}.\end{aligned}$$

Thus, using the Schwarz inequality, we obtain for $z \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$

$$\tilde u_n \le \frac{1}{|z|}\left(\tilde u_n^{3/4}\tilde v_n^{1/4} + \tilde u_n^{3/4}\tilde w_n^{1/4}\right) \le \alpha_{12}\tilde u_n^{3/4}\tilde v_n^{1/4} + \alpha_{13}\tilde u_n^{3/4}\tilde w_n^{1/4},$$

where α_{12} and α_{13} are given in (2.17). Besides, using the assertion of Lemma 2.2 with $F_A(z_1,z_2) = \left(\delta^\circ_{A_n}(z_1)\right)^2 \delta^\circ_{A_n}(z_2)$, $z_1 = z$, $z_2 = \bar z$, we obtain

$$\begin{aligned}\mathbf{E}\{|\delta^\circ_{A_n}(z)|^4\} &= \mathbf{Cov}\{\left(\delta^\circ_{A_n}(z)\right)^2 \delta^\circ_{A_n}(\bar z), g_{A_n}(h_{B_n}(\bar z))\}h_{B_n,}(\bar z) \\ &+\frac{\tilde I_1 - \tilde I_2}{\mathbf{E}\{g_n(\bar z)\}} + \frac{\tilde\gamma_{A_nB_n}(z,\bar z)}{n^2},\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}\tilde I_1 &= \mathbf{Cov}\{\left(\delta^\circ_{A_n}(z)\right)^2 \delta^\circ_{A_n}(\bar z), g_n^\circ(\bar z)\tilde k_{A_n}(\bar z)\}, \\ \tilde I_2 &= \mathbf{Cov}\{\left(\delta^\circ_{A_n}(z)\right)^2 \delta^\circ_{A_n}(\bar z), \delta^\circ_{B_n}(\bar z)\tilde p_{A_n}(\bar z)\}.\end{aligned}$$

On the other hand, the Schwarz inequality, (1.30), (2.11), (2.12) and (2.27) yield for $z \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$

$$\begin{aligned}|\mathbf{Cov}\{\left(\delta^\circ_{A_n}(z)\right)^2 \delta^\circ_{A_n}(\bar z), g_{A_n}(h_{B_n}(\bar z))\}| &\le \tilde v_n^{3/4}\tilde u_{A_n}^{1/4}(h_{B_n}(z)) \\ &= o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right),\end{aligned}$$

$$\begin{aligned}|\tilde I_1| &\le \frac{M\tilde v_n}{|\mathrm{Im}\,\bar z|}\left(\frac{|h_{B_n}(z)|}{|z|\,|\mathrm{Im}\,h_{B_n}(z)|} + \frac{1}{|\mathrm{Im}\,z|}\right) \\ &+\frac{M\tilde v_n^{3/4}\tilde w_n^{1/4}}{|\mathrm{Im}\,z|}\left(\frac{|h_{B_n}(z)|}{|z|\,|\mathrm{Im}\,h_{B_n}(z)|} + \frac{1}{|\mathrm{Im}\,z|}\right) \\ &+o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \\ |\tilde I_2| &\le \frac{M\tilde v_n^{3/4}\tilde w_n^{1/4}}{|\mathrm{Im}\,z||\mathrm{Im}\,h_{B_n}(z)|} + o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \\ |\tilde\gamma_{A_nB_n}(z,\bar z)| &\le \frac{8\mathbf{E}^{1/2}\{m_{B_n}^{(2)}\}\mathbf{E}^{1/2}\{m_{A_n}^{(4)}\}v_n^{1/2}}{|\mathrm{Im}\,z|^3\,\mathbf{E}\{g_n(z)\}||\mathrm{Im}\,h_{B_n}(z)|}, \\ \frac{|\tilde\gamma_{A_nB_n}(z,\bar z)|}{n^2} &= o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right).\end{aligned}$$

These inequalities and (2.28) imply

$$\tilde{v}_n \leq \alpha_{23}\tilde{v}_n^{3/4}\tilde{w}_n^{1/4} + \tilde{\gamma}_{2n},$$

where $\alpha_{23} = 3/4$ and $\tilde{\gamma}_{2n} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right)$, $n \to \infty$. Analogously, the version of assertion of Lemma 2.2 in which A_n and B_n are interchanged and $F_B(z_1, z_2) = \left(\delta^\circ_{B_n}(z_1)\right)^2 \delta^\circ_{B_n}(z_2)$, $z_1 = z$, $z_2 = \bar{z}$, we obtain the inequality

$$\tilde{w}_n \leq \alpha_{32}\tilde{w}_n^{3/4}\tilde{v}_n^{1/4} + \tilde{\gamma}_{3n},$$

where $\alpha_{32} = 3/4$, and $\tilde{\gamma}_{3,n} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right)$, $n \to \infty$. Now, introducing again the new variables

$$\tilde{s}_1 = \tilde{u}_n^{1/4},\ \tilde{s}_2 = \tilde{v}_n^{1/4},\ \tilde{s}_3 = \tilde{w}_n^{1/4},$$

we obtain the system of inequalities of the degree four

$$\tilde{s}_i^4 \leq \sum_{j=1, j\neq i}^{3} \alpha_{ij}\tilde{s}_i^3\tilde{s}_j + \tilde{\gamma}_n,\ i = 1, 2, 3,\ \gamma_n = \max\{\tilde{\gamma}_{2n}, \tilde{\gamma}_{3n}\}.$$

Solving this system by the same arguments as in the case of (2.24), we obtain that $\tilde{s}_i = O(\gamma_n^{1/4})$ uniformly in n for $z \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$, which completes the proof of (iv). ∎

P r o o f o f T h e o r e m 1.5. Using the assertion of Lemma 2.2 with $F(z_1, z_2) = g_n(z_1)$, we obtain

$$\begin{aligned} C_n(z_1, z_2) &= \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} \\ &+ \frac{1}{\mathbf{E}\{g_n(z_2)\}}\left(\mathbf{E}\{g_n^\circ(z_1)g_n^\circ(z_2)k_{A_n}(z_2)\}\right. \\ &\left. - \mathbf{E}\{g_n^\circ(z_1)\delta^\circ_{B_n}(z_2)p_{A_n}(z_2)\}\right) + \frac{\gamma_{A_nB_n}(z_1, z_2)}{n^2}. \end{aligned}$$

Then, substituting in this relation

$$k_{A_n}(z_2) = k^\circ_{A_n}(z_2) + \mathbf{E}\{k_{A_n}(z_2)\},\ p_{A_n}(z_2) = p^\circ_{A_n}(z_2) + \mathbf{E}\{p_{A_n}(z_2)\}$$

and regrouping the terms, we obtain

$$\begin{aligned} C_n(z_1, z_2) &= \alpha_A(z_2)C_n(z_1, z_2) - \beta_A(z_2)\mathbf{Cov}\{g_n(z_1), \delta_{B_n}(z_2)\} \\ &+ \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} + n^{-2}\gamma_{A_nB_n}(z_1, z_2) + T_{A_nB_n}, \end{aligned} \quad (2.29)$$

where

$$\begin{aligned}\alpha_A(z_2) &= \frac{\mathbf{E}\{k_{A_n}(z_2)\}}{\mathbf{E}\{g_n(z_2)\}},\ \beta_A(z_2)=\frac{\mathbf{E}\{p_{A_n}(z_2)\}}{\mathbf{E}\{g_n(z_2)\}},\\ \gamma_{A_nB_n}(z_1,z_2) &= \frac{\mathbf{E}\{n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z))[U_n^\dagger B_n U_n, G_n^2(z_1)]G_n(z_2)\}}{\mathbf{E}\{g_n(z_2)\}},\\ T_{A_nB_n} &= \frac{\mathbf{E}\{g_n^\circ(z_1)g_n^\circ(z_2)k_{A_n}^\circ(z_2)\}-\mathbf{E}\{g_n^\circ(z_1)\delta_{B_n}^\circ(z_2)p_{A_n}^\circ(z_2)\}}{\mathbf{E}\{g_n(z_2)\}}.\end{aligned}$$

In [13] the following relations were proved for the case of nonrandom A_n and B_n

$$\begin{aligned}\alpha_A(z_2) &= \frac{h_{B_n}(z_2)-z}{r_n(z_2)}r'_{A_n}(h_{B_n}(z_2))+o(1),\ n\to\infty, \qquad (2.30)\\ \beta_A(z_2) &= -\frac{1}{r_n(z_2)}r'_{A_n}(h_{B_n}(z_2))+o(1),\ n\to\infty,\end{aligned}$$

$$\begin{aligned}\gamma_{A_nB_n}(z_1,z_2) &= \frac{\partial}{\partial z_1}\left(\frac{1}{z_1-z_2}-\frac{1}{h_{B_n}(z_1)-h_{B_n}(z_2)}\right. \qquad (2.31)\\ &- \left. r'_{A_n}(h_{B_n}(z_2))\frac{h_{B_n}(z_1)-h_{B_n}(z_2)-z_1+z_2}{(z_1-z_2)\,(r_n(z_1)-r_n(z_2))}\right),\\ &+ o(1),\ n\to\infty.\end{aligned}$$

where

$$r_{A_n,B_n}(z)=-\frac{1}{\mathbf{E}\{g_{A_n,B_n}(z)\}},\ r'_{A_n,B_n}(z)=\frac{\mathbf{E}\{g'_{A_n,B_n}(z)\}}{\mathbf{E}^2\{g_{A_n,B_n}(z)\}}. \qquad (2.32)$$

They can be easy generalized for the case of random A_n and B_n. We also have

$$|T_{A_n,B_n}|\le\frac{\tilde v_n^{1/2}\sqrt{\mathbf{Var}\{k_{A_n}(z_2)\}}+\tilde u_n^{1/4}\tilde w_n^{1/4}\sqrt{\mathbf{Var}\{p_{A_n}(z_2)\}}}{|\mathbf{E}\{g_n(z_2)\}|}.$$

This and Lemma 2.3 yield for $z_{1,2}\in K$-compact, $K\subset\Gamma_{\alpha,\beta}$ (1.32)

$$T_{A_nB_n}=o\left(\max\{n^{-2},u_{A_n},u_{B_n}\}\right),\ n\to\infty.$$

It follows from the assertion of Lemma 2.2 that we also have the symmetric to (2.29) with respect to A_n and B_n relation:

$$\begin{aligned}C_n(z_1,z_2) &= \alpha_B(z_2)C_n(z_1,z_2)-\beta_B(z_2)\mathbf{Cov}\{g_n(z_1),\delta_{A_n}(z_2)\} \qquad (2.33)\\ &+ \mathbf{Cov}\{g_n(z_1),g_{B_n}(h_{A_n}(z_2))\}+n^{-2}\gamma_{B_nA_n}(z_1,z_2)+T_{B_nA_n},\end{aligned}$$

where $\alpha_B(z_2)$, $\beta_B(z_2)$ and $\gamma_{B_nA_n}(z_1,z_2)$ are given by (2.30) and (2.31) with interchanged A_n and B_n and by the same arguments

$$T_{B_nA_n} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \ n \to \infty.$$

This and the identity (2.14), implying

$$z_2 C_n(z_1,z_2) = \mathbf{Cov}\{g_n(z_1), \delta_{A_n}(z_2)\} + \mathbf{Cov}\{g_n(z_1), \delta_{B_n}(z_2)\},$$

lead to the system

$$\left\{\begin{array}{lcl} (1-\alpha_A(z_2))\, C_n(z_1,z_2) + \beta_A(z_2) C_{\delta_B}(z_1,z_2) & = & \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} \\ & & +n^{-2}\gamma_{AB}(z_1,z_2) + T_{A_nB_n} \\ (1-\alpha_B(z_2))\, C_n(z_1,z_2) + \beta_B(z_2) C_{\delta_A}(z_1,z_2) & = & \mathbf{Cov}\{g_n(z_1), g_{B_n}(h_{A_n}(z_2))\} \\ & & +n^{-2}\gamma_{BA}(z_1,z_2) + T_{B_nA_n}, \\ z_2 C_n(z_1,z_2) - C_{\delta_A}(z_1,z_2) - C_{\delta_B}(z_1,z_2) & = & 0, \end{array}\right. \tag{2.34}$$

where

$$\big(C_n(z_1,z_2); C_{\delta_A}(z_1,z_2) = \mathbf{Cov}\{g_n(z_1), \delta_{A_n}(z_2)\};$$
$$C_{\delta_B}(z_1,z_2) = \mathbf{Cov}\{g_n(z_1), \delta_{B_n}(z_2)\}\big).$$

It was shown in [13] that the determinant $D(z_2)$ of the system satisfies the following relation:

$$D(z_2) = \frac{1}{r(z_2)} J(z_2) + o(1), \ n \to \infty,$$

where

$$J(z) = r'_A(h_B(z)) + r'_B(h_A(z)) - r'_A(h_B(z)) r'_B(h_A(z)) = 1 + o(1), \ z \to \infty. \tag{2.35}$$

Thus, (2.34) is uniquely solvable for sufficiently large n and z_2 and its solution is

$$\begin{aligned} C_n(z_1,z_2) &= -\frac{\mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\}\beta_B(z_2)}{D(z_2)} \\ &\quad -\frac{\mathbf{Cov}\{g_n(z_1), g_{B_n}(h_{A_n}(z_2))\}\beta_A(z_2)}{D(z_2)} \\ &\quad -\frac{1}{n^2}\frac{\gamma_{A_nB_n}(z_1,z_2)\beta_B(z_2) + \gamma_{B_nA_n}(z_1,z_2)\beta_A(z_2)}{D(z_2)} + T_n \\ &= \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} h'_{B_n}(z_2) \\ &\quad +\mathbf{Cov}\{g_n(z_1), g_{B_n}(h_{A_n}(z_2))\} h'_{A_n}(z_2) \\ &\quad +\frac{1}{n^2}\left(\gamma_{A_nB_n}(z_1,z_2) h'_{B_n}(z_2) + \gamma_{B_nA_n}(z_1,z_2) h'_{A_n}(z_2)\right) + T_n, \end{aligned} \tag{2.36}$$

where

$$T_n = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \ n \to \infty.$$

To find $\mathbf{Cov}\{g_n(z_1), g_{A_n,B_n}(h_{B_n,A_n}(z_2))\}$ we use a simpler version of the above scheme. It follows from Proposition 2.1 with $\Phi = G_{ac}(z_1)$ and (2.2) that

$$\left\langle \left(G(z_1) \left[X, U^\dagger BU \right] G(z_1) \right)_{ac} \right\rangle = 0.$$

Choosing here $X = E^{(a,b)}$ and applying the operation $n^{-1} \sum\limits_{a=1}^{n}$ to the result, we obtain

$$\langle \delta_{B_n}(z_1) G(z_1) \rangle = \left\langle g_n(z_1) U^\dagger BUG(z_1) \right\rangle.$$

Then the same procedure as in the proof of Lemma 2.2, i.e.,the regrouping of the terms, the using of the centered quantities $g_n^\circ(z_1)$ and $\delta_{B_n}^\circ(z_1)$, the multiplying from the left by $G_{A_n}(h_{B_n}(z_2))$ and then the applying of $n^{-1}\mathrm{Tr}$, yields

$$\langle g_n(z_1) \rangle = g_{A_n}(h_{B_n}(z_1)) + \frac{\langle g_n^\circ(z_1) k_{A_n}(z_1) \rangle - \left\langle \delta_{B_n}^\circ(z_1) p_{A_n}(z_1) \right\rangle}{\mathbf{E}\{g_n(z_1)\}}.$$

Multiplying this relation by $g_{A_n}^\circ(h_{B_n}(z_2))$ and taking the expectation $\mathbf{E}\{...\}$, we obtain (cf (2.29))

$$\begin{aligned} \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} &= \alpha_A(z_1) \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} \\ &- \beta_A(z_1) \mathbf{Cov}\{\delta_{B_n}(z_1), g_{A_n}(h_{B_n}(z_2))\} \\ &+ C_{A_n}(h_{B_n}(z_1), h_{B_n}(z_2)) + \widehat{T}_{A_n B_n}, \end{aligned}$$

where $\alpha_A(z)$ and $\beta_A(z)$ are the same as in the (2.29) and

$$\widehat{T}_{A_n B_n} = \frac{\mathbf{E}\{g_{A_n}^\circ(h_{B_n}(z_2)) g_n^\circ(z_1) k_{A_n}^\circ(z_1)\} - \mathbf{E}\{g_{A_n}^\circ(h_{B_n}(z_2)) \delta_{B_n}^\circ(z_1) p_{A_n}^\circ(z_1)\}}{\mathbf{E}\{g_n(z_1)\}},$$

$$|\widehat{T}_{A_n B_n}| \le \frac{\tilde{u}_{A_n}^{1/4}(h_{B_n}(z_2)) \tilde{u}_n^{1/4} \sqrt{\mathbf{Var}\{k_{A_n}(z_1)\}} + \tilde{u}_{A_n}^{1/4}(h_{B_n}(z_2)) \tilde{w}_n^{1/4} \sqrt{\mathbf{Var}\{p_{A_n}(z_1)\}}}{|\mathbf{E}\{g_n(z_1)\}|}.$$

This and Lemma 2.3 yield for for $z_{1,2} \in K$-compact, $K \subset \Gamma_{\alpha,\beta}$ (1.32)

$$\widehat{T}_{A_n B_n} = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \ n \to \infty.$$

Next, it can be shown that the covariance triple

$$\begin{aligned} C_{g_A} &= \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\}, \\ C_{g_A\delta_A} &= \mathbf{Cov}\{\delta_{A_n}(z_1), g_{A_n}(h_{B_n}(z_2))\}, \\ C_{g_A\delta_B} &= \mathbf{Cov}\{\delta_{B_n}(z_1), g_{A_n}(h_{B_n}(z_2))\} \end{aligned}$$

satisfies the uniquely solvable system (cf. (2.34))

$$\begin{cases} (1-\alpha_A(z_1))\, C_{g_A} + \beta_A(z_1) C_{g_A\delta_B} &= C_{A_n}(h_{B_n}(z_1), h_{B_n}(z_2)) + \widehat{T}_{A_n B_n} \\ (1-\alpha_B(z_1))\, C_{g_A} + \beta_B(z_1) C_{g_A\delta_A} &= \widehat{T}_{B_n A_n} \\ z_1 C_{g_A} - C_{g_A\delta_A} - C_{g_A\delta_B} &= 0 \end{cases}$$

and that its solution is

$$\begin{aligned} \mathbf{Cov}\{g_n(z_1), g_{A_n}(h_{B_n}(z_2))\} &= -\frac{C_{A_n}(h_{B_n}(z_1), h_{B_n}(z_2))\beta_B(z_1)}{D(z_1)} + \widehat{T}_n \quad (2.37) \\ &= C_{A_n}(h_{B_n}(z_1), h_{B_n}(z_2)) h'_{B_n}(z_1) + \widehat{T}_n, \end{aligned}$$

where

$$\widehat{T}_n = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \ n \to \infty.$$

Analogously, we obtain the relation with interchanged A_n and B_n

$$\mathbf{Cov}\{g_n(z_1), g_{B_n}(h_{A_n}(z_2))\} = C_{B_n}(h_{A_n}(z_1), h_{A_n}(z_2)) h'_{A_n}(z_1) + \widetilde{T}_n, \qquad (2.38)$$

where

$$\widetilde{T}_n = o\left(\max\{n^{-2}, u_{A_n}, u_{B_n}\}\right), \ n \to \infty.$$

Substituting (2.37) and (2.38) in (2.36) and using (2.31), we obtain (1.31). ■

3. Central Limit Theorem

In Theorem 1.5 we did not suppose any convergence of n^{-1}-asymptotics leading term of the covariances $C_{A_n}(z_1, z_2)$ and $C_{B_n}(z_1, z_2)$. This makes Theorem 1.5 applicable to the general case of matrix models having limiting NCMs supported on more than one interval. In this case n^{-1}-asymptotics leading terms of the covariances of Stieltjes transforms of its NCMs do not have fixed limits. Now we will study the case of compactly supported measures N_{A_n,B_n} and N_n, which allows us to prove the central limit theorems for the linear eigenvalue statistics of ensemble (1.1). Consider now the linear eigenvalues statistics of H_n (1.1), defined by a test (measurable and bounded) function $\varphi : \mathbb{R} \to \mathbb{C}$ as follows:

$$\mathcal{N}_n[\varphi] := \operatorname{Tr} \varphi(H_n) = \sum_{l=1}^{n} \varphi(\lambda_l^{H_n}) = n \int \varphi(\lambda) N_n(d\lambda).$$

Theorem 3.1. *Consider the random matrices of the form (1.1). Assume (1.3) for nonrandom uniformly bounded* A_n *and* B_n*, supp* $N_{A_n,B_n} \subset [-T,T]$*, and the function* $\varphi : \mathbb{R} \to \mathbb{C}$ *to be analytic in the domain* D *such that*

$$[-2T, 2T] \subset \mathbb{C} \setminus D_T \subset D, \; D_T = \{z \in \mathbb{C} : \rho = \min dist(z, [-2T, 2T]) > 4T\}$$

and $\mathcal{N}_n[\varphi]$ *to be corresponding linear statistics. Then the random variable*

$$\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$$

converges in distribution to the Gaussian random variable with zero mean and the variance

$$V[\varphi] = \frac{1}{\pi^2} \int_{C_1} \int_{C_2} \varphi(z_1)\varphi(z_2) S(z_1, z_2) dz_1 dz_2,$$

where $C_{1,2} \subset D$ *are closed contours encircluing* $[-2T, 2T]$ *and*

$$S(z_1, z_2) \;\; = \;\; \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{(h_A(z_1) - h_A(z_2))(h_B(z_1) - h_B(z_2))}{(z_1 - z_2)(f(z_1) - f(z_2))}.$$

P r o o f. Since *supp* $N_{A_n,B_n} \subset [-T,T]$, we have *supp* $N_{H_n} \subset [-2T, 2T]$. Note that due to the Cauchy theorem

$$\begin{aligned}
\mathcal{N}_n^\circ[\varphi] &= \sum_{l=1}^{n} \left(\varphi(\lambda_l^{H_n}) - \mathbf{E}\left\{ \varphi(\lambda_l^{H_n}) \right\} \right) \\
&= n \int_{-2T}^{2T} \varphi(\lambda) N_n(d\lambda) - n\mathbf{E}\left\{ \int_{-2T}^{2T} \varphi(\lambda) N_n(d\lambda) \right\} \\
&= \frac{n}{2\pi i} \int_{\Gamma} \varphi(z) \left(\int_{-2T}^{2T} \frac{N_n(d\lambda)}{z - \lambda} - \mathbf{E}\left\{ \int_{-2T}^{2T} \frac{N_n(d\lambda)}{z - \lambda} \right\} \right) dz \\
&= -\frac{n}{2\pi i} \int_{\Gamma} \varphi(z) g_n^\circ(z) dz,
\end{aligned}$$

where $\Gamma \subset D$ is any closed contour in the complex plane encircling the segment $[-2T, 2T]$ in the real axis. Define the characteristic function

$$Z_n(x) = \mathbf{E}\left\{ e_n(x) \right\}, \; x \in \mathbb{R},$$

where

$$e_n(x) = e^{ix\mathcal{N}_n^\circ[\varphi]} = \exp\left\{ -\frac{nx}{2\pi} \int_{\Gamma} \varphi(z) g_n^\circ(z) dz \right\}.$$

Since $Z_n(0) = 1$ and

$$e_n(x) = 1 + \int_0^x e_n'(y)dy, \ Z_n(x) = 1 + \int_0^x Z_n'(y)dy, \tag{3.1}$$

it is suffices to prove that there exist subsequences $\{Z_{n_j}(x)\}$ and $\{Z_{n_j}'(x)\}$ that converge uniformly on any finite interval and

$$\lim_{n_j\to\infty} Z_{n_j}(x) = Z(x), \ \lim_{n_j\to\infty} Z_{n_j}'(x) = -xV[\varphi]Z(x).$$

Besides, due to the Cauchy theorem

$$\begin{aligned} \frac{d}{dx}e_n(x) &= -\frac{n}{2\pi}e_n(x)\int_\Gamma \varphi(z)g_n^\circ(z)dz \\ &= -\frac{n}{2\pi}\int_{\Gamma_1} \varphi(z_1)e_n(x)g_n^\circ(z_1)dz_1, \\ Z_n'(x) &= -\frac{1}{2\pi}\int_{\Gamma_1} \varphi(z_1)\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\}dz_1, \end{aligned} \tag{3.2}$$

where we choose the contour $\Gamma_1 \subset D_T \cap D$. To find $\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\}$, we apply the same procedure as in the previous section and obtain for the triple

$$\left(\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\}, \mathbf{E}\left\{ne_n^\circ(x)\delta_{A_n}(z_1)\right\}, \mathbf{E}\left\{ne_n^\circ(x)\delta_{B_n}(z_1)\right\}\right)$$

the uniquely solvable system

$$\left\{\begin{array}{rcl} (1-\alpha_A(z_1))\,\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\} + \beta_A(z_1)\mathbf{E}\left\{ne_n^\circ(x)\delta_{B_n}(z_1)\right\} &=& C_{AB} \\ (1-\alpha_B(z_1))\,\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\} + \beta_B(z_1)\mathbf{E}\left\{ne_n^\circ(x)\delta_{A_n}(z_1)\right\} &=& C_{BA} \\ z_1\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\} - \mathbf{E}\left\{ne_n^\circ(x)\delta_{A_n}(z_1)\right\} - \mathbf{E}\left\{ne_n^\circ(x)\delta_{B_n}(z_1)\right\} &=& 0, \end{array}\right. \tag{3.3}$$

where

$$\begin{aligned} C_{AB} &= -\frac{xZ_n(x)}{2\pi}\int_{\Gamma_2} \varphi(z_2)\gamma_{AB}(z_2,z_1)dz_2 + nT_{A_nB_n}(z_1) - \tau_{A_nB_n}(z_1,z_2), \\ T_{A_nB_n}(z_1) &= \frac{\mathbf{E}\{e_n^\circ(x)g_n^\circ(z_1)k_{A_n}^\circ(z_1)\} - \mathbf{E}\{e_n^\circ(x)\delta_{B_n}^\circ(z_1)p_{A_n}^\circ(z_1)\}}{\mathbf{E}\{g_n(z_1)\}}, \end{aligned}$$

$$\tau_{A_nB_n}(z_1,z_2)$$

$$= \frac{\int_{\Gamma_2} x\varphi(z_2)\mathbf{Cov}\{e_n(x), n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z_1))[U_n^\dagger B_nU_n, G_n^2(z_2)]G_n(z_1)\}dz_2}{2\pi\mathbf{E}\{g_n(z_1)\}},$$

contour $\Gamma_2 \subset D_T \cap D$, and C_{BA} is defined analogously to the C_{AB} with interchanged A and B. Besides, in view of (2.10) we have for $z \in D_T$ the following bounds:

$$\begin{aligned} |\delta_{A_n,B_n}(z)| &\leq \frac{1}{4},\ |g_n(z)| \geq \frac{1}{2|z|},\ \min dist(h_{A_n,B_n}(z), [-T,T]) \geq 2T, \\ ||G_n(z)|| &\leq \frac{1}{4T},\ ||G_{A_n,B_n}(h_{B_n,A_n}(z))|| \leq \frac{1}{2T},\ |e_n(x)| \leq 1. \end{aligned}$$

Moreover, by using the procedure from the previous section it can be shown that uniformly for $z_{1,2} \in K$, K-compact, $K \subset D_T$

$$\mathbf{Var}\left\{n^{-1}\mathrm{Tr}G_{A_n}(h_{B_n}(z_1))[U_n^\dagger B_n U_n, G_n^2(z_2)]G_n(z_1)\right\} \leq O(n^{-2}).$$

Thus, using these bounds, the bounds for variances of k_{A_n} and p_{A_n}

$$\mathbf{Var}\left\{k_{A_n}(z)\right\} = O(n^{-2}),\ \mathbf{Var}\left\{p_{A_n}(z)\right\} = O(n^{-2}),\ z \in K$$

analogously to those obtained in Lemma 2.3 and Schwarz inequality, we obtain uniformly in x on any finite interval and in $z_{1,2} \in K$, K-compact, $K \subset D_T$

$$nT_{A_nB_n}(z_1) = O(n^{-1}),\ \tau_{A_nB_n}(z_1,z_2) = O(n^{-1}).$$

Then, solving (3.3), we obtain uniformly in x on any finite interval

$$\begin{aligned} &\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\} \\ =&\frac{xZ_n(x)}{2\pi}\int\limits_{\Gamma_2}\varphi(z_2)\left(\frac{\gamma_{AB}(z_2,z_1)\beta_B(z_1)+\gamma_{BA}(z_2,z_1)\beta_A(z_1)}{D(z_1)}\right)dz_2 + O(n^{-1}) \\ =&\frac{xZ_n(x)}{2\pi}\int\limits_{\Gamma_2}\varphi(z_2)S_n(z_1,z_2)dz_2 + O(n^{-1}). \end{aligned}$$

Substituting this into (3.2), we obtain uniformly in x on any finite interval in view of finiteness of the contours $\Gamma_{1,2}$

$$Z_n'(x) = -\frac{xZ_n(x)}{4\pi}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}\varphi(z_1)\varphi(z_2)S_n(z_1,z_2)dz_1dz_2 + O(n^{-1}),$$

which completes the proof, due to the analyticity of $\varphi(z_1)\varphi(z_2)S_n(z_1,z_2)$ in $z_{1,2}$ for $z_{1,2} \in \mathbb{C}\backslash[-2T,2T]$. ∎

Acknowledgement. This work was supported by Franco-Ukrainian programme PICS (2009–2011) of NAS of Ukraine and CNRS. The author is thankful to Prof L. Pastur for helpful discussions.

References

[1] *N.I. Akhiezer and I.M. Glazman*, Theory of Linear Operators in Hilbert Space. Dover, New York, 1993.

[2] *L.H.Y. Chen*, An Inequality Involving Normal Distribution. — *J. Multivariate Anal.* **50** (1982) 213–223, 585–604.

[3] *S. Chatterjee and A. Bose*, A New Method for Bounding Rates of Convergence of Empirical Spectral Distributions. — *J. Theoret. Probab.* **17** (2004) 1003–1019.

[4] *G. Chistyakov and F. Götze*, The Arithmetic of Distributions in Free Probability Theory. Preprint ArXiv:mathOA/0508245.

[5] *A. Khorunzhy, B. Khoruzhenko, and L. Pastur*, $1/n$-Corrections to the Green Functions of Random Matrices with Independent Entries. — *J. Phys. A. Math. and General* **28** (1995) L31–L35.

[6] *M. Ledoux*, Concentration of Measure Phenomenon. AMS, Providence, RI, 2001.

[7] *V. Marchenko and L. Pastur*, The Eigenvalue Distribution in Some Ensembles of Random Matrices. — *Math. USSR Sb.* **1** (1967) 457–483.

[8] *M.L. Mehta*, Random Matrices. Acad. Press, Boston, 1991.

[9] *L. Pastur*, A Simple Approach to the Global Regime of the Random Matrix Theory. Mathematical Results in Statistical Mechanics. (S. Miracle-Sole, J. Ruiz, V. Zagrebnov, Eds.). World Sci., Singapore 1999.

[10] *L. Pastur*, A Simple Approach to the Global Regime of Gaussian Ensembles of Random Matrices. — *Ukr. Math. J.* **57** (2005) 936–966.

[11] *L. Pastur and M. Shcherbina*, Bulk Universality and Related Properties of Hermitian Matrix Models. — *J. Stat. Phys.* **130** (2008) 205–250.

[12] *L. Pastur and V. Vasilchuk*, On the Law of Addition of Random Matrices. — *Comm. Math. Phys.* **47** (2000) 1–30.

[13] *L. Pastur and V. Vasilchuk*, On the Law of Addition of Random Matrices: Covariance and the Central Limit Theorem for Traces of Resolvent. — *CRM Proc. Lect. Notes* **42** (2007) 399–416.

[14] *A. Stojanovic*, Une Majoration des Cumulants de la Statistique Linéaire des Valeura Propres d'une Classe de Matrices Aléatoires. — *C.R. Acad. Sci. Paris Sér. I.* **326** (1998) 99–104.

[15] *D. Voiculescu (Ed.)*, Free Probability Theory. Fields Institute Communications. **12**. AMS, Providence, RI, 1997.

[16] *D. Voiculescu*, Lectures on Free Probability Theory. — *Lect. Notes Math.* **1738** (2000) 279–349.

[17] *B. Collins, J. Mingo, P. Sniady, and R. Speicher*, Second Order Freeness and Fluctuations of Random Matrices. III. Higher Order Freeness and Free Cumulants. — *Doc. Math.* **12** (2007), 1–70.

[18] *J. Mingo, P. Sniady, and R. Speicher*, Second Order Freeness and Fluctuations of Random Matrices: II. Unitary Random Matrices. — *Adv. Math.* **209** (2007), 212–240.