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On the Boundary Value Problem with the Operator in Boundary Conditions for the Operator-Differential Equation of the Third Order

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A boundary value problem for a class of the operator-differential equations of the third order on a semi-axis, where one of the boundary conditions is perturbed by some linear operator, is studied. There are obtained sufficient conditions on the operator coefficients of the considered boundary value problem providing its correct and univalent resolvability in Sobolev type space.

Key words: boundary value problem, operator-differential equation, Hilbert space, self-adjoint operator, regular solvability.

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A number of problems in mathematical physics and mechanics can be reduced to the boundary value problems for the differential equations with operators in boundary conditions. In T. Kato's book [1, Ch. 7] one can find the statements of such problems. In particular, the non-local problem is one of them. We note that in works of many mathematicians similar problems for differential equations of the second order are studied in details. Among these works it is possible to specify, for example, the works by M.G. Gasymov and S.S. Mirzoev [2], V.A. Ilin and A.F. Filippov [3], M.L. Gorbachuk [4], F.S. Rofe-Beketov [5], S.Y. Yakubov and B.A. Aliev [6], S.S. Mirzoev and Kh.V. Yagubova [7], A.R. Aliev [8]. But we think that there are few works devoted to the equations of the third order which model the currents of a liquid in viscoelastic deformable tubes. In this paper, we try to fill this gap. Moreover, in comparison with the differential equations of even order, there are few works in which the equations of odd order with the scalar boundary conditions on semi-axis are studied (see, for example, [9–13]).

In this paper, we study a boundary value problem for the operator-differential equation of the third order on semi-axis, where the equation and one of the boundary conditions are perturbed.

1. Let A be a self-adjoint positive defined operator in a separable Hilbert space H, and H_γ be a scale of Hilbert spaces generated by the operator A, i.e., $D(A^\gamma) = H_\gamma, (x,y)_\gamma = (A^\gamma x, A^\gamma y), x, y \in H_\gamma, (\gamma \geq 0)$. If $\gamma = 0$, we assume that $H_0 = H$. We denote by $L_2\left((a;b)\,;H\right),\ -\infty \leq a < b \leq +\infty$, a Hilbert space of the vector functions $f(t)$, defined in $(a;b)$ almost everywhere, with values in H, measurable, quadratically integrable in the sense of Bochner:

$$\|f\|_{L_2((a;b);H)} = \left(\int_a^b \|f(t)\|_H^2\, dt\right)^{1/2}.$$

For $R = (-\infty;+\infty)$ and $R_+ = (0;+\infty)$, we assume that

$$L_2((-\infty;+\infty);\ H) \equiv L_2(R;H),\ \ L_2((0;+\infty);\ H) \equiv L_2(R_+;H).$$

Further, for the vector functions $u(t)$ that almost everywhere belong to $D\left(A^3\right)$ and have the derivative $u'''(t)$, we determine the Hilbert space [14, Ch. 1]

$$W_2^3\left(R_+;H;A\right) = \left\{u:\ u''' \in L_2\left((a;b)\,;\ H\right),\ A^3u \in L_2\left((a;b)\,;\ H\right)\right\}$$

with the norm

$$\|u\|_{W_2^3(R_+;H;A)} = \left(\left\|u'''\right\|^2_{L_2((a;b);\,H)} + \left\|A^3u\right\|^2_{L_2((a;b);\,H)}\right)^{1/2}$$

We assume that

$$W_2^3((-\infty;+\infty);H;A) \equiv W_2^3(R;H;A)\,,\quad W_2^3((0,+\infty);H;A) \equiv W_2^3(R_+;H;A).$$

Here all derivatives $u^{(j)} \equiv \frac{d^ju}{dt^j}$, $(j = \overline{1,3})$ are understood in the sense of the theory of distributions [14, Ch. 1].

Let us consider the trace operators

$$\Gamma_0 u = u(0), \Gamma_1 u = u'(0), u \in W_2^3\left(R_+;H;A\right).$$

From the theorem of traces [14, Ch. 1], it follows that $\Gamma_0 : W_2^3\left(R_+;H;A\right) \to H_{5/2}$, $\Gamma_1 : W_2^3\left(R_+;H;A\right) \to H_{3/2}$ are continuous operators. We denote

$$\overset{o}{W}{}_2^3\left(R_+;H;A\right) = \left\{u : u \in W_2^3\left(R_+;H;A\right), \Gamma_0 u = u(0) = 0, \Gamma_1 u = u'(0) = 0\right\}.$$

Let $L(X,Y)$ be a space of the bounded operators acting from space X to space Y.

We also assume that an operator $K \in L\left(W_2^3(R_+;H;A), H_{3/2}\right)$, and we denote

$$\overset{o}{W}{}^3_{2;K}(R_+;H;A) = \left\{u : u \in W_2^3(R_+;H;A), \Gamma_0 u = u(0) = 0, \Gamma_1 u = u'(0) = Ku\right\}.$$

Obviously, $\overset{o}{W}{}^3_2(R_+;H;A)$ and $\overset{o}{W}{}^3_{2;K}(R_+;H;A)$ are complete subspaces of $W_2^3(R_+;H;A)$.

Now in space H we consider the boundary value problem

$$u'''(t) - A^3u(t) + \sum_{j=1}^{3} A_j u^{(3-j)}(t) = f(t), \quad t \in R_+, \tag{1}$$

$$u(0) = 0, \quad u'(0) - Ku = 0, \tag{2}$$

where $f(t) \in L_2(R_+;H)$, $u(t) \in W_2^3(R_+;H;A)$, A_j, $j = \overline{1,3}$, are linear, in general, unbounded operators. Moreover, A is a self-adjoint positive defined operator, and the operator $K \in L\left(W_2^3(R_+;H;A), H_{3/2}\right)$, i.e. $\|Ku\|_{H_{3/2}} \le \kappa \|u\|_{W_2^3(R_+;H;A)}$.

Directly from equation (1) and boundary conditions (2) we can see that the main part of equation (1)

$$P_0(d/dt)\,u(t) = u'''(t) - A^3u(t)$$

is perturbed,

$$P_1(d/dt)\,u(t) = \sum_{j=1}^{3} A_j u^{(3-j)}(t),$$

and the second boundary condition from (2)

$$u'(0) = 0$$

is perturbed by some operator

$$u'(0) - Ku = 0, K \in L\left(W_2^3(R_+;H;A), H_{3/2}\right).$$

Definition 1. *If the vector function $u(t) \in W_2^3(R_+;H;A)$ satisfies equation (1) almost everywhere in R_+, then we say that u(t) is a regular solution of equation (1).*

Definition 2. *If for any $f(t) \in L_2(R_+;H)$ there is a regular solution of equation (1) which satisfies boundary conditions (2) in sense*

$$\lim_{t\to 0} \|u(t)\|_{H_{5/2}} = 0, \lim_{t\to 0} \left\|u'(t) - Ku\right\|_{H_{3/2}} = 0,$$

and the inequality

$$\|u\|_{W_2^3(R_+;H;A)} \le const\,\|f\|_{L_2(R_+;H)}$$

is fulfilled, then we say that the boundary value problem (1), (2) is regularly solvable.

In this paper we study conditions on the coefficients $A, A_j, j = \overline{1,3}$, of the operator-differential equation (1) and on the operator K, participating in boundary conditions (2), which provide regular resolvability of the problem (1), (2). The boundary value problem (1), (2) for $K = 0$ is studied in various aspects in [9, 11].

2. First of all, we consider the main part of the boundary value problem (1), (2) in H

$$u'''(t) - A^3 u(t) = f(t), t \in R_+, \tag{3}$$

$$u(0) = 0, \quad u'(0) - Ku = 0, \tag{4}$$

where $f(t) \in L_2(R_+; H)$, $u(t) \in W_2^3(R_+; H; A)$.

Denoting by

$$\mathrm{P}_0 u = P_0(d/dt)\,u,\ u \in \overset{o}{W}{}_{2;K}^{3}(R_+; H; A),$$

and using a technique from [15], we will be able to prove some auxiliary statements.

Lemma 1. *Let $\alpha > 0, \beta \in R$. Then for $x \in H_{5/2}$ the inequality*

$$\left\|A^3 e^{-\alpha A t}\sin\beta A t\; x\right\|^2_{L_2(R_+;H)} \le \left(\frac{1}{4\alpha} - \frac{\alpha}{4(\alpha^2+\beta^2)}\right)\|x\|^2_{H_{5/2}}$$

takes place.

P r o o f. Let $y = A^{5/2}x \in H$. Then,

$$\left\|A^3 e^{-\alpha A t}\sin\beta A t\; x\right\|^2_{L_2(R_+;H)} = \left\|A^{1/2} e^{-\alpha A t}\sin\beta A t\; y\right\|^2_{L_2(R_+;H)}$$

$$= \int_0^{+\infty}(A^{1/2}e^{-\alpha A t}\sin\beta A t\; y, A^{1/2}e^{-\alpha A t}\sin\beta A t\; y)dt = \int_0^{+\infty}(Ae^{-2\alpha A t}\sin^2\beta A t\; y, y)dt. \tag{5}$$

Using a spectral decomposition of the operator A in equality (5), we have

$$\int_0^{+\infty}\left(Ae^{-2\alpha A t}\sin^2\beta A t\; y, y\right)dt = \int_0^{+\infty}\left(\int_\mu^{+\infty}\sigma e^{-2\sigma\alpha t}\sin^2\beta\sigma t\,(dE_\sigma y, y)\right)dt$$

$$= \int\limits_{\mu}^{+\infty} \sigma \left(\int\limits_{0}^{+\infty} e^{-2\sigma\alpha t} \sin^2 \beta\sigma t dt \right) (dE_\sigma y, y).$$

Applying the formula of integration by parts, we get

$$\int\limits_{0}^{+\infty} e^{-2\sigma\alpha t} \sin^2 \beta\sigma t dt = \frac{1}{4\sigma\alpha} - \frac{1}{2} \int\limits_{0}^{+\infty} e^{-2\sigma\alpha t} \cos 2\beta\sigma t dt. \tag{6}$$

Taking into consideration that $\int\limits_{0}^{+\infty} e^{-2\sigma\alpha t} \cos 2\beta\sigma t dt = \frac{\alpha}{2\sigma(\alpha^2+\beta^2)}$, from (6) we obtain

$$\int\limits_{0}^{+\infty} e^{-2\sigma\alpha t} \sin^2 \beta\sigma t dt = \frac{1}{4\sigma\alpha} - \frac{\alpha}{4\sigma(\alpha^2 + \beta^2)}. \tag{7}$$

Substituting the value of integral (7) into expression (5), we have

$$\left\| A^3 e^{-\alpha A t} \sin \beta A t \; x \right\|^2_{L_2(R_+;H)} = \int\limits_{0}^{+\infty} (A e^{-2\alpha A t} \sin^2 \beta A t \; y, y) dt$$

$$= \int\limits_{\mu}^{+\infty} \sigma \left(\frac{1}{4\sigma\alpha} - \frac{\alpha}{4\sigma(\alpha^2 + \beta^2)} \right) (dE_\sigma y, y) = \left(\frac{1}{4\alpha} - \frac{\alpha}{4(\alpha^2 + \beta^2)} \right) \|y\|^2_H$$

$$= \left(\frac{1}{4\alpha} - \frac{\alpha}{4(\alpha^2 + \beta^2)} \right) \left\| A^{5/2} x \right\|^2_H = \left(\frac{1}{4\alpha} - \frac{\alpha}{4(\alpha^2 + \beta^2)} \right) \|x\|^2_{H_{5/2}},$$

i.e.,

$$\left\| A^3 e^{-\alpha A t} \sin \beta A t \; x \right\|^2_{L_2(R_+;H)} \le \left(\frac{1}{4\alpha} - \frac{\alpha}{4(\alpha^2 + \beta^2)} \right) \|x\|^2_{H_{5/2}}.$$

The lemma is proved.

Corollary 1. *Taking* $\alpha = \frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$ *in Lemma 1, we obtain the estimation*

$$\left\| A^3 e^{-\frac{1}{2} A t} \sin \frac{\sqrt{3}}{2} A t \; x \right\|^2_{L_2(R_+;H)} \le \frac{3}{8} \|x\|^2_{H_{5/2}}.$$

Lemma 2. *Let* $\kappa = \|K\|_{W_2^3(R_+;H;A) \to H_{3/2}} < 1$. *Then the equation* $\mathrm{P}_0 u = 0$ *has a unique trivial solution in the space* $\overset{o}{W}{}^3_{2;K}(R_+;H;A)$.

P r o o f. Let $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. The general solution of the equation $P_0(d/dt)\,u(t) = 0$ from the space $W_2^3(R_+; H; A)$ has the form [9, 15]:

$$u_0(t) = e^{\omega_1 At}x_1 + e^{\omega_2 At}x_2, \quad x_1,\ x_2 \in H_{5/2}.$$

From the condition $u(0) = 0$ we obtain that $x_1 = -x_2$. From the second boundary condition it follows that $(\omega_1 - \omega_2)Ax_1 = K\left(e^{\omega_1 At} - e^{\omega_2 At}\right)x_1$. From here we find

$$x_1 = \frac{1}{i\sqrt{3}}A^{-1}K\left(e^{\omega_1 At} - e^{\omega_2 At}\right)x_1 \equiv \Phi x_1,$$

and also we have

$$\|\Phi x_1\|_{H_{5/2}} = \left\|A^{5/2}\frac{1}{i\sqrt{3}}\left(A^{-1}K\left(e^{\omega_1 At} - e^{\omega_2 At}\right)x_1\right)\right\|_H$$

$$\leq \frac{1}{\sqrt{3}}\|K\|_{W_2^3(R_+;H;A)\to H_{3/2}}\left\|e^{\omega_1 At}x_1 - e^{\omega_2 At}x_1\right\|_{W_2^3(R_+;H;A)}. \tag{8}$$

Applying Corollary 1, we get

$$\left\|e^{\omega_1 At}x_1 - e^{\omega_2 At}x_1\right\|^2_{W_2^3(R_+;H;A)} = \left\|A^3\left(e^{\omega_1 At}x_1 - e^{\omega_2 At}x_1\right)\right\|^2_{L_2(R_+;H)}$$

$$+\left\|\omega_1^3 A^3 e^{\omega_1 At}x_1 - \omega_2^3 A^3 e^{\omega_2 At}x_1\right\|^2_{L_2(R_+;H)} = 2\left\|A^3\left(e^{\omega_1 At}x_1 - e^{\omega_2 At}x_1\right)\right\|^2_{L_2(R_+;H)}$$

$$= 2\left\|A^3\left(e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)At}x_1 - e^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)At}x_1\right)\right\|^2_{L_2(R_+;H)}$$

$$= 2\left\|A^3 e^{-\frac{1}{2}At}\left(e^{\frac{\sqrt{3}}{2}iAt}x_1 - e^{-\frac{\sqrt{3}}{2}iAt}x_1\right)\right\|^2_{L_2(R_+;H)}$$

$$= 8\left\|A^3 e^{-\frac{1}{2}At}\sin\frac{\sqrt{3}}{2}At\ x_1\right\|^2_{L_2(R_+;H)} \leq 8\cdot\frac{3}{8}\|x_1\|^2_{H_{5/2}} = 3\|x_1\|^2_{H_{5/2}}.$$

From here we have

$$\left\|e^{\omega_1 At}x_1 - e^{\omega_2 At}x_1\right\|_{W_2^3(R_+;H;A)} \leq \sqrt{3}\,\|x_1\|_{H_{5/2}}. \tag{9}$$

Considering inequality (9) from equality (8), we get

$$\|\Phi x_1\|_{H_{5/2}} \leq \frac{\kappa}{\sqrt{3}}\cdot\sqrt{3}\,\|x_1\|_{H_{5/2}} = \kappa\,\|x_1\|_{H_{5/2}}.$$

As $\kappa < 1$, then the operator $E - \Phi$ is invertible in $H_{5/2}$, and it follows that $x_1 = 0$, i.e. $u_0(t) = 0$. The lemma is proved.

Now we pass to the basic results of the problem (3), (4).

Theorem 1. *If* $u \in \overset{o}{W}{}^3_{2;K}(R_+; H; A)$ *and* $\kappa = \|K\|_{W_2^3(R_+;H;A)\to H_{3/2}} < 1$, *then the inequality*

$$\|P_0 u\|^2_{L_2(R_+;H)} \ge (1-\kappa)\,\|u\|^2_{W_2^3(R_+;H;A)} \tag{10}$$

takes place.

P r o o f. Let $u(t) \in \overset{o}{W}{}^3_{2;K}(R_+; H; A)$. Then we have

$$\|P_0 u\|^2_{L_2(R_+;H)} = \left\|\frac{d^3u}{dt^3} - A^3u\right\|^2_{L_2(R_+;H)}$$

$$= \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)} - 2\mathrm{Re}\left(\frac{d^3u}{dt^3}, A^3u\right)_{L_2(R_+;H)}. \tag{11}$$

Applying the formula of integration by parts, we obtain

$$\left(\frac{d^3u}{dt^3}, A^3u\right)_{L_2(R_+;H)} = -\left(A^{1/2}u''(0), A^{5/2}u(0)\right) + \left(A^{3/2}u'(0), A^{3/2}u'(0)\right)$$

$$-\left(A^{5/2}u(0), A^{1/2}u''(0)\right) - \left(A^3u, \frac{d^3u}{dt^3}\right)_{L_2(R_+;H)},$$

i.e.,

$$2\mathrm{Re}\left(\frac{d^3u}{dt^3}, A^3u\right)_{L_2(R_+;H)} = \left\|u'(0)\right\|^2_{H_{3/2}}. \tag{12}$$

So, for $\kappa = \|K\|_{W_2^3(R_+;H;A)\to H_{3/2}} < 1$ in view of (12) from equality (11), we have

$$\|P_0 u\|^2_{L_2(R_+;H)} = \|u\|^2_{W_2^3(R_+;H;A)} - \left\|u'(0)\right\|^2_{H_{3/2}}$$

$$= \|u\|^2_{W_2^3(R_+;H;A)} - \|Ku\|^2_{H_{3/2}} \ge (1-\kappa)\,\|u\|^2_{W_2^3(R_+;H;A)}.$$

The theorem is proved.

Theorem 2. *Let* A *be the positive defined self-adjoint operator in* H $(A = A^* \ge \mu_0 E)$, $\kappa = \|K\|_{W_2^3(R_+;H;A)\to H_{3/2}} < 1$. *Then the operator* $P_0 : \overset{o}{W}{}^3_{2;K}(R_+; H; A)$ $\to L_2(R_+; H)$ *isomorphically represents* $\overset{o}{W}{}^3_{2;K}(R_+; H; A)$ *on* $L_2(R_+; H)$.

P r o o f. From Lemma 2 it follows that $Ker P_0 = \{0\}$. We will prove that for any $f(t) \in L_2(R_+; H)$ there exists $u(t) \in \overset{o}{W}{}^3_{2;K}(R_+; H; A)$ such that $P_0 u = f$, i.e. $im P_0 = L_2(R_+; H)$.

Let us denote by $f_1(t) = \begin{cases} f(t), t > 0, \\ 0, t < 0, \end{cases}$ and $\widehat{f}_1(\xi)$ — the Fourier transformation of the vector function $f_1(t) \in L_2(R; H)$. Then the vector function

$$u_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(-i\xi^3 E - A^3\right)^{-1} \widehat{f}_1(\xi) e^{i\xi t} d\xi, \;\; t \in R,$$

satisfies the equation $P_0(d/dt)u(t) = f(t)$ in R_+ almost everywhere. We will also prove that $u_0(t) \in W_2^3(R; H; A)$. From the Plansharel theorem it follows that it is sufficient to prove that $A^3\widehat{u}_0(\xi)$, $\xi^3\widehat{u}_0(\xi) \in L_2(R; H)$, where

$$\widehat{u}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(t) e^{-i\xi t} d\xi.$$

It is obvious that

$$\left\|A^3\widehat{u}_0(\xi)\right\|^2_{L_2(R;H)} = \int_{-\infty}^{+\infty} \left\|A^3\widehat{u}_0(\xi)\right\|^2_H d\xi = \int_{-\infty}^{+\infty} \left\|A^3 \left(-i\xi^3 E - A^3\right)^{-1} \widehat{f}_1(\xi)\right\|^2_H d\xi$$

$$\leq \sup_{\xi \in R} \left\|A^3 \left(-i\xi^3 E - A^3\right)^{-1}\right\|^2 \int_{-\infty}^{+\infty} \left\|\widehat{f}_1(\xi)\right\|^2_H d\xi$$

$$= \sup_{\xi \in R} \left\|A^3 \left(i\xi^3 E + A^3\right)^{-1}\right\|^2 \left\|\widehat{f}_1(\xi)\right\|^2_{L_2(R;H)}$$

$$= \sup_{\xi \in R} \left\|A^3 \left(i\xi^3 E + A^3\right)^{-1}\right\|^2 \|f_1\|^2_{L_2(R;H)} = \sup_{\xi \in R} \left\|A^3 \left(i\xi^3 E + A^3\right)^{-1}\right\|^2 \|f\|^2_{L_2(R_+;H)}.$$

Further, from a spectral decomposition of the operator A it follows that for any $\xi \in R$,

$$\left\|A^3 \left(i\xi^3 E + A^3\right)^{-1}\right\| = \sup_{\mu \in \sigma(A)} \left|\mu^3 \left(i\xi^3 + \mu^3\right)^{-1}\right| \leq \sup_{\mu \geq \mu_0} \left|\mu^3 \left(\xi^6 + \mu^6\right)^{-1/2}\right| \leq 1$$

and $A^3\widehat{u}_0(\xi) \in L_2(R; H)$. In a similar way, one can prove that $\xi^3\widehat{u}_0(\xi) \in L_2(R; H)$. Hence $u_0(t) \in W_2^3(R; H; A)$.

Let us denote by $q(t)$ a narrowing of the vector function $u_0(t)$ on $[0; +\infty)$, i.e. $q(t) = u_0(t)|_{[0;+\infty)}$. It is obvious that $q(t) \in W_2^3(R_+; H; A)$. Therefore from the theorem of traces [14, Ch. 1], $q(0) \in H_{5/2}$, $q'(0) \in H_{3/2}$, $q''(0) \in H_{1/2}$. The solution of the equation $\mathrm{P}_0 u = f$ is searched in the form

$$u(t) = q(t) + e^{\omega_1 A t} x_1 + e^{\omega_2 A t} x_2,$$

where $\omega_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\omega_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, and $x_1, x_2 \in H_{5/2}$ are unknown vectors to be determined. From the condition $u(t) \in W^3_{2;K}(R_+; H; A)$ it follows that

$$\begin{cases} q(0) + x_1 + x_2 = 0, \\ q'(0) + \omega_1 A x_1 + \omega_2 A x_2 - K\left(q(t) + e^{\omega_1 A t} x_1 + e^{\omega_2 A t} x_2\right) = 0. \end{cases}$$

From here $(E - \Phi)x_1 = \psi$, where $\psi = \frac{1}{i\sqrt{3}}\left[\omega_2 q(0) - A^{-1}q'(0) + A^{-1}K\left(q(t) - q(0)e^{\omega_2 A t}\right)\right] \in H_{5/2}$. From the condition of the theorem we get $\|\Phi\|_{H_{5/2} \to H_{5/2}} < 1$, so $x_1 = (E - \Phi)^{-1}\psi \in H_{5/2}$. Now we can find $x_2 = -q(0) - (E - \Phi)^{-1}\psi \in H_{5/2}$. Consequently, $u \in \overset{o}{W}{}^3_{2;K}(R_+; H; A)$ and $\mathrm{P}_0 u = f$. And on the other hand,

$$\|\mathrm{P}_0 u\|^2_{L_2(R_+;H)} = \|P_0(d/dt)\,u\|^2_{L_2(R_+;H)}$$

$$= \left\|\frac{d^3u}{dt^3} - A^3 u\right\|^2_{L_2(R_+;H)} \le 2\,\|u\|^2_{W^3_2(R_+;H;A)}.$$

Therefore, by the Banach theorem, there is an inverse operator P_0^{-1} and it is bounded. From here it follows that $\|u\|_{W^3_2(R_+;H;A)} \le const\,\|f\|_{L_2(R_+;H)}$. The theorem is proved.

3. As it is clear from Theorem 2, the norms $\|u\|_{W^3_2(R_+;H;A)}$ and $\|\mathrm{P}_0 u\|_{L_2(R_+;H)}$ are equivalent in the space $\overset{o}{W}{}^3_{2;K}(R_+; H; A)$. Therefore, it is possible to estimate the norms of the operators of intermediate derivatives $A^{3-j}\frac{d^j}{dt^j} : \overset{o}{W}{}^3_{2;K}(R_+; H; A) \to L_2(R_+; H)$, $j = \overline{0,2}$, concerning $\|\mathrm{P}_0 u\|_{L_2(R_+;H)}$. We note that the methods of solving the equations without perturbed boundary conditions in the problems with perturbed boundary conditions are actually inapplicable. For example, in [9], for the estimation of the norms of operators of the intermediate derivatives, having great value for obtaining the conditions of resolvability of boundary value problems, the method of factorization is offered, which is inapplicable for studying boundary value problems with nonlocal boundary conditions or with the perturbed boundary conditions. Here, for our estimations we will follow [10] and use the inequalities known from analysis in combination with inequality (10).

The following theorem is true.

Theorem 3. *Let* $\kappa = \|K\|_{W^3_2(R_+;H;A) \to H_{3/2}} < 1$. *Then for any* $u \in \overset{o}{W}{}^3_{2;K}(R_+; H; A)$ *the following estimations take place:*

$$\left\|A^3 u\right\|_{L_2(R_+;H)} \le C_0(\kappa)\,\|\mathrm{P}_0 u\|_{L_2(R_+;H)}, \tag{13}$$

$$\left\|A^2 u'\right\|_{L_2(R_+;H)} \le C_1(\kappa)\,\|\mathrm{P}_0 u\|_{L_2(R_+;H)}, \tag{14}$$

$$\left\|Au''\right\|_{L_2(R_+;H)} \leq C_2(\kappa)\left\|\mathrm{P}_0 u\right\|_{L_2(R_+;H)}, \tag{15}$$

where

$$C_0(\kappa) = (1-\kappa)^{-1/2},\ C_1(\kappa) = \frac{2^{1/3}}{3^{1/2}}\left(1+\frac{3\kappa^{2/3}}{2^{1/3}}\right)^{1/2}(1-\kappa)^{-1/2},$$

$$C_2(\kappa) = \frac{2^{1/3}}{3^{1/2}}\cdot\frac{1+3^{1/2}\kappa^{2/3}}{(1-\kappa)^{1/2}}.$$

P r o o f. The validity of estimation (13) explicitly follows from inequality (10).

As $u \in \overset{o}{W}{}^3_{2;K}(R_+;H;A)$, then by the formula of integration by parts we get

$$\left\|A^2u'\right\|^2_{L_2(R_+;H)} = \int_0^\infty \left(A^2u', A^2u'\right)dt = -\int_0^\infty \left(A^3u, Au''\right)dt$$

$$= -\left(A^3u, Au''\right)_{L_2(R_+;H)} \leq \left\|A^3u\right\|_{L_2(R_+;H)}\left\|Au''\right\|_{L_2(R_+;H)}. \tag{16}$$

Similarly, we obtain

$$\left\|Au''\right\|^2_{L_2(R_+;H)} = \int_0^\infty \left(Au'', Au''\right)dt = -\left(A^{3/2}u'(0), A^{1/2}u''(0)\right)$$

$$-\int_0^\infty \left(A^2u', u'''\right)dt \leq \left\|A^{3/2}u'(0)\right\|_H \left\|A^{1/2}u''(0)\right\|_H + \left\|A^2u'\right\|_{L_2(R_+;H)}\left\|u'''\right\|_{L_2(R_+;H)}$$

$$= \left\|Ku\right\|_{H_{3/2}}\left\|A^{1/2}u''(0)\right\|_H + \left\|A^2u'\right\|_{L_2(R_+;H)}\left\|u'''\right\|_{L_2(R_+;H)}$$

$$\leq \kappa\left\|u\right\|_{W_2^3(R_+;H;A)}\left\|A^{1/2}u''(0)\right\|_H + \left\|A^2u'\right\|_{L_2(R_+;H)}\left\|u'''\right\|_{L_2(R_+;H)}. \tag{17}$$

On the other hand,

$$\left\|A^{1/2}u''(0)\right\|^2_H = 2\mathrm{Re}\int_0^\infty \left(Au'', u'''\right)dt = 2\mathrm{Re}\left(Au'', u'''\right)_{L_2(R_+;H)},$$

i.e.,

$$\left\|A^{1/2}u''(0)\right\|_H \leq 2^{1/2}\left\|Au''\right\|^{1/2}_{L_2(R_+;H)}\left\|u'''\right\|^{1/2}_{L_2(R_+;H)}. \tag{18}$$

Considering inequalities (16) and (18) in (17), we have

$$\left\|Au''\right\|^2_{L_2(R_+;H)} \le \kappa \left\|u\right\|_{W_2^3(R_+;H;A)} 2^{1/2} \left\|Au''\right\|^{1/2}_{L_2(R_+;H)} \left\|u'''\right\|^{1/2}_{L_2(R_+;H)}$$

$$+ \left\|Au''\right\|^{1/2}_{L_2(R_+;H)} \left\|A^3u\right\|^{1/2}_{L_2(R_+;H)} \left\|u'''\right\|_{L_2(R_+;H)},$$

i.e.,

$$\left\|Au''\right\|^{3/2}_{L_2(R_+;H)} \le 2^{1/2}\kappa \left\|u\right\|_{W_2^3(R_+;H;A)} \left\|u'''\right\|^{1/2}_{L_2(R_+;H)}$$

$$+ \left\|A^3u\right\|^{1/2}_{L_2(R_+;H)} \left\|u'''\right\|_{L_2(R_+;H)}.$$

Taking into consideration that $\left\|u'''\right\|_{L_2(R_+;H)} \le \left\|u\right\|_{W_2^3(R_+;H;A)}$, we obtain

$$\left\|Au''\right\|_{L_2(R_+;H)} \le 2^{1/3}\kappa^{2/3} \left\|u\right\|_{W_2^3(R_+;H;A)} + \left\|A^3u\right\|^{1/3}_{L_2(R_+;H)} \left\|u'''\right\|^{2/3}_{L_2(R_+;H)}. \quad (19)$$

Thus,

$$\left(\left\|Au''\right\|_{L_2(R_+;H)} - 2^{1/3}\kappa^{2/3} \left\|u\right\|_{W_2^3(R_+;H;A)}\right)^2 \le \left\|A^3u\right\|^{2/3}_{L_2(R_+;H)} \left\|u'''\right\|^{4/3}_{L_2(R_+;H)}.$$

Then for any $\varepsilon > 0$, applying the Young inequality, we get

$$\left(\left\|Au''\right\|_{L_2(R_+;H)} - 2^{1/3}\kappa^{2/3} \left\|u\right\|_{W_2^3(R_+;H;A)}\right)^2$$

$$\le \left(\varepsilon \left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{1/3} \left(\frac{1}{\varepsilon^{1/2}} \left\|u'''\right\|^2_{L_2(R_+;H)}\right)^{2/3}$$

$$\le \frac{1}{3}\varepsilon \left\|A^3u\right\|^2_{L_2(R_+;H)} + \frac{2}{3\varepsilon^{1/2}} \left\|u'''\right\|^2_{L_2(R_+;H)}.$$

Supposing $\frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon^{-1/2}$, we obtain $\varepsilon = 2^{2/3}$. Thus,

$$\left(\left\|Au''\right\|_{L_2(R_+;H)} - 2^{1/3}\kappa^{2/3} \left\|u\right\|_{W_2^3(R_+;H;A)}\right)^2 \le \frac{2^{2/3}}{3} \left\|u\right\|^2_{W_2^3(R_+;H;A)}.$$

Hence,

$$\left\|Au''\right\|_{L_2(R_+;H)} \le \left(\frac{2^{1/3}}{3^{1/2}} + 2^{1/3}\kappa^{2/3}\right) \left\|u\right\|_{W_2^3(R_+;H;A)}$$

$$= \frac{2^{1/3}}{3^{1/2}} \left(1 + 3^{1/2}\kappa^{2/3}\right) \left\|u\right\|_{W_2^3(R_+;H;A)} \le \frac{2^{1/3}}{3^{1/2}} \cdot \frac{1 + 3^{1/2}\kappa^{2/3}}{(1-\kappa)^{1/2}} \left\|\mathrm{P}_0 u\right\|_{L_2(R_+;H)}.$$

Finally, the estimation (15) is true. Now we will prove (14). Considering inequality (19) from (16), by the same reasoning as before, we get

$$\left\|A^2u'\right\|^2_{L_2(R_+;H)} \le 2^{1/3}\kappa^{2/3}\left\|u\right\|_{W_2^3(R_+;H;A)}\left\|A^3u\right\|_{L_2(R_+;H)}$$

$$+\left\|A^3u\right\|^{4/3}_{L_2(R_+;H)}\left\|u'''\right\|^{2/3}_{L_2(R_+;H)}$$

$$\le 2^{1/3}\kappa^{2/3}\left\|u\right\|^2_{W_2^3(R_+;H;A)} + \left(\varepsilon\left\|u'''\right\|^2_{L_2(R_+;H)}\right)^{1/3}\left(\frac{1}{\varepsilon^{1/2}}\left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{2/3}$$

$$\le 2^{1/3}\kappa^{2/3}\left\|u\right\|^2_{W_2^3(R_+;H;A)} + \frac{1}{3}\varepsilon\left\|u'''\right\|^2_{L_2(R_+;H)} + \frac{2}{3\varepsilon^{1/2}}\left\|A^3u\right\|^2_{L_2(R_+;H)}.$$

Also supposing here that $\varepsilon = 2^{2/3}$, we have

$$\left\|A^2u'\right\|^2_{L_2(R_+;H)} \le 2^{1/3}\kappa^{2/3}\left\|u\right\|^2_{W_2^3(R_+;H;A)} + \frac{2^{2/3}}{3}\left\|u\right\|^2_{W_2^3(R_+;H;A)}$$

$$= \frac{2^{2/3}}{3}\left(1+\frac{3\kappa^{2/3}}{2^{1/3}}\right)\left\|u\right\|^2_{W_2^3(R_+;H;A)} \le \frac{2^{2/3}}{3}\left(1+\frac{3\kappa^{2/3}}{2^{1/3}}\right)(1-\kappa)^{-1}\left\|\mathrm{P}_0u\right\|^2_{L_2(R_+;H)}.$$

So, estimation (14) is also proved. The theorem is proved.

The estimations of the norms of operators of intermediate derivatives in Theorem 3 are of separate mathematical interest. Similar problems for numerical functions can be found, for example, in [16] and references therein.

4. Before passing to the setting of conditions of regular resolvability for the boundary value problem (1), (2), we will prove the following statement.

Lemma 3. *Let $B_j = A_jA^{-j}$, $j = \overline{1,3}$, be bounded operators in H. Then an operator $\mathrm{P} = \mathrm{P}_0 + \mathrm{P}_1$, where P_1 is the operator acting in the following way:*

$$\mathrm{P}_1u = P_1\left(d/dt\right)u, \;\; u \in \overset{o}{W}{}^3_{2;K}\left(R_+;H;A\right),$$

is the bounded operator from $\overset{o}{W}{}^3_{2;K}\left(R_+;H;A\right)$ to $L_2(R_+;H)$.

P r o o f. In fact, for any $u(t) \in \overset{o}{W}{}^3_{2;K}\left(R_+;H;A\right)$,

$$\left\|\mathrm{P}u\right\|_{L_2(R_+;H)} \le \left\|\mathrm{P}_0u\right\|_{L_2(R_+;H)} + \left\|\mathrm{P}_1u\right\|_{L_2(R_+;H)} \le \left\|\mathrm{P}_0u\right\|_{L_2(R_+;H)}$$

$$+\sum_{j=1}^{3}\left\|A_ju^{(3-j)}\right\|_{L_2(R_+;H)} \le \left\|\mathrm{P}_0u\right\|_{L_2(R_+;H)} + \sum_{j=1}^{3}\left\|B_j\right\|_{H\to H}\left\|A^ju^{(3-j)}\right\|_{L_2(R_+;H)}.$$

Then from this inequality, taking into consideration Theorem 2 and the theorem of intermediate derivatives [14, ch.1], we get

$$\|\mathrm{P}u\|_{L_2(R_+;H)} \leq const\, \|u\|_{W_2^3(R_+;H;A)} .$$

The lemma is proved.

And now we formulate the basic theorem of regular solvability of the problem (1), (2).

Theorem 4. *Let the conditions of Theorem 2 be satisfied, and the operators* $B_j = A_j A^{-j}$, $j = \overline{1,3}$, *be bounded in* H, *and the inequality*

$$\alpha(\kappa) = \sum_{j=0}^{2} C_j(\kappa) \|B_{3-j}\|_{H\to H} < 1$$

take place, where $C_j(\kappa)$, $j = \overline{0,2}$, *are defined in Theorem 3. Then the problem (1), (2) is regularly solvable.*

P r o o f. By Theorem 2, the operator $\mathrm{P}_0 : \overset{o}{W}{}_{2;K}^3(R_+;H;A) \to L_2(R_+;H)$ is isomorphism. Then there is a bounded inverse operator P_0^{-1}. We rewrite the problem (1), (2) in the form of the operator equation $\mathrm{P}u = \mathrm{P}_0 u + \mathrm{P}_1 u = f$, where $f \in L_2(R_+;H)$, $u \in \overset{o}{W}{}_{2;K}^3(R_+;H;A)$. After replacement $\mathrm{P}_0 u = v$, we obtain the equation $v + \mathrm{P}_1\mathrm{P}_0^{-1} v = f$ from $L_2(R_+;H)$. But for any $v \in L_2(R_+;H)$, considering Theorem 3,

$$\left\|\mathrm{P}_1\mathrm{P}_0^{-1}v\right\|_{L_2(R_+;H)} = \|\mathrm{P}_1 u\|_{L_2(R_+;H)} = \left\|\sum_{j=0}^{2} A_{3-j} u^{(j)}\right\|_{L_2(R_+;H)}$$

$$\leq \sum_{j=0}^{2} \|B_{3-j}\| \left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)} \leq \sum_{j=0}^{2} C_j(\kappa) \|B_{3-j}\|_{H\to H} \|\mathrm{P}_0 u\|_{L_2(R_+;H)}$$

$$= \alpha(\kappa)\, \|v\|_{L_2(R_+;H)} .$$

Thus, the operator $E + \mathrm{P}_1\mathrm{P}_0^{-1}$ is invertible in $L_2(R_+;H)$. Then $v = \left(E + \mathrm{P}_1\mathrm{P}_0^{-1}\right)^{-1} f$ and $u = \mathrm{P}_0^{-1}\left(E + \mathrm{P}_1\mathrm{P}_0^{-1}\right)^{-1} f$. From the above, it follows that

$$\|u\|_{W_2^3(R_+;H;A)} \leq const\, \|f\|_{L_2(R_+;H)} .$$

The theorem is proved.

Corollary 2. *Let* $K = 0$. *If the inequality*

$$\alpha(0) = \frac{2^{1/3}}{3^{1/2}} \left(\|B_1\|_{H\to H} + \|B_2\|_{H\to H}\right) + \|B_3\|_{H\to H} < 1$$

takes place, the problem (1), (2) is regularly solvable.

We must note that for $K = 0$ from Theorem 4 we obtain the results similar to those of works [9] and [10] if we take coefficient $\rho(t)$ of the constant term in an equation as a unit.

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