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Controllability Problems for the Non-Homogeneous String that is Fixed at the Right End Point and has the Dirichlet Boundary Control at the Left End Point

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In the paper, the necessary and sufficient conditions of null-controllability and approximate null-controllability are obtained for the control system $w_{tt}(x,t) = w_{xx}(x,t) - q(x)w(x,t)$, $w(0,t) = u(t)$, $w(d,t) = 0$, $x \in (0,d)$, $d > 0$, $t \in (0,T)$, $0 < T \le d$, where $q(x) \in C^1[0,d]$, $q(x) \ge 0$, $q'_+(0) = q'_-(d) = 0$, u is a control, $|u(t)| \le 1$ on $(0,T)$. The problems for the control system are considered in the modified Sobolev spaces. The control that solves these problems is found explicitly. The bang-bang controls solving the approximate null-controllability problem are constructed as the solutions of the Markov trigonometric moment problem.

Key words: wave equation, controllability problem, Dirichlet control bounded by a hard constant, modified Sobolev space, Sturm–Liouville problem, transformation operator.

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1. Introduction

In the paper, the wave equation for a non-homogeneous string on a finite segment is considered. The string is fixed at the right end point. At the left end point we consider a control that is bounded by a hard constant. We study the problems of the null- and approximate null-controllability of this system in the space $\mathcal{H}_Q^s$, $s \le 0$ (the modified Sobolev space under the operator $(1+D^2+q(x))^{s/2}$ instead of $(1+D^2)^{s/2}$, $D = -id/dx$). First we consider the Sturm–Liouville problem on a given segment, find its eigenfunctions that form an orthonormal basis and expand the functions describing the control system in terms of this basis.

Then we apply the transformation operators for the Sturm–Liouville problem. This method allows to obtain the necessary and sufficient conditions of the null- and approximate null-L^∞-controllability and an explicit formula for the control.

Note that the controllability problems for a hyperbolic partial differential equation were studied in a number of papers (see, e.g., the references in [1]). Gugat and Leugering [2] considered the wave equation for the homogeneous string $\frac{1}{c^2}y_{tt}(x,t)-y_{xx}(x,t)=0$ on the segment $(0,L)$. The string was fixed at the left end point. At the right end point there was considered a control that had a minimal norm in $L^\infty(0,T)$. The problems of exact and approximate null-controllability at the time $T=2\frac{L}{c}$ were analyzed. A bang-bang control solving both problems for all initial states from $L^\infty(0,L)\times W^{-1,\infty}(0,L)$ was constructed.

The boundary L^p-controllability ($2\le p\le\infty$) for a homogeneous string on a finite segment is well studied in [3]–[8] and other papers. Gugat et al. [4] and Fardigola, Khalina [6] considered the boundary L^∞-controllability for the wave equation for a homogeneous string on a finite segment. Moreover, controls were bounded by a hard constant for practical purposes in [6]. We should note that some results obtained in [4, 6] can be found in the present paper, exactly when the potential $q(x)$ is equal to zero. But if $q(x)\ne 0$ on the considered segment, then the studying of the null-controllability problems for the wave equation becomes more complicated. That is why we use the transformation operators for the Sturm–Liouville problem on a segment. These operators describe the connection among the initial state of the controlled system, the control and the corresponding steering state. We introduce and study the transformation operators in $\mathcal{H}_Q^s$, $s\le 0$.

Most of [9] is devoted to the controllability and observability theories for the linear hyperbolic systems of the form $w_t=A(x)w_x+B(x)w$, $w\in E^n$, $t\ge 0$, $x\in(0,1)$. For these systems, a boundary value control is considered and the conditions of exact and approximate controllability are obtained when a control, the initial and steering functions are from L^2. The author also considers a problem of the boundary L^2-controllability for the wave equation $\rho(x)\frac{\partial^2 w}{\partial t^2}-\sum_{i,j=1}^n\frac{\partial}{\partial x^i}\left(a_j^i(x)\frac{\partial w}{\partial x^j}\right)=0$ in the bounded open connected set Ω in $\mathbb{R}^n$. The problem is considered in the Sobolev spaces $H^m(\Omega)$, $m=0,1,2$. There are also obtained the conditions of approximate controllability for this equation and the conditions of exact controllability for the equation when $\rho(x)$ and $a_j^i(x)$ are constants, $i,j=\overline{1,n}$. We should note that the wave equation for the non-homogeneous string, studied in the present paper, cannot be reduced to the systems above.

Emanuilov [10] investigated the controllability problems for the linear hyperbolic partial differential second-order equation in the bounded domain $\Omega\subset\mathbb{R}^n$. The Dirichlet control is extended on a part of the boundary. The problems are considered in $L^2(\Omega) \times H^{-1}(\Omega)$. A sufficient condition of the exact L^2-controllability is obtained. Note that the restrictions imposed on the coefficients of the wave equation in the present paper are weaker than those in [10].

Il'in and Moiseev [11] studied the problem of the boundary controllability for a string on $[0, l]$ in the case of $q = const > 0$. A control from $W_2^1(0, T)$ is considered at the left end of the string, and the right end is fixed. The problem is considered in the class $\widehat{W}_2^1(0, l)$, namely in the class $W_2^1(0, l)$ with an additional condition of the functions smoothness on the boundary of the considered domain. The time $T = 2l$ is proved to be minimal when the system is controllable under the described conditions. The control solving this problem is found in terms of the initial and steering functions. It is also pointed out that additional conditions should be imposed on the initial functions when $T < 2l$. In the present paper, the potential q is not a constant generally speaking. A boundary control is bounded by a hard constant. Therefore it is of the class L^∞. The problem is considered in the space $\mathcal{H}_Q^s$, $s \le 0$. In particular, the solution smoothness is smaller than that in [11]. In the present paper, the condition $q(x) \ne const$ essentially differs from that in [11]. That is why we have to apply the transformation operators for the Sturm–Liouville problem on a segment. We also consider only $T \in (0, l]$ in contrast to [11]. We prove that at the time $0 < T \le l$ (from not an arbitrary initial state) the system becomes (approximately) null-controllable. The description of these initial functions is given in the present paper. The necessary and sufficient conditions of the null- and approximate null-controllability for the considered system are also obtained. For the control, an explicit formula is found in terms of the initial state by using the transformation operator. Note that for $q = const$ the results of the present paper are not contained in [11].

The structure of the paper is the following.

In Section 3 we formulate the obtained results: the formula that connects the initial state, the control and the steering state; the necessary and sufficient conditions of the null-controllability and approximate null-controllability for a given control system. The control solving these problems is found explicitly. Moreover, it is proved that the boundedness of the initial state is the necessary and sufficient condition of the boundedness of the corresponding control. In this section the explicit representations for the kernels of the transformation operators are also obtained in the case when the potential of the wave equation is constant. The example for this case is given.

In Section 4 the proofs of the theorems formulated in Section 3 are given.

In Section 5 we construct the bang-bang controls that solve the approximate null-controllability problem. We reduce this problem to the Markov trigonometric moment problem which can be solved by the algorithm given in [12]. We show that the solutions of the Markov trigonometric moment problem are the solutions of the approximate null-controllability problem for $s < -1/2$.

In Appendix some auxiliary statements are proved. We describe in detail the spaces introduced in Section 2. We also introduce and study the transformation operators for the Sturm–Liouville problem on a segment in these spaces.

2. Notation

Consider the wave equation on a finite segment

$$w_{tt}(x,t) = w_{xx}(x,t) - q(x)w(x,t), \quad x \in (0,d),\, t \in (0,T), \tag{2.1}$$

controlled by the boundary conditions

$$w(0,t) = u(t),\, w(d,t) = 0, \qquad t \in (0,T), \tag{2.2}$$

where $d > 0$, $0 < T \le d$.

Let us introduce the spaces used in this paper. First we consider the Sturm–Liouville problem on the segment $(0,d)$

$$Gv \equiv -v''(x) + q(x)v(x) = \lambda^2 v(x), \quad v(0) = v(d) = 0, \qquad x \in (0,d), \tag{2.3}$$

where $q \in \mathcal{E}(0,d) = \left\{ r \in C^1[0,d] : r(x) \ge 0, r'_+(0) = r'_-(d) = 0 \right\}$.

It is well known that the operator $G \equiv -\left(\frac{d}{dx}\right)^2 + q(x)$ has a countable set of eigenvalues $\{\mu_n = \lambda_n^2\}_{n=1}^\infty$ that are real, nonnegative and simple, and $\lambda_n \ne 0$, $n = \overline{1,\infty}$ (see, e.g., [13]). Let $\{y_n(\lambda_n,x)\}_{n=1}^\infty$ be a system of corresponding eigenfunctions. They are real and form the orthonormal basis in $L^2[0,d]$. We have $Gy_n(\lambda_n,x) = \lambda_n^2 y_n(\lambda_n,x)$ for $n = \overline{1,\infty}$.

Let $\mathcal{S}$ be the Schwartz space ([14])

$$\mathcal{S} = \left\{ \varphi \in C^\infty(\mathbb{R}) : \forall m,l \in \mathbb{N}\cup 0\, \exists C_{ml} > 0\, \forall x \in \mathbb{R} \left| \varphi^{(m)}(x) \left(1 + |x|^2\right)^l \right| \le C_{ml} \right\}$$

and $\mathcal{S}'$ be the dual space.

A distribution $f \in \mathcal{S}'$ is said to be *odd* if $(f, \varphi(-x)) = -(f, \varphi(x))$, $\varphi \in \mathcal{S}$. A distribution $f \in \mathcal{S}'$ is said to be *even* if $(f, \varphi(-x)) = (f, \varphi(x))$, $\varphi \in \mathcal{S}$.

Let $\Omega : \mathcal{S}' \to \mathcal{S}'$ be the odd extension operator, $\Xi : \mathcal{S}' \to \mathcal{S}'$ be the even extension operator. Thus $(\Omega f)(x) = f(x) - f(-x)$, $(\Xi f)(x) = f(x) + f(-x)$ when $f \in \mathcal{S}'$. Let $\mathcal{T}_h$ be the translation operator: $\mathcal{T}_h\varphi(x) = \varphi(x+h)$, $\varphi \in \mathcal{S}$ and $(\mathcal{T}_h f, \varphi) = (f, \mathcal{T}_{-h}\varphi)$, $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

Let us assume that $q(x)$ and $y_n(\lambda_n,x)$, $n = \overline{1,\infty}$, are defined on $\mathbb{R}$ and are equal to 0 on $\mathbb{R} \setminus [0,d]$. Denote by $Q(x)$ the even $2d$-periodic extension of $q(x)$: $Q = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\Xi q$. It is obvious that $Q \in C^1(\mathbb{R})$. Denote by $Y_n(\lambda_n,x)$ the odd $2d$-periodic extension of $y_n(\lambda_n,x)$ with respect to x, $n = \overline{1,\infty}$. Thus $Y_n(\lambda_n,\cdot) = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\Omega y_n(\lambda_n,\cdot)$. Introduce the operator $D_Q^2 = Q(x) + D^2$, where $D = -id/dx$. Then $D_Q^2 Y_n(\lambda_n,x) = \lambda_n^2 Y_n(\lambda_n,x)$, $n = \overline{1,\infty}$.

Denote

$$\mathcal{H}^s_Q = \left\{ f \in \mathcal{S}' : f \text{ is odd and } 2d\text{–periodic, } \left(1 + D_Q^2\right)^{s/2} f \in L^2_{loc}(\mathbb{R}) \right\}, \qquad s \in \mathbb{R}$$

with the norm $\|f\|^s_Q = \left(\int_{-d}^{d} \left| \left(1 + D_Q^2\right)^{s/2} f(x) \right|^2 dx \right)^{1/2}$. We use the norm $|||f|||^s_Q = \left(\left(\|f_1\|^s_Q \right)^2 + \left(\|f_2\|^{s-1}_Q \right)^2 \right)^{1/2}$ for $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^s_Q \times \mathcal{H}^{s-1}_Q$.

Notice that, in fact, the space $\mathcal{H}^s_Q$ coincides with the Sobolev space of odd periodic functions $H^s_{0,per}$. It is shown in the proof of Lemma A.2.

It is easy to see that the system $\left\{ \frac{1}{\sqrt{2}} Y_n(\lambda_n, x) \right\}_{n=1}^{\infty}$ forms the orthonormal basis in $\mathcal{H}^0_Q$. It is proved in Lemma A.2 (Appendix A) that

$$\begin{aligned} \mathcal{H}^s_Q &= \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n Y_n(\lambda_n, x) \text{ and } \left\{ f_n (1 + \lambda_n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2 \right\} \\ &= \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n^1 \sin \frac{\pi n x}{d} \text{ and } \left\{ f_n^1 (1 + n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2 \right\}, \ s \in \mathbb{R} \end{aligned}$$

with the equivalent norms

$$\|f\|^s_Q = \left(\sum_{n=1}^{\infty} \left| f_n (1 + \lambda_n^2)^{s/2} \right|^2 \right)^{1/2}, \quad [\![f]\!]^s_Q = \left(\sum_{n=1}^{\infty} \left| f_n^1 \left(1 + n^2\right)^{s/2} \right|^2 \right)^{1/2}.$$

It follows from the above that $\mathcal{H}^s_Q \subset \mathcal{H}^{s'}_Q$ when $s' < s$. Denote for $s \in \mathbb{R}$

$$\mathcal{H}^s_Q(a, b) = \left\{ f \in \mathcal{S}' : \operatorname{supp} f \subset [a, b] \text{ and } \left(1 + D_Q^2\right)^{s/2} f \in L^2(\mathbb{R}) \right\}.$$

One can see that if $f \in \mathcal{H}^s_Q(0, d)$, then $F = \sum_{k \in \mathbb{Z}} \mathcal{T}_{2dk} \Omega f \in \mathcal{H}^s_Q$ for $s \in \mathbb{R}$. Evidently,

$$\|F\|^s_Q = \sqrt{2}\, \|f\|^s_Q .$$

We also use the following spaces:

$$\begin{aligned} \mathbb{H}^s_0(-d, d) = \Bigg\{ f \in \mathcal{S}' : \operatorname{supp} f \subset [-d, d],\ f(x) = \sum_{n=1}^{\infty} f_n e^{-i\lambda_n x} \\ \text{and } \left\{ f_n (1 + \lambda_n^2)^{s/2} \right\}_{n=1}^{\infty} \in l^2 \Bigg\}, \quad s \in \mathbb{R} \end{aligned}$$

with the norm $\|f\|^s_0 = \left(\sum_{n=1}^{\infty} \left| \left(1 + \lambda_n^2\right)^{s/2} f_n \right|^2 \right)^{1/2}$, where λ_n are the arithmetical roots of eigenvalues of the Sturm–Liouville problem (2.3). For convenience in further reasoning we will number $\{\lambda_n\}_{n=1}^{\infty}$ in ascending order.

Note that the series of exponentials in the definition of $\mathbb{H}_0^s(-d,d)$ converges with respect to the norm of the standard Sobolev space H_0^s that is proved in Lemma A.4. Note also that $\mathbb{H}_0^s(-d,d) \subset \mathbb{H}_0^{s'}(-d,d)$ when $s' < s$. In Lemma A.7 it is proved that if $f \in \mathbb{H}_0^s(-d,d)$, then $f' \in \mathbb{H}_0^{s-1}(-d,d)$.

The properties of the functions from the space $\mathbb{H}_0^s(-d,d)$ can be found in Appendix A.

R e m a r k 2.1. The system $\left\{e^{i\lambda_n x}\right\}_{n=1}^{\infty}$ is the Riesz basis in the space $L^2(-d,d)$ (it is proved in Lemma A.1). Hence $\mathcal{H}_Q^0(-d,d) = \left\{f \in \mathbb{H}_0^0(-d,d) : f \text{ is odd}\right\}$.

Further, throughout the paper we will assume that $s \le 0$.

3. Main Results

Consider control system (2.1), (2.2) with the initial conditions

$$w(x,0) = w_0^0(x), \qquad w_t(x,0) = w_1^0(x), \qquad x \in (0,d), \tag{3.1}$$

where $w^0 = \begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix} \in \mathcal{H}_Q^s(0,d) \times \mathcal{H}_Q^{s-1}(0,d)$. We assume that $q \in \mathcal{E}(0,d)$ and the control u satisfies the restriction

$$u \in \mathcal{B}(0,T) = \left\{v \in L^\infty(0,T) : |v(t)| \le 1 \text{ a. e. on } (0,T)\right\}.$$

We consider the solutions of system (2.1), (2.2), (3.1) in the space $\mathcal{H}_Q^s$.

Extend $w(x,t)$ and the initial functions from the segment $(0,d)$ on the whole axis. Consider the odd $2d$-periodic extensions (with respect to x)

$$W(\cdot,t) = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\Omega w(\cdot,t), \;\; W^0 = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\Omega w^0, \quad t \in (0,T).$$

One can see that $W^0 \in \mathcal{H}_Q^s \times \mathcal{H}_Q^{s-1}$, $W(\cdot,t) \in \mathcal{H}_Q^s$ $(t \in (0,T))$.

It is easy to prove that control problem (2.1), (2.2), (3.1) is equivalent to the following problem:

$$W_{tt}(x,t) = W_{xx}(x,t) - Q(x)W(x,t) - 2u(t)\sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\delta'(x), \quad x \in \mathbb{R},\, t \in (0,T), \tag{3.2}$$

$$W(x,0) = W_0^0(x), \qquad W_t(x,0) = W_1^0(x), \quad x \in \mathbb{R}, \tag{3.3}$$

where δ is the Dirac distribution. Consider (3.2), (3.3) with the steering condition

$$W(x,T) = W_0^T(x), \qquad W_t(x,T) = W_1^T(x), \quad x \in \mathbb{R}. \tag{3.4}$$

The solutions of (3.2)–(3.4) are considered in the space $\mathcal{H}^s_Q$.

For the given $T > 0$, $w^0 \in \mathcal{H}^s_Q(0,d) \times \mathcal{H}^{s-1}_Q(0,d)$ denote by $\mathcal{R}_T(w^0)$ the set of the states $W^T \in \mathcal{H}^s_Q \times \mathcal{H}^{s-1}_Q$ for which there exists a control $u \in \mathcal{B}(0,T)$ such that problem (3.2)–(3.4) has a unique solution in $\mathcal{H}^s_Q$.

Definition 3.1. *A state $w^0 \in \mathcal{H}^s_Q(0,d) \times \mathcal{H}^{s-1}_Q(0,d)$ is called null-controllable at a given time $T > 0$ if 0 belongs to $\mathcal{R}_T(w^0)$ and approximately null-controllable at a given time $T > 0$ if 0 belongs to the closure of $\mathcal{R}_T(w^0)$ in $\mathcal{H}^s_Q \times \mathcal{H}^{s-1}_Q$.*

Definition 3.2. *Denote by $\mathrm{S}_T : \mathcal{H}^p_Q \times \mathcal{H}^{p-1}_Q \to \mathcal{H}^p_Q \times \mathcal{H}^{p-1}_Q$, $p \in \mathbb{R}$, the operator*

$$(\mathrm{S}_T f)(x) = \sum_{n=1}^{\infty} \begin{pmatrix} \cos\lambda_n T & \frac{\sin\lambda_n T}{\lambda_n} \\ -\lambda_n \sin\lambda_n T & \cos\lambda_n T \end{pmatrix} \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} Y_n(\lambda_n, x),$$

where $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \sum_{n=1}^{\infty} \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} Y_n(\lambda_n, x)$, $D(\mathrm{S}_T) = R(\mathrm{S}_T) = \mathcal{H}^p_Q \times \mathcal{H}^{p-1}_Q$.

The properties of the operator S_T are studied in Lemma A.3.

To proceed further, we define the transformation operators for the Sturm–Liouville problem on a segment. It is known [15] that the integral operator $\mathcal{K}$, given by the formula $(\mathcal{K}f)(x) = f(x) + \int_0^x \mathtt{K}(x,t;\infty) f(t)\,dt$, transfers the solution of the equation $y'' + \lambda^2 y = 0$ on a segment $[-d,d]$ with the initial conditions $y(0) = 0$, $y'(0) = 1$ to the solution of the equation $y'' - q(x)y + \lambda^2 y = 0$ with the same initial conditions at the point $x = 0$. Due to [15], the operator $\mathcal{K}$ has an inverse which is denoted by $\mathcal{L}$ (see Appendix B for details). We determine these operators for $p \in \mathbb{R}$ in the following spaces: $\mathcal{K} : \mathbb{H}_0^{-p}(-d,d) \longrightarrow \mathcal{H}_Q^{-p}(-d,d)$, $\mathcal{L} : \mathcal{H}_Q^{-p}(-d,d) \longrightarrow \mathbb{H}_0^{-p}(-d,d)$, where $D(\mathcal{K}) = \left\{ f \in \mathbb{H}_0^{-p}(-d,d) : f \text{ is odd} \right\}$, $D(\mathcal{L}) = \mathcal{H}_Q^{-p}(-d,d)$, $R(\mathcal{K}) = D(\mathcal{L})$, $R(\mathcal{L}) = D(\mathcal{K})$ (see Lemma B.1). In Lemma B.2 it is proved that these operators are linear and continuous. In Appendix B we also determine the adjoint operators $\mathcal{K}^* : \mathcal{H}^p_Q(-d,d) \to \mathbb{H}^p_0(-d,d)$, $\mathcal{L}^* : \mathbb{H}^p_0(-d,d) \to \mathcal{H}^p_Q(-d,d)$, where $D(\mathcal{K}^*) = \mathcal{H}^p_Q(-d,d)$, $D(\mathcal{L}^*) = \{f \in \mathbb{H}^p_0(-d,d) : f \text{ is odd}\}$, $R(\mathcal{K}^*) = D(\mathcal{L}^*)$, $R(\mathcal{L}^*) = D(\mathcal{K}^*)$ that are linear and continuous. The properties of the operators $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}^*$, $\mathcal{L}^*$ are studied in Appendix B.

The main results of the work are the following theorems on the null- and approximate null-controllability of the initial state of (2.1), (2.2), (3.1).

Theorem 3.1. *Let $0 < T \le d$, $w^0 \in \mathcal{H}^s_Q(0,d) \times \mathcal{H}^{s-1}_Q(0,d)$, $s \le 0$. Then*

$$\mathcal{R}_T(w^0) = \left\{ \mathrm{S}_T \left[W^0 - \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd} \left(\mathcal{L}^* \begin{pmatrix} \Omega U \\ \Omega U' \end{pmatrix} \right) \right] : u \in \mathcal{B}(0,T) \right\}, \tag{3.5}$$

where $U(t) = u(t)$ *on* $[0, T]$ *and* $U(t) = 0$ *on* $\mathbb{R} \setminus [0, T]$.

Theorem 3.2. *Let* $0 < T \le d$, $w^0 \in \mathcal{H}_Q^s(0, d) \times \mathcal{H}_Q^{s-1}(0, d)$, $s \le 0$. *Then the following statements are equivalent:*

(i) *the state* w^0 *is null-controllable at the time* T*;*

(ii) *the state* w^0 *is approximately null-controllable at the time* T*;*

(iii) *the following conditions hold:*

$$\operatorname{supp} w_0^0 \subset [0, T], \tag{3.6}$$

$$w_0^0 \in L^\infty(0, d) \text{ and } \left|\left(\mathcal{K}^* \Omega w_0^0\right)(x)\right| \le 1 \text{ a.e. on } [-d, d], \tag{3.7}$$

$$\Omega w_1^0 = \mathcal{L}^* \left(\operatorname{sign} t \mathcal{K}^* \Omega w_0^0\right)' \quad \text{on } [-d, d]. \tag{3.8}$$

In addition, the solution of the controllability problem (the control u*) is unique and given by the formula*

$$u(t) = w_0^0(t) + \int_t^T K(x, t; \infty) w_0^0(x)\, dx, \quad t \in [0, T]. \tag{3.9}$$

We will prove these theorems in Section 4.

R e m a r k 3.1. From (3.9) we have that there exists $U > 0$ such that $|u(x)| \le U$ on $(0, T)$ iff there exists $V > 0$ such that $\left|w_0^0(x)\right| \le V$ on $(0, d)$.

R e m a r k 3.2. Consider the case when $q(x) \equiv q = const > 0$ on $(0, d)$. Obviously, $Q(x) \equiv q$ on $(-d, d)$. Find the kernels $\mathtt{L}(t, x; \infty)$ and $\mathtt{K}(t, x; \infty)$ of the transformation operators on $(-d, d) \times (-d, d)$. We have $\mathtt{K}(x, t; \infty) = \mathtt{K}(x, t) - \mathtt{K}(x, -t)$, $\mathtt{L}(x, t; \infty) = \mathtt{L}(x, t) - \mathtt{L}(x, -t)$, where $\mathtt{K}(x, t)$ and $\mathtt{L}(x, t)$ are the solutions of the following systems (see Appendix B):

$$\mathtt{K}_{xx}(x, t) - \mathtt{K}_{tt}(x, t) = q\mathtt{K}(x, t), \qquad \mathtt{L}_{xx}(x, t) - \mathtt{L}_{tt}(x, t) = -q\mathtt{L}(x, t),$$

$$\mathtt{K}(x, x) = \frac{1}{2} qx, \quad \mathtt{K}(x, -x) = 0, \qquad \mathtt{L}(x, x) = -\frac{1}{2} qx, \quad \mathtt{L}(x, -x) = 0.$$

We reduce these systems to the Gurs problems and solve by the convergence method (see, e.g., [13]). As a result, for the kernels we obtain the following formulas:

$$\mathtt{K}(x, t; \infty) = qt \frac{I_1\left(\sqrt{q(x^2 - t^2)}\right)}{\sqrt{q(x^2 - t^2)}}, \quad |t| < |x|, \quad \mathtt{K}(x, t; \infty) = 0, \quad |t| \ge |x|;$$

$$\mathtt{L}(x, t; \infty) = -qt \frac{J_1\left(\sqrt{q(x^2 - t^2)}\right)}{\sqrt{q(x^2 - t^2)}}, \quad |t| < |x|, \quad \mathtt{L}(x, t; \infty) = 0, \quad |t| \ge |x|;$$

where $J_m(z)$ is the Bessel function, $I_m(z) = i^{-m}J_m(iz)$ is the modified Bessel function, $m \in \mathbb{Z}$. Thus the explicit formula for a control in the case of $q(x) \equiv q > 0$ is $u(t) = w_0^0(t) + qt\int_t^T \frac{I_1\left(\sqrt{q(x^2-t^2)}\right)}{\sqrt{q(x^2-t^2)}} w_0^0(x)\,dx$, $t \in (0,T)$. The inverse formula is $w_0^0(t) = u(t) - qt\int_t^T \frac{J_1\left(\sqrt{q(x^2-t^2)}\right)}{\sqrt{q(x^2-t^2)}} u(x)\,dx$, $t \in (0,T)$. We remark that the similar formulas for a control and an initial state were obtained for the semi-infinite string [16] in the case when $q = const$ and the boundary control was of the Neumann type. It is easy to prove that if $\left|w_0^0(t)\right| \le C_w$ on $(0,d)$, then $|u(t)| \le C_w I_0(\sqrt{q}T)$ and if $|u(t)| \le C_u$ on $(0,T)$, then $\left|w_0^0(t)\right| \le C_u\left(1 + \sqrt{q}T\right)$.

E x a m p l e 3.1. Set $d = 20$, $T = 15$, $q(x) \equiv q > 0$, $x \in (0,20)$. Set $w_0^0(x) = \frac{x^3}{2\cdot 15^3 I_0(15\sqrt{q})}$ on $(0,15)$, $w_0^0(x) = 0$ on $(15,20)$, $w_1^0(x)$, $x \in (0,20)$ such that $\Omega w_1^0 = \left(\mathcal{L}^*\left(\operatorname{sign} t\mathcal{K}^*\Omega w_0^0\right)'\right)$ on $(-20,20)$. Then $\left|w_0^0(x)\right| \le \frac{15^3}{2\cdot 15^3 I_0(15\sqrt{q})} = \frac{1}{2I_0(15\sqrt{q})}$. Therefore, due to Remark 3.2, $\left|\mathcal{K}^*\Omega w_0^0(x)\right| \le \frac{1}{2I_0(15\sqrt{q})} I_0(15\sqrt{q}) = \frac{1}{2}$.

Thus, assertion (iii) of Theorem 3.2 holds. Hence the state w^0 is null-controllable at the time $T = 15$. The control is determined by the formula obtained in Remark 3.2. Consequently,

$$u(t) = \frac{t\left[15^2 q I_0(\sqrt{q(15^2-t^2)}) - 2\sqrt{q(15^2-t^2)}\,I_1(\sqrt{q(15^2-t^2)})\right]}{2\cdot 15^3 q I_0(15\sqrt{q})}, \quad t \in (0,15).$$

4. Proofs of Theorems 3.1, 3.2

4.1. Proof of Theorem 3.1

Due to Lemma A.2, we have

$$W(x,t) = \sum_{n=1}^{\infty} w_n(t) Y_n(\lambda_n, x), \qquad \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\delta'(x) = \sum_{n=1}^{\infty} \delta_n Y_n(\lambda_n, x), \tag{4.1}$$

where $w_n(t) = \frac{1}{2}\left(W(\cdot,t), Y_n(\lambda_n,\cdot)\right) = (w(\cdot,t), y_n(\lambda_n,\cdot))$, $\delta_n = \frac{1}{2}\left(\delta', Y_n(\lambda_n,\cdot)\right) = -\frac{1}{2}Y_n'(\lambda_n,0) = -\frac{1}{2}y_n'(\lambda_n,0)$. Note that $y_n'(\lambda_n,0) \neq 0$, $n = \overline{1,\infty}$. Substituting (4.1) into (3.2), we obtain

$$\sum_{n=1}^{\infty} w_n''(t) Y_n(\lambda_n,x) = \sum_{n=1}^{\infty} w_n(t) Y_n''(\lambda_n,x) - \sum_{n=1}^{\infty} w_n(t) Q(x) Y_n(\lambda_n,x) + u(t)\sum_{n=1}^{\infty} y_n'(\lambda_n,0) Y_n(\lambda_n,x).$$

By applying the operator D_Q^2, we get

$$\sum_{n=1}^{\infty} w_n''(t) Y_n(\lambda_n, x) = -\sum_{n=1}^{\infty} w_n(t)\lambda_n^2 Y_n(\lambda_n, x) + u(t)\sum_{n=1}^{\infty} y_n'(\lambda_n, 0) Y_n(\lambda_n, x).$$

From here we obtain the equation for the coefficients of $Y_n(\lambda_n, x)$

$$w_n''(t) + w_n(t)\lambda_n^2 = u(t) y_n'(\lambda_n, 0), \quad t \in [0, T], \quad n = \overline{1, \infty}. \tag{4.2}$$

Find the initial and steering conditions for this equation. We have $W_0^\gamma(x) = \sum_{n=1}^{\infty} w_{0n}^\gamma Y_n(\lambda_n, x)$ and $W_1^\gamma(x) = \sum_{n=1}^{\infty} w_{1n}^\gamma Y_n(\lambda_n, x)$, where for $\gamma = 0, T$ $w_{0n}^\gamma = (w_0^\gamma(\cdot), y_n(\lambda_n, \cdot))$, $w_{1n}^\gamma = (w_1^\gamma(\cdot), y_n(\lambda_n, \cdot))$. From (3.3) and (3.4) we have

$$W(x,\gamma) = \sum_{n=1}^{\infty} w_n(\gamma) Y_n(\lambda_n, x) = W_0^\gamma(x) = \sum_{n=1}^{\infty} w_{0n}^\gamma Y_n(\lambda_n, x),$$
$$W_t(x,\gamma) = \sum_{n=1}^{\infty} w_n'(\gamma) Y_n(\lambda_n, x) = W_1^\gamma(x) = \sum_{n=1}^{\infty} w_{1n}^\gamma Y_n(\lambda_n, x), \qquad \gamma = 0, T.$$

Thus we obtain the initial and steering conditions for equation (4.2)

$$\begin{cases} w_n(0) &= w_{0n}^0 \\ \frac{\partial w_n}{\partial t}(0) &= w_{1n}^0 \end{cases}, \qquad \begin{cases} w_n(T) &= w_{0n}^T \\ \frac{\partial w_n}{\partial t}(T) &= w_{1n}^T \end{cases}.$$

We reduce (4.2) to the linear system ($n = \overline{1, \infty}$)

$$v_n'(t) = \mathtt{A}_n v_n(t) + \mathtt{b}_n(t), \; v_n(0) = \begin{pmatrix} w_{0n}^0 \\ w_{1n}^0 \end{pmatrix}, \; v_n(T) = \begin{pmatrix} w_{0n}^T \\ w_{1n}^T \end{pmatrix}, \quad t \in [0, T], \tag{4.3}$$

where $v_n = \begin{pmatrix} w_n \\ w_n' \end{pmatrix}$, $\mathtt{A}_n = \begin{pmatrix} 0 & 1 \\ -\lambda_n^2 & 0 \end{pmatrix}$, $\mathtt{b}_n(t) = \begin{pmatrix} 0 \\ y_n'(\lambda_n, 0) \cdot u(t) \end{pmatrix}$. The solution of this system is ($n = \overline{1, \infty}$),

$$v_n(t) = \begin{pmatrix} \cos\lambda_n t & \frac{\sin\lambda_n t}{\lambda_n} \\ -\lambda_n \sin\lambda_n t & \cos\lambda_n t \end{pmatrix} \left(v_n(0) + \begin{pmatrix} -y_n'(\lambda_n, 0)\int_0^t \frac{\sin\lambda_n\tau}{\lambda_n} u(\tau)\,d\tau \\ y_n'(\lambda_n, 0)\int_0^t \cos\lambda_n\tau \cdot u(\tau)\,d\tau \end{pmatrix} \right).$$

Taking into account the steering condition in (4.3), for $n = \overline{1, \infty}$ we have

$$\begin{pmatrix} \cos\lambda_n T & \frac{\sin\lambda_n T}{\lambda_n} \\ -\lambda_n \sin\lambda_n T & \cos\lambda_n T \end{pmatrix} \begin{pmatrix} w_{0n}^0 - y_n'(\lambda_n, 0)\int_0^T \frac{\sin\lambda_n t}{\lambda_n} u(t)\,dt \\ w_{1n}^0 + y_n'(\lambda_n, 0)\int_0^T \cos\lambda_n t \cdot u(t)\,dt \end{pmatrix} = \begin{pmatrix} w_{0n}^T \\ w_{1n}^T \end{pmatrix}. \tag{4.4}$$

Since $u \in \mathcal{B}(0, T) \subset L^2(0, d)$, then $U \in L^2(-d, d)$. According to Lemmas A.5, A.6 and A.7, $U \in \mathbb{H}_0^0(-d, d)$, $\Omega U \in \mathbb{H}_0^0(-d, d)$, $\Xi U \in \mathbb{H}_0^0(-d, d)$, $(\Xi U)' = \Omega U' \in$

$\mathbb{H}_0^{-1}(-d,d)$. Now we can apply the transformation operators to $U(t)$. Denote $\widetilde{y}_n(\lambda_n,x)=\frac{\Omega y_n(\lambda_n,x)}{y_n'(\lambda_n,0)}$ (see Appendix B)

$$y_n'(\lambda_n,0)\int_0^T \frac{\sin\lambda_n t}{\lambda_n}u(t)\,dt=\frac{y_n'(\lambda_n,0)}{2}\int_{-d}^d \frac{\sin\lambda_n t}{\lambda_n}\Omega U(t)\,dt$$
$$=\frac{y_n'(\lambda_n,0)}{2}\left(\Omega U,\mathcal{L}(\widetilde{y}_n)\right)=\frac{y_n'(\lambda_n,0)}{2}\left(\mathcal{L}^*\Omega U,\widetilde{y}_n\right)=\frac{1}{2}\left(\mathcal{L}^*\Omega U,\Omega y_n\right)=\left(\mathcal{L}^*\Omega U,y_n\right),$$
$$y_n'(\lambda_n,0)\int_0^T \cos\lambda_n t u(t)\,dt=\frac{y_n'(\lambda_n,0)}{2}\int_{-d}^d \cos\lambda_n t\Xi U(t)\,dt$$
$$=\frac{y_n'(\lambda_n,0)}{2}\left(\Xi U,\left(\frac{\sin\lambda_n t}{\lambda_n}\right)'\right)=-\frac{y_n'(\lambda_n,0)}{2}\left((\Xi U)',\mathcal{L}(\widetilde{y}_n)\right)$$
$$=-\frac{y_n'(\lambda_n,0)}{2}\left(\mathcal{L}^*\Omega U',\widetilde{y}_n\right)=-\frac{1}{2}\left(\mathcal{L}^*\Omega U',\Omega y_n\right)=-\left(\mathcal{L}^*\Omega U',y_n\right).$$

Denote $u_n=(\mathcal{L}^*\Omega U,y_n)$, $u_n'=(\mathcal{L}^*\Omega U',y_n)$. According to Lemma A.2, we have

$$\sum_{k\in\mathbb{Z}}\mathcal{T}_{2kd}\left(\mathcal{L}^*\Omega U\right)(x)=\sum_{n=1}^{\infty}u_nY_n(\lambda_n,x),\ \sum_{k\in\mathbb{Z}}\mathcal{T}_{2kd}\left(\mathcal{L}^*\Omega U'\right)(x)=\sum_{n=1}^{\infty}u_n'Y_n(\lambda_n,x).$$

For $n=\overline{1,\infty}$, (4.4) is equivalent to the following equation:

$$\begin{pmatrix}\cos\lambda_nT & \frac{\sin\lambda_nT}{\lambda_n}\\ -\lambda_n\sin\lambda_nT & \cos\lambda_nT\end{pmatrix}\begin{pmatrix}w_{0n}^0-u_n\\ w_{1n}^0-u_n'\end{pmatrix}=\begin{pmatrix}w_{0n}^T\\ w_{1n}^T\end{pmatrix}. \tag{4.5}$$

From the coefficient equality (4.5) we get the function equality

$$S_T\left[W^0-\sum_{k\in\mathbb{Z}}\mathcal{T}_{2kd}\left(\mathcal{L}^*\begin{pmatrix}\Omega U\\ \Omega U'\end{pmatrix}\right)\right]=W^T. \tag{4.6}$$

Hence (3.5) is true. The theorem is proved.

4.2. Proof of Theorem 3.2

Let assertion (iii) of Theorem 3.2 holds. Put $\widetilde{U}(t)=\left(\mathcal{K}^*\Omega w_0^0\right)(t)$. It follows from (3.7) and Definition B.1 (see Appendix B) that $\widetilde{U}\in L^\infty(-d,d)$ and $\widetilde{U}$ is odd. By using (3.6) and Lemma B.5, we have that $\operatorname{supp}\widetilde{U}\subset[-T,T]$. Denote by $u(t)$ the restriction of $\widetilde{U}(t)$ on $[0,T]$. From (3.7) we get that $u\in\mathcal{B}(0,T)$. Put $U(t)=u(t)$ on $[0,T]$ and $U(t)=0$ on $\mathbb{R}\setminus[0,T]$. We have

$$\Omega U=\mathcal{K}^*\Omega w_0^0,\quad \left(\Omega U'\right)=(\Xi U)'=(\operatorname{sign}t\Omega U)'=\left(\operatorname{sign}t\mathcal{K}^*\Omega w_0^0\right)',\qquad \text{on } [-d,d].$$

Due to (3.8), from (4.6) we may conclude that the state w^0 is null-controllable at the time T. In addition, due to Lemma B.5, (3.9) holds.

Let the state w^0 be approximately null-controllable at the time T. This implies that there exists $(W^T)^m$ such that $(W^T)^m \in \mathcal{R}_T(w^0)$ and $\left|\left|\left|\left(W^T\right)^m\right|\right|\right|_Q^s < \frac{1}{m}$, $m \in \mathbb{N}$. Thus there exists $u^m \in \mathcal{B}(0,T)$ such that

$$S_T\left[\sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\Omega w^0 - \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\left(\mathcal{L}^*\Omega\begin{pmatrix} U^m \\ (U^m)' \end{pmatrix}\right)\right] = \left(W^T\right)^m, \qquad m = \overline{1,\infty},$$

where $U^m(t) = u^m(t)$ on $[0,T]$ and $U^m(t) = 0$ on $\mathbb{R}\setminus[0,T]$.

Obviously, $S_{-T}S_T f = S_T S_{-T} f = f$ for $f \in \mathcal{H}_Q^s \times \mathcal{H}_Q^{s-1}$. Therefore, $S_T^{-1} = S_{-T}$ is inverse to S_T. Hence S_T^{-1} is continuous, and we have $\mathcal{L}^*\Omega U^m \to \Omega w_0^0$ in $\mathcal{H}_Q^s(-d,d)$ and $\mathcal{L}^*\Omega\,(U^m)' \to \Omega w_1^0$ in $\mathcal{H}_Q^{s-1}(-d,d)$ as $m \to \infty$. According to Lemma B.4, $\operatorname{supp}\Omega w_0^0 \in [-T,T]$, i.e., (3.6) holds. Since $\mathcal{L}^*$ is continuous, we have

$$\begin{pmatrix} \Omega U^m \\ \Omega (U^m)' \end{pmatrix} \to \begin{pmatrix} \mathcal{K}^*\Omega w_0^0 \\ \mathcal{K}^*\Omega w_1^0 \end{pmatrix} \qquad \text{as } m\to\infty \text{ in } \mathbb{H}_0^s(-d,d)\times\mathbb{H}_0^{s-1}(-d,d). \qquad (4.7)$$

Put $\widetilde{U}(t) = \left(\mathcal{K}^*\Omega w_0^0\right)(t)$. According to Lemma B.5 and Definition B.1, $\operatorname{supp}\widetilde{U} \in [-T,T]$ and $\widetilde{U}$ is odd.

Since $\mathbb{H}_0^{-s}(-d,d)$ is dense in $\mathbb{H}_0^0(-d,d)$, $s \le 0$ (Lemma A.8), then $\Omega U^m \to \mathcal{K}^*\Omega w_0^0$ as $m \to \infty$ in $\left(\mathbb{H}_0^0(-d,d)\right)'$. According to the Riesz theorem, $\widetilde{U} \in \mathbb{H}_0^0(-d,d)$. Therefore, due to Lemma A.5, $\widetilde{U} \in L^2(-d,d)$. Since $u^m \in \mathcal{B}(0,T)$, $m \in \mathbb{N}$, then we have $\left|\widetilde{U}(t)\right| \le 1$ a.e. on $[-d,d]$. Thus $\widetilde{U} \in L^\infty(-d,d)$. Since $\mathcal{K}^*$ is continuous, then (3.7) holds.

Put $U(t) = \widetilde{U}(t)$ on $[0,d]$ and $U(t) = 0$ on $\mathbb{R}\setminus[0,d]$. Taking into account (4.7), we have

$$\begin{pmatrix} \Omega U^m \\ \Omega (U^m)' \end{pmatrix} \to \begin{pmatrix} \Omega U \\ \Omega U' \end{pmatrix} = \begin{pmatrix} \mathcal{K}^*\Omega w_0^0 \\ \mathcal{K}^*\Omega w_1^0 \end{pmatrix} \quad \text{as } m\to\infty \text{ in } \mathbb{H}_0^s(-d,d)\times\mathbb{H}_0^{s-1}(-d,d).$$

From the above we obtain (3.8). Thereby, if the state $w^0(t)$ is approximately null-controllable at the time T, then assertion (iii) of Theorem 3.2 holds. Denote by $u(t)$ the restriction of $U(t)$ on $[0,T]$. It is easy to see that $u(t) \in \mathcal{B}(0,T)$. According to Lemma B.5, formula (3.9) holds. The theorem is proved.

5. The Moment Problem

The control obtained in Theorem 3.2 for the null-controllability problem may be too complicated for practical use. In this section we consider the Markov trigonometric moment problem on $(0,T)$ and prove that there are bang-bang functions among its solutions. We show that they are the solutions of the approximate null-controllability problem for $s < -1/2$.

Consider control system (2.1), (2.2), (3.1). Set $0 < T \le d$ and $w^0 \in \mathcal{H}_Q^s(0,d) \times \mathcal{H}_Q^{s-1}(0,d)$, $s \le 0$. Assume that assertion (iii) of Theorem 3.2 holds. According to Theorem 3.2, there exists $\widetilde{u} \in \mathcal{B}(0,T)$ which is a solution of the null-controllability problem. Let us denote $\widetilde{U}(t) = \widetilde{u}(t)$, $t \in [0,T]$, $\widetilde{U}(t) = 0$, $t \in \mathbb{R} \setminus [0,T]$. Then $W^0 = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\mathcal{L}^* \begin{pmatrix} \Omega\widetilde{U} \\ \Omega\widetilde{U}' \end{pmatrix}$. Consider $u \in \mathcal{B}(0,T)$ and denote $U(t) = u(t)$, $t \in [0,T]$, $U(t) = 0$, $t \in \mathbb{R} \setminus [0,T]$. By using (3.5), we get

$$W^T = S_T \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\mathcal{L}^*\Omega \begin{pmatrix} \widetilde{U} - U \\ \widetilde{U}' - U' \end{pmatrix},$$

where W is the solution of (3.2)–(3.4). Applying Lemmas A.3, A.6, A.7 and B.3, we conclude that

$$\left|\left|\left| W^T \right|\right|\right|_Q^s \le 2\sqrt{2} C_S C_{\mathcal{L}^*} \left\| \left(\widetilde{U} - U\right) \right\|_0^s. \tag{5.1}$$

Put

$$\omega_m = \int_0^T e^{i\frac{\pi m x}{d}} \left(\mathcal{K}^*\Omega w_0^0\right)(x)\,dx, \qquad m = \overline{-\infty,\infty}. \tag{5.2}$$

The problem of determination of a function $u \in \mathcal{B}(0,T)$ such that

$$\int_0^T e^{i\frac{\pi m x}{d}} u(x)\,dx = \omega_m, \qquad m = \overline{-\infty,\infty} \tag{5.3}$$

for the given $\{\omega_m\}_{m=-\infty}^{\infty}$ and $T > 0$ is called the Markov trigonometric moment problem on $(0,T)$ for the infinite sequence $\{\omega_m\}_{m=-\infty}^{\infty}$.

Theorem 5.1. *Assume that $0 < T \le d$, $w^0 \in \mathcal{H}_Q^s(0,d) \times \mathcal{H}_Q^{s-1}(0,d)$, $s \le 0$. Assume that assertion (iii) of Theorem 3.2 holds. Assume also that $\{\omega_m\}_{m=-\infty}^{\infty}$ is defined by (5.2). Then the state w^0 is null-controllable at the time T iff the Markov trigonometric moment problem (5.3) has a unique solution on $(0,T)$. Moreover, this solution is of the form (3.9).*

P r o o f. Let the state w^0 be null-controllable at the time T. Then due to Theorem 3.2, the control $u(t) = \left(\mathcal{K}^*\Omega w_0^0\right)(t)$ on $(0,T)$ is the unique solution of controllability problem of system (2.1), (2.2), (3.1). Consequently, the Markov trigonometric moment problem (5.3) has a unique solution on $(0,T)$ of the form (3.9). Let the Markov trigonometric moment problem (5.3) has a unique solution on $(0,T)$. Hence $u(t) = \left(\mathcal{K}^*\Omega w_0^0\right)(t)$ on $(0,T)$. Since assertion (iii) of Theorem 3.2 holds, then the state w^0 is null-controllable at the time T. The theorem is proved.

R e m a r k 5.1. Theorem 5.1 is close to Theorem 3.2. The system $\{e^{-i\frac{\pi mx}{d}}\}_{m=-\infty}^{\infty}$ is an orthonormal basis in $L^2(-T,T)$ when $T=d$. Hence the moment problem (5.3) has a unique solution in $L^2(-d,d)$. If $T<d$, then the system $\{e^{-i\frac{\pi mx}{d}}\}_{m=-\infty}^{\infty}$ is complete and the solution is unique if it exists. Therefore the solution of (5.3) is in $\mathcal{B}(0,T)$. Moreover, it is unique if it exists and coincides with $\mathcal{K}^*\Omega w_0^0$.

Consider (5.3) for a finite set of m

$$\int_0^T e^{i\frac{\pi mx}{d}} u(x)\,dx = \omega_m, \qquad m=\overline{-M,M}, \qquad M\in\mathbb{N}. \tag{5.4}$$

The problem of determination of a function $u\in\mathcal{B}(0,T)$ such that (5.4) holds for the given $\{\omega_m\}_{m=-M}^{M}$, and $T>0$ is called the Markov trigonometric moment problem on $(0,T)$ for the finite sequence $\{\omega_m\}_{m=-M}^{M}$.

Obviously, u of the form (3.9) is a solution of this problem for $\{\omega_m\}_{m=-\infty}^{\infty}$ given by (5.2), but it is not unique.

Theorem 5.2. *Assume that* $0<T\le d$, $w^0\in\mathcal{H}_Q^s(0,d)\times\mathcal{H}_Q^{s-1}(0,d)$, $s<-1/2$. *Assume that assertion (iii) of Theorem 3.2 holds. Assume also that* $\{\omega_m\}_{m=-\infty}^{\infty}$ *is defined by (5.2). If* $u_M\in\mathcal{B}(0,T)$ *is a solution of the Markov trigonometric moment problem (5.4) for some* $M\in\mathbb{N}$, *then the corresponding solution* W *of control system (3.2) – (3.4) satisfies the estimate*

$$|||W^T|||_Q^s \le \frac{8\pi^s C_{\mathrm{S}} C_{\mathcal{L}^*} PTM^{s+\frac12}}{d^{s+1}\sqrt{-2s-1}} \longrightarrow 0 \text{ as } M\to\infty,$$

where $P>0$.

P r o o f. Let $\widetilde{u}\in\mathcal{B}(0,T)$ be a solution of the controllability problem of system (2.1), (2.2), (3.1) and $u\in\mathcal{B}(0,T)$ be a solution of the Markov trigonometric moment problem (5.4) on $(0,T)$ for the finite sequence $\{\omega_m\}_{m=-M}^{M}$. Set $\widetilde{U}(t)=\widetilde{u}(t)$ on $[0,T]$, $\widetilde{U}(t)=0$ on $\mathbb{R}\setminus[0,T]$, $U(t)=u(t)$ on $[0,T]$, $U(t)=0$ on $\mathbb{R}\setminus[0,T]$. Consider the following series expansions: $\widetilde{U}(x)=\frac1d\sum_{m=-\infty}^{\infty}\omega_m e^{-i\frac{\pi mx}{d}}$, $U(x)=\frac1d\sum_{m=-\infty}^{\infty}\nu_m e^{-i\frac{\pi mx}{d}}$ on $(-d,d)$, where

$$\omega_m=\int_{-d}^{d} e^{i\frac{\pi mx}{d}}\widetilde{U}(x)\,dx=\int_0^T e^{i\frac{\pi mx}{d}}\left(\mathcal{K}^*\Omega w_0^0\right)(x)\,dx,$$

$$\nu_m=\int_{-d}^{d} e^{i\frac{\pi mx}{d}}U(x)\,dx=\int_0^T e^{i\frac{\pi mx}{d}}u(x)\,dx.$$

Consequently,

$$\widetilde{U}(x) - U(x) = \frac{1}{d} \sum_{m=-\infty}^{\infty} (\omega_m - \nu_m)\, e^{-i\frac{\pi m x}{d}}. \tag{5.5}$$

According to Remark A.1 (see Appendix A), we have

$$\left\|\widetilde{U} - U\right\|_0^s = \left\|\left(1+D^2\right)^{s/2}\left(\widetilde{U} - U\right)\right\|_0^0 \le P\left\|\left(1+D^2\right)^{s/2}\left(\widetilde{U} - U\right)\right\|_{L^2}, \tag{5.6}$$

where $P > 0$.

Since $\nu_m = \omega_m$, $m = \overline{-M, M}$, then from (5.5) and (5.6) we have

$$\left\|\widetilde{U} - U\right\|_0^s \le \sqrt{2}\frac{P}{d}\left(\sum_{m=M+1}^{+\infty}\left(1+\left(\frac{m\pi}{d}\right)^2\right)^s |\omega_m - \nu_m|^2\right)^{1/2}.$$

Since $\widetilde{u} \in \mathcal{B}(0,T)$ and $u \in \mathcal{B}(0,T)$, then $|\omega_m - \nu_m| \le 2T$. It is easy to see that $\left(1+\left(\frac{m\pi}{d}\right)^2\right)^s \le \left(\frac{m\pi}{d}\right)^{2s}$ for $s \le 0$. Using these two inequalities and the estimate $\sum_{m=M+1}^{\infty} m^{2s} \le \int_M^\infty x^{2s}\, dx$, for $s < -1/2$ we get

$$\left\|\widetilde{U} - U\right\|_0^s \le \frac{2\sqrt{2}\pi^s P T M^{s+\frac{1}{2}}}{d^{s+1}\sqrt{-2s-1}}.$$

By continuing estimate (5.1), we have

$$|||W^T|||_Q^s \le \frac{8\pi^s C_S C_{\mathcal{L}^*} P T M^{s+\frac{1}{2}}}{d^{s+1}\sqrt{-2s-1}}.$$

The theorem is proved.

Denote

$$\begin{aligned}\mathcal{B}_M(0,T) = \; & \{u \in \mathcal{B}(0,T) | \exists T_* \in (0,T) : (|u(t)| = 1 \text{ a.e. on } (0,T_*)), \\ & (u(t) = 0 \text{ a.e. on } (T_*,T)), \\ & \text{and } (u(t) \text{ has no more than } M \text{ discontinuities on } (0,T_*))\}.\end{aligned}$$

Theorem 5.3. *Assume that $0 < T \le d$, $w^0 \in \mathcal{H}_Q^s(0,d) \times \mathcal{H}_Q^{s-1}(0,d)$, $s < -1/2$. Assume that assertion (iii) of Theorem 3.2 holds. Assume also that $\{\omega_m\}_{m=-\infty}^{\infty}$ is defined by (5.2). Then for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for this M there is a solution $u_M \in \mathcal{B}_M(0,T)$ of the Markov trigonometric moment problem (5.4). Moreover, for this u_M the corresponding solution W of control system (3.2)–(3.4) satisfies the estimate $|||W^T|||_Q^s \le \varepsilon$. The number M is defined from the condition*

$$\frac{8\pi^s C_S C_{\mathcal{L}^*} P T M^{s+\frac{1}{2}}}{d^{s+1}\sqrt{-2s-1}} < \varepsilon. \tag{5.7}$$

P r o o f. It is well known [12, 17] that if the Markov trigonometric moment problem (5.4) is solvable, then there exists its bang-bang solution $u \in \mathcal{B}_M(0,T)$. By given $\varepsilon > 0$ determine $M \in \mathbb{N}$ from (5.7). Since the conditions of Theorems 3.2 and 5.2 hold, then one can find a solution $u_M \in \mathcal{B}_M(0,T)$ of the Markov trigonometric moment problem (5.4) for the obtained M. For all M, these solutions $\{u_M\}_{M=1}^{\infty}$ give us the bang-bang controls solving the approximate null-controllability problem. The theorem is proved.

A. The Spaces $\mathcal{H}_Q^s$ and $\mathbb{H}_0^s(-d,d)$, $s \in \mathbb{R}$

Lemma A.1. *The system $\left\{e^{i\lambda_n x}\right\}_{n=1}^{\infty}$ is the Riesz basis in $L^2(-d,d)$.*

P r o o f. To prove the lemma we use Levin–Golovin's theorem ([18]):

Theorem (Levin–Golovin). *If the set $\{\lambda_n\}_{n=1}^{\infty}$ such that $\inf_{n\neq m}|\lambda_m - \lambda_n| > 0$ is the set of simple zeros of the sine-type function of an exponential type d, then $\left\{e^{i\lambda_n x}\right\}_{n=1}^{\infty}$ is the Riesz basis in $L^2(-d,d)$.*

As noted in Section 2, λ_n are real, positive, simple and numbered in ascending order, $n = \overline{1,\infty}$. We also use the asymptotic expression $\lambda_n = n\frac{\pi}{d} + \underline{O}\left(\frac{1}{n}\right)$, $n = \overline{1,\infty}$ obtained in [19, chap. V]. Thus,

$$\inf_{n\neq m} |\lambda_m - \lambda_n| = \inf_{n} \left|(n+1)\frac{\pi}{d} + \underline{O}\left(\frac{1}{n+1}\right) - n\frac{\pi}{d} - \underline{O}\left(\frac{1}{n}\right)\right| = \frac{\pi}{d} > 0.$$

To find a sine-type function in question we use the corollary to Theorem 1 in [20].

Corollary to Theorem 1. *Let $\widetilde{S}(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\mu_n+\psi_n}\right)$, where $\{\mu_n\}_{n=1}^{\infty}$ is the set of zeros of the sine-type function $S(z)$ of an exponential type σ. If $\{\psi_n\}_{n=1}^{\infty} \in l^p$, $p > 1$, then $\widetilde{S}(z)$ is the sine-type function of an exponential type σ.*

It is well known that $S(z) = \sin dz$ is a sine-type function of an exponential type d and $\{\mu_n = \frac{\pi n}{d}\}_{n=-\infty}^{\infty}$ is a set of its zeros. Put $\psi_n = -\frac{\pi n}{d} + \lambda_n$, $n = \overline{1,\infty}$. Then

$$|\psi_n| = \left|-\frac{\pi n}{d} + \frac{\pi n}{d} + \underline{O}\left(\frac{1}{n}\right)\right| = \left|\underline{O}\left(\frac{1}{n}\right)\right|, \qquad n = \overline{1,\infty}.$$

Hence, $\{\psi_n\}_{n=1}^{\infty} \in l^p$, where $p > 1$. Therefore

$$\widetilde{S}(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\mu_n + \psi_n}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\frac{\pi n}{d} - \frac{\pi n}{d} + \lambda_n}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n}\right)$$

is the sine-type function of an exponential type d. The lemma is proved.

Lemma A.2. *The equivalent definitions of the space $\mathcal{H}^s_Q$, $s \in \mathbb{R}$ are the following:*

$$\mathcal{H}^s_Q = \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n Y_n(\lambda_n, x) \text{ and } \left\{ f_n(1+\lambda_n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2 \right\}$$
$$= \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n^1 \sin\frac{\pi n x}{d} \text{ and } \left\{ f_n^1(1+n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2 \right\}, \quad s \in \mathbb{R}.$$

P r o o f. Since $\left\{ \frac{Y_n(\lambda_n,x)}{\sqrt{2}} \right\}_{n=1}^{\infty}$ is the orthonormal basis in $\mathcal{H}^0_Q$, then for any $f \in \mathcal{H}^s_Q$, $s \in \mathbb{R}$ we have $(1+D_Q^2)^{s/2} f(x) = \sum_{n=1}^{+\infty} \widetilde{f}_n Y_n(\lambda_n, x) < \infty$, where $\widetilde{f}_n \in l^2$. Use $(1+D_Q^2)^{s/2} Y_n(\lambda_n, \cdot) = (1+\lambda_n^2)^{s/2} Y_n(\lambda_n, \cdot)$, $n = \overline{1,\infty}$ for the coefficient $\widetilde{f}_n$ to get $\widetilde{f}_n = \frac{1}{\|Y_n\|^2} \left((1+D_Q^2)^{s/2} f, Y_n(\lambda_n, \cdot) \right) = \frac{1}{2} \left(f, (1+D_Q^2)^{s/2} Y_n(\lambda_n, \cdot) \right) = \frac{1}{2} \left(1+\lambda_n^2\right)^{s/2} (f, Y_n(\lambda_n, \cdot)) = \left(1+\lambda_n^2\right)^{s/2} f_n$. Here $f_n = \frac{1}{2}(f, Y_n(\lambda_n, \cdot))$ on $(-d, d)$. Thus,

$$f(x) = \sum_{n=1}^{+\infty} \widetilde{f}_n (1+D_Q^2)^{\frac{-s}{2}} Y_n(\lambda_n, x) = \sum_{n=1}^{+\infty} f_n Y_n(\lambda_n, x).$$

Since for $f \in \mathcal{H}^0_Q$ the norm can be calculated as $\|f\|^0_Q = \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2}$, then the equivalent norm in the space $\mathcal{H}^s_Q$ is $\|f\|^s_Q = \left(\sum_{n=1}^{\infty} \left| \left(1+\lambda_n^2\right)^{s/2} f_n \right|^2 \right)^{1/2}$.

It is well known that a function from the space $\mathcal{H}^0_Q$ can be represented as the convergent series $f(x) = \sum_{n=-\infty}^{+\infty} f_n^1 \sin\frac{\pi n x}{d}$.

Notice that since $Q \in C^1(\mathbb{R})$ and it is even and $2d$-periodic, then $(1 + D_Q^2)^{s/2} f \in \mathcal{H}^0_Q$ iff $(1+D^2)^{s/2} f \in \mathcal{H}^0_Q$ when $f \in \mathcal{H}^s_Q$. In other words, $f \in \mathcal{H}^s_Q$ iff $f \in \{\varphi \in H^s_{0,per} : \varphi \text{ is odd}\}$, where $H^s_{0,per}$ is the Sobolev space of periodic functions. Reasoning in the same way as above, with $(1+D^2)^{s/2} f$ we find that

$$f(x) = \sum_{n=1}^{+\infty} f_n^1 \sin\frac{\pi n x}{d} \quad \text{and} \quad \left\{ f_n^1 (1+n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2, \quad s \in \mathbb{R},$$

where $f_n^1 = \frac{1}{d}\left(f, \sin\frac{\pi n x}{d}\right)$ on $(-d, d)$, and $[]f[]^s_Q = \left(\sum_{n=1}^{\infty} \left| \left(1+n^2\right)^{s/2} f_n^1 \right|^2 \right)^{1/2}$ is the equivalent norm in the space $\mathcal{H}^s_Q$. The lemma is proved.

Lemma A.3. *The operator $\mathrm{S}_T : \mathcal{H}^s_Q \times \mathcal{H}^{s-1}_Q \to \mathcal{H}^s_Q \times \mathcal{H}^{s-1}_Q$, $s \in \mathbb{R}$ (see Definition 3.2) is linear and continuous.*

P r o o f. Set $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}_Q^s \times \mathcal{H}_Q^{s-1}$. Then $f(x) = \sum_{n=1}^{\infty} \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} Y_n(\lambda_n, x)$, $\left\{f_n^1(1+\lambda_n^2)^{s/2}\right\}_{n=1}^{+\infty} \in l^2$ and $\left\{f_n^2(1+\lambda_n^2)^{(s-1)/2}\right\}_{n=1}^{+\infty} \in l^2$. By the definition of S_T, we have

$$(S_T f)(x) = \sum_{n=1}^{\infty} \begin{pmatrix} f_n^1 \cos\lambda_n T + f_n^2 \frac{\sin\lambda_n T}{\lambda_n} \\ -\lambda_n f_n^1 \sin\lambda_n T + f_n^2 \cos\lambda_n T \end{pmatrix} Y_n(\lambda_n, x).$$

Taking into account the estimate $\left|\frac{\sin\lambda_n T}{\lambda_n}\right| \le \frac{\sqrt{2}T}{\sqrt{1+\lambda_n^2}}$ and the trivial inequality $(|a|+|b|)^2 \le 2(a^2+b^2)$, we obtain $\left\{(1+\lambda_n^2)^{s/2}\left(f_n^1 \cos\lambda_n T + f_n^2 \frac{\sin\lambda_n T}{\lambda_n}\right)\right\}_{n=1}^{+\infty} \in l^2$, $\left\{(1+\lambda_n^2)^{(s-1)/2}\left(-\lambda_n f_n^1 \sin\lambda_n T + f_n^2 \cos\lambda_n T\right)\right\}_{n=1}^{+\infty} \in l^2$. Therefore, $R(S_T) = \mathcal{H}_Q^s \times \mathcal{H}_Q^{s-1}$. We also get

$$|||S_T f|||_Q^s = \left(\sum_{n=1}^{\infty}(1+\lambda_n^2)^s \left|f_n^1 \cos\lambda_n T + f_n^2 \frac{\sin\lambda_n T}{\lambda_n}\right|^2 \right.$$
$$\left. + \sum_{n=1}^{\infty}(1+\lambda_n^2)^{(s-1)} \left|-\lambda_n f_n^1 \sin\lambda_n T + f_n^2 \cos\lambda_n T\right|^2 \right)^{1/2} \le C_S\, |||f|||_Q^s\,,$$

where $C_S^2 = 2\max\{2, 2T^2+1\}$. The linearity of the operator S_T is obvious. The lemma is proved.

Lemma A.4. *$f \in \mathbb{H}_0^s(-d,d)$, $s \in \mathbb{R}$ iff $\left(1+D^2\right)^{s/2} f \in \mathbb{H}_0^0(-d,d)$.*

P r o o f. Let $s \in \mathbb{R}$ and $f \in \mathbb{H}_0^s(-d,d)$. Hence $f(x) = \sum_{n=1}^{\infty} f_n e^{-i\lambda_n x}$ and $\left\{(1+\lambda_n^2)^{s/2} f_n\right\}_{n=1}^{\infty} \in l^2$. Then $\left(1+D^2\right)^{s/2} f(x) = \sum_{n=1}^{\infty} f_n (1+D^2)^{s/2} e^{-i\lambda_n x} = \sum_{n=1}^{\infty} (1+\lambda_n^2)^{s/2} f_n e^{-i\lambda_n x} = \sum_{n=1}^{\infty} \widehat{f}_n e^{-i\lambda_n x}$, where $\widehat{f}_n = (1+\lambda_n^2)^{s/2} f_n$, $n = \overline{1,\infty}$. Thus $\left\{\widehat{f}_n\right\}_{n=1}^{\infty} \in l^2$. Therefore, $\left(1+D^2\right)^{s/2} f \in \mathbb{H}_0^0(-d,d)$.

The converse part of the lemma is proved similarly. The lemma is proved.

Lemma A.5. *$f \in \mathbb{H}_0^0(-d,d)$ iff $f \in L^2(-d,d)$.*

P r o o f. Let $f \in \mathbb{H}_0^0(-d,d)$. As it is shown in [19, chap. V], $\lambda_n = n\frac{\pi}{d} + \varepsilon_n$, where $\varepsilon_n = \underline{O}\left(\frac{1}{n}\right)$, $n = \overline{1,\infty}$. Therefore, $f(x) = \sum_{n=1}^{\infty} f_n e^{-i\lambda_n x} = \sum_{n=1}^{\infty} f_n e^{-i\frac{n\pi}{d}x} e^{-i\varepsilon_n x}$, where $\{f_n\}_{n=1}^{\infty} \in l^2$. As it is known $\sum_{n=1}^{\infty} f_n e^{-i\frac{n\pi}{d}x} = \widetilde{f}(x)$, where $\widetilde{f}$ is a certain function from $L^2(-d,d)$. Thus $f \in L^2(-d,d)$ iff $\left(f - \widetilde{f}\right) \in L^2(-d,d)$. We have

$$f(x) - \widetilde{f}(x) = \sum_{n=1}^{\infty} f_n e^{-i\frac{n\pi}{d}x}\left(e^{-i\varepsilon_n x} - 1\right) = \sum_{n=1}^{\infty} f_n e^{-i\frac{n\pi}{d}x}\left(-i\varepsilon_n x + \gamma_n(x)\right),$$

where $\gamma_n(x) = \bar{o}(\varepsilon_n x)$. Hence $|\gamma_n(x)| \le C|\varepsilon_n x|$ as $n \to \infty$, where $C > 0$.

Consider $\left\| f - \widetilde{f} \right\|^2_{L^2(-d,d)}$. We have

$$\left\| f - \widetilde{f} \right\|^2_{L^2(-d,d)} = \int_{-d}^{d} \left| \sum_{n=1}^{\infty} f_n e^{-i\frac{n\pi}{d}x} \left(-i\varepsilon_n x + \gamma_n(x)\right) \right|^2 dx$$

$$\le \sum_{n,m=1}^{\infty} |f_n| \cdot |f_m| \int_{-d}^{d} |-i\varepsilon_n x + \gamma_n(x)| \cdot |-i\varepsilon_m x + \gamma_m(x)|\, dx$$

$$\le (1+C)^2 \frac{2d^3}{3} \sum_{n,m=1}^{\infty} |f_n| \cdot |f_m| \cdot |\varepsilon_n| \cdot |\varepsilon_m| = (1+C)^2 \frac{2d^3}{3} \left(\sum_{n=1}^{\infty} |f_n| \cdot |\varepsilon_n| \right)^2.$$

Since $\{f_n\}_{n=1}^{\infty} \in l^2$ and $\varepsilon_n \sim \frac{1}{n}$ as $n \to \infty$, then the last series converges. Therefore, $\left\| f - \widetilde{f} \right\|^2_{L^2(-d,d)} < \infty$. The lemma is proved.

R e m a r k A.1. From Lemma A.4 it follows that $\|f\|_0^s = \left\|(1+D^2)^{s/2} f\right\|_0^0$ for $f \in \mathbb{H}_0^s(-d,d)$, $s \in \mathbb{R}$. From Lemma A.5 it follows that there exist $P, P_1 > 0$ such that $\|f\|_0^0 \le P \|f\|_{L^2}$ and $\|f\|_{L^2} \le P_1 \|f\|_0^0$ for $f \in \mathbb{H}_0^0(-d,d)$.

Lemma A.6. *Let $g \in \mathbb{H}_0^s(-d,d)$, $s \in \mathbb{R}$. Then we have $\Omega g \in \mathbb{H}_0^s(-d,d)$, $\Xi g \in \mathbb{H}_0^s(-d,d)$ and $\|\Omega g\|_0^s = \|\Xi g\|_0^s = 2\|g\|_0^s$.*

P r o o f. Since $g \in \mathbb{H}_0^s(-d,d)$, then $g(x) = \sum_{n=1}^{\infty} g_n e^{-i\lambda_n x}$ and $\left\{(1+\lambda_n^2)^{s/2} g_n\right\}_{n=1}^{\infty} \in l^2$. Therefore,

$$(\Omega g)(x) = \sum_{n=1}^{\infty} g_n \Omega e^{-i\lambda_n x} = \sum_{n=1}^{\infty} g_n \left(e^{-i\lambda_n x} - e^{i\lambda_n x}\right) = \sum_{n=1}^{\infty} g_n^{sin} \sin \lambda_n x,$$

$$(\Xi g)(x) = \sum_{n=1}^{\infty} g_n \Xi e^{-i\lambda_n x} = \sum_{n=1}^{\infty} g_n \left(e^{-i\lambda_n x} + e^{i\lambda_n x}\right) = \sum_{n=1}^{\infty} g_n^{cos} \cos \lambda_n x,$$

where $g_n^{sin} = -2ig_n$, $g_n^{cos} = 2g_n$. Since $\left\{(1+\lambda_n^2)^{s/2} g_n\right\}_{n=1}^{\infty} \in l^2$, then $\left\{(1+\lambda_n^2)^{s/2} g_n^{sin}\right\}_{n=1}^{\infty} \in l^2$ and $\left\{(1+\lambda_n^2)^{s/2} g_n^{cos}\right\}_{n=1}^{\infty} \in l^2$. Hence $\Omega g \in \mathbb{H}_0^s(-d,d)$ and $\Xi g \in \mathbb{H}_0^s(-d,d)$. Finally we obtain

$$\|\Omega g\|_0^s = \left(\sum_{n=1}^{\infty} \left| \left(1+\lambda_n^2\right)^{s/2} g_n^{sin} \right|^2 \right)^{1/2} = 2 \left(\sum_{n=1}^{\infty} \left| \left(1+\lambda_n^2\right)^{s/2} g_n \right|^2 \right)^{1/2} = 2\|g\|_0^s,$$

$$\|\Xi g\|_0^s = \left(\sum_{n=1}^{\infty} \left| \left(1+\lambda_n^2\right)^{s/2} g_n^{cos} \right|^2 \right)^{1/2} = 2 \left(\sum_{n=1}^{\infty} \left| \left(1+\lambda_n^2\right)^{s/2} g_n \right|^2 \right)^{1/2} = 2\|g\|_0^s.$$

The lemma is proved.

Lemma A.7. *If $g \in \mathbb{H}_0^s(-d,d)$, then $g' \in \mathbb{H}_0^{s-1}(-d,d)$ and $\|g'\|_0^{s-1} < \|g\|_0^s$, $s \in \mathbb{R}$.*

P r o o f. Let $g \in \mathbb{H}_0^s(-d,d)$, $s \in \mathbb{R}$. Hence $g(x) = \sum_{n=1}^{\infty} g_n e^{-i\lambda_n x}$ and $\left\{g_n(1+\lambda_n^2)^{\frac{s}{2}}\right\} \in l^2$, $s \in \mathbb{R}$. Taking into account that $|\lambda_n| < \sqrt{1+\lambda_n^2}$, we get $\|g'\|_0^{s-1} = \left(\sum_{n=1}^{\infty} \left|(1+\lambda_n^2)^{(s-1)/2} i\lambda_n g_n\right|^2\right)^{1/2} < \|g\|_0^s$. The lemma is proved.

Lemma A.8. *$\mathbb{H}_0^s(-d,d)$ is dense in $\mathbb{H}_0^0(-d,d)$, $s \geq 0$.*

P r o o f. Let $f \in \mathbb{H}_0^0(-d,d)$. Then $f(x) = \sum_{n=1}^{\infty} f_n e^{-i\lambda_n x}$ and $\{f_n\}_{n=1}^{\infty} \in l^2$. Consider a sequence of the functions $\{f^m(x)\}_{m=1}^{\infty}$ such that $f^m(x) = \sum_{n=1}^{\infty} f_n^m e^{-i\lambda_n x}$, where $f_n^m = \frac{\sin(n^s/m)}{n^s/m} f_n$, $n, m = \overline{1,\infty}$. One can see that $\left\{(1+\lambda_n^2)^{s/2} f_n^m\right\}_{n=1}^{\infty} \in l^2$, $m = \overline{1,\infty}$. Therefore $f^m \in \mathbb{H}_0^s(-d,d)$, $m = \overline{1,\infty}$, $s \geq 0$. Since $f_n^m \to f_n$ as $m \to \infty$, we have $\|f - f^m\|_0^0 = \left(\sum_{n=1}^{\infty} |f_n - f_n^m|^2\right)^{1/2} \to 0$ as $m \to \infty$. The lemma is proved.

B. The Transformation Operators for the Sturm–Liouville Problem on a Segment

We would like to recall some definitions of transformation operators used in [15] as well as the statements proved there. Denote $\widetilde{y}_n(\lambda_n, x) = \frac{\Omega y_n(\lambda_n,x)}{y_n'(\lambda_n,0)}$, $n = \overline{1,\infty}$. Obviously, $\widetilde{y}_n(\lambda_n,x)$ satisfies the following Cauchy problem for $n = \overline{1,\infty}$:

$$\begin{aligned} &-\widetilde{y}_n''(\lambda_n,x) + Q(x)\widetilde{y}_n(\lambda_n,x) = \lambda_n^2 \widetilde{y}_n(\lambda_n,x), \qquad x \in (-d,d), \\ &\widetilde{y}_n(\lambda_n,0) = 0, \quad \widetilde{y}_n'(\lambda_n,0) = 1. \end{aligned}$$

According to [15], we have

$$\widetilde{y}_n(\lambda_n,x) = \mathcal{K}\left(\frac{\sin\lambda_n t}{\lambda_n}\right)(x) = \frac{\sin\lambda_n x}{\lambda_n} + \int_0^x \mathtt{K}(x,t;\infty)\frac{\sin\lambda_n t}{\lambda_n}\,dt, \quad n = \overline{1,\infty},$$

$$\frac{\sin\lambda_n x}{\lambda_n} = \mathcal{L}\left(\widetilde{y}_n(\lambda_n,t)\right)(x) = \widetilde{y}_n(\lambda_n,x) + \int_0^x \mathtt{L}(x,t;\infty)\widetilde{y}_n(\lambda_n,t)\,dt,\ n = \overline{1,\infty},$$

where $\mathtt{K}(x,t;\infty) = \mathtt{K}(x,t) - \mathtt{K}(x,-t)$, $\mathtt{L}(x,t;\infty) = \mathtt{L}(x,t) - \mathtt{L}(x,-t)$. Under the condition $Q \in C^1[-d,d]$ the continuous functions $\mathtt{K}(x,t)$ and $\mathtt{L}(x,t)$ are the solutions of the following systems on $[-d,d]\times[-d,d]$:

$$\begin{aligned}
&\mathtt{K}_{xx}(x,t)-\mathtt{K}_{tt}(x,t)=Q(x)\mathtt{K}(x,t), && \mathtt{L}_{xx}(x,t)-\mathtt{L}_{tt}(x,t)=-Q(x)\mathtt{L}(x,t),\\
&\mathtt{K}(x,x)=\frac{1}{2}\int\limits_0^x Q(\xi)\,d\xi, && \mathtt{L}(x,x)=-\frac{1}{2}\int\limits_0^x Q(\xi)\,d\xi,\\
&\mathtt{K}(x,-x)=0, && \mathtt{L}(x,-x)=0.
\end{aligned}$$

It is also known [15] that the kernels $\mathtt{K}(x,t)$ and $\mathtt{L}(x,t)$ are bounded functions with respect to the both arguments on $[-d,d]\times[-d,d]$ and $\mathtt{K}(x,t)=\mathtt{L}(x,t)=0$ when $|t|\geq|x|$.

Let us determine the transformation operators in the spaces $\mathbb{H}_0^{-s}(-d,d)$ and $\mathcal{H}_Q^{-s}(-d,d)$ via series.

Lemma B.1. *The operators $\mathcal{K}$ and $\mathcal{L}$ act in the spaces*

$$\mathcal{K}:\mathbb{H}_0^{-s}(-d,d)\longrightarrow\mathcal{H}_Q^{-s}(-d,d),\quad \mathcal{L}:\mathcal{H}_Q^{-s}(-d,d)\longrightarrow\mathbb{H}_0^{-s}(-d,d),\quad s\in\mathbb{R},$$

where $D(\mathcal{K})=\left\{f\in\mathbb{H}_0^{-s}(-d,d): f \text{ is odd}\right\}$, $D(\mathcal{L})=\mathcal{H}_Q^{-s}(-d,d)$, $R(\mathcal{K})=D(\mathcal{L})$, $R(\mathcal{L})=D(\mathcal{K})$.

P r o o f. Set $s\in\mathbb{R}$. Consider $\psi\in\mathcal{H}_Q^{-s}(-d,d)$, $\varphi\in\mathbb{H}_0^{-s}(-d,d)$ (φ is odd). Then we have $\psi(x)=\sum_{n=1}^{\infty}\psi_n\Omega y_n(\lambda_n,x)$, $\left\{\left(1+\lambda_n^2\right)^{\frac{-s}{2}}\psi_n\right\}_{n=1}^{\infty}\in l^2$, $\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin\lambda_n x$, $\left\{\left(1+\lambda_n^2\right)^{\frac{-s}{2}}\varphi_n\right\}_{n=1}^{\infty}\in l^2$. Let $\widetilde{\psi}_n=\frac{\lambda_n\varphi_n}{y_n'(\lambda_n,0)}$, $\widetilde{\varphi}_n=\frac{\psi_n y_n'(\lambda_n,0)}{\lambda_n}$. Applying the transformation operators, we get

$$\begin{aligned}
(\mathcal{K}\varphi)(x)&=\sum_{n=1}^{\infty}\lambda_n\varphi_n\mathcal{K}\left(\frac{\sin\lambda_n t}{\lambda_n}\right)(x)=\sum_{n=1}^{\infty}\lambda_n\varphi_n\widetilde{y}_n(\lambda_n,x)\\
&=\sum_{n=1}^{\infty}\lambda_n\varphi_n\frac{\Omega y_n(\lambda_n,x)}{y_n'(\lambda_n,0)}=\sum_{n=1}^{\infty}\widetilde{\psi}_n\Omega y_n(\lambda_n,x)=\widetilde{\psi}(x),\\
(\mathcal{L}\psi)(x)&=\sum_{n=1}^{\infty}\psi_n\mathcal{L}\left(\Omega y_n(\lambda_n,t)\right)(x)=\sum_{n=1}^{\infty}\psi_n y_n'(\lambda_n,0)\mathcal{L}\left(\widetilde{y}_n(\lambda_n,t)\right)(x)\\
&=\sum_{n=1}^{\infty}\psi_n y_n'(\lambda_n,0)\frac{\sin\lambda_n x}{\lambda_n}=\sum_{n=1}^{\infty}\widetilde{\varphi}_n\sin\lambda_n x=\widetilde{\varphi}(x).
\end{aligned}$$

The following asymptotic expressions for the solutions of (2.3) and for their derivatives are obtained in [19, chap. V]: $y_n(t)=\sqrt{\frac{2}{d}}\sin n\frac{\pi}{d}t+\underline{O}\left(\frac{1}{n}\right)$, $y_n'(t)=n\frac{\pi}{d}\sqrt{\frac{2}{d}}\cos n\frac{\pi}{d}t+\underline{O}\left(1\right)$, $n=\overline{1,\infty}$. We also use the expression $\lambda_n=n\frac{\pi}{d}+\underline{O}\left(\frac{1}{n}\right)$.

Therefore, $y_n'(\lambda_n,0)\sim n$ and $\lambda_n\sim n$ as $n\to\infty$. Thus, $\widetilde{\psi}_n\sim\frac{n\varphi_n}{n}=\varphi_n$ and $\widetilde{\varphi}_n\sim\frac{n\psi_n}{n}=\psi_n$. Hense, $\left\{\left(1+\lambda_n^2\right)^{-s/2}\widetilde{\psi}_n\right\}_{n=1}^{\infty}\in l^2$ and $\left\{\left(1+\lambda_n^2\right)^{-s/2}\widetilde{\varphi}_n\right\}_{n=1}^{\infty}\in l^2$. Therefore, $\widetilde{\varphi}\in\mathbb{H}_0^{-s}(-d,d)$ and $\widetilde{\psi}\in\mathcal{H}_Q^{-s}(-d,d)$. The lemma is proved.

Lemma B.2. *The operators $\mathcal{K}$ and $\mathcal{L}$ are linear and continuous on their domains.*

P r o o f. The linearity of the operators is obvious. Let us prove that the operators $\mathcal{K}$ and $\mathcal{L}$ are continuous. Let $s\in\mathbb{R}$, $\varphi\in\mathbb{H}_0^{-s}(-d,d)$ (φ is odd). Then $\mathcal{K}\varphi\in\mathcal{H}_Q^{-s}(-d,d)$. We have $\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin\lambda_n x$, $(\mathcal{K}\varphi)(x)=\sum_{n=1}^{\infty}\psi_n\Omega y_n(\lambda_n,x)$, $\left(\|\varphi\|_0^{-s}\right)^2=\sum_{n=1}^{\infty}\left|\varphi_n\left(1+\lambda_n^2\right)^{-s/2}\right|^2$, $\left(\|\mathcal{K}\varphi\|_Q^{-s}\right)^2=\sum_{n=1}^{\infty}\left|\psi_n\left(1+\lambda_n^2\right)^{-s/2}\right|^2$. Since $\psi_n=\frac{\lambda_n\varphi_n}{y_n'(\lambda_n,0)}$ (see Lemma B.1), then

$$\left(\|\mathcal{K}\varphi\|_Q^{-s}\right)^2=\sum_{n=1}^{\infty}\left|\frac{\lambda_n\varphi_n}{y_n'(\lambda_n,0)}\left(1+\lambda_n^2\right)^{\frac{-s}{2}}\right|^2=\sum_{n=1}^{\infty}\left|\frac{\lambda_n}{y_n'(\lambda_n,0)}\right|^2\left|\varphi_n\left(1+\lambda_n^2\right)^{\frac{-s}{2}}\right|^2.$$

It is evident that there exists $C>0$ such that the estimate $\left|\frac{\lambda_n}{y_n'(\lambda_n,0)}\right|=\frac{\left|n\frac{\pi}{d}+O\left(\frac{1}{n}\right)\right|}{\left|n\frac{\pi}{d}\sqrt{\frac{2}{d}}+O(1)\right|}\le C$ is valid. Therefore, $\left(\|\mathcal{K}\varphi\|_Q^{-s}\right)^2\le C^2\sum_{n=1}^{\infty}\left|\varphi_n\left(1+\lambda_n^2\right)^{-s/2}\right|^2=C^2\left(\|\varphi\|_0^{-s}\right)^2$. Thus the operator $\mathcal{K}$ is continuous from $\mathbb{H}_0^{-s}(-d,d)$ to $\mathcal{H}_Q^{-s}(-d,d)$. Its inverse operator $\mathcal{L}$ is also continuous from $\mathcal{H}_Q^{-s}(-d,d)$ to $\mathbb{H}_0^{-s}(-d,d)$. The lemma is proved.

Definition B.1. *Define by $\mathcal{K}^*$ and $\mathcal{L}^*$ the adjoint operators for $\mathcal{K}$ and $\mathcal{L}$: $(\mathcal{K}^*f,\varphi)=(f,\mathcal{K}\varphi)$, $(\mathcal{L}^*g,\psi)=(g,\mathcal{L}\psi)$, where $f\in D(\mathcal{K}^*)=\mathcal{H}_Q^s(-d,d)$, $\varphi\in D(\mathcal{K})$, $g\in D(\mathcal{L}^*)=\{f\in\mathbb{H}_0^s(-d,d): f$ is odd$\}$, $\psi\in D(\mathcal{L})$, $s\in\mathbb{R}$.*

Thus, $\mathcal{K}^*:\mathcal{H}_Q^s(-d,d)\to\mathbb{H}_0^s(-d,d)$, $\mathcal{L}^*:\mathbb{H}_0^s(-d,d)\to\mathcal{H}_Q^s(-d,d)$ and they are linear and continuous. Moreover, $(\mathcal{K}^*f)(x)=\sum_{n=1}^{\infty}K_n\sin\lambda_n x$, $(\mathcal{L}^*g)(x)=\sum_{n=1}^{\infty}L_n\widetilde{y}_n(\lambda_n,x)$. Evidently, $R(\mathcal{K}^*)=D(\mathcal{L}^*)$, $R(\mathcal{L}^*)=D(\mathcal{K}^*)$. It is obvious that $\mathcal{K}^*f$ and $\mathcal{L}^*g$ are odd when f and g are odd.

Lemma B.3. *Let $f\in\mathbb{H}_0^s(-d,d)\times\mathbb{H}_0^{s-1}(-d,d)$, f be odd, $s\in\mathbb{R}$. Then there exists $C_{\mathcal{L}^*}>0$ such that $|||\mathcal{L}^*f|||_Q^s\le C_{\mathcal{L}^*}\,|||f|||_0^s$.*

P r o o f. Since $\mathcal{L}^*$ is continuous, then there exist $B_1>0$, $B_2>0$ such that

$$|||\mathcal{L}^*f|||_Q^s=\left(\left(\|\mathcal{L}^*f_1\|_Q^s\right)^2+\left(\|\mathcal{L}^*f_2\|_Q^{s-1}\right)^2\right)^{1/2}\le\left(\left((B_1)^2\|f_1\|_0^s\right)^2\right.$$

$$\left.+\left((B_2)^2\|f_2\|_0^{s-1}\right)^2\right)^{1/2}\le C_{\mathcal{L}^*}\left(\left(\|f_1\|_0^s\right)^2+\left(\|f_2\|_0^{s-1}\right)^2\right)^{1/2}=C_{\mathcal{L}^*}\,|||f|||_0^s,$$

where $C_{\mathcal{L}^*} = \max\{B_1, B_2\}$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. The lemma is proved.

Lemma B.4. *Let* $f \in L^2(-d,d)$ *be odd,* $\operatorname{supp} f \subset [-T,T]$. *Then* $\operatorname{supp}(\mathcal{L}^* f) \subset [-T,T]$ *and* $(\mathcal{L}^* f)(t) = f(t) + \int_{|t|}^{T} \mathtt{L}(x,t;\infty) f(x)\,dx$.

P r o o f. Let $\psi \in L^2(-d,d)$ be odd. Then

$$(\mathcal{L}^* f, \psi) = (f, \mathcal{L}\psi) = \int\limits_{-d}^{d} f(x) \left\{ \psi(x) + \int\limits_{0}^{x} \mathtt{L}(x,t;\infty)\psi(t)\,dt \right\} dx$$

$$= \int\limits_{-d}^{d} \psi(x) \left\{ f(t) + \int\limits_{t}^{d \operatorname{sign} t} \mathtt{L}(x,t;\infty) f(x)\,dx \right\} dt.$$

Taking into account that $\widetilde{y}_n(\lambda_n, x)$ and $\frac{\sin \lambda_n x}{\lambda_n}$ are odd on x, $n = \overline{1,\infty}$, and $\mathtt{L}(x,t;\infty)$ is odd on t, it is easy to get $\mathtt{L}(-x,t;\infty) = \mathtt{L}(x,t;\infty)$. Therefore, due to the oddness of f, for $t < 0$ we have

$$\int\limits_{t}^{d \operatorname{sign} t} \mathtt{L}(x,t;\infty) f(x)\,dx = \int\limits_{t}^{-d} \mathtt{L}(x,t;\infty) f(x)\,dx = \int\limits_{-t}^{d} \mathtt{L}(-x,t;\infty) f(x)\,dx$$

$$= \int\limits_{|t|}^{d} \mathtt{L}(x,t;\infty) f(x)\,dx.$$

Obviously, the support of the last function is contained in $[-T,T]$. Consequently, $(\mathcal{L}^* f)(t) = f(t) + \int_{|t|}^{T} \mathtt{L}(x,t;\infty) f(x)\,dx$ and we get the assertion of the lemma. The lemma is proved.

Lemma B.5. *Let* $f \in L^2(-d,d)$ *be odd,* $\operatorname{supp} f \subset [-T,T]$. *Then* $\operatorname{supp}(\mathcal{K}^* f) \subset [-T,T]$ *and* $(\mathcal{K}^* f)(t) = f(t) + \int_{|t|}^{T} K(x,t;\infty) f(x)\,dx$.

The proof of this lemma is similar to the proof of the previous lemma.

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