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L- and M-structure in lush spaces

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Let X be a Banach space which is lush. It is shown that if a subspace of X is either an L-summand or an M-ideal then it is also lush.

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Introduction

Toeplitz defined [1] the *numerical range* of a square matrix A over the field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$), i.e. $A \in \mathbb{F}^{n \times n}$ for some $n \geq 0$, to be the set

$$W(A) = \{\langle Ax, x\rangle : \|x\| = 1, \quad x \in \mathbb{F}^n\},$$

which easily extends to operators on Hilbert spaces. In the 1960s, Lumer [2] and Bauer [3] independently extended this notion to arbitrary Banach spaces. For a Banach space X whose unit sphere we denote by S_X and an operator $T \in B(X) = \{T \colon X \to X : T \text{ linear, continuous}\}$, we thus call

$$V(T) = \{x^*(Tx) \colon x^*(x) = 1,\ x^* \in S_{X^*},\ x \in S_X\} \text{ and } v(T) = \sup\{|\lambda| \colon \lambda \in V(T)\}$$

the *numerical range* and *radius* of T, respectively. By construction, we have $v(T) \leq \|T\|$ for all $T \in B(X)$. The greatest number $m \geq 0$ that satisfies

$$m\|T\| \leq v(T) \quad \text{for every } T \in B(X)$$

is called the *numerical index* of X and denoted by $n(X)$. A summary of what is and what is not known about the numerical index can be found in [4] and [5]. In the special case $n(X) = 1$ the operator norm and the numerical radius coincide on $B(X)$.

Several attempts have been made to characterize the spaces with numerical index one among all Banach spaces geometrically, one of them in [6]. We denote by

$$S(B_X, x^*, \alpha) := \{x \in B_X \colon \operatorname{Re} x^*(x) > 1 - \alpha\}$$

for any $x^* \in S_{X^*}$ and $\alpha > 0$ an open slice of the unit ball. Setting $\mathbb{T} := \{\omega \in \mathbb{F} \colon |\omega| = 1\}$ and writing $\operatorname{co}(F)$ for the convex hull of a subset $F \subseteq X$ allows us to write the absolutely convex hull of F as $\operatorname{co}(\mathbb{T}F)$.

Definition. Let X be a Banach space. If for every two points u, $v \in S_X$ and $\varepsilon > 0$ there is a functional $x^* \in S_{X^*}$ that satisfies

$$u \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}(v, \operatorname{co}(\mathbb{T}\, S(B_X, x^*, \varepsilon))) < \varepsilon,$$

the space X is said to be *lush*.

Unfortunately, whilst lush spaces do have numerical index one, spaces with numerical index one need not be lush [7, Rem. 4.2]. Lushness has proved invaluable in constructing a Banach space whose dual has strictly smaller numerical index — answering a question that up until then had been open for decades. Consequently, the property deserves attention.

Let us recall some results about sums of Banach spaces.

Proposition (M. Martín and P. Payá [8, Prop. 1]). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then*

$$n\big(c_0((X_n)_{n\in\mathbb{N}})\big) = n\big(\ell^1((X_n)_{n\in\mathbb{N}})\big) = n\big(\ell^\infty((X_n)_{n\in\mathbb{N}})\big) = \inf_{n\in\mathbb{N}} n(X_n).$$

In particular, the following statements are equivalent:

(i) *every X_n has numerical index one,*

(ii) *the space $c_0\big((X_n)_{n\in\mathbb{N}}\big)$ has numerical index one,*

(iii) *the space $\ell^1\big((X_n)_{n\in\mathbb{N}}\big)$ has numerical index one, and*

(iv) *the space $\ell^\infty\big((X_n)_{n\in\mathbb{N}}\big)$ has numerical index one.*

A notion that has been introduced in [9] is that of a CL space. Originally defined for real spaces, it has proven inappropriate for complex spaces. Thus we will deal with a weakening introduced in [10] that had previously been used in [11] but remained unnamed.

Definition. Let X be a Banach space. If for every convex subset $F \subseteq S_X$ that is maximal in S_X with respect to convexity, $\overline{\operatorname{co}}(\mathbb{T}F) = B_X$ holds, then X is called an *almost-CL* space.

Almost-CL spaces are easily seen to be lush spaces but the converse does not hold [6, Ex. 3.4(c)]. With regard to sums, the following result has been obtained.

Proposition (M. Martín and P. Payá [12, Prop. 8 & 9]). *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of Banach spaces. Then the following are equivalent:*

(i) *every* X_n *is an almost-CL space,*

(ii) *the space* $c_0\big((X_n)_{n\in\mathbb{N}}\big)$ *is almost-CL, and*

(iii) *the space* $\ell^1\big((X_n)_{n\in\mathbb{N}}\big)$ *is almost-CL.*

For the recently introduced lushness property, however, only part of the corresponding equivalence has been shown.

Proposition (Boyko et al. [13, Prop. 5.3]). *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of Banach spaces. If every* X_n *is lush, then so are the spaces*

$$c_0((X_n)_{n\in\mathbb{N}}),\quad \ell^1((X_n)_{n\in\mathbb{N}}),\quad \textit{and}\quad \ell^\infty((X_n)_{n\in\mathbb{N}}).$$

We seek to improve this result, bringing it up to par with what has been proved for almost-CL spaces and spaces with numerical index one.

Inheritance of Lushness

To this end we will show that if X and Y are arbitrary Banach spaces and one of the two spaces $X \oplus_1 Y$ or $X \oplus_\infty Y$ is lush, then X and Y are lush themselves.

Such a relation between the spaces X, Y, and their sum can also be expressed in terms of projections.

Definition. Let Z be a Banach space and $P\colon Z \to Z$ a linear projection that satisfies $\|z\| = \max\{\|Pz\|, \|z - Pz\|\}$ for every $z \in Z$. Then P and $\operatorname{ran} P$ are called an *M-projection* and an *M-summand*, respectively.

Definition. Let Z be a Banach space and $P\colon Z \to Z$ a linear projection that satisfies $\|z\| = \|Pz\| + \|z - Pz\|$ for every $z \in Z$. Then P and $\operatorname{ran} P$ are called an *L-projection* and an *L-summand*, respectively.

Basic results of L- and M-structure theory that will be used from here on can be found in [14, Sec. I.1]. If a subspace $X \subseteq Z$ is an M-summand, its annihilator $X^\perp$ is an L-summand in Z^*. However, an L-summand of Z^* need not be the annihilator of any space $X \subseteq Z$, nor must subspaces $X \subseteq Z$ for which $X^\perp$ is an L-summand in Z^* be M-summands. Subspaces $X \subseteq Z$ for which $X^\perp$ is an L-summand in Z^* are referred to as *M-ideals*.

M-summands

We can now proceed to show that M-summands inherit lushness.

Proposition 1. *Let X be an M-summand in a lush space Z. Then X is lush.*

P r o o f. Let u, $v \in S_X$ and $\varepsilon \in (0,1)$ be arbitrary. Since X is an M-summand there is an M-projection $P\colon Z \to Z$ with $\operatorname{ran}(P) = X$. Because Z is lush, there is a functional $z^* \in S_{Z^*}$ satisfying $u \in S(B_Z, z^*, \varepsilon/2)$ and

$$\operatorname{dist}(v, \operatorname{co}(\mathbb{T}\, S(B_z, z^*, \varepsilon/2))) < \varepsilon/2.$$

Hence there are points $z_1, \ldots, z_n \in S(B_Z, z^*, \varepsilon/2)$ and corresponding $\theta_1, \ldots, \theta_n \in \mathbb{F}$ that satisfy $\sum_{k=1}^n |\theta_k| \leq 1$ such that $\|\sum_{k=1}^n \theta_k z_k - v\| < \varepsilon/2$ holds. The projection P allows us to split these points up into

$$x_k := Pz_k \quad \text{and} \quad y_k := Px_k - x_k,$$

of which the x_k appear to approximate v mostly by themselves:

$$\left\|\sum_{k=1}^n \theta_k z_k - v\right\| = \max\left\{\left\|\sum_{k=1}^n \theta_k y_k\right\|, \left\|\sum_{k=1}^n \theta_k x_k - v\right\|\right\}.$$

By $\operatorname{Re} z^*(x) > 1 - \varepsilon/2$ and $\|z^*\| = 1$ we clearly have $\operatorname{Re} z^*(y_k) \leq \varepsilon/2\|x_k\| \leq \varepsilon/2$ for every k and thus

$$\operatorname{Re} z^*(x_k) = \operatorname{Re} z^*(z_k) - \operatorname{Re} z^*(y_k) > 1 - \varepsilon,$$

leaving us with $x_k \in S(B_X, z^*, \varepsilon)$, and therefore

$$\operatorname{dist}(v, \operatorname{co}(\mathbb{T}\, S(B_X, z^*, \varepsilon))) < \varepsilon.$$

By restricting z^* to X and normalizing the restriction, we obtain the desired functional. ∎

M-ideals

The celebrated principle of local reflexivity due to Lindenstrauss and Rosenthal [15] can be used to extend Proposition 1 to M-ideals. More precisely, we require a refined statement.

Theorem (Johnson et al. [16, Sec. 3]). *Let X be a Banach space, $E \subseteq X^{**}$ and $F \subseteq X^*$ finite dimensional and $\varepsilon > 0$ arbitrary. Then there is an operator $T\colon E \to X$ with $||T||||T^{-1}|| \leq 1 + \varepsilon$ that satisfies $(T \circ i_X)(x) = x$ for every $x \in X$ with $i_X(x) \in E$ and $x^{**}(x^*) = x^*(Tx^{**})$ for every $x^* \in F$, $x^{**} \in E$.*

An elementary proof is given in [17, Th. 2].

Remark 1. *We shall only be concerned with the case* $X \neq \{0\}$ *in the above theorem. Without loss of generality, we can then assume* $E \cap i_X(X) \neq \{0\}$*. Consequently, the* ε*-isometry* T *can be chosen to satisfy*

$$1 - \varepsilon \leq ||Tz^{**}|| \leq 1 + \varepsilon \quad \text{for every } z^{**} \in S_E.$$

With that in mind extending Proposition 1 to M-ideals is straightforward.

Theorem 2. *Let* X *be an M-ideal in a lush space* Z*. Then* X *is lush as well.*

P r o o f. Let the points u, $v \in S_X$ be arbitrary and $\varepsilon > 0$. The lushness of Z now guarantees that there is a functional $z^* \in S_{Z^*}$ with $u \in S(B_Z, z^*, \varepsilon/2)$ as well as an absolutely convex combination of points $z_1, \ldots, z_n \in S(B_Z, z^*, \varepsilon/2)$ and corresponding scalars $\theta_1, \ldots, \theta_n \in \mathbb{F}$ such that $\|\sum_{k=1}^n \theta_k z_k - v\| < \varepsilon/2$ and $\sum_{k=1}^n |\theta_k| \leq 1$. We observe $Z^{**} = X^{\perp\perp} \oplus_\infty M$ for some subspace $M \subseteq Z^{**}$. For $k \in \{1, \ldots, n\}$ we can now find a decomposition $i_Z(z_k) = x_k^{**} + y_k^{**}$ with $x_k^{**} \in X^{\perp\perp}$ and $y_k^{**} \in M$. By

$$\mathrm{Re}\big(i_{Z^*}(z^*)\big)\big(i_Z(u)\big) = \mathrm{Re}\, z^*(u) > 1 - \varepsilon/2,$$

we clearly have

$$|y^{**}(z^*)| \leq \varepsilon/2 \quad \text{for every } y^{**} \in S_M.$$

The functionals x_k^{**} satisfy

$$\mathrm{Re}\, x_k^{**}(z^*) = \mathrm{Re}\, z^*(z_k) - \mathrm{Re}\, y_k^{**}(z^*) > 1 - \varepsilon$$

and in particular

$$1 - \varepsilon \leq \|x_k^{**}\| \leq \|z_k\| = 1.$$

We also remark

$$\left\|\sum_{k=1}^n \theta_k z_k - v\right\| = \max\left\{\left\|\sum_{k=1}^n \theta_k y_k^{**}\right\|, \left\|\sum_{k=1}^n \theta_k x_k^{**} - i_Z(v)\right\|\right\}.$$

Since $X^{\perp\perp}$ and X^{**} can be identified, we have shown that the functionals x_k^{**} meet the requirements of lushness for $i_X(u)$ and $i_X(v)$ in X^{**}.

In applying the principle of local reflexivity to the finite dimensional subspace $E := \mathrm{lin}\{x_1^{**}, \ldots, x_n^{**}, i_Z(v)\} \subseteq X^{**}$, we obtain an operator $T\colon E \to X$ that satisfies

- $(T \circ i_X)x = x$ for every $x \in X$ with $i_X(x) \in E$,

- $z^*(Tz^{**}) = z^{**}(z^*)$ for $z^{**} \in E$ and
- $1 - \varepsilon/2 \leq \|Tz^{**}\| \leq 1 + \varepsilon/2$ for $z^{**} \in S_E$ (as per Remark 1).

We can now project x_k^{**} onto X with any relevant structure preserved. For $x_k := Tx_k^{**} \in X$ we observe

$$\left\|\sum_{k=1}^{n} \theta_k x_k - v\right\| = \left\|\sum_{k=1}^{n} \theta_k T x_k^{**} - (T \circ i_Z)v\right\| \leq (1+\varepsilon/2)\left\|\sum_{k=1}^{n} \theta_k x_k^{**} - i_Z(v)\right\| < \varepsilon$$

and $\operatorname{Re} z^*(x_k) = \operatorname{Re} x_k^{**}(z^*) > 1 - \varepsilon$. What remains to be done is normalizing. We thus continue to set $\tilde{x}_k := x_k/||x_k||$ and obtain

$$\begin{aligned}\|x_k - \tilde{x}_k\| &= |\|x_k\| - 1| \\ &\leq |\|x_k\| - \|x_k^{**}\|| + |\|x_k^{**}\| - 1| \\ &\leq |\|Tx_k^{**}\| - \|x_k^{**}\|| + \varepsilon/2 \\ &= \varepsilon\|x_k^{**}\|/2 + \varepsilon/2 \\ &\leq \varepsilon,\end{aligned}$$

and therefore

$$\left\|\sum_{k=1}^{n} \theta_k \tilde{x}_k - v\right\| \leq \left\|\sum_{k=1}^{n} \theta_k (x_k - \tilde{x}_k)\right\| + \left\|\sum_{k=1}^{n} \theta_k x_k - v\right\| \leq \max_{k \leq n} \|x_k - \tilde{x}_k\| + \varepsilon \leq 2\varepsilon$$

as well as

$$\operatorname{Re} z^*(\tilde{x}_k) \geq \operatorname{Re} z^*(x_k) - \|x_k - \tilde{x}_k\| > 1 - 2\varepsilon.$$

■

L-summands

Lushness is also inherited by L-summands. To see this we replace the complementary parts y_k of z_k with elements $\xi_k \in X$ on which the functional z^* nearly attains its norm, such that the $\theta_k \xi_k$ nearly add up to zero.

Theorem 3. *Let X be an L-summand of a lush space Z. Then X is lush.*

P r o o f. Let $u, v \in S_X$ and $\varepsilon > 0$ be arbitrary. Since Z is lush, for any $\eta > 0$ there is a functional $z^* \in S_{Z^*}$ as well as $z_1, \dots, z_n \in S(B_Z, z^*, \eta)$ and $\theta_1, \dots, \theta_n \in \mathbb{F}$ with $\sum_{k=1}^n |\theta_k| \leq 1$ satisfying $u \in S(B_Z, z^*, \eta)$ and $\|\sum_{k=1}^n \theta_k z_k - v\| < \eta$. Let P be the L-projection onto X. We set $x_k := Pz_k$, $y_k := z_k - x_k$ and note

$$\left\|\sum_{k=1}^{n} \theta_k z_k - v\right\| = \left\|\sum_{k=1}^{n} \theta_k x_k - v\right\| + \left\|\sum_{k=1}^{n} \theta_k y_k\right\|.$$

In particular, this gives $\|\sum_{k=1}^n \theta_k x_k - v\| < \eta$ and $\|\sum_{k=1}^n \theta_k y_k\| < \eta$. Replacing y_k with $\xi_k := \|y_k\|/\|u\|u$ by setting $\tilde{x}_k := x_k + \xi_k$ yields $\|\tilde{x}_k\| \le \|z_k\| \le 1$ and

$$\begin{aligned}\operatorname{Re} z^*(\tilde{x}_k) &= \operatorname{Re} z^*(z_k - y_k + \xi_k)\\ &> (1-\eta) - \|y_k\| + (1-\eta)\|y_k\|\\ &= 1 - \eta - \eta\|y_k\|\\ &\ge 1 - 2\eta.\end{aligned}$$

We observe

$$\operatorname{Re} z^*(y_k) = \operatorname{Re} z^*(z_k) - \operatorname{Re} z^*(x_k) \ge (1-\eta) - \|x_k\| \ge \|y_k\| - \eta, \tag{1}$$

which we will utilize to prove

$$\big(\operatorname{Im} z^*(y_k)\big)^2 \le 2\|y_k\|\eta. \tag{2}$$

Since (2) trivially holds if $\|y_k\| \le \eta$ is satisfied, we shall assume $\|y_k\| > \eta$, leaving us with

$$\begin{aligned}\big(\operatorname{Im} z^*(y_k)\big)^2 &\le (\operatorname{Re} z^*(y_k))^2 + (\operatorname{Im} z^*(y_k))^2 - (\|y_k\| - \eta)^2\\ &= |z^*(y_k)|^2 - \|y_k\|^2 + 2\|y_k\|\eta - \eta^2\\ &\le 2\|y_k\|\eta - \eta^2\\ &< 2\|y_k\|\eta.\end{aligned}$$

We therefore have

$$\begin{aligned}\left|\sum_{k=1}^n \theta_k \operatorname{Re} z^*(y_k)\right| &= \left|\sum_{k=1}^n \theta_k z^*(y_k) - i\sum_{k=1}^n \theta_k \operatorname{Im} z^*(y_k)\right|\\ &\le \left\|\sum_{k=1}^n \theta_k y_k\right\| + \max_{k\le n} |\operatorname{Im} z^*(y_k)|\\ &\le \eta + \max_{k\le n} \sqrt{2\|y_k\|\eta}\\ &\le \eta + 2\sqrt{\eta}.\end{aligned}$$

Applying (1) to $\delta_k := \|y_k\| - \operatorname{Re} z^*(y_k)$ yields $|\delta_k| \le \eta$; we conclude

$$\begin{aligned}\left\|\sum_{k=1}^n \theta_k \xi_k\right\| &\le \left|\sum_{k=1}^n \theta_k \operatorname{Re} z^*(y_k)\right| + \left|\sum_{k=1}^n \theta_k \delta_k\right|\\ &\le 2\eta + 2\sqrt{\eta}\end{aligned}$$

and thus

$$\begin{aligned}\left\|\sum_{k=1}^{n}\theta_k\tilde{x}_k - v\right\| &= \left\|\sum_{k=1}^{n}\theta_k\left(x_k+\xi_k\right) - v\right\| \\ &\le \left\|\sum_{k=1}^{n}\theta_k x_k - v\right\| + \left\|\sum_{k=1}^{n}\theta_k\xi_k\right\| \\ &\le 3\eta + 2\sqrt{\eta}.\end{aligned}$$

Going back and choosing η such that $3\eta+2\sqrt{\eta}<\varepsilon$ and $2\eta<\varepsilon$ are satisfied yields

$$\operatorname{Re} z^*(\tilde{x}_k) > 1-\varepsilon \quad \text{for every } k\in\{1,\dots,n\}$$

and

$$\operatorname{dist}(v, \operatorname{co}(\mathbb{T}\, S(B_X, z^*, \varepsilon))) < \varepsilon$$

as desired. ■

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