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On the Neumann Boundary Controllability for the Non-Homogeneous String on a Half-Axis

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In the paper, the equation of a vibrating non-homogeneous string, whose potential is not equal to a constant, is considered on a half-axis. The Neumann control of the class L^∞ is considered at a point $x = 0$. The control problem is studied in the Sobolev spaces. The sufficient conditions for null-controllability and approximate null-controllability at a free time $T > 0$ are obtained for the given system. The controls solving these problems are found explicitly.

Key words: wave equation, controllability problem, Neumann control, Sobolev space, Sturm–Liouville equation, transformation operator.

Mathematics Subject Classification 2010: 93B05, 35B37, 35L05, 34B24.

1. Introduction

In the paper, the controllability problems for a vibrating non-homogeneous string on a half-axis are studied. The control system under consideration is

$$w_{tt}(x,t) = w_{xx}(x,t) - q(x)w(x,t), \qquad x \in (0,+\infty),\ t \in (0,T), \tag{1.1}$$

$$w_x(0,t) = u(t), \qquad t \in (0,T), \tag{1.2}$$

where $T > 0$, $u \in L^\infty(0,T)$ is a control, q is a potential under the conditions

$$q \in C[0,\infty) \cap L^\infty[0,\infty), \qquad \int_0^\infty x|q(x)|\,dx < \infty. \tag{1.3}$$

This control system is considered in the Sobolev spaces H_0^s. A time $T > 0$ is not fixed.

Controllability problems for hyperbolic partial differential equations were studied in a number of papers (see, e.g., [1–20]). The boundary controllability of the wave equation on bounded domains in the context of L^p-controls ($2 \le p \le \infty$) is well studied. Some results for a homogeneous string were obtained in [1–8] and other papers. The results for a non-homogeneous string were obtained in [9–14]. It should be noted that only L^∞-controls can be implemented practically.

The controllability problems for the wave equation on unbounded domains have not been studied as extensively as on bounded domains. The boundary controllability of the wave equation on a half-axis in the context of L^∞-controls was studied in [15–20]. In particular, the controllability for a homogeneous string with the Dirichlet control was investigated in [15, 16], and with the Neumann control in [17]. In [18] and [19], the controllability for a non-homogeneous string was studied for the case when $q \equiv \mathrm{const} \ge 0$. In [18], a time $T > 0$ was fixed. In [19], both cases with fixed and free time were studied. The Neumann control was considered in [18], and the Dirichlet control was considered in [19]. In [20], the controllability for a non-homogeneous string was studied for the case when the potential q was not generally speaking a constant. A control system was considered in the class of functions with bounded supports, and a time $T > 0$ was fixed in [20]. The case of the Dirichlet control and the case of the Neumann control were studied there. In papers [15–20], the control systems were considered in the Sobolev spaces H_0^s. The necessary and sufficient conditions for null-controllability and approximate null-controllability were obtained. The controls solving these problems were found explicitly.

In the present paper, unlike in [15–19], the potential q is not a constant, which makes the studying of controllability problems more complicated. To solve these problems, we apply the transformation operators for the Sturm–Liouville equation that do not change a solution asymptotic at infinity. We extend these operators to the Sobolev spaces and prove their continuity under conditions (1.3). Notice that in contrast to [20], in the present paper a time $T > 0$ is free and there are no restrictions on the functions supports, but stronger restrictions on the potential q are required. We prove that the application of the transformation operator to the control system with $q \ne \mathrm{const}$ reduces it to the similar control system studied in [17] with $q \equiv 0$. The converse is also correct: the application of the inverse transformation operator to the control system with $q \equiv 0$ reduces it to the control system with $q \ne \mathrm{const}$. A one-to-one correspondence between the solutions of these systems is proved. Moreover, the control u of the system is transformed to the control p of the system with $q \equiv 0$. We also prove that if a state of the control system with $q \equiv 0$ is approximately null-controllable, then a state of the control system with $q \ne \mathrm{const}$ is approximately null-controllable. All the above makes it possible to study the control system under consideration by using the results obtained in [17].

Thus, in the paper, the sufficient conditions for null-controllability and approximate null-controllability are obtained for the given control system at a free time. There is obtained the explicit formula for the control depending on the initial state of the given system and on the control p of the system with $q \equiv 0$. It should be noticed that the sufficient conditions obtained for null-controllability and approximate null-controllability of the system are also necessary when a time $T > 0$ is fixed and a control system is considered in the class of functions with bounded supports.

2. Notation and the Problem Definition

Consider control system (1.1), (1.2) with the initial conditions

$$w(x,0) = \mathrm{V}_0^0(x), \quad w_t(x,0) = \mathrm{V}_1^0(x), \qquad x \in (0,+\infty). \tag{2.1}$$

The aim of the paper is to study the null-controllability and approximate null-controllability problems for system (1.1), (1.2), (2.1), namely, to find the control of the class $L^\infty(0,T)$ which transfers a semi-infinite string from the given initial state to the origin and to a given neighborhood of the origin at time T. In addition, time T is free and may depend on the neighborhood.

Introduce the spaces used in the paper. Let $\mathcal{S}$ be the Schwartz space [21],

$$\mathcal{S} = \{\varphi \in C^\infty(\mathbb{R}) : \forall m,l \in \mathbb{N}\cup\{0\}\ \exists C_{ml} > 0 : \forall x \in \mathbb{R} \left|\varphi^{(m)}(x)(1+|x|^2)^l\right| \le C_{ml}\},$$

and let $\mathcal{S}'$ be the dual space. Denote by H_l^s ($s,l \in \mathbb{R}$) the Sobolev spaces [22, Chap. 1]

$$H_l^s = \left\{ f \in \mathcal{S}' : (1+x^2)^{l/2}(1+|D|^2)^{s/2} f \in L^2(\mathbb{R}) \right\},$$

$$\|f\|_l^s = \left(\int_{-\infty}^{+\infty} \left|(1+x^2)^{l/2}(1+|D|^2)^{s/2} f(x)\right|^2 dx \right)^{1/2},$$

where $D = -id/dx$. The norm $|||f|||_l^s = \left((\|f_0\|_l^s)^2 + \left(\|f_1\|_l^{s-1}\right)^2 \right)^{1/2}$ is used for $f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in H_l^s \times H_l^{s-1}$. A distribution $f \in \mathcal{S}'$ is said to be odd if $(f,\varphi(x)) = -(f,\varphi(-x))$, $\varphi \in \mathcal{S}$. A distribution $f \in \mathcal{S}'$ is said to be even if $(f,\varphi(x)) = (f,\varphi(-x))$, $\varphi \in \mathcal{S}$.

We also use the following subspaces of the Sobolev spaces ($s,l \in \mathbb{R}$):

$$H_{l,o}^s = \{f \in H_l^s : f \text{ is odd}\}, \quad H_{l,e}^s = \{f \in H_l^s : f \text{ is even}\},$$
$$\mathbb{H}_{l,p}^s = H_{l,p}^s \times H_{l,p}^{s-1}, \quad p = o, e.$$

Obviously, if $f \in H_{l,o}^s$, then $f' \in H_{l,e}^{s-1}$ and if $f \in H_{l,e}^s$, then $f' \in H_{l,o}^{s-1}$.

We assume that $\mathrm{V}^0 = \begin{pmatrix} \mathrm{V}_0^0 \\ \mathrm{V}_1^0 \end{pmatrix} \in \mathbb{H}_{0,e}^1$. The solutions of system (1.1), (1.2), (2.1) are considered in $H_{0,e}^1$.

Denote by $\Omega : \mathcal{S}' \to \mathcal{S}'$ and $\Xi : \mathcal{S}' \to \mathcal{S}'$, $D(\Omega) = D(\Xi) = \mathcal{S}'$ the operators $(\Omega f)(x) = f(x) - f(-x)$ and $(\Xi f)(x) = f(x) + f(-x)$, $f \in \mathcal{S}'$. Notice that these operators coincide with the odd and the even extension operators, respectively, for functions $f \in \mathcal{S}'$ such that $\operatorname{supp} f \subset (0, \infty)$. Assume that q is defined on $\mathbb{R}$ and $q \equiv 0$ on $(-\infty, 0)$. Denote $Q = \Xi q$, $\mathrm{V}(\cdot, t) = \Xi w(\cdot, t)$, $t \in (0, T)$. Evidently, $\mathrm{V}(\cdot, t) \in H_{0,e}^1$, $t \in (0, T)$.

Let w be the solution of control problem (1.1), (1.2), (2.1). It is easy to see that V is the solution of the problem

$$\mathrm{V}_{tt}(x,t) = \mathrm{V}_{xx}(x,t) - Q(x)\mathrm{V}(x,t) - 2u(t)\delta(x), \qquad x \in \mathbb{R},\ t \in (0,T), \tag{2.2}$$

$$\mathrm{V}(x,0) = \mathrm{V}_0^0(x), \qquad \mathrm{V}_t(x,0) = \mathrm{V}_1^0(x), \qquad x \in \mathbb{R}. \tag{2.3}$$

Consider some steering conditions for (2.2), (2.3):

$$\mathrm{V}(x,T) = \mathrm{V}_0^T(x), \quad \mathrm{V}_t(x,T) = \mathrm{V}_1^T(x), \qquad x \in \mathbb{R}, \tag{2.4}$$

where $\mathrm{V}^T = \begin{pmatrix} \mathrm{V}_0^T \\ \mathrm{V}_1^T \end{pmatrix} \in \mathbb{H}_{0,e}^1$. Let $T > 0$. For a given $\mathrm{V}^0 \in \mathbb{H}_{0,e}^1$, denote by $\mathcal{R}_T^e(\mathrm{V}^0)$ a set of the states $\mathrm{V}^T \in \mathbb{H}_{0,e}^1$ for which there exists a control $u \in L^\infty(0,T)$ such that problem (2.2)–(2.4) has a unique solution in $H_{0,e}^1$.

Definition 2.1. *A state $\mathrm{V}^0 \in \mathbb{H}_{0,e}^1$ is called null-controllable with respect to system* (2.2), (2.3) *if* 0 *belongs to $\bigcup_{T>0} \mathcal{R}_T^e(\mathrm{V}^0)$, and it is called approximately null-controllable with respect to system* (2.2), (2.3) *if* 0 *belongs to the closure of $\bigcup_{T>0} \mathcal{R}_T^e(\mathrm{V}^0)$ in $\mathbb{H}_{0,e}^1$.*

To study the controllability problems for system (2.2), (2.3), we use the transformation operators for the Sturm–Liouville equation that do not change a solution asymptotic at infinity. These operators were studied, e.g., in [23, Chap. 3]. In the present paper, the operators are extended to $H_{0,e}^s$, $s = 1, 0$ and proved to be continuous (see Sec. 5). Determine the operators $\mathcal{M}, \mathcal{M}^{-1} : H_{0,e}^0 \to H_{0,e}^0$, $D(\mathcal{M}) = D(\mathcal{M}^{-1}) = H_{0,e}^0$ by the formulas

$$(\mathcal{M}f)(x) = f(x) + \int_{|x|}^{\infty} \mathtt{M}(|x|, t) f(t)\,dt, \qquad x \in \mathbb{R}, \tag{2.5}$$

$$(\mathcal{M}^{-1}g)(x) = g(x) + \int_{|x|}^{\infty} \mathtt{N}(|x|, t) g(t)\,dt, \quad x \in \mathbb{R}, \tag{2.6}$$

where $f, g \in H^0_{0,e}$, $\mathtt{M}(\xi,\eta)$ and $\mathtt{N}(\xi,\eta)$ are the kernels of the operators, $(\xi,\eta) \in (0,\infty)\times(0,\infty)$. The properties of the kernels as well as the method used to find them are described at the beginning of Sec 5. In Lemma 5.2, we prove that the operators are continuous from $H^0_{0,e}$ to $H^0_{0,e}$, and $R(\mathcal{M}) = R(\mathcal{M}^{-1}) = H^0_{0,e}$. Consider the restrictions of the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ to $H^1_{0,e}$, $D(\mathcal{M}) = D(\mathcal{M}^{-1}) = H^1_{0,e}$. In Lemma 5.3, we prove that they are continuous from $H^1_{0,e}$ to $H^1_{0,e}$, and $R(\mathcal{M}) = R(\mathcal{M}^{-1}) = H^1_{0,e}$. In Lemma 5.4, the formulas for the adjoint operators $\mathcal{M}^*, (\mathcal{M}^{-1})^*$, $D(\mathcal{M}^*) = D((\mathcal{M}^{-1})^*) = H^0_{0,e}$ are obtained and they are proved to be continuous from $H^0_{0,e}$ to $H^0_{0,e}$, and $R(\mathcal{M}^*) = R((\mathcal{M}^{-1})^*) = H^0_{0,e}$. Consider the restrictions of $\mathcal{M}^*, (\mathcal{M}^{-1})^*$ to $H^1_{0,e}$, $D(\mathcal{M}^*) = D((\mathcal{M}^{-1})^*) = H^1_{0,e}$. In Lemma 5.5, $\mathcal{M}^*, (\mathcal{M}^{-1})^*$ are proved to be continuous from $H^1_{0,e}$ to $H^1_{0,e}$, and $R(\mathcal{M}^*) = R((\mathcal{M}^{-1})^*) = H^1_{0,e}$. Therefore, we can extend the operators $\mathcal{M}, \mathcal{M}^{-1}$ to $H^{-1}_{0,e}$ by the rule

$$(\mathcal{M}f, \psi) = (f, \mathcal{M}^*\psi), \tag{2.7}$$

$$(\mathcal{M}^{-1}g, \varphi) = (g, (\mathcal{M}^{-1})^*\varphi), \tag{2.8}$$

where $f, g \in H^{-1}_{0,e}$, $\varphi, \psi \in H^1_{0,e}$, $D(\mathcal{M}) = D(\mathcal{M}^{-1}) = H^{-1}_{0,e}$. In Lemma 5.5, we establish that these operators are continuous from $H^{-1}_{0,e}$ to $H^{-1}_{0,e}$, and $R(\mathcal{M}) = R(\mathcal{M}^{-1}) = H^{-1}_{0,e}$.

3. Null- and Approximate Null-Controllability Conditions

Consider the auxiliary control system with $Q \equiv 0$,

$$\mathcal{V}_{tt}(x,t) = \mathcal{V}_{xx}(x,t) - 2p(t)\delta(x), \qquad x \in \mathbb{R},\ t \in (0,T), \tag{3.1}$$

$$\mathcal{V}(x,0) = \mathcal{V}^0_0(x), \qquad \mathcal{V}_t(x,0) = \mathcal{V}^0_1(x), \qquad x \in \mathbb{R}, \tag{3.2}$$

with some steering conditions

$$\mathcal{V}(x,T) = \mathcal{V}^T_0(x), \quad \mathcal{V}_t(x,T) = \mathcal{V}^T_1(x), \qquad x \in \mathbb{R}, \tag{3.3}$$

where $\mathcal{V}(\cdot,t) \in H^1_{0,e}$, $\mathcal{V}^0 = \begin{pmatrix}\mathcal{V}^0_0\\ \mathcal{V}^0_1\end{pmatrix} \in \mathbb{H}^1_{0,e}$, $\mathcal{V}^T = \begin{pmatrix}\mathcal{V}^T_0\\ \mathcal{V}^T_1\end{pmatrix} \in \mathbb{H}^1_{0,e}$, $p \in L^\infty(0,T)$ is a control. Let $T > 0$. For a given $\mathcal{V}^0 \in \mathbb{H}^1_{0,e}$, denote by $\mathcal{Z}^e_T(\mathcal{V}^0)$ a set of the states $\mathcal{V}^T \in \mathbb{H}^1_{0,e}$ for which there exists a control $p \in L^\infty(0,T)$ such that problem (3.1)–(3.3) has a unique solution in $H^1_{0,e}$.

Definition 3.1. *A state $\mathcal{V}^0 \in \mathbb{H}^1_{0,e}$ is called null-controllable with respect to system* (3.1), (3.2) *if* 0 *belongs to $\bigcup_{T>0} \mathcal{Z}^e_T(\mathcal{V}^0)$, and it is called approximately null-controllable with respect to system* (3.1), (3.2) *if* 0 *belongs to the closure of $\bigcup_{T>0} \mathcal{Z}^e_T(\mathcal{V}^0)$ in $\mathbb{H}^1_{0,e}$.*

The controllability problems for system (3.1), (3.2) were well studied in [17]. The following assertions are special cases of the results obtained in [17]:

Statement 3.1 (Fardigola, [17]). *A solution of system* (3.1), (3.2) *is described by the formula*

$$\begin{pmatrix} \mathcal{V}(\cdot,t) \\ \mathcal{V}_t(\cdot,t) \end{pmatrix} = \mathcal{E}(\cdot,t) * \left[\mathcal{V}^0 - \begin{pmatrix} \partial^{-1}\Omega P^t \\ \Xi P^t \end{pmatrix} \right], \quad t \in (0,T), \tag{3.4}$$

where $P^t(x) = p(x)[H(x) - H(x-t)]$, $\partial^{-1}\Omega P^t(x) = \int_{-\infty}^x \Omega P^t(\xi)d\xi$,

$$\mathcal{E}(x,t) = \frac{1}{2}\begin{pmatrix} \delta(x+t)+\delta(x-t) & \frac{1}{2}(\operatorname{sign}(x+t) - \operatorname{sign}(x-t)) \\ \delta'(x+t) - \delta'(x-t) & \delta(x+t)+\delta(x-t) \end{pmatrix}, \quad x,t \in \mathbb{R}.$$

Theorem 3.1 (Fardigola, [17]). *A state* $\mathcal{V}^0 \in \mathbb{H}^1_{0,e}$ *is approximately null-controllable with respect to system* (3.1), (3.2) *iff the conditions below hold*

$$\mathcal{V}_1^0 \in L^\infty(\mathbb{R}), \tag{3.5}$$

$$\mathcal{V}_1^0 = \operatorname{sign} x (\mathcal{V}_0^0)'. \tag{3.6}$$

Under these conditions there exists a sequence $\{T_n\}_{n=1}^\infty$ *such that* $T_n|\mathcal{V}_0^0(T_n)|^2 \to 0$ *as* $n\to\infty$. *For this sequence the controls* $p_n(t) = \mathcal{V}_1^0(t)$ *a.e. on* $(0,T_n)$, $n \in \mathbb{N}$, *solve the approximate null-controllability problem for system* (3.1), (3.2).

Theorem 3.2 (Fardigola, [17]). *A state* $\mathcal{V}^0 \in \mathbb{H}^1_{0,e}$ *is null-controllable with respect to system* (3.1), (3.2) *iff conditions* (3.5), (3.6) *hold and there exists* $T>0$ *such that* $\operatorname{supp}\mathcal{V}_1^0 \subset (-T,T)$. *Under these conditions the control solving the null-controllability problem for system* (3.1), (3.2) *is of the form* $p = \mathcal{V}_1^0$ *a.e. on* $(0,T)$.

We first prove an auxiliary lemma for system (3.1), (3.2).

Lemma 3.1. *Let* $\mathcal{V}(x,t)$ *be the solution of* (3.1), (3.2). *Then* $\mathcal{V}_x(+0,t) = p(t)$, $t \in (0,T)$.

P r o o f. From (3.4) it follows that

$$\begin{aligned}\mathcal{V}(x,t) = \frac{1}{2}\Big\{&\mathcal{V}_0^0(x+t) + \mathcal{V}_0^0(x-t) + \widetilde{\mathcal{V}}_1^0(x+t) - \widetilde{\mathcal{V}}_1^0(x-t) - (\partial^{-1}\Omega P^t)(x+t) \\ &- (\partial^{-1}\Omega P^t)(x-t) - (\partial^{-1}\Xi P^t)(x+t) + (\partial^{-1}\Xi P^t)(x-t)\Big\},\end{aligned} \tag{3.7}$$

where $x \in \mathbb{R}$, $t \in (0,T)$, $\widetilde{\mathcal{V}}_1^0 \in H^1_{0,o}$ such that $(\widetilde{\mathcal{V}}_1^0)' = \mathcal{V}_1^0$. Differentiating (3.7) with respect to x, we obtain

$$\begin{aligned}\mathcal{V}_x(x,t) = \frac{1}{2}\big\{&(\mathcal{V}_0^0)'(x+t) + (\mathcal{V}_0^0)'(x-t) + \mathcal{V}_1^0(x+t) - \mathcal{V}_1^0(x-t) - 2P^t(x+t) \\ &+ 2P^t(-x+t)\big\}, \qquad x \in \mathbb{R},\ t \in (0,T).\end{aligned} \tag{3.8}$$

Therefore,

$$\mathcal{V}_x(+0,t) = \frac{1}{2}\lim_{x\to+0}\left\{(\mathcal{V}_0^0)'(x+t) + (\mathcal{V}_0^0)'(x-t) + \mathcal{V}_1^0(x+t) - \mathcal{V}_1^0(x-t)\right.$$
$$\left. -2P^t(x+t) + 2P^t(-x+t)\right\}, \qquad x\in\mathbb{R},\ t\in(0,T).$$

For any $f\in L^2(\mathbb{R})$ we may set $\lim_{x\to+0} f(x) = \lim_{x\to 0} f(|x|)$. Hence, taking into account the supports of $P^t(x+t)$ and $P^t(-x+t)$, we obtain

$$\mathcal{V}_x(+0,t) = \frac{1}{2}\left\{(\mathcal{V}_0^0)'(t) + (\mathcal{V}_0^0)'(-t) + \mathcal{V}_1^0(t) - \mathcal{V}_1^0(-t) + 2p(t)\right\}, \qquad t\in(0,T).$$

We remark that the values $(\mathcal{V}_0^0)'(t)$, $\mathcal{V}_1^0(t)$, $(\mathcal{V}_0^0)'(-t)$, and $\mathcal{V}_1^0(-t)$ exist a.e. on $(0,T)$, whereas $(\mathcal{V}_0^0)'$, $\mathcal{V}_1^0$ are locally integrable. Taking into account that $(\mathcal{V}_0^0)'$ is odd and $\mathcal{V}_1^0$ is even, we obtain the assertion of the lemma. The lemma is proved.

Theorem 3.3. *Let $\mathcal{V}(x,t)$ be the solution of* (3.1), (3.2). *Let* $\mathrm{V}(\cdot,t) = \mathcal{M}\mathcal{V}(\cdot,t)$, $t\in(0,T)$, $\mathrm{V}_j^0 = \mathcal{M}\mathcal{V}_j^0$, $j=0,1$. *Determine the function u by the formula*

$$u(t) = p(t) + \int_0^\infty M_x(0,\xi)\mathcal{V}(\xi,t)d\xi - \frac{1}{2}\mathcal{V}(0,t)\int_0^\infty q(\xi)d\xi, \qquad t\in(0,T), \tag{3.9}$$

where $\mathcal{V}(\xi,t)$ is defined by (3.7), *p is the control of system* (3.1), (3.2). *Then* $\mathrm{V}(x,t)$ *is the solution of system* (2.2), (2.3) *with the control u determined by* (3.9).

P r o o f. Apply the operator $\mathcal{M}$ to system (3.1), (3.2). Thus, conditions (2.3) hold immediately, and equation (3.1) takes the form

$$\mathcal{M}\mathcal{V}_{tt}(\cdot,t) = \mathcal{M}\mathcal{V}_{xx}(\cdot,t) - 2p(t)\mathcal{M}\delta, \qquad t\in(0,T). \tag{3.10}$$

Using (5.29), it is easy to get $(\mathcal{M}\delta,\psi) = (\mathcal{M}^*\psi)(0) = \psi(0) = (\delta,\psi)$ for any even $\psi\in\mathcal{S}$. Hence, $\mathcal{M}\delta = \delta$. Due to Lemma 5.6, equation (3.10) takes the form

$$\frac{d^2}{dt^2}\mathcal{M}\mathcal{V}(\cdot,t) = \frac{d^2}{dx^2}\mathcal{M}\mathcal{V}(\cdot,t) - Q\mathcal{M}\mathcal{V}(\cdot,t) - 2\delta\int_0^\infty \mathtt{M}_x(0,\xi)\mathcal{V}(\xi,t)d\xi$$
$$+\delta\mathcal{V}(0,t)\int_0^\infty q(\xi)d\xi - 2p(t)\delta, \qquad t\in(0,T).$$

Taking into account (3.9), we can see that the equation above is reduced to (2.2). The theorem is proved.

Theorem 3.4. *Let* $\mathrm{V}(x,t)$ *be the solution of system* (2.2), (2.3). *Let also* $\mathcal{V}(\cdot,t)=\mathcal{M}^{-1}\mathrm{V}(\cdot,t)$, $t\in(0,T)$, $\mathcal{V}_j^0=\mathcal{M}^{-1}\mathrm{V}_j^0$, $j=0,1$. *Suppose that the function* p *is connected with the control* u *of system* (2.2), (2.3) *by the following formula:*

$$p(t)=u(t)+\int_0^\infty \mathtt{N}_x(0,\xi)\mathrm{V}(\xi,t)d\xi+\frac{1}{2}\mathrm{V}(0,t)\int_0^\infty q(\xi)d\xi,\qquad t\in(0,T).\tag{3.11}$$

Then $\mathcal{V}(x,t)$ *is the solution of system* (3.1), (3.2) *with the control* p *determined by* (3.11).

P r o o f. Apply the operator $\mathcal{M}^{-1}$ to system (2.2), (2.3). Evidently, (2.3) is reduced to (3.2). Equation (2.2) takes the form

$$\mathcal{M}^{-1}\mathrm{V}_{tt}(\cdot,t)=\mathcal{M}^{-1}\mathrm{V}_{xx}(\cdot,t)-\mathcal{M}^{-1}(Q\mathrm{V})(\cdot,t)-2u(t)\mathcal{M}^{-1}\delta,\qquad t\in(0,T).\tag{3.12}$$

Since $\mathcal{M}\delta=\delta$, we have $\mathcal{M}^{-1}\delta=\delta$. Using Lemma 5.7, from (3.12) we get

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{M}^{-1}\mathrm{V}(\cdot,t)=\frac{d^2}{dx^2}\mathcal{M}^{-1}\mathrm{V}(\cdot,t)-2\delta\int_0^\infty \mathtt{N}_x(0,\xi)\mathrm{V}(\xi,t)d\xi\\ -\delta\mathrm{V}(0,t)\int_0^\infty q(\xi)d\xi-2\delta u(t),\qquad t\in(0,T).\end{aligned}\tag{3.13}$$

Taking into account (3.11), it is easy to see that (3.13) is reduced to (2.2). The theorem is proved.

R e m a r k 3.1. Theorems 3.3 and 3.4 establish a one-to-one correspondence between the solutions of systems (2.2), (2.3) and (3.1), (3.2) under the condition that the controls are connected by the corresponding relations.

Lemma 3.2. *Let* $\mathrm{V}(x,t)$ *be the solution of system* (2.2), (2.3). *Let also* $\mathcal{V}(\cdot,t)=\mathcal{M}^{-1}\mathrm{V}(\cdot,t)$, $t\in(0,T)$, $\mathcal{V}_j^0=\mathcal{M}^{-1}\mathrm{V}_j^0$, $j=0,1$, *and* (3.11) *holds. Then* $\mathrm{V}_x(+0,t)=u(t)$, $t\in(0,T)$.

P r o o f. Applying the operator $\mathcal{M}^{-1}$ to equation (2.2), we get (3.13). Taking into account expression (2.6) after differentiation and (5.17), we obtain

$$\begin{aligned}&\int_0^\infty \mathtt{N}_x(0,\xi)\mathrm{V}(\xi,t)d\xi+\mathrm{V}(0,t)\frac{1}{2}\int_0^\infty q(\xi)d\xi=\Bigg(\operatorname{sign}x\int_{|x|}^\infty \mathtt{N}_x(|x|,\xi)\mathrm{V}(\xi,t)d\xi\\ &+\operatorname{sign}x\mathrm{V}(|x|,t)\frac{1}{2}\int_{|x|}^\infty q(\xi)d\xi\Bigg)\Bigg|_{x=+0}=\left[\frac{d}{dx}(\mathcal{M}^{-1}\mathrm{V})(x,t)-\mathrm{V}_x(x,t)\right]\Bigg|_{x=+0}\\ &=\mathcal{V}_x(+0,t)-\mathrm{V}_x(+0,t),\qquad t\in(0,T).\end{aligned}\tag{3.14}$$

Substituting (3.14) in (3.13), we get

$$\mathcal{V}_{tt}(x,t) = \mathcal{V}_{xx}(x,t) - 2\delta[\mathcal{V}_x(+0,t) - \mathrm{V}_x(+0,t)] - 2\delta u(t), \quad x \in \mathbb{R}, t \in (0,T). \tag{3.15}$$

Since the conditions of Theorem 3.4 hold, $\mathcal{V}(x,t)$ is the solution of system (3.1), (3.2). Hence, due to Lemma 3.1, $\mathcal{V}_x(+0,t) = p(t)$, $t \in (0,T)$. Thus equation (3.15) takes the form

$$\mathcal{V}_{tt}(x,t) = \mathcal{V}_{xx}(x,t) - 2\delta(x)p(t) + 2\delta(x)\mathrm{V}_x(+0,t) - 2\delta(x)u(t), \quad x \in \mathbb{R},\ t \in (0,T).$$

From the above, it is seen that $\mathcal{V}(x,t)$ is the solution of system (3.1), (3.2) whenever $\mathrm{V}_x(+0,t) = u(t)$, $t \in (0,T)$. The lemma is proved.

R e m a r k 3.2. Let V be the solution of system (2.2), (2.3). By Lemma 3.2, it follows that the restriction of $\mathrm{V}(\cdot,t)$ to $[0,\infty)$, $t \in (0,T)$, is the solution of system (1.1), (1.2), (2.1). Thus we prove that control systems (1.1), (1.2), (2.1) and (2.2), (2.3) are equivalent.

Lemma 3.3. *Formulas* (3.9) *and* (3.11) *are equivalent.*

Pr o o f. Let $\mathrm{V}(\cdot,t) = \mathcal{M}\mathcal{V}(\cdot,t)$, $t \in (0,T)$, and (3.11) be valid. We prove that (3.9) is also valid. From (3.11), we have

$$u(t) = p(t) - \int_0^\infty \mathtt{N}_x(0,\xi)\mathrm{V}(\xi,t)d\xi - \frac{1}{2}\mathrm{V}(0,t)\int_0^\infty q(\xi)d\xi, \qquad t \in (0,T).$$

Using (2.5), (2.6), (5.10) and (5.17), we obtain

$$\begin{aligned}
u(t) &= p(t) - \left.\left[\operatorname{sign} x \int_{|x|}^\infty \mathtt{N}_x(|x|,\xi)\mathrm{V}(\xi,t)d\xi - \operatorname{sign} x\mathrm{V}(|x|,t)\mathtt{N}(|x|,|x|)\right]\right|_{x=+0} \\
&= p(t) - \left[d/dx(\mathcal{M}^{-1}\mathrm{V})(x,t) - \mathrm{V}_x(x,t)\right]\big|_{x=+0} \\
&= p(t) - [\mathcal{V}_x(x,t) - d/dx(\mathcal{M}\mathcal{V})(x,t)]|_{x=+0} \\
&= p(t) - \left.\left[-\operatorname{sign} x \int_{|x|}^\infty \mathtt{M}_x(|x|,\xi)\mathcal{V}(\xi,t)d\xi + \operatorname{sign} x\mathcal{V}(|x|,t)\mathtt{M}(|x|,|x|)\right]\right|_{x=+0} \\
&= p(t) + \int_0^\infty \mathtt{M}_x(0,\xi)\mathcal{V}(\xi,t)d\xi - \frac{1}{2}\mathcal{V}(0,t)\int_0^\infty q(\xi)d\xi, \qquad t \in (0,T).
\end{aligned}$$

Analogously, it can be proved that (3.11) is valid whenever (3.9) holds. The lemma is proved.

Lemma 3.4. *Let* (3.9) *hold for the controls* u *and* p *of systems* (2.2), (2.3) *and* (3.1), (3.2), *respectively. Let* $p \in L^\infty(0,T)$. *Let also a state* $\mathcal{V}^0$ *of control system* (3.1), (3.2) *be approximately null-controllable with respect to* (3.1), (3.2). *Then* $u \in L^\infty(0,T)$.

P r o o f. Let $p \in L^\infty(0,T)$. Taking into account (3.9), we have to prove that $\mathcal{V}(\xi,\cdot) \in L^\infty(0,T)$, $\xi \in \mathbb{R}$. Due to (3.7), it remains to show that $\mathcal{V}_0^0(x \pm t)$, $\widetilde{\mathcal{V}}_1^0(x \pm t)$, $(\partial^{-1}\Omega P^t)(x \pm t)$, $(\partial^{-1}\Xi P^t)(x \pm t) \in L^\infty(0,T)$ when x is fixed. Since $p \in L^\infty(0,T)$, then $P^t \in L^\infty(\mathbb{R})$. Therefore, $\Omega P^t \in L^\infty(\mathbb{R})$ and $\Xi P^t \in L^\infty(\mathbb{R})$. Hence, $(\partial^{-1}\Omega P^t)(x \pm t) \in L^\infty(0,T)$, $(\partial^{-1}\Xi P^t)(x \pm t) \in L^\infty(0,T)$.

Since the control p solves the approximate null-controllability problem for system (3.1), (3.2), then conditions (3.5) and (3.6) hold. Thus, $\mathcal{V}_1^0 \in L^\infty(\mathbb{R})$. Hence, $\widetilde{\mathcal{V}}_1^0(x \pm t) = (\partial^{-1}\mathcal{V}_1^0)(x \pm t) \in L^\infty(0,T)$. It follows from (3.6) that $\mathcal{V}_0^0 = \partial^{-1}(\operatorname{sign} x \mathcal{V}_1^0)$. Therefore, $\mathcal{V}_0^0(x \pm t) \in L^\infty(0,T)$. The lemma is proved.

Theorem 3.5. *Let* (3.9) *hold and* $\mathrm{V}(\cdot,t) = \mathcal{M}\mathcal{V}(\cdot,t)$, $t \in (0,T)$, $\mathrm{V}_j^0 = \mathcal{M}\mathcal{V}_j^0$, $j = 0,1$. *Let a state* $\mathcal{V}^0$ *of control system* (3.1), (3.2) *be approximately null-controllable with respect to* (3.1), (3.2). *Then a state* V^0 *of control system* (2.2), (2.3) *is approximately null-controllable with respect to* (2.2), (2.3).

P r o o f. Let a state $\mathcal{V}^0$ be approximately null-controllable with respect to (3.1), (3.2). Therefore, for each $m \in \mathbb{N}$ there exist $T_m > 0$ and $p^m \in L^\infty(0,T_m)$ such that $|||\mathcal{V}(\cdot,T_m)|||_0^1 \to 0$ as $m \to \infty$. Here $\mathcal{V}$ is the solution of (3.1), (3.2) with the control p^m. Since $\mathcal{M}\mathcal{V}(\cdot,T_m) = \mathrm{V}(\cdot,T_m)$, $m = \overline{1,\infty}$, and the operator $\mathcal{M}$ is continuous in the spaces $H_{0,e}^s$, $s = 1,0$, we obtain $|||\mathrm{V}(\cdot,T_m)|||_0^1 \to 0$ as $m \to \infty$. Thus, for each $m \in \mathbb{N}$ there exist $T_m > 0$ and $u^m = p^m + \int_0^\infty \mathtt{M}_x(0,\xi)\mathcal{V}^m(\xi,\cdot)d\xi - \frac{1}{2}\mathcal{V}^m(0,\cdot)\int_0^\infty q(\xi)d\xi$ such that $u^m \in L^\infty(0,T_m)$ (due to Lemma 3.4), moreover, $|||\mathrm{V}(\cdot,T_m)|||_0^1 \to 0$ as $m \to \infty$. This implies that a state V^0 is approximately null-controllable with respect to (2.2), (2.3). The theorem is proved.

Due to Theorems 3.3, 3.5, Lemma 5.8 and Theorems 3.1, 3.2, we obtain the controllability conditions for system (2.2), (2.3) and, consequently, for system (1.1), (1.2), (2.1).

Theorem 3.6. *Suppose the conditions below hold:*

$$\mathrm{V}_1^0 \in L^\infty(\mathbb{R}), \tag{3.16}$$

$$\mathrm{V}_1^0 = \mathcal{M} \operatorname{sign}\xi(\mathcal{M}^{-1}\mathrm{V}_0^0)'. \tag{3.17}$$

Then a state $\mathrm{V}^0 \in \mathbb{H}_{0,e}^1$ *is approximately null-controllable with respect to* (2.2), (2.3). *Under conditions* (3.16), (3.17) *there exists a sequence* $\{T_n\}_{n=1}^\infty$ *such that*

$T_n|\mathcal{M}^{-1}\mathrm{V}_0^0(T_n)|^2 \to 0$ *as* $n \to \infty$, *and for this sequence the controls*

$$u_n(t) = p_n(t) + \int_0^\infty M_x(0,\xi)\mathcal{V}(\xi,t)d\xi - \frac{1}{2}\mathcal{V}(0,t)\int_0^\infty q(\xi)d\xi$$

$$= (\mathcal{M}^{-1}\mathrm{V}_1^0)(t) + \int_0^\infty M_x(0,\xi)\mathcal{V}(\xi,t)d\xi - \frac{1}{2}\mathcal{V}(0,t)\int_0^\infty q(\xi)d\xi \tag{3.18}$$

a.e. on $(0,T_n)$, $n \in \mathbb{N}$, *solve the approximate null-controllability problem for system* (2.2), (2.3), *where* $\mathcal{V}(\xi,t)$ *is defined by* (3.7).

Theorem 3.7. *Suppose conditions* (3.16), (3.17) *hold and there exists* $T > 0$ *such that* $\operatorname{supp}\mathcal{M}^{-1}\mathrm{V}_1^0 \subset (-T,T)$. *Then a state* $\mathrm{V}^0 \in \mathbb{H}_{0,e}^1$ *is null-controllable with respect to* (2.2), (2.3). *In addition, the control solving the null-controllability problem for system* (2.2), (2.3) *is of the form* (3.18) *a.e. on* $(0,T)$.

R e m a r k 3.3. Unfortunately, we can not prove the necessity of conditions (3.16), (3.17) for approximate null-controllability of the state $\mathrm{V}^0 \in \mathbb{H}_{0,e}^1$ as it is not proved that $\mathrm{V}(x,\cdot) \in L^\infty(0,T)$ in general. Nevertheless, in the following theorem we will prove the necessity of these conditions under some restrictions.

Theorem 3.8. *Let conditions* (1.3) *hold. Let a time* $T > 0$ *be fixed and* $\operatorname{supp}\mathrm{V}_j^0 \subset (-T,T)$, $j = 0,1$. *Then conditions* (3.16), (3.17) *are not only sufficient, but also necessary for approximate null-controllability and null-controllability of a state* $\mathrm{V}^0 \in \mathbb{H}_{0,e}^1$ *at a fixed time.*

P r o o f. In [20], the controllability problems at a fixed time for system (1.1), (1.2), (2.1) were considered in the class of functions with bounded supports. Let $u \in L^\infty(0,T)$ and a state V^0 be approximately null-controllable at a time T with respect to (2.2), (2.3). One can conclude from [20, Lemma 4.1] that $\operatorname{supp}\mathrm{V}(\cdot,t) \subset (-2T,2T)$ and $\mathrm{V}(x,\cdot) \in L^\infty(0,T)$ in this case. Due to these facts, the proof of the following statement is trivial.

A) *Let* (3.11) *hold for the controls* u *and* p *of systems* (2.2), (2.3) *and* (3.1), (3.2), *respectively. Let* $u \in L^\infty(0,T)$. *Let also a state* V^0 *of control system* (2.2), (2.3) *be approximately null-controllable with respect to* (2.2), (2.3). *Then* $p \in L^\infty(0,T)$.

It is obvious that $\operatorname{supp}\mathcal{M}^{-1}\mathrm{V}(\cdot,t) \subset (-2T,2T)$, $t \in (0,T)$, and $\operatorname{supp}\mathcal{M}^{-1}\mathrm{V}_j^0 \subset (-T,T)$, $j = 0,1$. The following statement is proved in a similar way as that to Theorem 3.5 but for each $m \in \mathbb{N}$ the fixed time T is taken instead of T_m.

B) *Let* (3.11) *hold and* $\mathcal{V}(\cdot,t) = \mathcal{M}^{-1}\mathrm{V}(\cdot,t)$, $t \in (0,T)$, $\mathcal{V}_j^0 = \mathcal{M}^{-1}\mathrm{V}_j^0$, $j = 0,1$. *Let a state* V^0 *of control system* (2.2), (2.3) *be approximately null-controllable at a time* T *with respect to* (2.2), (2.3). *Then a state* $\mathcal{V}^0$ *of control system* (3.1), (3.2) *is approximately null-controllable at a time* T *with respect to* (3.1), (3.2).

Thus, using statements **A)**, **B)** and Theorems 3.1, 3.2, we may conclude that conditions (3.16), (3.17) hold. The theorem is proved.

R e m a r k 3.4. In practice, to find the controls u_n, $n \in \mathbb{N}$, solving the approximate null-controllability problem for system (2.2), (2.3), another formula is more convenient than (3.18). Let us transform (3.18) using (5.10) and (2.5). Let $t \in (0, T_n)$, $n \in \mathbb{N}$. Then

$$\begin{aligned} u_n(t) &= p_n(t) + \left.\left[\operatorname{sign} x \int_{|x|}^{\infty} \mathtt{M}_x(|x|, \xi)\mathcal{V}(\xi, t)d\xi - \operatorname{sign} x\mathtt{M}(|x|, |x|)\mathcal{V}(|x|, t)\right]\right|_{x=+0} \\ &= p_n(t) + \left.\left[\frac{d}{dx}\int_{|x|}^{\infty} \mathtt{M}(|x|, \xi)\mathcal{V}(\xi, t)d\xi\right]\right|_{x=+0} \\ &= p_n(t) + \left.\left[\frac{d}{dx}(\mathcal{M}\mathcal{V})(x, t)\right]\right|_{x=+0} - \mathcal{V}_x(+0, t). \end{aligned}$$

Using Lemma 3.1, we get

$$u_n(t) = \left.\left[\frac{d}{dx}(\mathcal{M}\mathcal{V})(x, t)\right]\right|_{x=+0}, \qquad t \in (0, T_n),\ n \in \mathbb{N}. \tag{3.19}$$

R e m a r k 3.5. Let conditions (1.3) hold, a time $T > 0$ be fixed, and $\operatorname{supp} \mathrm{V}_j^0 \subset (-T, T)$, $j = 0, 1$. We have proved that conditions (3.16) and (3.17) are necessary and sufficient for null-controllability and approximate null-controllability of system (1.1), (1.2), (2.1) at a fixed time. On the other hand, in [20] it is proved that the conditions

$$\mathrm{V}_1^0 \in L^\infty(\mathbb{R}), \tag{3.20}$$

$$\mathrm{V}_1^0 = \mathcal{K}_e^{-1} \operatorname{sign} x(\mathcal{K}_e \mathrm{V}_0^0)' \tag{3.21}$$

are necessary and sufficient for null-controllability and approximate null-controllability of the system at a fixed time. Here $\mathcal{K}_e$ and $\mathcal{K}_e^{-1}$ are other transformation operators with other kernels.

One can see that conditions (3.16) and (3.20) coincide. Since under consideration are *necessary and sufficient conditions*, we obtain that conditions (3.17) and (3.21) are different forms of the same relation between the initial functions.

4. Examples

Let $q(x) = e^{-x}$, $x > 0$. It is obvious that conditions (1.3) are valid. In this section, the kernels of the transformation operators will be found explicitly for the given q, and thus relation (3.17) will be rewritten in a simpler form. The controls solving the approximate null-controllability problem for system (1.1), (1.2), (2.1) with given initial functions will also be found.

E x a m p l e 4.1. Find the kernel $\mathtt{M}(x,t)$ of the operator $\mathcal{M}$. System (5.9)–(5.11) takes the form

$$\mathtt{M}_{xx}(x,t) - \mathtt{M}_{tt}(x,t) = e^{-x}\mathtt{M}(x,t), \qquad 0 < x < t, \tag{4.1}$$

$$\mathtt{M}(x,x) = \frac{1}{2}e^{-x}, \qquad x > 0, \tag{4.2}$$

$$\lim_{x+t\to\infty} \mathtt{M}_x(x,t) = \lim_{x+t\to\infty} \mathtt{M}_t(x,t) = 0. \tag{4.3}$$

Put $\xi = e^{-\frac{x+t}{2}}, \eta = e^{\frac{t-x}{2}} - 1$ and denote $\mathtt{A}(\xi,\eta) = \mathtt{M}(x,t)$. It is easy to see that system (4.1)–(4.3) is equivalent to the system

$$\begin{aligned} &\mathtt{A}_{\xi\eta}(\xi,\eta) = \mathtt{A}(\xi,\eta), \qquad 0 < \eta < \xi^{-1} - 1,\\ &\mathtt{A}(\xi,0) = \frac{\xi}{2}, \qquad 0 < \xi < 1,\\ &\mathtt{A}_\eta(0,\eta) = 0, \qquad \eta > 0. \end{aligned}$$

Then $\mathtt{A}(\xi,\eta) = \frac{\xi}{2}\frac{I_1(2\sqrt{\xi\eta})}{\sqrt{\xi\eta}}$ is the unique solution of this system. Here $I_1(z)$ is the modified Bessel function of order one, $I_1(z) = \frac{1}{i}J_1(iz)$, where $J_1(y)$ is the Bessel function of order one. Thus the kernel of the operator $\mathcal{M}$ is

$$\mathtt{M}(x,t) = \frac{e^{-\frac{x+t}{2}}}{2}\frac{I_1\left(2\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}\right)}{\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}}, \qquad 0 < x < t. \tag{4.4}$$

For the kernel $\mathtt{N}(x,t)$ of the operator $\mathcal{M}^{-1}$ we have the system

$$\begin{aligned} &\mathtt{N}_{xx}(x,t) - \mathtt{N}_{tt}(x,t) = -e^{-t}\mathtt{N}(x,t), \qquad 0 < x < t,\\ &\mathtt{N}(x,x) = -\frac{1}{2}e^{-x}, \qquad x > 0,\\ &\lim_{x+t\to\infty} \mathtt{N}_x(x,t) = \lim_{x+t\to\infty} \mathtt{N}_t(x,t) = 0. \end{aligned}$$

Putting $\mu = e^{-\frac{x+t}{2}}$, $\nu = e^{\frac{x-t}{2}} - 1$ and denoting $\mathtt{B}(\mu,\nu) = \mathtt{N}(x,t)$, we reduce this system to the form

$$\begin{aligned} &\mathtt{B}_{\mu\nu}(\mu,\nu) = \mathtt{B}(\mu,\nu), \qquad 0 < \nu < \mu^{-1} - 1,\\ &\mathtt{B}(\mu,0) = -\frac{\mu}{2}, \qquad 0 < \mu < 1,\\ &\mathtt{B}_\nu(0,\nu) = 0, \qquad \nu > 0. \end{aligned}$$

Then $\mathtt{B}(\mu,\nu) = -\frac{\mu}{2}\frac{I_1(2\sqrt{\mu\nu})}{\sqrt{\mu\nu}}$ is its unique solution. Hence, for $0<x<t$, we have

$$\mathtt{N}(x,t) = -\frac{e^{-\frac{x+t}{2}}}{2}\frac{I_1\left(2\sqrt{e^{-t}-e^{-\frac{x+t}{2}}}\right)}{\sqrt{e^{-t}-e^{-\frac{x+t}{2}}}} = -\frac{e^{-\frac{x+t}{2}}}{2}\frac{J_1\left(2\sqrt{e^{-\frac{x+t}{2}}-e^{-t}}\right)}{\sqrt{e^{-\frac{x+t}{2}}-e^{-t}}}. \quad (4.5)$$

Thus the kernels of the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ are of the forms (4.4) and (4.5), respectively, when $q(x) = e^{-x}$, $x>0$.

E x a m p l e 4.2. Consider (3.17). Substituting (4.5) in (2.6), we obtain

$$(\mathcal{M}^{-1}\mathrm{V}_0^0)(\xi) = -\operatorname{sign}\xi\frac{d}{d\xi}\int\limits_{|\xi|}^{\infty} J_0\left(2\sqrt{e^{-\frac{|\xi|+y}{2}}-e^{-y}}\right)\mathrm{V}_0^0(y)dy = -\operatorname{sign}\xi G'(\xi),$$

where $\xi\in\mathbb{R}$ and

$$G(\xi) = \int\limits_{|\xi|}^{\infty} J_0\left(2\sqrt{e^{-\frac{|\xi|+y}{2}}-e^{-y}}\right)\mathrm{V}_0^0(y)dy. \quad (4.6)$$

For any $\varphi\in H^0_{0,e}$, we have

$$\begin{aligned}\big(\operatorname{sign}\xi(\mathcal{M}^{-1}\mathrm{V}_0^0)'(\xi),\varphi(\xi)\big) &= \big(\operatorname{sign}\xi(-\operatorname{sign}\xi G'(\xi))',\varphi(\xi)\big)\\ &= \big(\operatorname{sign}\xi G'(\xi), 2\delta(\xi)\varphi(\xi)+\operatorname{sign}\xi\varphi'(\xi)\big)\\ &= 2\varphi(0)\big(\operatorname{sign}\xi G'(\xi),\delta(\xi)\big)+\big(\operatorname{sign}\xi G'(\xi),\operatorname{sign}\xi\varphi'(\xi)\big)\\ &= \big(2\delta(\xi)G'(+0)-G''(\xi),\varphi(\xi)\big).\end{aligned}$$

Thus, $\operatorname{sign}\xi(\mathcal{M}^{-1}\mathrm{V}_0^0)'(\xi) = 2\delta(\xi)G'(+0)-G''(\xi)$, $\xi\in\mathbb{R}$. Substituting this equality in (3.17) and taking into account that $\mathcal{M}\delta=\delta$, we get

$$\mathrm{V}_1^0(x) = 2\delta(x)G'(+0)-(\mathcal{M}G'')(x), \qquad x\in\mathbb{R}.$$

Using Lemma 5.6, we have

$$\begin{aligned}\mathrm{V}_1^0(x) &= 2\delta(x)G'(+0)-(\mathcal{M}G)''(x)+e^{-|x|}(\mathcal{M}G)(x)+2\delta(x)\int\limits_0^{\infty}\mathtt{M}_x(0,\xi)G(\xi)d\xi\\ &+\delta(x)G(0)\int\limits_0^{\infty}e^{-y}dy = 2\delta(x)G'(+0)-(\mathcal{M}G)''(x)+e^{-|x|}(\mathcal{M}G)(x)\\ &+2\delta(x)\left.\left[\frac{d}{dx}\int\limits_{|x|}^{\infty}\mathtt{M}(|x|,\xi)G(\xi)d\xi\right]\right|_{x=+0}\\ &= -(\mathcal{M}G)''(x)+e^{-|x|}(\mathcal{M}G)(x)+2\delta(x)(\mathcal{M}G)'(+0).\end{aligned} \quad (4.7)$$

Consider $(\mathcal{M}G)(x)$. Substituting (4.4) and (4.6) in (2.5) and changing the order of integration, we obtain

$$(\mathcal{M}G)(x) = \int_{|\xi|}^{\infty} J_0\left(2\sqrt{e^{-\frac{|\xi|+y}{2}} - e^{-y}}\right) \mathrm{V}_0^0(y)dy$$
$$+ \int_{|x|}^{\infty} \mathrm{V}_0^0(y) \int_{|x|}^{y} \frac{e^{-\frac{x+t}{2}}}{2} \frac{I_1\left(2\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}\right)}{\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}} J_0\left(2\sqrt{e^{-\frac{|\xi|+y}{2}} - e^{-y}}\right) dt\, dy.$$

Consider the inner integral. Putting $e^{-t/2} = z$, $e^{-|x|/2} = h$, $e^{-y/2} = g$, we reduce it to the form

$$\int_{|x|}^{y} \frac{e^{-\frac{x+t}{2}}}{2} \frac{I_1\left(2\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}\right)}{\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}} J_0\left(2\sqrt{e^{-\frac{|\xi|+y}{2}} - e^{-y}}\right) dt$$
$$= h \int_g^h \frac{I_1(2\sqrt{h(h-z)})}{\sqrt{h(h-z)}} J_0(2\sqrt{g(z-g)})dz.$$

After expanding $I_1(\tau)$ and $J_0(\tau)$ into series over τ^n, $n = \overline{0,\infty}$, and integrating, we obtain

$$\int_{|x|}^{y} \frac{e^{-\frac{x+t}{2}}}{2} \frac{I_1\left(2\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}\right)}{\sqrt{e^{-x} - e^{-\frac{x+t}{2}}}} J_0\left(2\sqrt{e^{-\frac{|\xi|+y}{2}} - e^{-y}}\right) dt$$
$$= I_0\left(2\left(e^{-\frac{|x|}{2}} - e^{-\frac{y}{2}}\right)\right) - J_0\left(2\sqrt{e^{-\frac{|x|+y}{2}} - e^{-y}}\right).$$

Thus,

$$(\mathcal{M}G)(x) = \int_{|x|}^{\infty} I_0\left(2\left(e^{-\frac{|x|}{2}} - e^{-\frac{y}{2}}\right)\right) \mathrm{V}_0^0(y)dy, \qquad x \in \mathbb{R}. \tag{4.8}$$

Differentiating (4.8), we get

$$(\mathcal{M}G)'(+0) = -\int_0^{\infty} I_1\left(2\left(1 - e^{-\frac{y}{2}}\right)\right) \mathrm{V}_0^0(y)dy - \mathrm{V}_0^0(+0); \tag{4.9}$$

$$(\mathcal{M}G)''(x) = -2\delta(x)\int\limits_0^\infty I_1\left(2\left(1-e^{-\frac{y}{2}}\right)\right)\mathrm{V}_0^0(y)dy - 2\delta(x)\mathrm{V}_0^0(+0)$$

$$-\operatorname{sign} x\left(\mathrm{V}_0^0\right)'(x) - \int\limits_{|x|}^\infty \frac{e^{-\frac{|x|+y}{2}}}{2}\frac{I_1\left(2\left(e^{-\frac{|x|}{2}}-e^{-\frac{y}{2}}\right)\right)}{e^{-\frac{|x|}{2}}-e^{-\frac{y}{2}}}\mathrm{V}_0^0(y)dy$$

$$+ e^{-|x|}\int\limits_{|x|}^\infty I_0\left(2\left(e^{-\frac{|x|}{2}}-e^{-\frac{y}{2}}\right)\right)\mathrm{V}_0^0(y)dy. \tag{4.10}$$

Substituting (4.8)–(4.10) in (4.7), we have

$$\mathrm{V}_1^0(x) = \operatorname{sign} x\left(\mathrm{V}_0^0\right)'(x) + \int\limits_{|x|}^\infty \frac{e^{-\frac{|x|+y}{2}}}{2}\frac{I_1\left(2\left(e^{-\frac{|x|}{2}}-e^{-\frac{y}{2}}\right)\right)}{e^{-\frac{|x|}{2}}-e^{-\frac{y}{2}}}\mathrm{V}_0^0(y)dy, \qquad x\in\mathbb{R}. \tag{4.11}$$

Thus, condition (3.17) is of the form (4.11) when $q(x)=e^{-x}$, $x>0$.

E x a m p l e 4.3. Let $q(x)=e^{-x}$, $x>0$; $\mathrm{V}_0^0(x)=I_1(2e^{-|x|/2})$, $\mathrm{V}_1^0(x)=-\frac{1}{2}I_1(2e^{-|x|/2})$, $x\in\mathbb{R}$. Consider the approximate null-controllability problem for system (1.1), (1.2), (2.1). Evidently, (3.16) is valid. One can see that (4.11) is also valid. Therefore, due to Theorem 3.6, the initial state V^0 is approximately null-controllable with respect to system (1.1), (1.2), (2.1). To find the controls u_n, $n\in\mathbb{N}$, solving the approximate null-controllability problem for this system, we reduce the given system to a system with $Q=0$ applying the operator $\mathcal{M}^{-1}$. Putting $e^{-y/2}=z$, $e^{-|x|/2}=h$, we have

$$\mathcal{M}^{-1}\mathrm{V}_0^0(x) = I_1(2e^{-\frac{|x|}{2}}) - \int\limits_{|x|}^\infty \frac{e^{-\frac{|x|+y}{2}}}{2}\frac{J_1\left(2\sqrt{e^{-\frac{|x|+y}{2}}-e^{-y}}\right)}{2\sqrt{e^{-\frac{|x|+y}{2}}-e^{-y}}}I_1\left(2e^{-\frac{y}{2}}\right)dy$$

$$= I_1(2h) - h\int\limits_0^h I_1(2z)\frac{J_1(2\sqrt{z(h-z)})}{\sqrt{z(h-z)}}dz.$$

After expanding $I_1(\tau)$ and $J_1(\tau)$ into series over τ^n, $n=\overline{0,\infty}$, and integrating, we obtain

$$\mathcal{V}_0^0(x) = \mathcal{M}^{-1}\mathrm{V}_0^0(x) = h\sum_{k=0}^\infty\sum_{n=0}^\infty \frac{(-1)^n h^{2k+2n}(2k+n)!}{k!(k+1)!n!(2k+2n)!}.$$

It is easy to check that $\mathcal{V}_0^0(x)=h=e^{-|x|/2}$. Hence, $\mathcal{V}_1^0(x)=-1/2e^{-|x|/2}$. It is evident that conditions (3.5), (3.6) are valid. Therefore the state $\mathcal{V}^0$ is approximately null-controllable with respect to system (3.1), (3.2), and the controls $p_n(t) = \mathcal{V}_1^0(t) = -1/2e^{-t/2}$ a.e. on $(0,n)$, $n \in \mathbb{N}$, solve the approximate null-controllability problem for system (3.1), (3.2). To find $\mathcal{V}(x,t)$, we substitute the explicit expressions for $\mathcal{V}_0^0$, $\mathcal{V}_1^0$ and p_n, $n \in \mathbb{N}$, in (3.8) and obtain $\mathcal{V}_x(x,t) = \mathcal{V}_1^0(x+t)H(x) - \mathcal{V}_1^0(x-t)H(-x)$, $x \in \mathbb{R}$, $t \in (0,T_n)$. Hence,

$$\mathcal{V}(x,t) = \int\limits_{-\infty}^{x} \left[\mathcal{V}_1^0(\xi+t)H(\xi) - \mathcal{V}_1^0(\xi-t)H(-\xi)\right] d\xi.$$

Obviously, $\mathcal{V}_x(x,t)$ is odd on x. Since $\int_{-\infty}^{x} f(\xi)d\xi = \int_{-\infty}^{-|x|} f(\xi)d\xi$ for any odd function f, we have

$$\mathcal{V}(x,t) = \int\limits_{-\infty}^{-|x|} \mathcal{V}_\xi(\xi,t)d\xi = -\int\limits_{-\infty}^{-|x|} \mathcal{V}_1^0(\xi-t)d\xi = -\int\limits_{-\infty}^{-|x|-t} \mathcal{V}_1^0(y)dy = e^{-\frac{|x|-t}{2}},$$

where $x \in \mathbb{R}$, $t \in (0,T_n)$. To find the controls u_n, $n \in \mathbb{N}$, we use (3.19). Thus,

$$(\mathcal{M}\mathcal{V})(x,t) = e^{-t/2}\mathcal{M}(e^{-\xi/2})(x,t) = e^{-t/2}I_1(2e^{-|x|/2}), \qquad x \in \mathbb{R},\ t \in (0,T_n).$$

Hence,

$$\frac{d}{dx}(\mathcal{M}\mathcal{V})(x,t) = \operatorname{sign} x e^{-\frac{t}{2}}\left[\frac{1}{2}I_1\left(2e^{-\frac{|x|}{2}}\right) - e^{-\frac{|x|}{2}}I_0\left(2e^{-\frac{|x|}{2}}\right)\right],\ x \in \mathbb{R},\ t \in (0,T_n).$$

Thus the controls

$$u_n(t) = e^{-\frac{t}{2}}\left[I_1(2)/2 - I_0(2)\right] \text{ a.e. on } (0,n),\ n \in \mathbb{N},$$

solve the approximate null-controllability problem for system (1.1), (1.2), (2.1) with the given initial state.

5. The Transformation Operators for the Sturm–Liouville Equation that do not Change a Solution Asymptotic at Infinity

At the beginning of the section we recall definitions and some properties of the transformation operators from [23, Chap. 3]. Further, we will extend these operators to the Sobolev spaces and prove their continuity. Consider two differential equations

$$-y''(x) = \lambda^2 y(x), \qquad x \in (0,+\infty),\ \lambda \in \mathbb{C}, \tag{5.1}$$

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \qquad x \in (0,+\infty),\ \lambda \in \mathbb{C}. \tag{5.2}$$

As it is known [23, Chap. 3], the integral operator $(\mathbf{I}+\mathbf{K})f = f(x)+\int_x^\infty \mathtt{M}(x,t)f(t)dt$ transfers the solution of (5.1) to the solution of (5.2), and it is the transformation operator that does not change a solution asymptotic at infinity. Due to [23, Chap. 3], this operator is a bijection of $L^2[0,\infty)$ onto $L^2[0,\infty)$, and the inverse operator $(\mathbf{I}+\mathbf{K})^{-1} = \mathbf{I}+\mathbf{L}$ is of the same form: $(\mathbf{I}+\mathbf{L})f = f(x)+\int_x^\infty \mathtt{N}(x,t)f(t)dt$.

For the operators kernels $\mathtt{M}(x,t)$ and $\mathtt{N}(x,t)$, the following estimates were obtained in [23, Chap. 3]:

$$|\mathtt{M}(x,t)| \le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)-\sigma_1((x+t)/2)}, \qquad (x,t)\in(0,\infty)\times(0,\infty), \tag{5.3}$$

$$|\mathtt{N}(x,t)| \le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1((x+t)/2)-\sigma_1(t)}, \qquad (x,t)\in(0,\infty)\times(0,\infty), \tag{5.4}$$

where $\sigma(x) = \int_x^\infty |q(\xi)|d\xi$, $\sigma_1(x) = \int_x^\infty \sigma(\xi)d\xi$. It is also known that

$$\mathtt{M}(x,t) = 0 \qquad \text{when } 0<t<x. \tag{5.5}$$

R e m a r k 5.1. The method of finding the kernel $\mathtt{M}(x,t)$ is obtained in [23, Chap. 3]. The function $\mathtt{M}(x,t)$ is the kernel of the operator $\mathbf{I}+\mathbf{K}$ iff the function $\widetilde{\mathtt{M}}(\alpha,\beta)$ is the solution of the following problem:

$$\widetilde{\mathtt{M}}_{\alpha\beta}(\alpha,\beta) = -q(\alpha-\beta)\widetilde{\mathtt{M}}(\alpha,\beta), \qquad 0<\beta<\alpha, \tag{5.6}$$

$$\widetilde{\mathtt{M}}(\alpha,0) = \frac{1}{2}\int\limits_\alpha^\infty q(\xi)d\xi, \qquad \alpha>0, \tag{5.7}$$

$$\lim_{\alpha\to\infty}\widetilde{\mathtt{M}}_\alpha(\alpha,\beta) = \lim_{\alpha\to\infty}\widetilde{\mathtt{M}}_\beta(\alpha,\beta) = 0. \tag{5.8}$$

Hence, $\mathtt{M}(x,t) = \widetilde{\mathtt{M}}(\frac{x+t}{2},\frac{t-x}{2})$ when $0<x<t$.

R e m a r k 5.2. Problem (5.6)–(5.8) is equivalent to the problem

$$\mathtt{M}_{xx}(x,t) - \mathtt{M}_{tt}(x,t) = q(x)\mathtt{M}(x,t), \qquad 0<x<t, \tag{5.9}$$

$$\mathtt{M}(x,x) = \frac{1}{2}\int\limits_x^\infty q(\xi)d\xi, \qquad x>0, \tag{5.10}$$

$$\lim_{x+t\to\infty}\mathtt{M}_x(x,t) = \lim_{x+t\to\infty}\mathtt{M}_t(x,t) = 0. \tag{5.11}$$

R e m a r k 5.3. Using the properties of the kernel $\mathtt{M}(x,t)$, from the obvious equation $\mathtt{N}(x,t)+\mathtt{M}(x,t)+\int_x^t \mathtt{N}(x,\xi)\mathtt{M}(\xi,t)d\xi = 0$ one can easily obtain the following statements:

a) $$\mathtt{N}(x,t) = 0, \quad \text{when } 0<t<x. \tag{5.12}$$

b) The function $\mathtt{N}(x,t)$ is the kernel of the operator $\mathbf{I}+\mathbf{L}$ iff the function $\widetilde{\mathtt{N}}(\alpha,\beta)$ is the solution of the following problem:

$$\widetilde{\mathtt{N}}_{\alpha\beta}(\alpha,\beta)=q(\alpha+\beta)\widetilde{\mathtt{N}}(\alpha,\beta),\qquad 0<\beta<\alpha, \tag{5.13}$$

$$\widetilde{\mathtt{N}}(\alpha,0)=-\frac{1}{2}\int\limits_{\alpha}^{\infty}q(\xi)d\xi,\qquad \alpha>0, \tag{5.14}$$

$$\lim_{\alpha\to\infty}\widetilde{\mathtt{N}}_{\alpha}(\alpha,\beta)=\lim_{\alpha\to\infty}\widetilde{\mathtt{N}}_{\beta}(\alpha,\beta)=0. \tag{5.15}$$

Hence, $\mathtt{N}(x,t)=\widetilde{\mathtt{N}}(\frac{x+t}{2},\frac{t-x}{2})$ when $0<x<t$.

c) Problem (5.13)–(5.15) is equivalent to the problem

$$\mathtt{N}_{xx}(x,t)-\mathtt{N}_{tt}(x,t)=-q(t)\mathtt{N}(x,t),\qquad 0<x<t, \tag{5.16}$$

$$\mathtt{N}(x,x)=-\frac{1}{2}\int\limits_{x}^{\infty}q(\xi)d\xi,\qquad x>0, \tag{5.17}$$

$$\lim_{x+t\to\infty}\mathtt{N}_{x}(x,t)=\lim_{x+t\to\infty}\mathtt{N}_{t}(x,t)=0. \tag{5.18}$$

Passing to the integral equations

$$\widetilde{\mathtt{M}}(\alpha,\beta)=\frac{1}{2}\int\limits_{\alpha}^{\infty}q(\xi)d\xi+\int\limits_{\alpha}^{\infty}\int\limits_{0}^{\beta}q(y-z)\widetilde{\mathtt{M}}(y,z)dz\,dy,\qquad 0<\beta<\alpha,$$

$$\widetilde{\mathtt{N}}(\alpha,\beta)=-\frac{1}{2}\int\limits_{\alpha}^{\infty}q(\xi)d\xi-\int\limits_{\alpha}^{\infty}\int\limits_{0}^{\beta}q(y+z)\widetilde{\mathtt{N}}(y,z)dz\,dy,\qquad 0<\beta<\alpha,$$

that are equivalent to boundary problems (5.6)–(5.8) and (5.13)–(5.15), respectively, we can find the estimates for $\widetilde{\mathtt{M}}_{\alpha}$, $\widetilde{\mathtt{M}}_{\beta}$, $\widetilde{\mathtt{N}}_{\alpha}$, $\widetilde{\mathtt{N}}_{\beta}$. Returning to variables x and t, we obtain the following estimates for $0<x<t$:

$$|\mathtt{M}_t(x,t)|\le\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|+\frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)-\sigma_1(\frac{x+t}{2})}, \tag{5.19}$$

$$|\mathtt{M}_x(x,t)|\le\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|+\frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)-\sigma_1(\frac{x+t}{2})}, \tag{5.20}$$

$$|\mathtt{N}_t(x,t)|\le\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|+\frac{1}{4}e^{\sigma_1(\frac{x+t}{2})-\sigma_1(t)}\sigma\left(\frac{x+t}{2}\right)\left[\sigma\left(\frac{x+t}{2}\right)+\sigma(t)\right], \tag{5.21}$$

$$|\mathtt{N}_x(x,t)|\le\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|+\frac{1}{4}e^{\sigma_1(\frac{x+t}{2})-\sigma_1(t)}\sigma\left(\frac{x+t}{2}\right)\left[\sigma\left(\frac{x+t}{2}\right)+\sigma(t)\right]. \tag{5.22}$$

In the following lemma, we obtain some properties of the functions $\sigma(x)$ and $\sigma_1(x)$ due to which estimates (5.3), (5.4), (5.19)–(5.22) will be somewhat simplified.

Lemma 5.1. *Let* $\sigma(x) = \int_x^\infty |q(\xi)|d\xi$, $\sigma_1(x) = \int_x^\infty \sigma(\xi)d\xi$, $x \in [0,\infty)$, *where conditions* (1.3) *hold for* q. *Then*
(*a*) σ *and* σ_1 *are decreasing functions;*
(*b*) $\sigma \le \sigma(0) < \infty$, $\sigma_1 \le \sigma_1(0) < \infty$ *on* $[0,\infty)$.

P r o o f. Assertion (*a*) is evident. Prove (*b*). Using (1.3), we get

$$\sigma(0) = \int\limits_0^\infty |q(\xi)|d\xi = \int\limits_0^1 |q(\xi)|d\xi + \int\limits_1^\infty |q(\xi)|d\xi \le C_q + \int_1^\infty x|q(\xi)|d\xi < \infty,$$

where $C_q > 0$ such that $|q| \le C_q$ a.e. on $(0,\infty)$. Consider $\sigma_1(0)$. Integrating the outer integral by parts, we get

$$\sigma_1(0) = \int\limits_0^\infty \int\limits_\xi^\infty |q(y)|dyd\xi = \left(\xi \int\limits_\xi^\infty |q(y)|dy \right)\Bigg|_{\xi=0}^{\xi=\infty} + \int\limits_0^\infty \xi|q(\xi)|d\xi$$

$$\le \left(\int\limits_\xi^\infty y|q(y)|dy \right)\Bigg|_{\xi=\infty} + \int\limits_0^\infty \xi|q(\xi)|d\xi < \infty$$

due to (1.3). The lemma is proved.

Using Lemma 5.1, one can make the following conclusions for $t > x > 0$: $\sigma\left(\frac{x+t}{2}\right) \le \sigma(x)$, $\sigma(t) \le \sigma(x)$, $e^{\sigma_1(x)-\sigma_1\left(\frac{x+t}{2}\right)} \le e^{2\sigma_1(0)}$, $e^{\sigma_1\left(\frac{x+t}{2}\right)-\sigma_1(t)} \le e^{2\sigma_1(0)}$. Therefore estimates (5.3), (5.4), (5.19)–(5.22) can be rewritten in the form

$$|\mathtt{M}(x,t)| \le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{2\sigma_1(0)}, \quad 0 < x < t, \tag{5.23}$$

$$|\mathtt{N}(x,t)| \le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{2\sigma_1(0)}, \quad 0 < x < t, \tag{5.24}$$

$$|\mathtt{M}_t(x,t)| \le \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + \frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right)e^{2\sigma_1(0)}, \quad 0 < x < t, \tag{5.25}$$

$$|\mathtt{M}_x(x,t)| \le \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + \frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right)e^{2\sigma_1(0)}, \quad 0 < x < t, \tag{5.26}$$

$$|\mathtt{N}_t(x,t)| \le \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + \frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right)e^{2\sigma_1(0)}, \quad 0 < x < t, \tag{5.27}$$

$$|\mathtt{N}_x(x,t)| \le \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + \frac{1}{2}\sigma(x)\sigma\left(\frac{x+t}{2}\right)e^{2\sigma_1(0)}, \quad 0 < x < t. \tag{5.28}$$

Further, consider the extensions of the operators $\mathbf{I}+\mathbf{K}$ and $\mathbf{I}+\mathbf{L}$, denoted by $\mathcal{M}$ and $\mathcal{M}^{-1}$, respectively, extended to $H^0_{0,e}$ by formulas (2.5), (2.6).

Lemma 5.2. *The operators* $\mathcal{M},\mathcal{M}^{-1}: H^0_{0,e}\to H^0_{0,e}$, $D(\mathcal{M})=D(\mathcal{M}^{-1})=H^0_{0,e}$ *defined by* (2.5), (2.6) *are continuous from* $H^0_{0,e}$ *to* $H^0_{0,e}$. *In addition,* $R(\mathcal{M})=R(\mathcal{M}^{-1})=H^0_{0,e}$.

P r o o f. Let $f\in H^0_{0,e}$. The evenness of $\mathcal{M}f$ is evident. Since $\mathbf{I}+\mathbf{K}$ is continuous from $L^2[0,\infty)$ to $L^2[0,\infty)$, we can see that $\mathcal{M}$ is continuous from $H^0_{0,e}$ to $H^0_{0,e}$. The assertion on the operator $\mathcal{M}^{-1}$ is proved in a similar way. From the continuity of the operators it follows that $R(\mathcal{M})=R(\mathcal{M}^{-1})=H^0_{0,e}$. The lemma is proved.

Lemma 5.3. *Let* $\varphi,\psi\in H^0_{0,e}$. *Then the adjoint operators* $\mathcal{M}^*,(\mathcal{M}^{-1})^*: H^0_{0,e}\to H^0_{0,e}$, $D(\mathcal{M}^*)=D((\mathcal{M}^{-1})^*)=H^0_{0,e}$ *are continuous from* $H^0_{0,e}$ *to* $H^0_{0,e}$ *and can be defined by the formulas*

$$(\mathcal{M}^*\varphi)(t)=\varphi(t)+\int_0^{|t|}\mathtt{M}(x,|t|)\varphi(x)dx,\qquad t\in\mathbb{R},\tag{5.29}$$

$$((\mathcal{M}^{-1})^*\psi)(t)=\psi(t)+\int_0^{|t|}\mathtt{N}(x,|t|)\psi(x)dx,\quad t\in\mathbb{R}.\tag{5.30}$$

In addition, $R(\mathcal{M}^*)=R((\mathcal{M}^{-1})^*)=H^0_{0,e}$.

P r o o f. Let $f\in H^0_{0,e}$. Substituting (2.5) into the known definition $(\mathcal{M}f,\varphi)=(f,\mathcal{M}^*\varphi)$ and changing the order of integration, we obtain (5.29). In the same way, we get (5.30). The continuity of the operators $(\mathcal{M})^*$ and $(\mathcal{M}^{-1})^*$ from $H^0_{0,e}$ to $H^0_{0,e}$ follows from the continuity of the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ from $H^0_{0,e}$ to $H^0_{0,e}$. The fact that $R(\mathcal{M}^*)=R((\mathcal{M}^{-1})^*)=H^0_{0,e}$ follows from the continuity of the operators. The lemma is proved.

Lemma 5.4. *The operators* $\mathcal{M},\mathcal{M}^{-1}: H^1_{0,e}\to H^1_{0,e}$, $D(\mathcal{M})=D(\mathcal{M}^{-1})=H^1_{0,e}$ *defined by* (2.5), (2.6) *are continuous from* $H^1_{0,e}$ *to* $H^1_{0,e}$, *and* $R(\mathcal{M})=R(\mathcal{M}^{-1})=H^1_{0,e}$.

P r o o f. Let $f\in H^1_{0,e}$. Taking into account that $\|y\|^1_0\le\|y\|^0_0+\|y'\|^0_0$ for any $y\in H^1_0$, we have

$$\|\mathcal{M}f\|^1_0\le\|f\|^1_0+\left\|\int_{|x|}^{\infty}\mathtt{M}(|x|,t)f(t)dt\right\|^0_0+\left\|\frac{d}{dx}\int_{|x|}^{\infty}\mathtt{M}(|x|,t)f(t)dt\right\|^0_0.\tag{5.31}$$

Taking into account (5.5), (5.23), using the Cauchy–Bunyakovsky–Schwartz inequality and the inequality $\|f\|_0^0 \le \|f\|_0^1$, we obtain the estimate for the second summand in (5.31),

$$\left\| \int_{|x|}^{\infty} \mathtt{M}(|x|,t)f(t)dt \right\|_0^0 = \sqrt{2}\left(\int_0^{\infty} \left| \int_0^{\infty} \mathtt{M}(x,t)f(t)dt \right|^2 dx \right)^{1/2}$$

$$\le \|f\|_0^0 \left(\int_0^{\infty}\int_x^{\infty} |\mathtt{M}(x,t)|^2 dtdx \right)^{1/2} \le \frac{e^{2\sigma_1(0)}}{2}\|f\|_0^1 \left(\int_0^{\infty}\int_x^{\infty} \left|\sigma\left(\frac{x+t}{2}\right)\right|^2 dtdx \right)^{1/2}$$

$$\le \frac{1}{2}e^{2\sigma_1(0)}\|f\|_0^1 \left(\int_0^{\infty} \sigma(x) \int_x^{\infty} \sigma\left(\frac{x+t}{2}\right) dt\, dx \right)^{1/2} \le \frac{\sigma_1(0)}{\sqrt{2}} e^{2\sigma_1(0)} \|f\|_0^1 . \qquad (5.32)$$

Using (5.10) and the evenness of f, we get the estimate for the third summand in (5.31),

$$\left\| \frac{d}{dx}\int_{|x|}^{\infty} \mathtt{M}(|x|,t)f(t)dt \right\|_0^0 \le \left\| \operatorname{sign} x \int_{|x|}^{\infty} \mathtt{M}_x(|x|,t)f(t)dt \right\|_0^0 + \frac{1}{2}\left\| \operatorname{sign} x f(x) \int_{|x|}^{\infty} q(\xi)d\xi \right\|_0^0 .$$

Taking into account Lemma 5.1 and the inequality $\|f\|_0^0 \le \|f\|_0^1$, we obtain

$$\frac{1}{2}\left\| \operatorname{sign} x f(x) \int_{|x|}^{\infty} q(\xi)d\xi \right\|_0^0 \le \frac{1}{2}\left(\int_{-\infty}^{\infty} \left| f(x) \int_{|x|}^{\infty} q(\xi)d\xi \right|^2 dx \right)^{1/2} \le \frac{1}{2}\sigma(0)\|f\|_0^1 .$$

Taking into account (5.5), and using the Cauchy–Bunyakovsky–Schwartz inequality and (5.26), we get

$$\left\| \operatorname{sign} x \int_{|x|}^{\infty} \mathtt{M}_x(|x|,t)f(t)dt \right\|_0^0 = \sqrt{2}\left(\int_0^{\infty} \left| \int_0^{\infty} \mathtt{M}_x(x,t)f(t)dt \right|^2 dx \right)^{1/2}$$

$$\le \sqrt{2}\left(\int_0^{\infty}\int_0^{\infty} |\mathtt{M}_x(x,t)|^2 dt \int_0^{\infty} |f(t)|^2 dtdx \right)^{1/2} = \|f\|_0^0 \left(\int_0^{\infty}\int_x^{\infty} |\mathtt{M}_x(x,t)|^2 dtdx \right)^{1/2}$$

$$\leq \|f\|_0^1 \left(\int_0^\infty \left[\frac{1}{16} \int_x^\infty \left| q\left(\frac{x+t}{2}\right)\right|^2 dt + \frac{1}{4} e^{2\sigma_1(0)} \sigma(x) \int_x^\infty \left| q\left(\frac{x+t}{2}\right)\right| \sigma\left(\frac{x+t}{2}\right) dt \right.\right.$$

$$\left.\left. + \frac{1}{4} e^{4\sigma_1(0)} (\sigma(x))^2 \int_x^\infty \left(\sigma\left(\frac{x+t}{2}\right)\right)^2 dt \right] dx \right)^{1/2}. \tag{5.33}$$

Let us estimate the last three summands. From (1.3) it follows that there exists $C_q > 0$ such that $|q| \leq C_q$ a.e. on $[0, \infty)$. Using Lemma 5.1, we obtain

$$\frac{1}{16} \int_0^\infty \int_x^\infty \left| q\left(\frac{x+t}{2}\right)\right|^2 dt dx = \frac{1}{8} \int_0^\infty \int_x^\infty |q(y)|^2 dy dx \leq \frac{C_q}{8} \int_0^\infty \int_x^\infty |q(y)| dy dx = \frac{C_q}{8} \sigma_1(0).$$

Then we use Lemma 5.1 again to obtain

$$\begin{aligned}
&\frac{1}{4} e^{2\sigma_1(0)} \int_0^\infty \sigma(x) \int_x^\infty \left| q\left(\frac{x+t}{2}\right)\right| \sigma\left(\frac{x+t}{2}\right) dt\, dx \\
&= \frac{1}{2} e^{2\sigma_1(0)} \int_0^\infty \sigma(x) \int_x^\infty |q(y)| \sigma(y) dy\, dx \leq \frac{1}{2} e^{2\sigma_1(0)} \sigma(0) \int_0^\infty \sigma(x)\sigma(x) dx \\
&\leq \frac{1}{2} e^{2\sigma_1(0)} (\sigma(0))^2 \sigma_1(0);
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{4} e^{4\sigma_1(0)} \int_0^\infty (\sigma(x))^2 \int_x^\infty \left(\sigma\left(\frac{x+t}{2}\right)\right)^2 dt\, dx \\
&= \frac{1}{2} e^{4\sigma_1(0)} \int_0^\infty (\sigma(x))^2 \int_x^\infty (\sigma(y))^2 dy\, dx \leq \frac{1}{2} e^{4\sigma_1(0)} (\sigma(0))^2 \int_0^\infty \sigma(x)\sigma_1(x) dx \\
&\leq \frac{1}{2} e^{4\sigma_1(0)} (\sigma(0))^2 \sigma_1(0) \int_0^\infty \sigma(x) dx = \frac{1}{2} e^{4\sigma_1(0)} (\sigma(0))^2 (\sigma_1(0))^2.
\end{aligned}$$

Continuing estimate (5.33), we get

$$\left\| \operatorname{sign} x \int_{|x|}^\infty \mathtt{M}_x(|x|, t) f(t) dt \right\|_0^0 \leq P \|f\|_0^1,$$

where $P = \left(\frac{C_q}{8}\sigma_1(0) + \frac{e^{2\sigma_1(0)}}{2}(\sigma(0))^2\sigma_1(0)\left[1 + e^{2\sigma_1(0)}\sigma_1(0)\right]\right)^{1/2} > 0$. Thus,

$$\left\| \frac{d}{dx} \int\limits_{|x|}^{\infty} \mathtt{M}(|x|, t) f(t) dt \right\|_0^0 \le \|f\|_0^1 \left(\frac{1}{2}\sigma(0) + P\right). \tag{5.34}$$

Substituting (5.32) and (5.34) in (5.31), we obtain that the operator $\mathcal{M}$ is continuous. Analogously, $\mathcal{M}^{-1}$ is continuous from $H^1_{0,e}$ to $H^1_{0,e}$. From the continuity of the operators it follows that $R(\mathcal{M}) = R(\mathcal{M}^{-1}) = H^1_{0,e}$ when $D(\mathcal{M}) = D(\mathcal{M}^{-1}) = H^1_{0,e}$. The lemma is proved.

Lemma 5.5. *The operators* $\mathcal{M}, \mathcal{M}^{-1} : H^{-1}_{0,e} \to H^{-1}_{0,e}$, $D(\mathcal{M}) = D(\mathcal{M}^{-1}) = H^{-1}_{0,e}$, *defined by* (2.7), (2.8), *are continuous from* $H^{-1}_{0,e}$ *to* $H^{-1}_{0,e}$, *and* $R(\mathcal{M}) = R(\mathcal{M}^{-1}) = H^{-1}_{0,e}$.

P r o o f. Let us prove that the restrictions of the adjoint operators to $H^1_{0,e}$ are continuous from $H^1_{0,e}$ to $H^1_{0,e}$ and their range is the space $H^1_{0,e}$ if the domain is $H^1_{0,e}$. The proof of the continuity of the operators $\mathcal{M}^*$ and $(\mathcal{M}^{-1})^*$ is similar to the proof of the previous lemma. Here the adjoint operator is considered instead of the original one. From the continuity of the operators it follows that $R(\mathcal{M}^*) = R((\mathcal{M}^{-1})^*) = H^1_{0,e}$. Thereby, the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ are well defined by formulas (2.7), (2.8). Since the adjoint operators are continuous, then $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous from $H^{-1}_{0,e}$ to $H^{-1}_{0,e}$, and thus $R(\mathcal{M}) = R(\mathcal{M}^{-1}) = H^{-1}_{0,e}$. The lemma is proved.

Lemma 5.6. *Let* $f \in H^1_{0,e}$. *Then*

$$\mathcal{M}(f'') = (\mathcal{M}f)'' - Q\mathcal{M}f - 2\delta \int\limits_0^{\infty} \mathtt{M}_x(0, \xi) f(\xi) d\xi + \delta f(0) \int\limits_0^{\infty} q(\xi) d\xi.$$

P r o o f. Let $\varphi \in H^1_{0,e}$. Let us transform the expressions $(\mathcal{M}(f''), \varphi)$ and $((\mathcal{M}f)'', \varphi)$ using (2.5), (5.29), (5.10) and the evenness of f and φ. Further, we will consider the difference $(\mathcal{M}(f''), \varphi) - ((\mathcal{M}f)'', \varphi)$. Thus,

$$(\mathcal{M}(f''), \varphi) = -(f', (\mathcal{M}^*\varphi)') =$$
$$-\left(f'(x), \varphi'(x) + \operatorname{sign} x \int\limits_0^{|x|} \mathtt{M}_x(y, |x|) \varphi(y) dy + \operatorname{sign} x \varphi(x) \frac{1}{2} \int\limits_{|x|}^{\infty} q(\xi) d\xi \right)$$

$$= (f'', \varphi) + \left(f(x), 2\delta(x) \int_0^{|x|} \mathtt{M}_x(y, |x|)\varphi(y)dy \right) + \left(f(x), \varphi(x)\, \mathtt{M}_x(y, |x|)\big|_{y=|x|} \right)$$

$$+ \left(f(x), \int_0^{|x|} \mathtt{M}_{xx}(y, |x|)\varphi(y)dy \right) - \left(f'(x) \operatorname{sign} x \frac{1}{2} \int_{|x|}^{\infty} q(\xi)d\xi, \varphi(x) \right)$$

$$= (f'', \varphi) + \left(f(x)\, \mathtt{M}_x(t, |x|)\big|_{t=|x|}, \varphi(x) \right) + \left(\int_{|x|}^{\infty} \mathtt{M}_{tt}(|x|, t) f(t)dt, \varphi(x) \right)$$

$$- \left(f'(x) \operatorname{sign} x \frac{1}{2} \int_{|x|}^{\infty} q(\xi)d\xi, \varphi(x) \right).$$

Taking into account that $q(|x|) = Q(x)$, $x \in \mathbb{R}$, we have

$$((\mathcal{M}f)'', \varphi) = -((\mathcal{M}f)', \varphi')$$

$$= -\left(f'(x) + \operatorname{sign} x \int_{|x|}^{\infty} \mathtt{M}_x(|x|, t) f(t)dt - \operatorname{sign} x f(x) \frac{1}{2} \int_{|x|}^{\infty} q(\xi)d\xi, \varphi'(x) \right)$$

$$= (f'', \varphi) + \left(2\delta(x) \int_0^{\infty} \mathtt{M}_x(0, t) f(t)dt, \varphi(x) \right) + \left(\int_{|x|}^{\infty} \mathtt{M}_{xx}(|x|, t) f(t)dt, \varphi(x) \right)$$

$$- \left(f(x)\, \mathtt{M}_x(|x|, t)\big|_{t=|x|}, \varphi(x) \right) - \left(\delta(x) f(0) \int_0^{\infty} q(\xi)d\xi, \varphi(x) \right)$$

$$- \left(f'(x) \operatorname{sign} x \frac{1}{2} \int_{|x|}^{\infty} q(\xi)d\xi, \varphi(x) \right) + \frac{1}{2}(f(x)Q(x), \varphi(x)).$$

Thus,

$$(\mathcal{M}(f''), \varphi) - ((\mathcal{M}f)'', \varphi) = \left(\int_{|x|}^{\infty} [\mathtt{M}_{tt}(|x|, |t|) - \mathtt{M}_{xx}(|x|, |t|)] f(t)dt, \varphi(x) \right)$$

$$+ \left(f(x) \left[\mathtt{M}_x(t, |x|)\big|_{t=|x|} + \mathtt{M}_x(|x|, t)\big|_{t=|x|} \right], \varphi(x) \right) - \frac{1}{2}(f(x)Q(x), \varphi(x))$$

$$-\left(2\delta(x)\int\limits_0^\infty \mathtt{M}_x(0,t)f(t)dt,\varphi(x)\right)+\left(\delta(x)f(0)\int\limits_0^\infty q(\xi)d\xi,\varphi(x)\right).$$

From (5.9), (5.10) it follows that $\mathtt{M}_{tt}(|x|,|t|)-\mathtt{M}_{xx}(|x|,|t|)=-q(|x|)\mathtt{M}(|x|,t)$ when $|x|<|t|$ and $\mathtt{M}_x(t,|x|)|_{t=|x|}+\mathtt{M}_x(|x|,t)|_{t=|x|}=\mathtt{M}'(|x|,|x|)=-1/2q(|x|)$. Consequently,

$$\begin{aligned}
&(\mathcal{M}(f'')-(\mathcal{M}f)'',\varphi)\\
&=-\left(Q(x)\int\limits_{|x|}^\infty \mathtt{M}(|x|,t)f(t)dt,\varphi(x)\right)-(f(x)Q(x),\varphi(x))\\
&-\left(2\delta(x)\int\limits_0^\infty \mathtt{M}_x(0,t)f(t)dt,\varphi(x)\right)+\left(\delta(x)f(0)\int\limits_0^\infty q(\xi)d\xi,\varphi(x)\right)\\
&=\left(-Q(x)(\mathcal{M}f)(x)-2\delta(x)\int\limits_0^\infty \mathtt{M}_x(0,t)f(t)dt+\delta(x)f(0)\int\limits_0^\infty q(\xi)d\xi,\varphi(x)\right)
\end{aligned}$$

from which the assertion of the lemma follows. The lemma is proved.

Lemma 5.7. *Let $g\in H^1_{0,e}$. Then*

$$\mathcal{M}^{-1}(g'')=(\mathcal{M}^{-1}f)''+\mathcal{M}^{-1}(Qf)-2\delta\int\limits_0^\infty N_x(0,\xi)f(\xi)d\xi-\delta f(0)\int\limits_0^\infty q(\xi)d\xi.$$

P r o o f. Consider any $\psi\in H^1_{0,e}$. As in the previous lemma, using (2.6), (5.30), (5.17) and the evenness of g and ψ, we obtain

$$\begin{aligned}
&(\mathcal{M}^{-1}(g''),\psi)=(g'',\psi)+\left(\int\limits_{|x|}^\infty \mathtt{N}_{tt}(|x|,t)g(t)dt,\psi(x)\right)\\
&+\left(g(x)\,\mathtt{N}_x(t,|x|)|_{t=|x|},\psi(x)\right)+\left(g'(x)\operatorname{sign}x\frac{1}{2}\int\limits_{|x|}^\infty q(\xi)d\xi,\psi(x)\right),\\
&((\mathcal{M}^{-1}g)'',\psi)=(g'',\psi)+\left(2\delta(x)\int\limits_0^\infty \mathtt{N}_x(0,t)g(t)dt,\psi(x)\right)-\frac{1}{2}(g(x)Q(x),\psi(x))
\end{aligned}$$

$$+\left(\int\limits_{|x|}^{\infty}\mathtt{N}_{xx}(|x|,t)g(t)dt,\psi(x)\right)-\left(g(x)\,\mathtt{N}_x(|x|,t)|_{t=|x|}\,,\psi(x)\right)$$
$$+\left(\delta(x)g(0)\int\limits_{0}^{\infty}q(\xi)d\xi,\psi(x)\right)+\left(g'(x)\operatorname{sign}x\frac{1}{2}\int\limits_{|x|}^{\infty}q(\xi)d\xi,\psi(x)\right).$$

Using (5.16), (5.17), we obtain

$$(\mathcal{M}^{-1}(g'')-(\mathcal{M}^{-1}g)'',\psi)=\left(\int\limits_{|x|}^{\infty}[\mathtt{N}_{tt}(|x|,|t|)-\mathtt{N}_{xx}(|x|,|t|)]g(t)dt,\psi(x)\right)$$
$$+\left(g(x)\left[\mathtt{N}_x(t,|x|)|_{t=|x|}+\mathtt{N}_x(|x|,t)|_{t=|x|}\right],\psi(x)\right)+\frac{1}{2}(g(x)Q(x),\psi(x))$$
$$-\left(2\delta(x)\int\limits_{0}^{\infty}\mathtt{N}_x(0,t)g(t)dt,\psi(x)\right)-\left(\delta(x)g(0)\int\limits_{0}^{\infty}q(\xi)d\xi,\psi(x)\right)$$
$$=\left(\int\limits_{|x|}^{\infty}Q(t)\mathtt{N}(|x|,t)g(t)dt,\psi(x)\right)+(g(x)Q(x),\psi(x))$$
$$-\left(2\delta(x)\int\limits_{0}^{\infty}\mathtt{N}_x(0,t)g(t)dt,\psi(x)\right)-\left(\delta(x)g(0)\int\limits_{0}^{\infty}q(\xi)d\xi,\psi(x)\right)$$
$$=\left((\mathcal{M}^{-1}(Qg))(x)-2\delta(x)\int\limits_{0}^{\infty}\mathtt{N}_x(0,t)g(t)dt-\delta(x)g(0)\int\limits_{0}^{\infty}q(\xi)d\xi,\psi(x)\right).$$

The lemma is proved.

Lemma 5.8. *Let $f\in H^0_{0,e}$ such that $f\in L^\infty(\mathbb{R})$. Then $\mathcal{M}f$, $\mathcal{M}^{-1}f\in L^\infty(\mathbb{R})$.*

P r o o f. Using (2.5) and (5.23), we get

$$|\mathcal{M}f|\le|f|\left(1+\left|\int\limits_{|x|}^{\infty}|\mathtt{M}(|x|,t)|dt\right|\right)\le|f|\left(1+\frac{e^{2\sigma_1(0)}}{2}\left|\int\limits_{|x|}^{\infty}\sigma\left(\frac{|x|+t}{2}\right)dt\right|\right)$$
$$\le|f|\left(1+e^{2\sigma_1(0)}\sigma_1(0)\right)<\infty.$$

The assertion about the operator $\mathcal{M}^{-1}$ is proved in a similar way. The lemma is proved.

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