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Homogenization of Spectral Problem on Small-Periodic Networks

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The homogenization of a spectral problem on small-periodic networks with periodic boundary conditions is considered. Asymptotic expansions for eigenfunctions and corresponding eigenvalues on the network are constructed. The theorem is proved which is a justification of the asymptotic expansions for some eigenvalues and eigenfunctions of the problem on the network.

Key words: homogenization, spectral problem, small-periodic network, string cross.

Mathematics Subject Classification 2000: 35B27, 34L20, 35B40.

Introduction

The development of modern technology leads to the research of various problems for differential equations with small parameters. Some problems analysis is based on the homogenization theory and asymptotic methods. In the paper, the homogenization of spectral problems for a second order operator on small-periodic networks is studied. The homogenization of the equation on a small-periodic network was studied in [1], whereas in [2] it was studied on a small-periodic framework with thin bars. The research of differential equations on the networks, which in the more complicated form can be transferred to stratified sets or geometric graphs [3, 4], is a new enough direction. The analysis of the processes in these complex systems leads to the consideration of ordinary second order differential operators on segment systems [5]. The homogenization of a spectral problem on domains was studied in [2, 6–9]. The homogenization of the problems on domains with a high-frequency spectrum was studied in [9].

The spectral problems on networks were considered in [3, 5], and the spectral problems on graphs, in [10].

In the paper, in Secs. 1, 2, we consider the spectral cell problem solved in [11], and the spectral problem on the ε-periodic networks in $\mathbb{R}^2$, where ε is a small positive parameter. The network, spanned on a rectangle, one side of which is defined by a real l and the other is equal to the unit, is constructed as a union of the same εY-periodic string crosses. According to [3], we can define the function spaces on the string cross by identifying it with a stratified set or a geometrical graph. In Sec. 2, the main theorem of the paper is formulated. In Sec. 3, we define a derivative for the network according to [1] and construct the asymptotic expansions by following the principle of homogenization from [12]. The expansions are approximate solutions for the spectral problem on the ε-periodic network. In the last section we provide the asymptotic expansion justification which is the proof of the main theorem.

1. Statement of the Cell Problem

Let Y be a union of four closed stretched strings that are in the rectangle $Q = [0, 1] \times [0, l]$, where Q is a subset of $\mathbb{R}^2$ and l is a fixed positive real number. Suppose that the network of such rectangles with the same set Y covers $\mathbb{R}^2$. The totality of all Y is called the network G. The set Y is called the periodical recurring cell. We denote by $\sigma_1, \sigma_2, \sigma_3$ and σ_4 the corresponding cell strings, whereas strings ends are called nodes. We define $T_{\partial Y}$ as nodes that are disposed on the rectangle boundary ∂Q intersecting with Y. For example, the points $\sigma_{01} = (0, l/2)$, $\sigma_{02} = (1, l/2)$, $\sigma_{03} = (1/2, 0)$ and $\sigma_{04} = (1/2, l)$ are the nodes for the string cross Y (Fig. 1). We define T_Y as the set of all nodes for the cell Y. Let T be the set of all nodes for the network G.

One of the cases of the union of strings in the periodically recurring cell is the string cross described in [5] and [11]. The strings of the cross under consideration are supposed to be homogeneous segments, which are right angle arranged and have a unit tension and mass distribution density [5]. Following [11], the two-dimensional string cross, which is represented as a pencil of four strings, can be considered as a string cross formed by two strings $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ connected in the complementary midpoint. For the statement of the problem on Y, we define the function spaces for the string cross with string lengths, which equal 1 and l (Fig. 1), by using the simple case of the problem for stratified sets that are considered, for example, in [3].

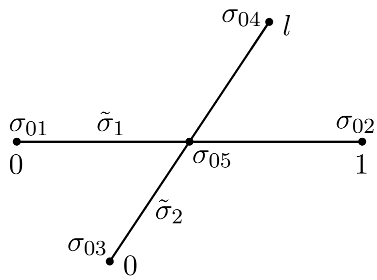

Fig. 1. The Y-periodic string cross.

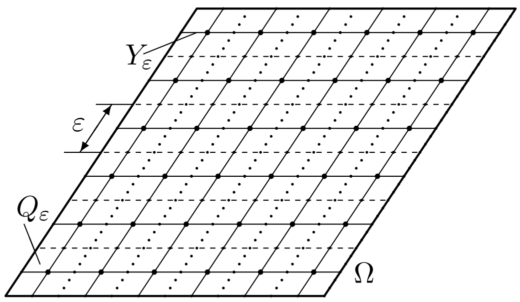

Fig. 2. The small-periodic network.

Let $C(Y)$ be the set of the functions $u : Y \to \mathbb{R}$, which are restrictions on Y of continuous functions defined on the rectangle Q. These functions define continuous restrictions to each string $\tilde{\sigma}_i$ denoted as u_i for $i = 1, 2$. An integral of the function is defined by the following equality:

$$\int\limits_Y u\,dy = \sum_{i=1,2} \int\limits_{\tilde{\sigma}_i} u_i\,dy_i,$$

where the right-hand side is the sum of the Riemann integrals over standard measures on the strings with the standard parametrizations $y_1 \in [0, 1]$ and $y_2 \in [0, l]$.

Thus, the function u on the string cross can be considered as a pair of continuous restrictions of the function to each string of the cross denoted by u_1 and u_2. Therefore, the integral is defined as the sum of the string integrals. However, it is useful to connect the cell integral with the "volume" of this cell. For example, we define the cross integral of the unit function as l for the Y-periodic string cross (Fig. 1). Thus, we introduce a normalizing factor $l/(l + 1)$ and denote

$$\int\limits_Y u(y)\,dy = \frac{l}{l+1}\left(\int\limits_0^1 u_1(y_1)\,dy_1 + \int\limits_0^l u_2(y_2)\,dy_2\right). \tag{1}$$

Let $C^1(Y)$ be the set of functions $u : Y \to \mathbb{R}$, which are the restrictions to Y of the functions from $C^1(Q)$ that are defined on the rectangle Q. These functions define the $C^1(\tilde{\sigma}_i)$ restrictions u_i to each string $\tilde{\sigma}_i$ for $i = 1, 2$. Let $L^2(Y)$ be the completion of the space $C(Y)$ with respect to the norm induced by the inner product

$$(u, v)_{L^2(Y)} = \int\limits_Y uv\,dy.$$

The function space $H^1(Y)$ is the completion of $C^1(Y)$ with respect to the norm $\|\cdot\|_{H^1(Y)}$ induced by the inner product

$$\langle u, v\rangle_{H^1(Y)} = \int\limits_Y uv\,dy + \int\limits_Y (\partial_y u)(\partial_y v)\,dy. \tag{2}$$

Taking into account (1), we can rewrite the last equality in details in the form

$$\langle u, v\rangle_{H^1(Y)} = \frac{l}{l+1}\left(\int\limits_0^1 u_1 v_1\,dy_1 + \int\limits_0^l u_2 v_2\,dy_2 + \right.$$
$$\left. + \int\limits_0^1 (\partial_{y_1} u_1)(\partial_{y_1} v_1)\,dy_1 + \int\limits_0^l (\partial_{y_2} u_2)(\partial_{y_2} v_2)\,dy_2\right).$$

To be precise, the last equality defines the integral in (2), since the given function $u \in C^1(Y)$ can be considered as a pair of functions $u_1(y_1)$ and $u_2(y_2)$ defined on the segments $[0,1]$ and $[0,l]$, respectively, because the coordinates on Y can be defined with more complexity (further details are to be found in [1, 3–5].

We denote by $C^1_{per}(Y)$ the space of functions $u \in C^1(Y)$ satisfying the periodicity condition

$$u(y_i) = u(y_i + l_i), \qquad \partial_{y_i} u(y_i) = \partial_{y_i} u(y_i + l_i),$$

where $y \in T_{\partial Y}$, $i = 1, 2$, $l_1 = 1$ and $l_2 = l$. We mean that the given equalities hold for $u_1(y_1)$ and $u_2(y_2)$, which are defined by $u \in C^1(Y)$. We denote by $H^1_{per}(Y)$ the completion of the periodic function space $C^1_{per}(Y)$ with respect to the norm $\|u\|_{H^1(Y)} = \int_Y u^2 dy + \int_Y (\partial_y u)^2\,dy$.

Consider the Y-periodic spectral cell problem: find $u \in H^1_{per}(Y)$ such that

$$\begin{gathered} -\partial_y^2 u(y) = \lambda u(y) \quad \text{for} \quad y \in Y; \\ u(y_i) = u(y_i + l_i), \quad \partial_{y_i} u(y_i) = \partial_{y_i} u(y_i + l_i) \quad \text{for} \quad y \in T_{\partial Y}, \end{gathered} \tag{3}$$

where $\|u\|_{L^2(Y)} = 1$, $i = 1, 2$, $l_1 = 1$ and $l_2 = l$, with the conditions of function and flux continuity in string nodes for u_1 and u_2, which are defined by u. The conditions have the following form:

$$u_1\left(\frac{1}{2}\right) = u_2\left(\frac{l}{2}\right),$$

$$\partial_{y_1} u_1\left(\frac{1}{2} + 0\right) - \partial_{y_1} u_1\left(\frac{1}{2} - 0\right) + \partial_{y_2} u_2\left(\frac{l}{2} + 0\right) - \partial_{y_2} u_2\left(\frac{l}{2} - 0\right) = 0.$$

To be definite, the first equation of (3) can be written as two equations

$$-\partial_{y_1}^2 u_1(y_1) = \lambda\, u_1(y_1) \ \text{ for } \ y_1 \in [0,1], \quad -\partial_{y_2}^2 u_2(y_2) = \lambda\, u_2(y_2) \ \text{ for } \ y_2 \in [0,l],$$

for u_1 and u_2, which are defined by u. It agrees with the definition of the derivative ∂_y by equality (2). We remark that the continuity and periodicity conditions are fulfilled automatically for $u \in C^1_{per}(Y)$ (and, in the known sense [1], for $u \in H^1_{per}(Y)$). Moreover, the solutions of problem (3) are smooth according to [11], where this problem was solved and the eigenvalues $\lambda^m = 4\pi^2 m^2$ with $m \in \mathbb{Z}^+ = \{0, 1, 2 \ldots\}$ and the corresponding eigenfunctions $N^0(y), N^1(y), \ldots$, which are combinations of $\cos 2\pi y_1 m$ and $\sin 2\pi y_2 m$ on each string, were obtained. In this paper, we consider only the eigenvalue $\lambda^0 = 0$ of problem (3) with the eigenfunction $N^0(y) = l^{-1/2}$ that means $N^0_1(y_1) = l^{-1/2}$ and $N^0_2(y_2) = l^{-1/2}$.

2. Problem on Networks

We shrink N times the rectangle Q and the string cross Y, where N is a given natural number. As a result, we obtain the sets Q_ε and Y_ε with coordinates $x' = \varepsilon\, y$ for $\varepsilon = 1/N$. If we repeat the rectangles Q_ε by periodicity, then we obtain a closed domain $\Omega = [0,1] \times [0,l] \subset \mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega$. The small-periodic network G_ε spans this domain in $\mathbb{R}^2$ (Fig. 2) and it is the union of N^2 string cross Y_ε. This Y_ε we call periodically recurring cell with edge lengths that are equal to ε and $l\varepsilon$. The parameter x' denotes the position of a point on the network G_ε. We denote by T_ε the whole node set of the network G_ε.

Similar networks with arbitrary arcs instead of stretched strings, which have the unit span and mass distribution density, was considered in [1]. Therefore, we will use some notation and calculations from this paper. According to [1], on the network G_ε, we define $H^1(G_\varepsilon)$ as the set of functions, which are continuous in nodes and absolutely continuous on each string, with the norm

$$\|u\|^2_{H^1(G_\varepsilon)} = \varepsilon \int\limits_{G_\varepsilon} \left(u^2 + (\partial_{x'} u)^2 \right) dx'. \tag{4}$$

The norm definition is equivalent to the norm definition induced by inner product (2), which is defined for G_ε. The space $H^1_{per}(G_\varepsilon) \subset H^1(G_\varepsilon)$ of periodic functions on $G_\varepsilon \cap \partial\Omega$ is defined similarly to the space on Y.

Consider the following spectral boundary problem for the network G_ε: find $u_\varepsilon \in H^1_{per}(G_\varepsilon)$ such that $\|u_\varepsilon\|_{L^2(G_\varepsilon)} = 1$ and

$$-\varepsilon^2 \partial^2_{x'} u_\varepsilon(x') = \lambda_\varepsilon u_\varepsilon(x') \quad \text{for} \quad x' \in G_\varepsilon \cap \Omega, \tag{5}$$

$$\begin{gathered} u_\varepsilon(x'_1, x'_2) = u_\varepsilon(x'_1 + 1, x'_2), \quad u_\varepsilon(x'_1, x'_2) = u_\varepsilon(x'_1, x'_2 + l), \\ \partial_{x'} u_\varepsilon(x'_1, x'_2) = \partial_{x'} u_\varepsilon(x'_1 + 1, x'_2), \\ \partial_{x'} u_\varepsilon(x'_1, x'_2) = \partial_{x'} u_\varepsilon(x'_1, x'_2 + l) \quad \text{for} \quad x' \in G_\varepsilon \cap \partial\Omega. \end{gathered} \tag{6}$$

This means that the coordinates x_1' and x_2' can change along the parallel lines, which are obtained by periodic extensions of the strings $\varepsilon\tilde{\sigma}_1$ and $\varepsilon\tilde{\sigma}_2$. To be precise, it is helpful to consider the function $u_\varepsilon \in H^1_{per}(G_\varepsilon)$ as a set of $2N$ functions $u^\varepsilon_{1j}(x'_{1j})$ and $u^\varepsilon_{2j}(x'_{2j})$ that are defined on the lines. The lines are periodic extensions of the strings $\varepsilon\tilde{\sigma}_1$ and $\varepsilon\tilde{\sigma}_2$, which are parametrized by coordinates $x'_{1j} \in [0,1]$ and $x'_{2j} \in [0,l]$ for $j = 1,\dots,N$. In this case, periodicity condition (6) and the function and flux continuity conditions in nodes of the network G_ε, which are defined for the functions $u^\varepsilon_{1j}(x'_{1j})$ and $u^\varepsilon_{2j}(x'_{2j})$, hold automatically, since $u_\varepsilon \in H^1_{per}(G_\varepsilon)$. Thus, problem (5)–(6) reduces to the $2N$ elliptical equations for the functions $u^\varepsilon_{1j}(x'_{1j})$ and $u^\varepsilon_{2j}(x'_{2j})$ with $j = 1,\dots,N$,

$$-\varepsilon^2\partial^2_{x'_{1j}}(u^\varepsilon_{1j}) = \lambda\, u^\varepsilon_{1j} \ \text{ for } \ x'_{1j} \in [0,1], \quad -\varepsilon^2\partial^2_{x'_{2j}}(u^\varepsilon_{2j}) = \lambda\, u^\varepsilon_{2j} \ \text{ for } \ x'_{2j} \in [0,l].$$

Besides, the functions u^ε_{1j} and u^ε_{2j} are smooth enough due to the ellipticity of these equations.

Problem (5)–(6) has the trivial solution $u_\varepsilon = l^{-1/2}$ for the eigenvalue $\lambda_\varepsilon = 0$. In order to exclude this solution from further consideration, we define the space $H^1_{per*}(G_\varepsilon) = \left\{u \in H^1_{per}(G_\varepsilon) : (u,1)_{L^2(G_\varepsilon)} = 0\right\}$ with the norm

$$\|u\|^2_{H^1_*(G_\varepsilon)} = \varepsilon \int\limits_{G_\varepsilon} (\partial_{x'}u)^2\, dx'$$

that is equivalent to norm (4) on $H^1_{per}(G_\varepsilon)$ by virtue of Poincare's inequality [3]. We define $L^2_*(G_\varepsilon) = \left\{u \in L^2(G_\varepsilon) : (u,1)_{L^2(G_\varepsilon)} = 0\right\}$ in the same way. Thus, the problem (5)–(6) can be considered as variational in the sense of integral identity: find $u \in H^1_{per*}(G_\varepsilon)$ such that

$$\varepsilon^2 \int\limits_{G_\varepsilon} (\partial_{x'}u)(\partial_{x'}v)\, dx' = \lambda_\varepsilon \int\limits_{G_\varepsilon} uv\, dx' \qquad \forall v \in H^1_{per*}(G_\varepsilon).$$

By the definition, the operator of problem (5)–(6) is negative definite and has a compact inverse operator on $L^2_*(G_\varepsilon)$ for a fixed ε (with $0 < \varepsilon \le \varepsilon_0$) which follows from the embedding compactness $H^1_{per*}(G_\varepsilon) \subset L^2_*(G_\varepsilon)$ [4]. Thus, there are countable sets of eigenvalues λ^1_ε, λ^2_ε, $\dots$ and orthonormalized eigenfunctions u^1_ε, $u^2_\varepsilon, \dots$ for the problem such that $\alpha\varepsilon^2 \le \lambda^1_\varepsilon \le \dots \le \lambda^s_\varepsilon \le \dots$ with the multiplicity being taken into account, where α is some positive constant and $\lim_{s\to\infty} \lambda^s_\varepsilon = \infty$.

The main aim of the paper is to construct and justify the asymptotic expansions for eigenvalues and eigenfunctions of problem (5)–(6) with insufficiently large numbers ($s \ll \varepsilon^{-2}$). The main statement of the paper is the following theorem.

Theorem 1. *For eigenvalues* λ_ε^s *and eigenfunctions* u_ε^s *of problem* (5)–(6) *there exists a constant* C *independent of* ε *and* s *such that*

$$|\lambda_\varepsilon^s - \varepsilon^2\lambda^s| \le C\varepsilon^3(\lambda^s)^{3/2} \quad \textit{and} \quad \|u_\varepsilon^s - v^s\|_{L^2(G_\varepsilon)} \le C\varepsilon(\lambda^s)^{1/2}$$

for $\lambda^s \ll \varepsilon^{-2}$ *and* $0 < \varepsilon \le \varepsilon_0$*, where* λ^s *and* v_s *are some eigenvalue and eigenfunction of the relevant homogenization problem which is to be determined later.*

Theorem 1 will be proved in Sec. 4. The estimate of this theorem is valid for all λ^s and u_ε^s such that $\lambda^s \le c\,\varepsilon^{-2+\sigma}$ with a positive constant c, where $0 < \sigma \le 2$ and $0 < \varepsilon \le \varepsilon_0$ (this means that $\lambda^s \ll \varepsilon^{-2}$ precisely). However, the proof of the theorem may be incorrect when $\lambda^s = c\,\varepsilon^{-2}$. This situation is natural, since there is a high-frequency spectrum for problem (5)–(6). The spectrum will be introduced and justified in further researches. The initial component construction of asymptotic expansions for λ_ε^s and u_ε^s as well as the homogenized problem will be considered in the next section. In order to construct the initial component of the asymptotic expansions for the solutions to problem (5)–(6), we will follow the principles of homogenization introduced in [12].

3. Construction of Asymptotic Expansions

We define the derivative of functions on the network in accordance with [1]. If a function $v(x, y)$ is an element of $C^1\left(\overline{\Omega}, H^1(Y\backslash T_Y)\right)$ (where $C^k\left(\overline{\Omega}, W\right)$ denotes the space of abstract functions on the domain $\overline{\Omega}$ with values in some Hilbert space W), then the following relation is valid:

$$\partial_{x'}v(x', \varepsilon^{-1}x') = \left(\nabla v(x, y) + \varepsilon^{-1}\partial_y v(x, y)\right)\big|_{x=x',\ y=x'/\varepsilon}, \tag{7}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$ and $\partial_y = (\partial/\partial y_1, \partial/\partial y_2)$. Notice that sometimes it is more convenient to consider the function on the string cross Y as two functions on the corresponding strings.

We will construct an expansion of the eigenfunction u_ε for problem (5)–(6) in the form of asymptotic sum (see [12] for more details). The components of the asymptotic sum are functions of the form $u_0(x, y), u_1(x, y), \ldots,$ where $(x, y) \in Q \times Y$ and $x = x',\ y = x'/\varepsilon$. The functions have separated variables and are Y-periodic in second variable. The asymptotic sum is in the following form:

$$u_a(x', \frac{x'}{\varepsilon}) = u_0(x', \frac{x'}{\varepsilon}) + \varepsilon u_1(x', \frac{x'}{\varepsilon}) + \varepsilon^2 u_2(x', \frac{x'}{\varepsilon}). \tag{8}$$

Consequently, the expansion for eigenvalue λ_ε of problem (5)–(6) should be of the form

$$\lambda_a = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2. \tag{9}$$

The function $u_a(x, y)$ is Y-periodic by the definition. Therefore, we can consider this function as the function on Y for fixed $x \in \Omega$, if necessary. Moreover, it is helpful to consider the function as two functions $u_a^1(x, y_1)$ and $u_a^2(x, y_2)$ on the corresponding strings. Using (7), we can rewrite equation (5) in the following form:

$$-\left\{\partial_y^2 + 2\varepsilon^1 \nabla \partial_y + \varepsilon^2 \nabla^2\right\} u_a(x, y) = \lambda_a u_a(x, y), \tag{10}$$

where $y = x'/\varepsilon,\ \ x = x'$, and the notation

$$\partial_y^2 = \left(\frac{\partial^2}{\partial y_1^2}, \frac{\partial^2}{\partial y_2^2}\right), \quad \nabla \partial_y = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}\frac{\partial}{\partial y_2}\right), \quad \nabla^2 = \left(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}\right)$$

is used for convenience.

Substituting asymptotic sum (8), (9) to (10) and setting equal coefficients with the same powers of ε, we obtain the equations for the functions u_0, u_1 and u_2. Taking into account [12], it is useful to choose the asymptotic expansion components in the form $u_i = N_i\,(y)\, v_i(x)$, where $N_i \in H^1_{per}(Y)$. As stated before, we consider only the eigenvalue $\lambda_0 = 0$ for cell problem (3) with the eigenfunction $N^0(y) = l^{-1/2}$.

Thus, setting equal coefficients for ε^0, we obtain the equation

$$-\partial_y^2 u_0 = 0,$$

which is a cell problem. Thus, we can choose $u_0 = A\, v(x)$, where the constant A and the function $v(x)$ are defined later. Next, we consider the equation for ε^1. The equation has the form

$$-\partial_y^2 u_1 = \lambda_1 u_0,$$

since $\lambda_0 = 0$ and $\partial_y \nabla u_0 = 0$ due to the definition of u_0. It is well known [1] and may be directly verified that the equation for $w \in H^1_{per}(Y)$ such that

$$-\partial_y^2 w(y) = g(y)$$

with a given function $g \in L^2(Y)$ is solvable if and only if the right-hand side is orthogonal to $N^0(y)$. The equation for ε^1 satisfies the condition only if $\lambda_1 = 0$ for $v(x) \neq 0$. The function $u_1 = A\, v_1(x)$ is a periodic solution of the equation, where the function $v_1(x)$ is chosen to be null for further simplicity.

Setting equal coefficients for ε^2, we obtain the cell equation which, by the definition, may be considered as the system of equations

$$-\partial_{y_1}^2 u_2^1(y_1, x) = A\partial_{x_1}^2 v + \lambda_2 A v,$$
$$-\partial_{y_2}^2 u_2^2(y_2, x) = A\partial_{x_2}^2 v + \lambda_2 A v.$$

Calculating the inner product of the right-hand side of the equation with the kernel element $N^0(y)$ of the cell equation and setting the product to be equal to zero, we get

$$A\int\limits_Y \left(\nabla^2 v(x) + \lambda_2 v(x)\right) N^0(y)\, dy = 0.$$

By the integration rule (1), for $A \neq 0$, we get the spectral homogenized problem for $v(x)$

$$\begin{gathered}\partial^2_{x_1} v(x) + l\partial^2_{x_2} v(x) + (l+1)\,\lambda_2 v(x) = 0 \quad \text{for} \quad x \in \Omega, \\ v(x) = v(x+l_i), \quad \partial_x v(x) = \partial_x v(x+l_i) \quad \text{for} \quad x \in \partial\Omega,\end{gathered} \tag{11}$$

where the periodicity conditions are conformed to (5)–(6), $v(x)$ is normalized by the condition $\|v\|_{L^2(\Omega)} = 1$, and $l_1 = 1$, $l_2 = l$. The solutions of the spectral problem are determined by the countable sets of eigenvalues

$$\lambda^s = 4\pi^2(l+1)^{-1}(n^2 + m^2 l^{-1})$$

for $m, n \in \mathbb{N} = \{1, 2, \dots\}$ to be such arranged that $0 < \lambda^1 \leq \lambda^2 \leq \dots$ (with the multiplicity being taken into account, which may be 2, 4 or 8 depending on l) and the corresponding eigenfunctions $v^s_0(x)$ have the form

$$\begin{gathered}2l^{-1/2}\cos 2\pi n x_1 \cos 2\pi m x_2 l^{-1}, \quad 2l^{-1/2}\sin 2\pi n x_1 \cos 2\pi m x_2 l^{-1}, \\ 2l^{-1/2}\cos 2\pi n x_1 \sin 2\pi m x_2 l^{-1} \quad \text{or} \quad 2l^{-1/2}\sin 2\pi n x_1 \sin 2\pi m x_2 l^{-1}\end{gathered}$$

for $m, n \in \mathbb{N}$. It is known [13, 14] that $\lambda^s = 4\pi s\, l^{-1} + O(s^{1/2})$ for large s.

Thus, fixing λ^s, v^s_0 and substituting the homogenized equation (11) in the equation for ε^2, we can define $A = 1$ and

$$u^s_0 = v^s_0(x), \qquad u^s_2 = N_2(y)\left(\partial^2_{x_1} v^s_0(x) + \lambda^s v^s_0(x)\right),$$

where $N_2(y) \in H^1_{per}(Y)$ satisfies the system of the equations

$$-\partial^2_{y_1} N^1_2(y_1) = 1, \qquad -\partial^2_{y_2} N^2_2(y_2) = -l^{-1}.$$

There exists a solution of the system (that is determined up to the constant function $AN^0(y)$) and the solution is normalized such that $\int_Y N^0(y) N_2(y) dy = 0$. Thus, we get the eigenvalue and the eigenfunction asymptotics in the following form:

$$\lambda^s_a = \varepsilon^2 \lambda^s, \qquad u^s_a(x, y) = v^s_0(x) + \varepsilon^2 N_2(y)\left(\partial^2_{x_1} v^s_0(x) + \lambda^s v^s_0(x)\right).$$

In what follows, we assume that $N_2(y)$ is periodically extended to the whole network. Thus, the function $u^s_a(x', \frac{x'}{\varepsilon})$ is well defined on G_ε. Moreover, the function $u_a^s(x', \frac{x'}{\varepsilon})$ satisfies automatically all periodicity conditions (6) and the function and flux continuity conditions in string nodes of the network G_ε, since the smooth function $v_0^s(x)$ is defined on the rectangle Ω and satisfies the periodic conditions on Ω whereas the function $N_2(y)$ is an element of $H^1_{per}(Y)$ and is regular enough on Y as a solution of the elliptical equation on the cross [1, 3].

4. Asymptotics Justification

Before justifying the asymptotic expansions, we must return to the network definition. We enumerate all N^2 cells Q_ε^{ij} and the same number of string crosses Y_ε^{ij} by the numbers $i, j = 1, \dots, N$. Then, for the whole domain $\Omega = \bigcup_{ij} Q_\varepsilon^{ij}$ and the network $G_\varepsilon = \bigcup_{ij} Y_\varepsilon^{ij}$, we obtain the equalities

$$\int_\Omega v(x)\,dx = \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{Q_\varepsilon^{ij}} v(x)\,dx, \qquad \int_{G_\varepsilon} v(x')\,dx' = \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{Y_\varepsilon^{ij}} v(y)\,dy,$$

where for the cell integral and the cross integral we have

$$\int_{Q_\varepsilon^{ij}} v(x)\,dx = \int_{\varepsilon(i-1)}^{i\varepsilon} \int_{l\varepsilon(j-1)}^{l\varepsilon j} v(x_1, x_2)\,dx_1 dx_2$$

and

$$\int_{Y_\varepsilon^{ij}} v(y)\,dy = \frac{l}{1+l}\left(\int_{\varepsilon(i-1)}^{i\varepsilon} v(y_1, jl\varepsilon - 2^{-1}l\varepsilon)\,dy_1 + \int_{l\varepsilon(j-1)}^{l\varepsilon j} v(i\varepsilon - 2^{-1}\varepsilon, y_2)\,dy_2 \right),$$

respectively.

We prove now the statement on the change of integrals over the small-periodic network G_ε by integrals over $\Omega = [0,1] \times [0,l]$ which we will need later.

Proposition 1. *For $v \in H^2(\Omega)$ there exists a constant C independent of ε such that the following inequality holds:*

$$\left| \int_\Omega v(x)\,dx - \varepsilon \int_{G_\varepsilon} v(x')\,dx' \right| \le C\varepsilon^2.$$

P r o o f. Consider the linear functional

$$l(v) = \int_\Omega v(x)dx - \varepsilon \int_{G_\varepsilon} v(x')dx'.$$

By the definition,

$$l(v) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int\limits_{Q_\varepsilon^{ij}} v(x)\,dx - \varepsilon \int\limits_{Y_\varepsilon^{ij}} v(y)\,dy\right).$$

We estimate the functional in brackets using the following lemma from [15].

Lemma (Bramble–Hilbert). *Suppose Ω is an open convex bounded domain in $\mathbb{R}^n$ and the linear functional $l(u)$ is bounded on $H^{m+1}(\Omega)$, where $m \in \mathbb{Z}^+$ is fixed, that is*

$$|l(u)| \le M\|u\|_{H^{m+1}(\Omega)}.$$

If $l(u)$ is equal to zero for every polynomial in variables $x_1, x_2, \ldots, x_n$ of degree m, then there exists a constant $\overline{M}$ dependent only on Ω such that the following inequality holds:

$$|l(u)| \le M\overline{M}|u|_{H^{m+1}(\Omega)},$$

where $|u|_{H^{m+1}(\Omega)} = \|\nabla^{m+1}u\|_{L_2(\Omega)}$ is a standard seminorm for $H^{m+1}(\Omega)$.

Before using the embedding estimate for $H^2(Q_\varepsilon^{ij}) \subset C(\overline{Q_\varepsilon^{ij}})$ with some constant independent of ε, we change the variable $y = x/\varepsilon$ and introduce the notation $\widetilde{v}(y) = v(\varepsilon y)$. Thus we get

$$l^{ij}(v) = l^{ij}(\widetilde{v}) = \varepsilon^2\left(\int\limits_{Q^{ij}} \widetilde{v}(y)\,dy - \int\limits_{Y^{ij}} \widetilde{v}(y)\,dy\right).$$

Taking into account the embedding estimate $\max\limits_{y\in Q^{ij}} |\widetilde{v}(y)| \le C\|\widetilde{v}(y)\|_{H^2(Q^{ij})}$ for the domain Q^{ij}, we have

$$|l^{ij}(\widetilde{v})| \le C\varepsilon^2\|\widetilde{v}(y)\|_{H^2(Q^{ij})}.$$

It is verified directly that the functional $l(\widetilde{v})$ becomes zero on the first-degree polynomials. In this case, by the Bramble–Hilbert lemma, we have the relevant estimate. In this estimate, we return again to the variable x and obtain

$$|l^{ij}(v)| \le M\varepsilon^2|\widetilde{v}(y)|_{H^2(Q^{ij})} = M\varepsilon^3|v(x)|_{H^2(Q_\varepsilon^{ij})}.$$

Therefore, we have

$$|l(v)| \le M\varepsilon^3\sum_{i=1}^{N}\sum_{j=1}^{N}|v(x)|_{H^2(Q_\varepsilon^{ij})} = M\varepsilon^3 N\left(\sum_{i=1}^{N}\sum_{j=1}^{N}|v(x)|^2_{H^2(Q_\varepsilon^{ij})}\right)^{1/2}$$
$$= M\varepsilon^2|v(x)|_{H^2(\Omega)}.$$

Thus, the integral change leads to the error $O(\varepsilon^2)$, and that is just what was to be proved.

This analogue of the Riemann–Lebesgue lemma will be used later on.

Proposition 2. *Let $U \in L_2(Y)$ be periodically extended to the whole network functions and $v \in H^2_{per}(\Omega)$ be such that $\| \partial^2_{x'} v\|_{L^2(G_\varepsilon)} \le c$ with the constant c independent of ε. Then the following inequality holds*:

$$\left| \varepsilon \int\limits_{G_\varepsilon} U\left(\frac{x'}{\varepsilon}\right) v(x')\,dx' - l^{-1} \int\limits_Y U(y)\,dy \int\limits_\Omega v(x)\,dx \right| \le C\varepsilon^2,$$

where the constant C is independent of ε.

P r o o f. Let $M(y) \in H^2_{per*}(Y)$ be a solution of the equation

$$\partial^2_y M(y) = U(y) - l^{-1}\int\limits_Y U(y)dy$$

and $M(y)$ be periodically extend to the whole network. Using the equality

$$\varepsilon^2 \partial^2_{x'} M\left(\frac{x'}{\varepsilon}\right) = \partial^2_y M(y)\big|_{y=\frac{x'}{\varepsilon}} = U\left(\frac{x'}{\varepsilon}\right) - l^{-1}\int\limits_Y U(y)dy,$$

we multiply it by the function $v(x)$ and integrate the result over the small-periodic network G_ε. Thus, we obtain

$$\varepsilon^3 \int\limits_{G_\varepsilon} \partial^2_{x'} M\left(\frac{x'}{\varepsilon}\right) v(x')\,dx' = \varepsilon \int\limits_{G_\varepsilon} U\left(\frac{x'}{\varepsilon}\right) v(x')\,dx' - \varepsilon l^{-1}\int\limits_Y U(y)dy \int\limits_{G_\varepsilon} v(x')\,dx'.$$

Consider the left-hand side of the equality separately. Integrating by parts twice, we use the Cauchy–Bunyakovsky inequality. Taking into account the function periodicity over the network, we get

$$\varepsilon^3 \int\limits_{G_\varepsilon} \partial^2_{x'} M\left(\frac{x'}{\varepsilon}\right) v(x')\,dx' = \varepsilon^3 \int\limits_{G_\varepsilon} M\left(\frac{x'}{\varepsilon}\right) \partial^2_{x'} v(x')\,dx'$$

$$\le \varepsilon^2 \sqrt{\varepsilon \int\limits_{G_\varepsilon} M^2\left(\frac{x'}{\varepsilon}\right) dx'}\ \sqrt{\varepsilon \int\limits_{G_\varepsilon} \left(\partial^2_{x'} v(x')\right)^2 dx'} \le c\,\varepsilon^2 \|M\|_{L^2(Y)},$$

since

$$\|M\left(\frac{x'}{\varepsilon}\right)\|^2_{L^2(G_\varepsilon)} = \sum_{i=1}^N \sum_{j=1}^N \varepsilon \int\limits_{Y^{ij}_\varepsilon} M^2\left(\frac{x'}{\varepsilon}\right) dx' = \sum_{i=1}^N \sum_{j=1}^N \varepsilon^2 \|M\|^2_{L^2(Y)} = \|M\|^2_{L^2(Y)}.$$

Using Proposition 1, we get the estimate

$$\left| \varepsilon \int\limits_{G_\varepsilon} U\left(\frac{x'}{\varepsilon}\right) v(x')\, dx' - l^{-1} \int\limits_{Y} U(y)\, dy \int\limits_{\Omega} v(x)\, dx \right| \le \varepsilon^2 C.$$

Thus, we obtained the error $O(\varepsilon^2)$ that is defined by the values $\|M\|_{L^2(Y)}$, $\|\partial^2_{x'} v\|_{L^2(G_\varepsilon)}$ and $\|\partial^2_x v\|_{L^2(\Omega)}$ (by Proposition 1). This concludes the proof.

Here we use $H^2_{per*}(Y)$, $H^2_{per}(\Omega)$, ...that are defined in the standard way according to Sec. 1, [3, 15]. In what follows, the constants independent of ε and s are denoted by C, although the constants may be different in different formulas.

P r o o f of Theorem1. The justification of the asymptotic expansions is realized by the minimax principle, the Rayleigh–Ritz method used from [14] and the known Vishik–Lyusternik theorem from [13, 16]. We recall the exact formulations of the statements.

Suppose H^1 and H^0 are Hilbert spaces. The space H^1 is embedded compactly in H^0, and H^{-1} is a dual space for H^1 with respect to the inner product of H^0. The operator $L : H^1 \to H^{-1}$ is continuous and such that $\langle Lv, v\rangle_{H^0} \ge \alpha \|v\|^2_{H^0}$ $\forall\, v \in H^1$, where α is a positive constant, and $L^* = L$. Then the eigenvalues λ_k of the operator L are real and define a nondecreasing sequence $\alpha \le \lambda_1 \le \lambda_2 \le ...$ (with the multiplicity being taken into account), and the eigenfunctions $u_1, u_2, \dots$ are orthonormalized in H^0. For the operator L, the following statements hold.

Theorem 2 (Minimax principle). *Let λ_k and u_k be (ordered) eigenvalues and eigenfunctions of the operator L. Then the following equalities hold:*

$$\lambda_1 = \min_{\substack{v \in H^0 \\ v \ne 0}} \frac{\langle Lv, v\rangle_{H^0}}{\|v\|^2_{H^0}}, \quad \lambda_k = \min_{\substack{v \in H^0_k \\ v \ne 0}} \frac{\langle Lv, v\rangle_{H^0}}{\|v\|^2_{H^0}},$$

where $H^0_k = \left\{ v \in H^0 : \langle v, u_1\rangle_{H^0} = 0, \dots, \langle v, u_{k-1}\rangle_{H^0} = 0 \right\}$ for $k = 2, 3, \dots$.

Theorem 3 (Rayleigh–Ritz method). *Let H_d be a d-dimensional subspace of H^0 and P_d be an orthogonal projector onto H_d with some natural number d. Then for the ordered eigenvalues $\mu_1, \dots, \mu_d$ of the operator $P_d L P_d$ on H_d and the eigenvalues $\lambda_1, \dots, \lambda_d$ of the operator L the following inequalities hold:*

$$\lambda_1 \le \mu_1, \; \dots, \; \lambda_d \le \mu_d.$$

Theorem 4 (Vishik–Lyusternik). *Let λ_k and u_k be eigenvalues and eigenfunctions of the operator L. Assume that there exists a real $\mu \in \mathbb{R}$ and $u \in H^1$ such that $\|u\|_{H^0} = 1$ and*

$$\|Lu - \mu u\|_{H^0} \le \beta.$$

Then there exists λ_s such that $|\mu - \lambda_s| \le \beta$, and for every $\sigma > \beta$ there exists $\widetilde{u} \in H^1$ such that $\|\widetilde{u}\|_{H^0} = 1$ and

$$\|u - \widetilde{u}\|_{H^0} \le 2\beta\sigma^{-1},$$

where $\widetilde{u}$ is a linear combination of the eigenvectors of the operator L, which correspond to eigenvalues from the interval $(\mu - \sigma, \mu + \sigma)$.

Applying the operator $L = -\varepsilon^2 \partial_{x'}^2$ to the approximate function $u_a^s = u_0^s + \varepsilon^2 u_2^s$ and using the differentiation rule (7), we obtain

$$\begin{aligned} Lu_a^s\left(x', x'/\varepsilon\right) &= -\varepsilon^2 \nabla^2 u_0^s - \varepsilon^2 \partial_y^2 u_2^s - 2\varepsilon^3 \nabla \partial_y u_2^s - \varepsilon^4 \nabla^2 u_2^s \\ = \varepsilon^2 \lambda^s u_0^s + \varepsilon^4 \lambda^s u_2^s &+ \varepsilon^3 K^s = \varepsilon^2 \lambda^s u_a^s\left(x', x'/\varepsilon\right) + \varepsilon^3 K^s(x', x'/\varepsilon), \end{aligned} \tag{12}$$

where for $w_{3/2}^s = \partial_x^3 v_0^s + \lambda^s \partial_x v_0^s$ and $w_2^s = \partial_x^4 v_0^s + 2\lambda^s \partial_x^2 v_0^s + \lambda^s \lambda^s v_0^s$ we denote

$$K^s(x, y) = -2\nabla \partial_y u_2^s - \varepsilon \nabla^2 u_2^s - \varepsilon \lambda^s u_2^s = -2(\partial_y N_2) w_{3/2}^s - \varepsilon (N_2) w_2^s.$$

For large λ^s, directly from (11) we have $\left\|\partial_x v_0^s\right\|_{L^2(\Omega)}^2 = O(\lambda^s)$ and

$$\left\|\partial_x^2 v_0^s\right\|_{L^2(\Omega)}^2 = O((\lambda^s)^2), \quad \left\|\partial_x^2 v_0^s + \lambda^s v_0^s\right\|_{L^2(\Omega)}^2 = O((\lambda^s)^2),$$

$$\left\|w_{3/2}^s\right\|_{L^2(\Omega)}^2 = O((\lambda^s)^3), \quad \left\|w_2^s\right\|_{L^2(\Omega)}^2 = O((\lambda^s)^4),$$

$$\left\|\partial_x^2((w_{3/2}^s)^2)\right\|_{L^2(\Omega)} = O((\lambda^s)^4), \qquad \left\|\partial_x^2((w_2^s)^2)\right\|_{L^2(\Omega)} = O((\lambda^s)^5).$$

The similar estimates $\left\|\partial_{x'} v_0^s\right\|_{L^2(G_\varepsilon)}^2 = O(\lambda^s), \dots$ are valid for the corresponding norms on G_ε, since a differentiation of v_0^s by $\partial_{x_1'}$ and $\partial_{x_2'}$ gives a multiplier equivalent to $(\lambda^s)^{1/2}$, $|v_0^s| \le 2l^{-1/2}$ by the definition, and, for example, we have $\varepsilon \int_{G_\varepsilon} 1\, dx' = l$.

Using Proposition 2 (with the constant dependence on v being taken into account), the regularity of functions $N_2(y)$, and the smoothness of functions $v_0^s(x)$, we get

$$\begin{aligned} \varepsilon \int_{G_\varepsilon} \left(K^s(x', x'/\varepsilon)\right)^2 dx' &\le 4\varepsilon \int_{G_\varepsilon} \left(\partial_y N_2 w_{3/2}^s\right)^2 dx' + 2\varepsilon^3 \int_{G_\varepsilon} \left(N_2 w_2^s\right)^2 dx' \\ &\le C \int_\Omega \left(w_{3/2}^s\right)^2 dx + \varepsilon^2 C \int_\Omega \left(w_2^s\right)^2 dx + \varepsilon^2 C (\lambda^s)^4 + \varepsilon^4 C (\lambda^s)^5, \end{aligned} \tag{13}$$

where the constant C is independent of ε and s that is essential for large λ^s.

Besides, according to Proposition 2, we obtain the following relations:

$$\begin{aligned} \Big| \|u_a^s(x', x'/\varepsilon)\|^2_{L^2(G_\varepsilon)} - 1 \Big| &\le C\varepsilon^2\lambda^s + \int\limits_\Omega v_0^s(x)^2 dx - 1 + C\varepsilon^4(\lambda^s)^2 \\ + 2\varepsilon^2 \int\limits_Y N^0(y)N_2(y)dy \int_\Omega v_0^s(x)\left(\partial_x^2 v_0^s(x) + \lambda^s v_0^s(x)\right)dx &+ C\varepsilon^6(\lambda^s)^3 \\ + \varepsilon^4 \int\limits_Y N_2(y)^2 dy \int\limits_\Omega &\left(\partial_x^2 v_0^s(x) + \lambda^s v_0^s(x)\right)^2 dx \\ &\le C\varepsilon^2\lambda^s + C\varepsilon^4(\lambda^s)^2 + C\varepsilon^6(\lambda^s)^3 \le C\varepsilon^2\lambda^s \end{aligned} \tag{14}$$

for $\lambda^s \ll \varepsilon^{-2}$ (that is for $\lambda^s \le C\,\varepsilon^{-2+\sigma}$ with $0 < \sigma \le 2$), where the constant C is independent of ε and s, since $\int_\Omega v_0^s(x)^2 dx = 1$ and $\int_Y N^0(y)N_2(y)dy = 0$ by the definition. Similarly, we can verify that

$$\left| \varepsilon \int\limits_{G_\varepsilon} u_a^s(x', x'/\varepsilon)\, u_a^j(x', x'/\varepsilon)\, dx' \right| \le C\varepsilon^2\lambda^s + C\varepsilon^2\lambda^j$$

for $s \ne j$ and $\lambda^s, \lambda^j \ll \varepsilon^{-2}$. This means that the functions u_a^1, $u_a^2, \dots, u_a^s$ are almost orthonormalized in $L^2(G_\varepsilon)$ (in the sense of last two inequalities) and are linearly independent for $\lambda^s \ll \varepsilon^{-2}$. Here, it is important that the system of the eigenfunctions $\{v_0^s\}_{s=1}^\infty$ is orthonormalized in $L^2(\Omega)$. According to (14), we get $\|u_a^s\|_{L^2(G_\varepsilon)} \ne 0$ for $\lambda^s \ll \varepsilon^{-2}$. Therefore, defining $\hat{u}_a^s = \|u_a^s\|^{-1}_{L^2(G_\varepsilon)} u_a^s$, we obtain $\|\hat{u}_a^s\|_{L^2(G_\varepsilon)} = 1$.

By shifting the term with the eigenvalue λ^s to the left-hand side of equality (12), raising the result to the second power, multiplying by ε and integrating the resulting relation over the network G_ε, for $\lambda^s \ll \varepsilon^{-2}$ we obtain

$$\varepsilon \int\limits_{G_\varepsilon} \left(Lu_a^s - \varepsilon^2\lambda^s u_a^s\right)^2 dx' = \left\|Lu_a^s - \varepsilon^2\lambda^s u_a^s\right\|^2_{L^2(G_\varepsilon)} = \varepsilon^7 \int\limits_{G_\varepsilon} \left(K^s(x', x'/\varepsilon)\right)^2 dx'$$

$$\le \varepsilon^6 C(\lambda^s)^3 + \varepsilon^8 C(\lambda^s)^4 + \varepsilon^{10} C(\lambda^s)^5 \le \varepsilon^6 C(\lambda^s)^3$$

in accordance with (13) and (14). Thus, we have the following inequality:

$$\left\|L\hat{u}_a^s - \varepsilon^2\lambda^s \hat{u}_a^s\right\|_{L^2(G_\varepsilon)} \le \varepsilon^3 C(\lambda^s)^{3/2},$$

where the constant C is independent of ε and s for $\lambda^s \ll \varepsilon^{-2}$. Therefore, by Theorem 4, there exists an eigenvalue $\lambda_\varepsilon^{k(s)}$ of problem (5)–(6) such that

$$\left|\lambda_\varepsilon^{k(s)} - \varepsilon^2\lambda_2^s\right| \le \varepsilon^3 C(\lambda^s)^{3/2}. \tag{15}$$

Thus, the estimate of Theorem 1 for eigenvalues of problem (5)–(6) will follow from inequality (15) if we verify that $k(s) = s$ for every $s = 1, 2 \ldots$.

By Theorem 2, for the first eigenvalue of problem (5)–(6) we have the relation

$$\lambda_\varepsilon^1 = \min_{0 \neq u \in H^1_{per*}(G_\varepsilon)} \frac{(Lu, u)_{L^2(G_\varepsilon)}}{\|u\|^2_{L^2(G_\varepsilon)}} \,. \tag{16}$$

By the definition, u_ε^1 satisfies the equality $\left(u_\varepsilon^1, 1\right)_{L^2(G_\varepsilon)} = 0$. The constructed asymptotic expansion u_a^1 for u_ε^1 may not comply with the orthogonality condition for a constant. Therefore, we subtract the constant $A_\varepsilon^1 = \varepsilon l^{-1} \int_{G_\varepsilon} u_a^1 dx'$ from u_a^1 to have $u_a^1 - A_\varepsilon^1 \in H^1_{per*}(G_\varepsilon)$. Then, according to Proposition 2, we get

$$\varepsilon l^{-1} \int\limits_{G_\varepsilon} u_a^1 dx' = \int\limits_{\Omega} v_0^1 dx + \varepsilon^2 \int\limits_{\Omega} \left(\partial_x^2 v_0^1 + \lambda^1 v_0^1\right) dx \int\limits_{Y} N^0 N_2 dy + O(\varepsilon^2) = O(\varepsilon^2),$$

since $\int_\Omega v_0^1(x)dx = 0$ and $\int_Y N^0 N_2(y)dy = 0$ by the definition. Thus, we can write that $A_\varepsilon^1 = \varepsilon^2 \tilde{A}_\varepsilon^1$, where $|\tilde{A}_\varepsilon^1| \leq C$ with C independent of ε. Furthermore, denoting $\tilde{u}_a^1 = u_a^1 - \varepsilon^2 \tilde{A}_\varepsilon^1$, we obtain $\tilde{u}_a^1 \in H^1_{per*}(G_\varepsilon)$ by the definition of the constant $\tilde{A}_\varepsilon^1$.

Thus, we can substitute the obtained function $\tilde{u}_a^1(x', x'/\varepsilon)$ into (16). Applying the operator L to $\tilde{u}_a^1$ and using the differentiation rule (7), we obtain the result (which is similar to (12) for $s = 1$) in the following form:

$$L\tilde{u}_a^1\left(x', x'/\varepsilon\right) = \varepsilon^2 \lambda^1 \left(u_0^1 + \varepsilon^2 u_2^1 - \varepsilon^2 \tilde{A}_\varepsilon^1\right) + \varepsilon^3 \tilde{K}^1(x', x'/\varepsilon),$$

where we denote $\tilde{K}^1(x, y) = K^1(x, y) + \varepsilon^2 \tilde{A}_\varepsilon^1 \lambda^1$ and use the relations from Section 3. Multiplying the obtained result by $\varepsilon \tilde{u}_a^1(x', x'/\varepsilon)$ and integrating over the network, we get

$$\left(L\tilde{u}_a^1, \tilde{u}_a^1\right)_{L^2(G_\varepsilon)} = \varepsilon \int\limits_{G_\varepsilon} L\tilde{u}_a^1 \tilde{u}_a^1 dx' = \varepsilon^2 \lambda^1 \, \|\tilde{u}_a^1\|^2_{L^2(G_\varepsilon)} + \varepsilon^4 \int\limits_{G_\varepsilon} \tilde{K}^1 \tilde{u}_a^1 dx'. \tag{17}$$

Consider the last term of relation (17). Using the estimate (13) and the Cauchy–Bunyakovsky inequality, we obtain

$$\varepsilon \int\limits_{G_\varepsilon} \tilde{K}^1 \tilde{u}_a^1 dx' \leq \left[\varepsilon \int\limits_{G_\varepsilon} (\tilde{K}^1)^2 dx'\right]^{1/2} \left[\varepsilon \int\limits_{G_\varepsilon} (\tilde{u}_a^1)^2 dx'\right]^{1/2} \leq C \|\tilde{u}_a^1\|_{L^2(G_\varepsilon)}.$$

The subtraction of the constant $\varepsilon^2 \tilde{A}_\varepsilon^1$ from the function u_a^1 does not influence on estimate (14) essentially, hence we can write $\|\tilde{u}_a^1\|_{L^2(G_\varepsilon)} = 1 + O(\varepsilon^2)$. Substituting the result in (16), multiplying the identity (17) by $\|\tilde{u}_a^1\|^{-2}_{L^2(G_\varepsilon)}$, and using the estimate for the last term of the identity, we get the estimate from above for the first eigenvalue of problem (5)–(6) in the following form:

$$\lambda^1_\varepsilon \le \varepsilon^2 \lambda^1 + C\varepsilon^3.$$

We fix a natural number $d > 1$ and denote $U_a^1 = \tilde{u}_a^1 \|\tilde{u}_a^1\|^{-1}_{L^2(G_\varepsilon)}$, then we have $\|U_a^1\|_{L^2(G_\varepsilon)} = 1$ and $U_a^1 \in H^1_{per*}(G_\varepsilon)$. In what follows, we orthonormalize the functions $U_a^1(x', x'/\varepsilon)$, $u_a^2(x', x'/\varepsilon)$, $\dots$, $u_a^d(x', x'/\varepsilon)$ in the space $L^2_*(G_\varepsilon)$.

Define the constant $A^i_\varepsilon = \varepsilon l^{-1} \int_{G_\varepsilon} u_a^i dx'$ for $i = 2, \dots, d$. Then, we can write $A^i_\varepsilon = \varepsilon^2 \tilde{A}^i_\varepsilon$ as in the case $i = 1$, where $|\tilde{A}^i_\varepsilon| \le C$ with C independent of ε. Thus, we have $\tilde{u}_a^i = u_a^i - \varepsilon^2 \tilde{A}^i_\varepsilon \in H^1_{per*}(G_\varepsilon)$ for $i = 2, \dots, d$. Define also the constant $A^{21}_\varepsilon = \varepsilon \int_{G_\varepsilon} \tilde{u}_a^2 U_a^1 dx'$. Then, according to Proposition 2, we obtain

$$A^{21}_\varepsilon = \int_\Omega v_0^2 v_0^1 \, dx + O(\varepsilon^2) = O(\varepsilon^2),$$

since $\int_\Omega v_0^2(x) v_0^1(x) \, dx = 0$ by the definition. Thus, $A^{21}_\varepsilon = \varepsilon^2 \tilde{A}^{21}_\varepsilon$, where $|\tilde{A}^{21}_\varepsilon| \le C$ with C independent of ε.

Denote $\breve{u}_a^2 = \tilde{u}_a^2 - \varepsilon^2 \tilde{A}^{21}_\varepsilon U_a^1$. Then $\breve{u}_a^2$ is orthogonal to U_a^1 and satisfies the relations similar to (12)–(14) and (17). Thus, $U_a^2 = \breve{u}_a^2 \, \|\breve{u}_a^2\|^{-1}_{L^2(G_\varepsilon)}$ is well defined and orthogonal to U_a^1, $\|U_a^2\|_{L^2(G_\varepsilon)} = 1$ and $U_a^2 \in H^1_{per*}(G_\varepsilon)$. Furthermore, by induction we can find the orthonormalized U_a^1, U_a^2,$\dots$, U_a^{d-1} and define the function

$$\breve{u}_a^d = \tilde{u}_a^d - \varepsilon^2 \tilde{A}^{d,d-1}_\varepsilon U_a^{d-1} - \dots - \varepsilon^2 \tilde{A}^{d1}_\varepsilon U_a^1.$$

The function is orthogonal to U_a^i when $\varepsilon^2 \tilde{A}^{di}_\varepsilon = \varepsilon \int_{G_\varepsilon} \tilde{u}_a^d U_a^i dx'$ with $|\tilde{A}^{di}_\varepsilon| \le C$ for $i = 1, \dots, d-1$ (it is helpful here that the system of the eigenfunctions v_0^1, $\dots$, v_0^d is orthonormalized) and it satisfies the relations similar to (12)–(14) and (17). Thus, $U_a^d = \breve{u}_a^d \, \|\breve{u}_a^d\|^{-1}_{L^2(G_\varepsilon)}$ is defined and orthogonal to the functions U_a^1, U_a^2, $\dots$, U_a^{d-1}. Moreover, we get $\|U_a^d\|_{L^2(G_\varepsilon)} = 1$ and $U_a^d \in H^1_{per*}(G_\varepsilon)$.

Define the d-dimensional subspace $H_d \subset H^1_{per*}(G_\varepsilon)$ as a linear span of the functions $U_a^1(x', x'/\varepsilon)$, $U_a^2(x', x'/\varepsilon)$, $\dots$, $U_a^d(x', x'/\varepsilon)$ and the orthogonal projector P_d onto H_d. By the definition, for $U \in H^1_{per*}(G_\varepsilon)$ we have

$$P_d U = \sum_{i=1}^d U_a^i \, (U_a^i, U)_{L^2_*(G_\varepsilon)}.$$

Therefore, $P_d U_a^i = U_a^i$ for $i = 1, \dots, d$. Thus, there exist d-element sets of the eigenvalues $\mu^1_\varepsilon, \mu^2_\varepsilon, \dots, \mu^d_\varepsilon$ and of the orthonormalized eigenfunctions $w^1_\varepsilon, w^2_\varepsilon, \dots, w^d_\varepsilon$ of the operator $L_d = P_d L P_d$ such that the inequality $\mu^1_\varepsilon \le \mu^2_\varepsilon \le \dots \le \mu^d_\varepsilon$ is valid (with the multiplicity being taken into account). Besides, according to Theorem 3, we obtain

$$\lambda_\varepsilon^1 \le \mu_\varepsilon^1, \dots, \lambda_\varepsilon^d \le \mu_\varepsilon^d.$$

Using the relations similar to (12)–(14) and (17) for the orthonormalized functions $U_a^1(x', x'/\varepsilon)$, $U_a^1(x', x'/\varepsilon)$, …, $U_a^d(x', x'/\varepsilon)$, we get

$$L_d U_a^i(x', x'/\varepsilon) = P_d L U_a^i = \varepsilon^2 \lambda^i U_a^i + O(\varepsilon^3)$$

and

$$\left\| L_d U_a^i - \varepsilon^2 \lambda^i U_a^i \right\|_{L^2(G_\varepsilon)} \le \varepsilon^3 C,$$

where $i = 1, \dots, d$, and the constant C is independent of ε. Therefore, according to Theorem 4, there exists an eigenvalue $\mu_\varepsilon^{j(i)}$ of the operator L_d such that

$$\left| \mu_\varepsilon^{j(i)} - \varepsilon^2 \lambda^i \right| \le \varepsilon^3 C, \tag{18}$$

where $i = 1, \dots, d$, and the constant C is independent of ε. Here the dependence of C on i (which is clear from the relations similar to (12)–(14)) is not essential, since in order to complete the proof of the estimate of Theorem 1 for the eigenvalues of problem (5)–(6), we have to verify that $k(s) = s$ in (15) for every $s = 1, 2, \dots, d$.

Following [13] and [16], we verify that $j(i) = i$ in (18) for every $i = 1, \dots, d$. Indeed, if the eigenvalues λ^1, …, λ^d are simple, then we have $j(i) = i$ for every $i = 1, \dots, d$, since d ordered values $\mu_\varepsilon^1, \dots, \mu_\varepsilon^d$ are in the ε^3-neighborhoods of d strictly ordered values $\varepsilon^2\lambda^1, \dots, \varepsilon^2\lambda^d$ which is possible only if $j(i) = i$. However, the multiplicity of the eigenvalue λ^1 is equal either to two or four, depending on l.

Consider the first case, for example, then λ^3 is separated from λ^1 by some positive constant δ (for example, $\delta = 1$ for $l = 1/2$). Choosing $d = 2$ in (18), we conclude that $j(1) = 1$ and $j(2) = 2$ (what is to be proved, since $\lambda^1 = \lambda^2$) or $j(1) = 2$ and $j(2) = 2$. In the latest case, there exists a constant $\sigma > 0$ such that $\mu_\varepsilon^1 < \mu_\varepsilon^2 - \varepsilon^2\sigma < \mu_\varepsilon^2 + \varepsilon^2\sigma < \mu_\varepsilon^3$, and on the interval $(\varepsilon^2\lambda^1 - \varepsilon^2\sigma, \varepsilon^2\lambda^1 + \varepsilon^2\sigma)$ there exists only one eigenvalue μ_ε^2 of the operator L_2. Thus, by Theorem 4, we have

$$\|U_a^1 - w_\varepsilon^2\|_{L^2(G_\varepsilon)} \le \varepsilon C, \qquad \|U_a^2 - w_\varepsilon^2\|_{L^2(G_\varepsilon)} \le \varepsilon C.$$

But, it is impossible [13] since the normalized function w_ε^2 approximates two orthonormalized functions U_a^1 and U_a^2 simultaneously.

Thus, the equality $j(i) = i$ for $i = 1, 2$ is proved. In the same way, we can prove that $k(s) = s$ in (15) for $s = 1, 2$ (when the multiplicity of λ^1 is 2). Indeed, there exist only two eigenvalues $\lambda_\varepsilon^1, \lambda_\varepsilon^2$ of problem (5)–(6) on the segment $[\alpha\varepsilon^2, \mu_\varepsilon^2]$, since $\alpha\varepsilon^2 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le \mu_\varepsilon^2$ due to Theorem 3. Moreover, inequality (15) is valid. Therefore, we have $k(1) = 1$ and $k(2) = 2$ or $k(1) = 2$ and $k(2) = 2$.

In the last case, there exists a constant $\sigma > 0$ such that only one eigenvalue λ_ε^2 of the operator L is on the interval $(\varepsilon^2\lambda^1 - \varepsilon^2\sigma, \varepsilon^2\lambda^1 + \varepsilon^2\sigma)$. Hence, according to Theorem 4, almost orthonormalized functions $\hat{u}_a^1$ and $\hat{u}_a^2$ (in the sense of (14)) are approximated by one normalized function u_ε^2, which is impossible.

Next, we consider, for example, the case when the multiplicities of λ^1 and λ^3 are equal to 2 and r, respectively. Choosing $d = 3 + r - 1$ in (18), we have that the eigenvalues $\mu_\varepsilon^1, \mu_\varepsilon^2$ are in a ε^3-neighborhood of $\varepsilon^2\lambda^1$, and μ_ε^{3+r-1} is in a ε^3-neighborhood of the value $\varepsilon^2\lambda^3$ at least. If μ_ε^3 is not in the ε^3-neighborhood of the value $\varepsilon^2\lambda^3$, then r orthonormalized functions $U_a^3, \ldots, U_a^{3+r-1}$ can be approximated by $(r-1)$ orthonormalized functions $w_\varepsilon^4, \ldots, w_\varepsilon^{3+r-1}$, which is impossible.

Thus, the equality $j(i) = i$ for $i = 1, \ldots, 3+r-1$ is proved. Similarly, we can prove that $k(s) = s$ in (15) for $s = 1, \ldots, 3 + r - 1$. Due to inequality (15) and Theorem 3, this proof can be continued by the induction over d for $\lambda^d \ll \varepsilon^{-2}$. This proves the estimate of Theorem 1 for the eigenvalues of problem (5)–(6). It is useful here that for every d and ε the relations

$$\alpha\varepsilon^2 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le \cdots \le \lambda_\varepsilon^d \le \mu_\varepsilon^d$$

hold, which provides a control over the number of eigenvalues for problem (5)–(6) on the concrete segment $[\alpha\varepsilon^2, \mu_\varepsilon^d] \subset [\alpha\varepsilon^2, \varepsilon^2\lambda^d + \varepsilon^2 C(\lambda^d)^{3/2}]$. It is useful, because the function $k(s)$ in (15) can depend on ε. To be definite, Theorem 4 guaranties only that the number $k(s)$ in (15) is defined for fixed s and ε.

Next, we consider some eigenvalue λ^s of problem (11) with the multiplicity r (which can be 2, 4 or 8). By the definition, we have the relations

$$\lambda^{s-1} < \lambda^s = \lambda^{s+1} = \cdots = \lambda^{s+r-1} < \lambda^{s+r}.$$

Denote by σ_s the smallest number of $(\lambda^{s-1} + \lambda^s)/2$ and $(\lambda^s + \lambda^{s+r})/2$. It follows from inequality (15) that only eigenvalues $\lambda_\varepsilon^s, \ldots, \lambda_\varepsilon^{s+r-1}$ of problem (5)–(6) are in the interval $(\varepsilon^2\lambda^s - \varepsilon^2\sigma_s, \varepsilon^2\lambda^s + \varepsilon^2\sigma_s)$. Thus, according to Theorem 4, there exist constants α_i^j (possibly, dependent on ε) for $i, j = s, s + r - 1$ such that

$$\begin{aligned} \left\| \hat{u}_a^s - \alpha_s^s u_\varepsilon^s - \cdots - \alpha_{s+r-1}^s u_\varepsilon^{s+r-1} \right\|_{L^2(G_\varepsilon)} &\le C\,\varepsilon\,(\lambda^s)^{3/2}\sigma_s^{-1}, \\ \cdots \qquad\qquad \cdots \qquad\qquad & \qquad\qquad \cdots, \\ \left\| \hat{u}_a^{s+r-1} - \alpha_s^{s+r-1} u_\varepsilon^s - \cdots - \alpha_{s+r-1}^{s+r-1} u_\varepsilon^{s+r-1} \right\|_{L^2(G_\varepsilon)} &\le C\,\varepsilon\,(\lambda^s)^{3/2}\sigma_s^{-1}. \end{aligned} \tag{19}$$

Moreover, by the conditions of Theorem 4, we have

$$\begin{aligned} \left\| \alpha_s^s u_\varepsilon^s + \cdots + \alpha_{s+r-1}^s u_\varepsilon^{s+r-1} \right\|_{L^2(G_\varepsilon)} &= 1, \ldots, \\ \left\| \alpha_s^{s+r-1} u_\varepsilon^s + \cdots + \alpha_{s+r-1}^{s+r-1} u_\varepsilon^{s+r-1} \right\|_{L^2(G_\varepsilon)} &= 1. \end{aligned} \tag{20}$$

The functions $u_\varepsilon^s, \ldots, u_\varepsilon^{s+r-1}$ are orthonormalized. Hence, we get

$$\left(\alpha_s^s\right)^2 + \cdots + \left(\alpha_{s+r-1}^s\right)^2 = 1, \ldots, \left(\alpha_s^{s+r-1}\right)^2 + \cdots + \left(\alpha_{s+r-1}^{s+r-1}\right)^2 = 1$$

in accordance with (20). This means that the matrix $\left\{\alpha_j^i\right\}_{i,j=s,s+r-1}$ is orthogonal.

Define the functions $\check{u}_a^s, \ldots, \check{u}_a^{s+r-1}$ as the orthogonal transform of the functions $\hat{u}_a^s, \ldots, \hat{u}_a^{s+r-1}$ by the matrix $\left\{\alpha_j^i\right\}_{i,j=s,s+r-1}$. Then, it follows from (19) that

$$\|\check{u}_a^s - u_\varepsilon^s\|_{L^2(G_\varepsilon)} \le C\varepsilon\,(\lambda^s)^{3/2}\sigma_s^{-1}, \ldots, \left\|\check{u}_a^{s+r-1} - u_\varepsilon^{s+r-1}\right\|_{L^2(G_\varepsilon)} \le C\varepsilon\,(\lambda^s)^{3/2}\sigma_s^{-1},$$

which concludes the proof of Theorem 1 (since it can be assumed that $\lambda^s = \sigma_s$ for large s). Here, as the eigenfunction v^s of problem (11) from the estimate of Theorem 1, we can take the relevant linear combination of the eigenfunctions

$$\begin{gathered} 2l^{-1/2}\cos 2\pi n x_1 \cos 2\pi m x_2 l^{-1}, \quad 2l^{-1/2}\sin 2\pi n x_1 \cos 2\pi m x_2 l^{-1}, \\ 2l^{-1/2}\cos 2\pi n x_1 \sin 2\pi m x_2 l^{-1}, \quad 2l^{-1/2}\sin 2\pi n x_1 \sin 2\pi m x_2 l^{-1} \end{gathered} \tag{21}$$

with the coefficients located in a line of the matrix $\left\{\alpha_j^i\right\}_{i,j=s,s+r-1}$ and the relevant n and m. We emphasize that for the eigenvalue λ^s of problem (11) with multiplicity r there exists some arbitrariness in choosing the eigenfunctions v_0^s, $\ldots, v_0^{s+r-1}$, which is determined by some orthogonal matrix. Thus, it is necessary to use the lines of the corresponding orthogonal matrix $\left\{\alpha_j^i\right\}_{i,j=s,s+r-1}$.

In conclusion, the asymptotics of the eigenvalues and eigenfunctions for problem (5)–(6) is constructed and Theorem 1 for the sth eigenvalue λ_ε^s and sth eigenfunction u_ε^s of this problem is proved, where λ^s is the eigenvalue of the homogenized problem (11) with the eigenfunction v^s, which is suitably orthonormalized, i.e., v^s is the relevant linear combination of the eigenfunctions from (21).

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