Journal of Mathematical Physics, Analysis, Geometry
2014, vol. 10, No. 3, pp. 300–308

The Warped Product of Hamiltonian Spaces

H. Attarchi and M.M. Rezaii

Department of Mathematics and Computer Science
Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran

E-mail: hassan.attarchi@aut.ac.ir
mmreza@aut.ac.ir

Received November 21, 2012, revised March 31, 2014

In this paper, the geometric properties of warped product Hamiltonian spaces are studied. It is shown there is a close geometrical relation between a warped product Hamiltonian space and its base Hamiltonian manifolds. For example, it is proved that for nonconstant warped function f, the Sasaki lifted metric G of Hamiltonian warped product space is bundle-like for its vertical foliation if and only if based Hamiltonian spaces are pseudo-Riemannian manifolds.

Key words: warped product, Hamiltonian space, bundle-like metric.

Mathematics Subject Classification 2010: 54B10, 37J99.

1. Introduction

The notion of warped product spaces was introduced to study manifolds with negative curvatures by Bishop and O'Neill [3]. Afterwards, the warped product was used to model the standard space-time, especially in the neighborhood of stars and black holes [10]. The notion of the warped product Finslerian manifolds was initially introduced by Kozma [5] in 2001. Recently, it was developed by one of the present authors [1, 4, 11]. In this work, the warped product of Hamiltonian spaces is introduced and it is shown that these spaces obtain Hamiltonian structure as well. Moreover, some geometric properties of warped product Hamilton spaces such as its nonlinear connections are studied.

The Lagrange space has been certified as an excellent model for some important problems in Relativity, Gauge Theory and Electromagnetism [6, 7]. The geometry of Lagrange spaces gives a model for both the gravitational and electromagnetic fields. Moreover, this structure plays a fundamental role in studying the geometry of the tangent bundle TM. The geometries of the cotangent bundle T^*M and the tangent bundle TM which follows the same outlines are related by the Legendre transformation. From this duality, the geometry of a Hamiltonian space can be obtained from that of certain Lagrangian space and vice versa. Using this duality, several important results in the Hamiltonian spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection, etc. Therefore, the theory of Hamiltonian spaces has the same symmetry and beauty as the Lagrangian geometry. Moreover, it gives a geometrical framework for the Hamiltonian theory of mechanics or physical fields. With respect to the importance of these spaces in physical areas, present work is formed to develop the concept of a warped product on Hamiltonian spaces. Aiming at our purpose, this paper is organized in the following way:

Let (M, H) be a warped Hamiltonian space of the Hamiltonian spaces (M_1, H_1) and (M_2, H_2). In Sec. 2, the notion of the warped product Hamiltonian spaces is presented and some natural geometrical properties of the cotangent bundle for a warped manifold are given. In Sec. 3, it is shown that (M, H) is a Hamiltonian space and its canonical nonlinear connections are calculated as well. Moreover, the Sasaki lifted metric G on T^*M is introduced. In Sec. 4, the Levi–Civita connection of pseudo-Riemannian metric G on T^*M is calculated. Finally, in Sec. 5, we prove some theorems that show close relation between the geometries of the warped product Hamiltonian manifolds and their base Hamiltonian spaces.

2. Preliminaries and Notations

Here, a Hamiltonian space is a pair (M, H), where M is a real n-dimensional manifold and $H : T^*M \longrightarrow I\!R$ is a smooth function whose Hessian with respect to the cotangent bundle coordinate is a d-tensor field of type $(2, 0)$ symmetric, nondegenerate and of constant signature on $T^*M\backslash\{0\}$. Let $\mathbb{H}_1^n = (M_1, H_1)$ and $\mathbb{H}_2^m = (M_2, H_2)$ be two Hamiltonian spaces with $\dim(\mathbb{H}_1^n) = n$ and $\dim(\mathbb{H}_2^m) = m$, respectively. The warped product of these spaces is denoted by $\mathbb{H} = (M, H)$, where

$$M = M_1 \times M_2 \qquad \textit{and} \qquad H = H_1 + fH_2 \tag{1}$$

for some smooth function $f : M_1 \longrightarrow \mathbb{R}^+$. Then a coordinate system on M is denoted by $\{(U \times V, \varphi \times \psi)\}$, where $\{(U, \varphi)\}$ and $\{(V, \psi)\}$ are coordinate systems on M_1 and M_2, respectively, such that each $\mathbf{x} = (x, z) \in M$ has the local expression (x^i, z^α). It is notable that throughout the paper, the indices $\{i, j, k, \ldots\}$ and $\{\alpha, \beta, \lambda, \ldots\}$ are used for the ranges $1, \ldots, n$ and $1, \ldots, m$, respectively. Moreover, the canonical projections of T^*M_1 on M_1 and T^*M_2 on M_2 are denoted by π_1 and π_2, respectively. The fibre of the cotangent bundle at $\mathbf{x} = (x, z) \in M$ is $T^*_{(x,z)}M = T^*_x M_1 \oplus T^*_z M_2$, therefore $T^*M = T^*M_1 \oplus T^*M_2$.

The induced coordinate systems on T^*M_1 and T^*M_2 are (x^i, p_i) and (z^α, q_α), respectively, whose coordinates p_i and q_α are called *momentum variables* [8]. The change of these coordinates on T^*M_1 and T^*M_2 are given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n), \\ \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j \,. \end{cases} \qquad \begin{cases} \tilde{z}^\alpha = \tilde{z}^\alpha(z^1, \ldots, z^m), \\ \text{rank}\left(\frac{\partial \tilde{z}^\alpha}{\partial z^\beta}\right) = m, \\ \tilde{q}_\alpha = \frac{\partial z^\beta}{\partial \tilde{z}^\alpha} q_\beta. \end{cases} \tag{2}$$

Let $(\mathbf{x}, \mathbf{p}) = (x, z, p, q) \in T^*M = T^*M_1 \oplus T^*M_2$. The tangent space at $(\mathbf{x}, \mathbf{p})$ to T^*M is denoted by $T_{(\mathbf{x},\mathbf{p})}T^*M$, that is, a $2(n+m)$-dimensional vector space. The natural basis induced on $T_{(\mathbf{x},\mathbf{p})}T^*M$ by the local coordinate of T^*M_1 and T^*M_2 is $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_\alpha}\}$. These coordinates are changed with respect to transformations (2) as follows:

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_j}{\partial x^i} \frac{\partial}{\partial \tilde{p}_j}, \\ \frac{\partial}{\partial z^\alpha} = \frac{\partial \tilde{z}^\beta}{\partial z^\alpha} \frac{\partial}{\partial \tilde{z}^\beta} + \frac{\partial \tilde{q}_\beta}{\partial z^\alpha} \frac{\partial}{\partial \tilde{q}_\beta}, \\ \frac{\partial}{\partial p_i} = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{p}_j}, \\ \frac{\partial}{\partial q_\alpha} = \frac{\partial z^\alpha}{\partial \tilde{z}^\beta} \frac{\partial}{\partial \tilde{q}_\beta}. \end{cases} \tag{3}$$

In the paper, the notations $\dot{\partial}^i$ and $\dot{\partial}^\alpha$ are used instead of $\frac{\partial}{\partial p_i}$ and $\frac{\partial}{\partial q_\alpha}$, respectively, similarly to the notations in [8]. The Jacobian matrix of transformations (3) is

$$\text{Jac} := \begin{pmatrix} \frac{\partial \tilde{x}^j}{\partial x^i} & 0 & 0 & 0 \\ 0 & \frac{\partial \tilde{z}^\beta}{\partial z^\alpha} & 0 & 0 \\ \frac{\partial \tilde{p}_j}{\partial x^i} & 0 & \frac{\partial x^i}{\partial \tilde{x}^j} & 0 \\ 0 & \frac{\partial \tilde{q}_\beta}{\partial z^\alpha} & 0 & \frac{\partial z^\alpha}{\partial \tilde{z}^\beta} \end{pmatrix}. \tag{4}$$

It follows that

$$\det(\text{Jac}) = 1.$$

By means of last equation, we have the following corollary.

Corollary 2.1. *The manifold $T^*M = T^*M_1 \oplus T^*M_2$ is orientable.*

Let $\bar{\partial}^a$ and $\frac{\partial}{\partial \mathbf{x}^a}$ be abbreviations for $\dot{\partial}^i \delta^a_i + \dot{\partial}^\alpha \delta^{\alpha+n}_a$ and $\frac{\partial}{\partial x^i} \delta^i_a + \frac{\partial}{\partial z^\alpha} \delta^a_{\alpha+n}$, respectively, where the indices $\{a, b, c, \ldots\}$ are used for the range $1, \ldots, n+m$. Throughout the paper, these notations and range of the indices are established.

We know that there are some natural structures that live on the cotangent bundle T^*M. It would be interesting to present them on the cotangent bundle of a warped product Hamiltonian space. First, the *Liouville–Hamilton vector field* of T^*M is given by

$$C^* := \mathbf{p}_a \bar{\partial}^a = p_i \dot{\partial}^i + q_\alpha \dot{\partial}^\alpha = C^*_1 + C^*_2, \tag{5}$$

where C_1^* and C_2^* denote the Liouville-Hamilton vector fields of T^*M_1 and T^*M_2, respectively.

Next, the *Liouville 1-form* θ on T^*M is defined by

$$\theta := \mathbf{p}_a d\mathbf{x}^a = p_i dx^i + q_\alpha dz^\alpha = \theta_1 + \theta_2, \tag{6}$$

where θ_1 and θ_2 are the Liouville 1-forms of T^*M_1 and T^*M_2, respectively.

And, the *canonical symplectic structure* ω on T^*M is defined by $\omega = d\theta$ and has the local expression

$$\omega := d\mathbf{p}_a \wedge d\mathbf{x}^a = dp_i \wedge dx^i + dq_\alpha \wedge dz^\alpha = \omega_1 + \omega_2, \tag{7}$$

where ω_1 and ω_2 are canonical symplectic structures of T^*M_1 and T^*M_2, respectively.

Finally, if the Poisson bracket on the cotangent bundles of T^*M_1, T^*M_2 and T^*M are denoted by $\{.,.\}_1$, $\{.,.\}_2$ and $\{.,.\}$, respectively, then they are related as follows:

$$\{g,h\} = \bar{\partial}^a g \frac{\partial h}{\partial \mathbf{x}^a} - \bar{\partial}^a h \frac{\partial g}{\partial \mathbf{x}^a} = \{g,h\}_1 + \{g,h\}_2, \tag{8}$$

where $g, h \in C^\infty(T^*M)$.

The *Hamilton vector field* of the Hamiltonian function H is denoted by X_H and satisfies the equation

$$\iota_{X_H}\omega = -dH.$$

Let X_{H_1} and X_{H_2} be Hamilton vector fields of the spaces $\mathbb{H}_1^n$ and $\mathbb{H}_2^m$, respectively, then the following theorem gives an expression of X_H.

Theorem 2.1. *Suppose that* $\mathbb{H} = (M, H)$ *is a warped product Hamiltonian space defined in* (1)*. Then the Hamilton vector field of* $\mathbb{H}$ *is given by*

$$X_H = X_{H_1} + fX_{H_2} - H_2 \frac{\partial f}{\partial x^i} \dot{\partial}^i.$$

P r o o f. By the definition of Hamilton vector fields, we have $\iota_{X_H}\omega = -dH$. It is a straightforward calculation to complete the prove. ■

3. Nonlinear Connection on Warped Product Hamiltonian Space

For the Hamiltonian spaces $\mathbb{H}_1^n$ and $\mathbb{H}_2^m$, the equations

$$\begin{cases} g^{ij} = \frac{1}{2}\dot{\partial}^i \dot{\partial}^j H_1, \\ g^{\alpha\beta} = \frac{1}{2}\dot{\partial}^\alpha \dot{\partial}^\beta H_2 \end{cases} \tag{9}$$

define the fundamental tensors of the spaces $\mathbb{H}_1^n$ and $\mathbb{H}_2^m$, respectively. The fundamental tensor of the warped product Hamiltonian space (M, H) is given by

$$(g^{ab}) = \left(\frac{1}{2}\bar{\partial}^a\bar{\partial}^b H\right) = \begin{pmatrix} g^{ij} & 0 \\ 0 & fg^{\alpha\beta} \end{pmatrix}. \tag{10}$$

Now, it is easy to check that (M, H) is a Hamilton space as well. By the definition of the canonical nonlinear connections of a Hamiltonian space presented in [8], the canonical nonlinear connections of $\mathbb{H}_1^n$, $\mathbb{H}_2^m$ and $\mathbb{H}$, respectively, are obtained as follows:

$$\begin{cases} N_{ij} = \frac{1}{4}\{g_{ij}, H_1\} - \frac{1}{4}\left(g_{ik}\frac{\partial^2 H_1}{\partial p_k \partial x^j} + g_{jk}\frac{\partial^2 H_1}{\partial p_k \partial x^i}\right), \\ N_{\alpha\beta} = \frac{1}{4}\{g_{\alpha\beta}, H_2\} - \frac{1}{4}\left(g_{\alpha\gamma}\frac{\partial^2 H_2}{\partial q_\gamma \partial z^\beta} + g_{\beta\gamma}\frac{\partial^2 H_2}{\partial q_\gamma \partial z^\alpha}\right), \\ \bar{N}_{ab} = \frac{1}{4}\{g_{ab}, H\} - \frac{1}{4}\left(g_{ac}\frac{\partial^2 H}{\partial \mathbf{p}_c \partial \mathbf{x}^b} + g_{bc}\frac{\partial^2 H}{\partial \mathbf{p}_c \partial \mathbf{x}^a}\right), \end{cases} \tag{11}$$

where (g_{ij}), $(g_{\alpha\beta})$ and (g_{ab}) are the inverse matrices of (g^{ij}), $(g^{\alpha\beta})$ and (g^{ab}), respectively. The relation of the nonlinear connections $\bar{N}_{ab}$ of the Hamiltonian space $\mathbb{H}$ and those of $\mathbb{H}_1^n$ and $\mathbb{H}_2^m$ are given by

$$\begin{cases} \bar{N}_{ij} = N_{ij} + \frac{1}{4}\dot{\partial}^k g_{ij}\frac{\partial f}{\partial x^k}H_2, \\ \bar{N}_{\alpha\beta} := \bar{N}_{(\alpha+n)(\beta+n)} = N_{\alpha\beta} - \frac{1}{4f^2}g_{\alpha\beta}\dot{\partial}^k H_1\frac{\partial f}{\partial x^k}, \\ \bar{N}_{i\alpha} := \bar{N}_{i(\alpha+n)} = -\frac{1}{4f}g_{\alpha\beta}\dot{\partial}^\beta H_2\frac{\partial f}{\partial x^i}. \end{cases} \tag{12}$$

Let π be the projection map

$$\pi := (\pi_1, \pi_2) : T^*M_1 \oplus T^*M_2 \longrightarrow M_1 \times M_2.$$

Then the kernel of π_* is known as the vertical bundle on T^*M and denoted by VT^*M. The local sections of VT^*M are given by

$$\{\frac{\partial}{\partial p_1}, ..., \frac{\partial}{\partial p_n}, \frac{\partial}{\partial q_1}, ..., \frac{\partial}{\partial q_m}\}.$$

Using the nonlinear connections $\bar{N}_{ij}$, $\bar{N}_{i\alpha}$ and $\bar{N}_{\alpha\beta}$, we can define the nonholomorphic vector fields

$$\begin{cases} \frac{\delta^*}{\delta^* x^i} := \frac{\delta^*}{\delta^* \mathbf{x}^i} = \frac{\partial}{\partial x^i} + \bar{N}_{ij}\dot{\partial}^j + \bar{N}_{i\alpha}\dot{\partial}^\alpha, \\ \frac{\delta^*}{\delta^* z^\alpha} := \frac{\delta^*}{\delta^* \mathbf{x}^{\alpha+n}} = \frac{\partial}{\partial z^\alpha} + \bar{N}_{\alpha i}\dot{\partial}^i + \bar{N}_{\alpha\beta}\dot{\partial}^\beta, \end{cases} \tag{13}$$

which generate the warped horizontal distribution on T^*M denoted by HT^*M. The dual 1-forms of these local vector fields are given by

$$\begin{cases} d\mathbf{x}^a = dx^i\delta_i^a + dz^\alpha\delta_{\alpha+n}^a, \\ \delta^* p_i := \delta\mathbf{p}_i = dp_i - \bar{N}_{ij}dx^j - \bar{N}_{i\alpha}dz^\alpha, \\ \delta^* q_\alpha := \delta\mathbf{p}_{\alpha+n} = dq_\alpha - \bar{N}_{\alpha i}dx^i - \bar{N}_{\alpha\beta}dz^\beta. \end{cases} \tag{14}$$

Moreover, the Sasaki metric G on T^*M of the Hamiltonian structure H is defined by

$$G = g_{ij}dx^i \otimes dx^j + \frac{g_{\alpha\beta}}{f}dz^\alpha \otimes dz^\beta + g^{ij}\delta^* p_i \otimes \delta^* p_j + f g^{\alpha\beta}\delta^* q_\alpha \otimes \delta^* q_\beta. \tag{15}$$

4. The Levi–Civita Connection of Metric G

The Lie brackets of the local vector fields given in previous section are presented as follows:

$$\begin{cases} [\frac{\delta^*}{\delta^* x^i}, \frac{\delta^*}{\delta^* x^j}] = \mathbf{R}_{ijk}\dot{\partial}^k + \mathbf{R}_{ij\alpha}\dot{\partial}^\alpha, \\ [\frac{\delta^*}{\delta^* x^i}, \frac{\delta^*}{\delta^* z^\alpha}] = \mathbf{R}_{i\alpha j}\dot{\partial}^j + \mathbf{R}_{i\alpha\beta}\dot{\partial}^\beta, \\ [\frac{\delta^*}{\delta^* z^\alpha}, \frac{\delta^*}{\delta^* z^\beta}] = \mathbf{R}_{\alpha\beta i}\dot{\partial}^i + \mathbf{R}_{\alpha\beta\gamma}\dot{\partial}^\gamma, \end{cases} \tag{16}$$

where

$$\begin{cases} \mathbf{R}_{ijk} = \frac{\delta^* \bar{N}_{jk}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{ik}}{\delta^* x^j}, \quad \mathbf{R}_{ij\alpha} = \frac{\delta^* \bar{N}_{j\alpha}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{i\alpha}}{\delta^* x^j}, \\ \mathbf{R}_{i\alpha k} = \frac{\delta^* \bar{N}_{\alpha k}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{ik}}{\delta^* z^\alpha}, \quad \mathbf{R}_{i\alpha\beta} = \frac{\delta^* \bar{N}_{\alpha\beta}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{i\beta}}{\delta^* z^\alpha}, \\ \mathbf{R}_{\alpha\beta k} = \frac{\delta^* \bar{N}_{\beta k}}{\delta^* z^\alpha} - \frac{\delta^* \bar{N}_{\alpha k}}{\delta^* z^\beta}, \quad \mathbf{R}_{\alpha\beta\gamma} = \frac{\delta^* \bar{N}_{\beta\gamma}}{\delta^* z^\alpha} - \frac{\delta^* \bar{N}_{\alpha\gamma}}{\delta^* z^\beta}. \end{cases} \tag{17}$$

The components $\mathbf{R}_{abc}$ are called the *curvature tensors* of the nonlinear connection $\bar{N}_{ab}$ and they are skew-symmetric with respect to the indices a and b. Moreover,

$$\begin{cases} [\dot{\partial}^i, \frac{\delta^*}{\delta^* x^j}] = \dot{\partial}^i(\bar{N}_{jk})\dot{\partial}^k, \\ [\dot{\partial}^\alpha, \frac{\delta^*}{\delta^* x^i}] = \dot{\partial}^\alpha(\bar{N}_{ik})\dot{\partial}^k + \dot{\partial}^\alpha(\bar{N}_{i\beta})\dot{\partial}^\beta, \\ [\dot{\partial}^i, \frac{\delta^*}{\delta^* z^\alpha}] = \dot{\partial}^i(\bar{N}_{\alpha\beta})\dot{\partial}^\beta, \\ [\dot{\partial}^\alpha, \frac{\delta^*}{\delta^* z^\beta}] = \dot{\partial}^\alpha(\bar{N}_{\beta k})\dot{\partial}^k + \dot{\partial}^\alpha(\bar{N}_{\beta\gamma})\dot{\partial}^\gamma. \end{cases} \tag{18}$$

Let ∇ be the Levi–Civita connection on (T^*M, G) which is given by

$$\begin{aligned} 2G(\nabla_X Y, Z) = XG(Y, Z) + YG(X, Z) - ZG(X, Y) \\ -G([X, Z], Y) - G([Y, Z], X) + G([X, Y], Z) \end{aligned} \tag{19}$$

for any $X, Y, Z \in \Gamma(TT^*M)$. Then the components of ∇ are given by

$$\begin{cases} \nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j} = \Gamma^k_{ij}\frac{\delta^*}{\delta^* x^k} - \frac{f}{2}\bar{N}_{\alpha k} g^k_{ij} g^{\alpha\beta}\frac{\delta^*}{\delta^* z^\beta} + \frac{1}{2}g_{ijk}\dot{\partial}^k + \frac{1}{2}\mathbf{R}_{ija}\bar{\partial}^a, \\ \nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* z^\alpha} = \nabla_{\frac{\delta^*}{\delta^* z^\alpha}} \frac{\delta^*}{\delta^* x^i} + \mathbf{R}_{i\alpha a}\bar{\partial}^a = -\frac{1}{2}\bar{N}_{\alpha j} g^{jk}_i \frac{\delta^*}{\delta^* x^k} \\ \qquad + \frac{1}{2}(\frac{\partial \ln f}{\partial x^i}\delta^\gamma_\alpha - \bar{N}_{i\beta} g^{\beta\gamma}_\alpha)\frac{\delta^*}{\delta^* z^\gamma} + \frac{1}{2}\mathbf{R}_{i\alpha a}\bar{\partial}^a, \\ \nabla_{\frac{\delta^*}{\delta^* z^\alpha}} \frac{\delta^*}{\delta^* z^\beta} = -\frac{1}{2}\frac{\delta^* f g_{\alpha\beta}}{\delta^* x^i} g^{ij}\frac{\delta^*}{\delta^* x^j} + \Gamma^\gamma_{\alpha\beta}\frac{\delta^*}{\delta^* z^\gamma} + \frac{1}{2f^2}g_{\alpha\beta\lambda}\dot{\partial}^\lambda + \frac{1}{2}\mathbf{R}_{\alpha\beta a}\bar{\partial}^a, \end{cases} \tag{20}$$

$$\begin{cases} \nabla_{\dot\partial^i}\dot\partial^\alpha = \nabla_{\dot\partial^\alpha}\dot\partial^i = \frac{1}{8}\dot\partial^\alpha H_2 g^{ikh}\frac{\partial f}{\partial x^k}\frac{\delta^*}{\delta^* x^h} \\ \qquad -\frac{1}{2}(f^2\dot\partial^i(\bar N_{\beta\gamma})g^{\gamma\alpha}g^{\beta\lambda} + f\dot\partial^\alpha(\bar N_{\beta k})g^{ki}g^{\beta\lambda})\frac{\delta^*}{\delta^* z^\lambda}\,, \\ \nabla_{\dot\partial^i}\dot\partial^j = -\frac{1}{2}(\frac{\delta^* g^{ij}}{\delta^* x^k} + \dot\partial^i(\bar N_{kt})g^{tj} + \dot\partial^j(\bar N_{kt})g^{ti})g^{kh}\frac{\delta^*}{\delta^* x^h} \\ \qquad +\frac{1}{8}\dot\partial^\beta H_2 g^{ijk}\frac{\partial f}{\partial x^k}\frac{\delta^*}{\delta^* z^\beta} + \frac{1}{2}g^{ij}_k\dot\partial^k, \\ \nabla_{\dot\partial^\alpha}\dot\partial^\beta = -\frac{1}{2}(\frac{\delta^* f g^{\alpha\beta}}{\delta^* x^k} + f\dot\partial^\alpha(\bar N_{k\gamma})g^{\gamma\beta} + f\dot\partial^\beta(\bar N_{k\gamma})g^{\gamma\alpha})g^{kh}\frac{\delta^*}{\delta^* x^h} \\ \qquad -\frac{f^2}{2}(\frac{\delta^* g^{\alpha\beta}}{\delta^* z^\gamma} + \dot\partial^\alpha(\bar N_{\gamma\theta})g^{\theta\beta} + \dot\partial^\beta(\bar N_{\gamma\theta})g^{\theta\alpha})g^{\gamma\lambda}\frac{\delta^*}{\delta^* z^\lambda} + \frac{1}{2}g^{\alpha\beta}_\gamma\dot\partial^\gamma, \end{cases} \tag{21}$$

$$\begin{cases} \nabla_{\frac{\delta^*}{\delta^* x^i}}\dot\partial^j = \nabla_{\dot\partial^j}\frac{\delta^*}{\delta^* x^i} - \dot\partial^j(\bar N_{ik})\dot\partial^k = -\frac{1}{2}\dot\partial^j(\bar N_{ik})\dot\partial^k \\ \qquad -\frac{1}{2}(g^{jh}_i + \mathbf{R}_{iks}g^{sj}g^{kh})\frac{\delta^*}{\delta^* x^h} - \frac{f}{2}\mathbf{R}_{i\alpha k}g^{kj}g^{\alpha\beta}\frac{\delta^*}{\delta^* z^\beta} \\ \qquad +\frac{1}{2}(\frac{\delta^* g^{jk}}{\delta^* x^i} + \dot\partial^k(\bar N_{is})g^{sj})g_{kh}\dot\partial^h + \frac{1}{2f}\dot\partial^\alpha(\bar N_{ik})g^{kj}g_{\alpha\beta}\dot\partial^\beta, \\ \nabla_{\frac{\delta^*}{\delta^* x^i}}\dot\partial^\alpha = \nabla_{\dot\partial^\alpha}\frac{\delta^*}{\delta^* x^i} - \dot\partial^\alpha(\bar N_{ia})\bar\partial^a = -\frac{1}{2}\dot\partial^\alpha(\bar N_{ia})\bar\partial^a \\ \qquad \frac{f}{2}\mathbf{R}_{ki\beta}g^{\beta\alpha}g^{kh}\frac{\delta^*}{\delta^* x^h} + \frac{f^2}{2}\mathbf{R}_{\beta i\gamma}g^{\gamma\alpha}g^{\beta\lambda}\frac{\delta^*}{\delta^* z^\lambda} \\ \qquad +\frac{1}{2}(\frac{1}{f}\frac{\delta^* f g^{\alpha\beta}}{\delta^* x^i} + \dot\partial^\beta(\bar N_{i\gamma})g^{\gamma\alpha})g_{\beta\lambda}\dot\partial^\lambda, \\ \nabla_{\frac{\delta^*}{\delta^* z^\alpha}}\dot\partial^i = \nabla_{\dot\partial^i}\frac{\delta^*}{\delta^* z^\alpha} - \dot\partial^i(\bar N_{\alpha\beta})\dot\partial^\beta = -\frac{1}{2}\dot\partial^i(\bar N_{\alpha\beta})\bar\partial^\beta \\ \qquad \frac{1}{2}\mathbf{R}_{k\alpha s}g^{si}g^{kh}\frac{\delta^*}{\delta^* x^h} + \frac{f}{2}\mathbf{R}_{\beta\alpha k}g^{ki}g^{\beta\gamma}\frac{\delta^*}{\delta^* z^\gamma} + \frac{1}{2}\frac{\delta^* g^{ik}}{\delta^* z^\alpha}g_{kh}\dot\partial^h \\ \qquad +\frac{1}{2f}\dot\partial^\beta(\bar N_{\alpha k})g^{ki}g_{\beta\gamma}\dot\partial^\gamma, \\ \nabla_{\frac{\delta^*}{\delta^* z^\alpha}}\dot\partial^\beta = \nabla_{\dot\partial^\beta}\frac{\delta^*}{\delta^* z^\alpha} - \dot\partial^\beta(\bar N_{\alpha a})\bar\partial^a = -\frac{1}{2}\dot\partial^\beta(\bar N_{\alpha a})\bar\partial^a \\ \qquad \frac{f}{2}\mathbf{R}_{k\alpha\gamma}g^{\gamma\beta}g^{kh}\frac{\delta^*}{\delta^* x^h} - \frac{1}{2}(g^{\beta\lambda}_\alpha + f^2\mathbf{R}_{\alpha\gamma\theta}g^{\theta\beta}g^{\gamma\lambda})\frac{\delta^*}{\delta^* z^\lambda} \\ \qquad -\frac{1}{4f}\delta^\beta_\alpha\frac{\partial f}{\partial x^j}\dot\partial^j + \frac{1}{2}(\frac{\delta^* g^{\beta\gamma}}{\delta^* z^\alpha}g_{\gamma\lambda} + \dot\partial^\gamma(\bar N_{\alpha\theta})g^{\theta\beta}g_{\gamma\lambda})\dot\partial^\lambda, \end{cases} \tag{22}$$

where

$$g^{abc} = \bar\partial^a g^{bc}, \quad g_{abc} = g_{cf}g^f_{ab} = g_{cf}g_{be}g^{ef}_a = g_{cf}g_{be}g_{ad}g^{def}$$

and

$$\Gamma^k_{ij} = \frac{g^{kh}}{2}\left(\frac{\delta^* g_{jh}}{\delta^* x^i} + \frac{\delta^* g_{ih}}{\delta^* x^j} - \frac{\delta^* g_{ij}}{\delta^* x^h}\right),$$

$$\Gamma^\gamma_{\alpha\beta} = \frac{g^{\gamma\lambda}}{2}\left(\frac{\delta^* g_{\beta\lambda}}{\delta^* z^\alpha} + \frac{\delta^* g_{\alpha\lambda}}{\delta^* z^\beta} - \frac{\delta^* g_{\alpha\beta}}{\delta^* z^\lambda}\right).$$

5. Foliations on Warped Product Hamiltonian Spaces

In this section, we study geometric properties of the vertical distribution VT^*M which is bundle-like with respect to the metric G and totally geodesic. The conditions which are equivalent to these properties show a close relation between the geometry of the warped Hamiltonian manifold and its base Hamiltonian spaces.

Theorem 5.1. *Let $\mathbb{H} = (M, H)$ be a warped product Hamiltonian space with nonconstant warped function f. Then the warped Sasaki metric G is bundle-like* *for the vertical foliation VT^*M if and only if $(M_1, (g_{ij}))$ and $(M_2, (g_{\alpha\beta}))$ are two pseudo-Riemannian manifolds.*

P r o o f. With respect to the bundle-like condition (see [2, 9]), G is bundle-like for VT^*M if and only if

$$G(\nabla_X Y + \nabla_Y X, Z) = 0, \quad \forall X, Y \in \Gamma(HT^*M), \quad Z \in \Gamma(VT^*M).$$

It is equivalent to the following equations:

$$G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j} + \nabla_{\frac{\delta^*}{\delta^* x^j}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^k) = G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j} + \nabla_{\frac{\delta^*}{\delta^* x^j}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^\alpha) = 0,$$

$$G(\nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* u^\beta} + \nabla_{\frac{\delta^*}{\delta^* u^\beta}} \frac{\delta^*}{\delta^* u^\alpha}, \dot{\partial}^i) = G(\nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* u^\beta} + \nabla_{\frac{\delta^*}{\delta^* u^\beta}} \frac{\delta^*}{\delta^* u^\alpha}, \dot{\partial}^\gamma) = 0,$$

$$G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* u^\alpha} + \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^j) = G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* u^\alpha} + \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^\beta) = 0.$$

By using (19)–(22), one can obtain that above equations are satisfied if and only if $g_{ijk} = g_{\alpha\beta\gamma} = 0$, and this completes the proof. ■

Theorem 5.2. *Let $\mathbb{H} = (M, H)$ be a warped product Hamiltonian space with nonconstant warped function f. Then, $\mathbb{H} = (M, H)$ is a Landsberg–Hamilton space if and only if the vertical foliation VT^*M is totally geodesic.*

P r o o f. With respect to the definition of the Landsberg–Hamilton space [8], (M, H) is a Landsberg–Hamilton space if and only if

$$g_{ab|_* c} = \frac{\delta^* g_{ab}}{\delta^* \mathbf{x}^c} + g^{bd} \dot{\partial}^a (\bar{N}_{dc}) + g^{ad} \dot{\partial}^b (\bar{N}_{dc}) = 0.$$

By using (19)–(22), one can check that

$$g_{ab|_* c} = 0$$

is satisfied if and only if VT^*M is totally geodesic, and this completes the proof. ■

Theorem 5.3. *Let $\mathbb{H} = (M, H)$ be a warped product Hamiltonian space with nonconstant warped function f. Then the horizontal distribution HT^*M is a totally geodesic one if and only if (M, H) is an Euclidean space.*

P r o o f. Suppose that HT^*M is a totally geodesic distribution, then

$$\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j},\ \nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* u^\alpha},\ \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* x^i},\ \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* u^\beta} \in \Gamma(HT^*M).$$

From (20), the above conditions are hold if and only if

$$\mathbf{R}_{abc} = g_{abc} = 0.$$

These equations mean that (M, H) is an Euclidean space (the pseudo-Riemannian space with zero curvature). ■

Combining Theorems 5.1 and 5.2, we have the following corollary.

Corollary 5.1. *Let the warped product Hamiltonian space (M, H) be a pseudo-Riemannian manifold with nonconstant warped function f, then the vertical distribution VT^*M is totally geodesic and the metric G is bundle-like for VT^*M.*

Acknowledgement. The first author would like to thank the INSF for the partially support on the grant number 92006616.

References

[1] *Y. Alipour-Fakhri and M.M. Rezaii*, The Warped Sasaki–Matsumoto Metric and Bundle-like Condition. — *J. Math. Phys.* **51** (2010), 122701–13.

[2] *A. Bejancu and H.R. Farran*, Foliations and Geometric Structures. Springer–Verlag, Netherlands, 2006.

[3] *R. Bishop and B. O'Neill*, Manifolds of Negative Curvature. — *Trans. Amer. Math. Soc.* **46** (1969), 1–49.

[4] *A.B. Hushmandi and M.M. Rezaii*, On Warped Product Finsler Spaces of Landsberg Type. — *J. Math. Phys.* **52** (2011), 093506–17.

[5] *L. Kozma, I.R. Peter, and C. Varga*, Warped Product of Finsler Manifolds. — *Ann. Univ. Sci. Pudapest* **44** (2001), 157–170.

[6] *R. Miron and M. Anastasiei*, Vector Bundles and Lagrange Spaces with Applications to Relativity. Geometry Balkan Press, No. 1, 1997.

[7] *R. Miron and M. Anastasiei*, The Geometry of Lagrange Spaces: Theory and Applications. Kluwer Acad. Publ., FTPH, No. 59, 1994.

[8] *R. Miron, D. Hrimiuc, H. Shimada, and S.V. Sabau*, The Geometry of Hamilton and Lagrange Spaces. Kluwer Acad. Publ., New York, 2002.

[9] *P. Molino*, Riemannian Foliations, Progress in Math. Birkhauser, Boston, 1988.

[10] *B. O'Neill*, Semi-Riemannian Geometry with Application to Relativity. Academic Press, New York, 1983.

[11] *M.M. Rezaii and Y. Alipour-Fakhri*, On Projectively Related Warped Product Finsler Manifolds. — *Int. J. Geom. Methods Modern Phys.* **8** (2011), 953–967.