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Distribution of Eigenvalues of Sample Covariance Matrices with Tensor Product Samples

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We consider the $n^2 \times n^2$ real symmetric and hermitian matrices M_n, which are equal to the sum m_n of tensor products of the vectors $X^\mu = B(Y^\mu \otimes Y^\mu)$, $\mu = 1, \dots, m_n$, where Y^μ are i.i.d. random vectors from $\mathbb{R}^n (\mathbb{C}^n)$ with zero mean and unit variance of components, and B is an $n^2 \times n^2$ positive definite non-random matrix. We prove that if $m_n/n^2 \to c \in [0, +\infty)$ and the Normalized Counting Measure of eigenvalues of BJB, where J is defined below in (2.6), converges weakly, then the Normalized Counting Measure of eigenvalues of M_n converges weakly in probability to a non-random limit, and its Stieltjes transform can be found from a certain functional equation.

Key words: random matrix, sample covariance matrix, tensor product, distribution of eigenvalues.

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1. Introduction

Sample covariance matrices appeared initially in statistics in the 1920s–1930s. Nowadays these random matrices are widely used in statistical mechanics, probability theory and statistics, combinatorics, operator theory and theoretical computer science in mathematics, as well as in telecommunication theory, qualitative finances, structural mechanics, etc. (see, e.g., [2]).

We consider the sample covariance matrices of the form

$$M_n = \frac{1}{n} X T X^*, \tag{1.1}$$

where X is an $n \times m$ matrix whose entries are i.i.d. random variables such that

$$\mathbf{E}\{X_{ij}\} = 0, \quad \mathbf{E}\{X_{ij}^2\} = 1, \tag{1.2}$$

and T is an $m \times m$ positive definite matrix. We begin considering the ensemble of random matrices with studying Normalized Counting Measure of eigenvalues which is defined by the formula

$$N_n(\Delta) = Card\{i \in [1,n] : \lambda_i \in \Delta\}/n,$$

where

$$-\infty < \lambda_1 \leq \ldots \leq \lambda_n < \infty$$

are the eigenvalues of M_n. Also let σ_m be the Normalized Counting Measure of eigenvalues $\{\tau_i\}_{i=1}^m$ of T.

The first rigorous result on the model (1.1) was obtained in [9], where it was proved that if $\{m_n\}$ is a sequence of positive integers such that

$$m_n \to +\infty,\ n \to +\infty,\ c_n = m_n/n \to c \in [0, +\infty),$$

and the sequence σ_m converges weakly to the probability measure σ,

$$\lim_{n\to\infty} \sigma_m = \sigma,$$

then the Normalized Counting Measure N_n of eigenvalues M_n converges weakly in probability to a non-random measure N $(N(\mathbb{R}) = 1)$. The Stieltjes transform f of N,

$$f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \quad \Im z \neq 0,$$

is uniquely determined by the equation

$$f(z) = \Big(c\int \frac{\tau\sigma(d\tau)}{1 + \tau f(z)} - z\Big)^{-1}.$$

By now, a number of ensembles have been considered. We mention two versions of ensembles of sample covariance matrices that are similar to (1.1). The first is

$$BXX^*B, \tag{1.3}$$

where X is an $n \times m$ matrix whose entries are i.i.d. random variables satisfying (1.2) and B is an $n \times n$ matrix. Note that while studying the eigenvalues of (1.3) we can consider the matrices X^*B^2X instead of (1.3) coinciding with (1.1) for $T = B^2$. The second version is

$$(R_n + aX_n)(R_n + aX_n)^*, \tag{1.4}$$

where X_n is an $n \times m$ matrix whose entries are i.i.d. random variables satisfying (1.2), $a > 0$ constant, and R_n is an $n \times m$ random matrix independent of X_n.

Numerous results and references on the eigenvalue distribution of these random matrices can be found in [3], [4].

The paper is organized as follows. In Section 2 we present our result. In Section 3, we give the proof of the main theorem and in Section 4 we prove all the technical results which we use in Section 3. We denote by C, c, etc., various constants appearing below, which can be different in different formulas.

2. Problem and Main Results

Let us define multi-indexes $\mathbf{i} = (i_1, i_2)$, where $i_1, i_2 = \overline{1,n}$, and inversion in multi-indexes $\bar{\mathbf{i}} = (i_2, i_1)$. Let

$$B = B_n = \{B_{\mathbf{i},\mathbf{j}}\} \tag{2.1}$$

be an $n^2 \times n^2$ real symmetric or hermitian matrix.

We consider the real symmetric or hermitian random matrices

$$M_n = \frac{1}{n^2} \sum_{\mu=1}^{m} X^\mu \otimes \bar{X}^\mu, \tag{2.2}$$

where the vectors X^μ are given by the formula (cf. (1.3))

$$X^\mu = B(Y^\mu \otimes Y^\mu), \mu = 1, \dots, m, \tag{2.3}$$

and $Y^\mu = \{Y_i^\mu\}_{i=1}^n$, $\mu = 1, \dots, m$, are the vectors of $\mathbb{R}^n$ (or $\mathbb{C}^n$) such that $\{Y_i^\mu\}$ (or $\{\Re Y_i^\mu, \Im Y_i^\mu\}$) are i.i.d. random variables for all $i = \overline{1,n}$, $\mu = \overline{1,m}$, and

$$\mathbf{E}\{Y_i^\mu\} = 0, \quad \mathbf{E}\{Y_i^\mu Y_k^\nu\} = \delta_{ik}\delta_{\mu\nu} \tag{2.4}$$

in the real symmetric case, and

$$\mathbf{E}\{Y_i^\mu\} = \mathbf{E}\{Y_i^\mu Y_k^\nu\} = 0, \quad \mathbf{E}\{Y_i^\mu \bar{Y}_k^\mu\} = \delta_{ik} \tag{2.5}$$

in the hermitian case. Introduce the $n^2 \times n^2$ matrix

$$J_{\mathbf{p},\mathbf{q}} = \delta_{\mathbf{pq}} + \delta_{\bar{\mathbf{p}}\mathbf{q}}, \tag{2.6}$$

and denote by N_n and σ_n the Normalized Counting Measure of eigenvalues of M_n and BJB, respectively.

In what follows, by saying that the matrix is bounded, we will mean that its euclidian (or hermitian) norm $| \dots | < c$ for some constant c. The main result of the paper is

Theorem 1. *Let M_n be a random matrix defined by (2.1)–(2.2). Assume that the sequence σ_n converges weakly to a probability measure σ,*

$$\lim_{n\to\infty} \sigma_n = \sigma,$$

B is bounded uniformly in n, and $\{m_n\}$ is a sequence of positive integers such that

$$m_n \to +\infty,\ n \to +\infty,\ c_n = m_n/n^2 \to c \in [0, +\infty).$$

Then the Normalized Counting Measures N_n of eigenvalues of M_n converge weakly in probability to a non-random probability measure N, and if $f^{(0)}$ is the Stieltjes transform of σ, then the Stieltjes transform f of N is uniquely determined by the equation

$$f(z) = f^{(0)}\left(\frac{z}{c - zf(z) - 1}\right)(c - zf(z) - 1)^{-1}$$

in the class of Stieltjes transforms of probability measures.

3. Proof of the Main Result

We will prove the theorem for the technically simpler case of hermitian matrices. The case of real symmetric matrices is analogous. The next Proposition sets a one-to-one correspondence between the finite nonnegative measures and their Stieltjes transforms.

Proposition 1. *Let f be the Stieltjes transform of a finite nonnegative measure m. Then:*

(i) f is analytic in $\mathbb{C}\backslash\mathbb{R}$, and $\overline{f(z)} = f(\overline{z})$;

(ii) $\Im f(z)\Im z > 0$ for $\Im z \neq 0$;

(iii) $|f(z)| \leq m(R)/|\Im z|$, in particular, $\lim\limits_{\eta\to+\infty} \eta|f(i\eta)| \leq \infty$;

(iv) for any function f possessing the above properties there exists a nonnegative finite measure m on $\mathbb{R}$ such that f is its Stieltjes transform, and

$$\lim_{\eta\to+\infty} \eta|f(i\eta)| = m(\mathbb{R}); \tag{3.1}$$

(v) if Δ is an interval of $\mathbb{R}$ whose edges are not atoms of the measure m, then we have the Stieltjes-Perron inversion formula

$$m(\Delta) = \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{\Delta} \Im f(\lambda + i\varepsilon)d\lambda;$$

(vi) the above one-to-one correspondence between the finite nonnegative measures and their Stieltjes transforms is continuous if we use the uniform convergence of analytic functions on a compact set of infinite cardinality of $\mathbb{C}\backslash\mathbb{R}$ for Stieltjes transforms and the vague convergence for measures in general and the weak convergence of probability measures if the r.h.s. of (3.1) is 1;

For the proofs of assertions see [1, Section 59] and [5]. Now recall some facts from linear algebra on the resolvent of real symmetric or hermitian matrix:

Proposition 2. *Let M be a real symmetric (hermitian) matrix and*

$$G_M(z) = (M - z)^{-1}, \Im z \neq 0,$$

be its resolvent. We have:

(i)

$$|G_M(z)| \leq |\Im z|^{-1}; \tag{3.2}$$

(ii) if $G_1(z)$ and $G_2(z)$ are the resolvents of real symmetric (hermitian) matrices M_1 and M_2, respectively, then

$$G_2(z) = G_1(z) - G_1(z)(M_2 - M_1)G_2(z); \tag{3.3}$$

(iii) if $Y \in \mathbb{R}^n(\mathbb{C}^n)$, then

$$G_{M+Y\otimes\bar{Y}} = G_M - \frac{G_M(Y \otimes \bar{Y})G_M}{1 + (G_M Y, Y)}, \quad \Im z \neq 0. \tag{3.4}$$

In what follows, we need

$$Y_i^{\mu(\tau)} = Y_i^{\mu}\mathbf{1}_{|Y_i^{\mu}|\leq\tau\sqrt{n}}, \quad Y_i^{\mu(\tau)\circ} = Y_i^{\mu(\tau)} - \mathbf{E}\{Y_i^{\mu(\tau)}\}.$$

It is easy to see that these random variables satisfy the conditions

$$\mathbf{E}\{Y_i^{\mu(\tau)\circ}\} = \mathbf{E}\{(Y_i^{\mu(\tau)\circ})^2\} = 0, \quad \mathbf{E}\{|Y_i^{\mu(\tau)\circ}|^2\} = 1 + o(1),\ n \to +\infty, \tag{3.5}$$

$$\mathbf{E}\{|Y_i^{\mu(\tau)\circ}|^k\} \leq n^{(k-2)/2}\tau^{k-2}. \tag{3.6}$$

Similarly to X^{μ} and M_n, we can define

$$X^{\mu(\tau)} = B(Y^{\mu(\tau)\circ} \otimes Y^{\mu(\tau)\circ}), \quad M_n^{\tau} = \frac{1}{n^2}\sum_{\mu=1}^{m} X^{\mu(\tau)} \otimes \bar{X}^{\mu(\tau)}.$$

Consider the $n^2 \times n^2$ matrices

$$K_n = \frac{1}{n^2}\sum_{\mu=1}^{m} C^{\mu} \otimes \bar{C}^{\mu}, \quad \widehat{K}_n = \frac{1}{n^2}\sum_{\mu=1}^{m} C^{\mu} \otimes \bar{X}^{\mu},$$

where

$$C^{\mu}_{\mathbf{i}} = \sum_{\mathbf{p}} B_{\mathbf{i},\mathbf{p}}(Y^{\mu}_{p_1}Y^{\mu}_{p_2}(1-\delta_{p_1,p_2}) + Y^{\mu(\tau)\circ}_{p_1}Y^{\mu(\tau)\circ}_{p_2}\delta_{p_1,p_2}). \tag{3.7}$$

Here and below $\sum\limits_{\mathbf{p}} = \sum\limits_{p_1=1}^{n}\sum\limits_{p_2=1}^{n}$.

We need the following simple fact, a version of the min-max principle of linear algebra (see [7], Section I.6.10).

Proposition 3. *Let M_1 and M_2 be the $n \times n$ hermitian matrices and N_1 and N_2 be the Normalized Counting Measures of their eigenvalues. Then for any interval $\Delta \subset \mathbb{R}$, we have*

$$|N_1(\Delta) - N_2(\Delta)| \le \mathrm{rank}(A_1 - A_2)/n. \tag{3.8}$$

Let N_n, $N_n^{(1)}$ and $\widehat{N}_n^{(1)}$ be the Normalized Counting Measures of eigenvalues of matrices M_n, K_n and $\widehat{K}_n$, respectively. Then, according to (3.8) and (3.7), we have

$$\begin{aligned}
|N_n - N_n^{(1)}| &\le |N_n - \widehat{N}_n^{(1)}| + |\widehat{N}_n^{(1)} - N_n^{(1)}| \\
&\le \mathrm{rank}(M_n - \widehat{K}_n)/n^2 + \mathrm{rank}(\widehat{K}_n - K_n)/n^2 \\
&\le \frac{1}{n^2}\Big(\mathrm{rank}\{\sum_{\mathbf{p}} B_{\mathbf{i},\mathbf{p}}\{\sum_{\mu=1}^{m}(Y^{\mu(\tau)\circ}_{p_1}Y^{\mu(\tau)\circ}_{p_2} - Y^{\mu}_{p_1}Y^{\mu}_{p_2})\delta_{p_1,p_2}\bar{X}^{\mu}_{\mathbf{q}}\}_{\mathbf{p},\mathbf{q}}\}_{\mathbf{i},\mathbf{q}} \\
&+ \mathrm{rank}\{\sum_{\mathbf{q}}\{\sum_{\mu=1}^{m} C^{\mu}_{\mathbf{p}}(\bar{Y}^{\mu(\tau)\circ}_{q_1}\bar{Y}^{\mu(\tau)\circ}_{q_2} - \bar{Y}^{\mu}_{q_1}\bar{Y}^{\mu}_{q_2})\delta_{q_1,q_2}\}_{\mathbf{p},\mathbf{q}}\bar{B}_{\mathbf{q},\mathbf{i}}\}_{\mathbf{p},\mathbf{i}}\Big) \\
&\le \frac{1}{n^2}\Big(\mathrm{rank}\{\sum_{\mu=1}^{m}(Y^{\mu(\tau)\circ}_{p_1}Y^{\mu(\tau)\circ}_{p_2} - Y^{\mu}_{p_1}Y^{\mu}_{p_2})\delta_{p_1,p_2}\bar{X}^{\mu}_{\mathbf{q}}\}_{\mathbf{p},\mathbf{q}} \\
&+ \mathrm{rank}\{\sum_{\mu=1}^{m} C^{\mu}_{\mathbf{p}}(\bar{Y}^{\mu(\tau)\circ}_{q_1}\bar{Y}^{\mu(\tau)\circ}_{q_2} - \bar{Y}^{\mu}_{q_1}\bar{Y}^{\mu}_{q_2})\delta_{q_1,q_2}\}_{\mathbf{p},\mathbf{q}}\Big) = \frac{2}{n}.
\end{aligned}$$

Lemma 1. *Let $G^{(1)}(z)$ and $G^{\tau}(z)$ be the resolvents of the matrices K_n and M_n^{τ}, respectively. Then*

$$\frac{1}{n^2}|\mathbf{E}\{\mathrm{Tr}(G^{(1)}(z) - G^{\tau}(z))\}| = o(1),\ n \to +\infty.$$

P r o o f. Consider the $(n^2+m)\times(n^2+m)$ block matrices $\widetilde{M}_n$ and $\widetilde{M}_n^{\tau}$ such that

$$\widetilde{M}_n = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad \widetilde{M}_n^{\tau} = \begin{pmatrix} 0 & (A^{\tau})^* \\ A^{\tau} & 0 \end{pmatrix}, \tag{3.9}$$

where A, A^τ are the $n^2\times m$ matrices, and

$$A_{\mathbf{i},\mu}=n^{-1}C_{\mathbf{i}}^{\mu},\quad A^\tau_{\mathbf{i},\mu}=n^{-1}X_{\mathbf{i}}^{\mu(\tau)}.$$

Denote by $\widetilde{G}(z)$ and $\widetilde{G}^\tau(z)$ the resolvents of the matrices $\widetilde{M}_n$ and $\widetilde{M}_n^\tau$, respectively. Using the formula of inversion of block matrix, we get

$$\mathrm{Tr}(G^{(1)}(z^2)-G^\tau(z^2))=-\frac{z}{2}\mathrm{Tr}(\widetilde{G}(z)-\widetilde{G}^\tau(z)). \tag{3.10}$$

Now we should estimate the last expression. From (3.3), we have

$$\begin{aligned}|\mathrm{Tr}(\widetilde{G}-\widetilde{G}^\tau)|&=|\mathrm{Tr}(\widetilde{G}\widetilde{G}^\tau(\widetilde{M}_n-\widetilde{M}_n^\tau))|\\&\le(\mathrm{Tr}(\widetilde{G}\widetilde{G}^\tau\widetilde{G}^*\widetilde{G}^{\tau*}))^{1/2}(\mathrm{Tr}(\widetilde{M}_n-\widetilde{M}_n^\tau)(\widetilde{M}_n^*-\widetilde{M}_n^{\tau*}))^{1/2}.\end{aligned}$$

Here and below we drop the argument z. Relations (3.2) and (3.9) imply

$$\begin{aligned}|\mathrm{Tr}(\widetilde{G}-\widetilde{G}^\tau)|&\le\frac{n}{\Im z^2}(\mathrm{Tr}(2(A-A^\tau)(A^*-(A^\tau)^*)))^{1/2}\\&=\frac{1}{n\Im z^2}\Big(2\sum_{\mu=1}^m\sum_{\mathbf{i}}(C_{\mathbf{i}}^\mu-X_{\mathbf{i}}^{\mu(\tau)})(\bar{C}_{\mathbf{i}}^\mu-\bar{X}_{\mathbf{i}}^{\mu(\tau)})\Big)^{1/2}\\&=\frac{n}{\Im z^2}\Big(2\sum_{\mu=1}^m\sum_{\mathbf{i},\mathbf{p},\mathbf{q}}B_{\mathbf{i},\mathbf{p}}(1-\delta_{p_1,p_2})(Y_{p_1}^\mu Y_{p_2}^\mu-Y_{p_1}^{\mu(\tau)\circ}Y_{p_2}^{\mu(\tau)\circ})\\&\times B_{\mathbf{q},\mathbf{i}}(1-\delta_{q_1,q_2})(\bar{Y}_{q_1}^\mu\bar{Y}_{q_2}^\mu-\bar{Y}_{q_1}^{\mu(\tau)\circ}\bar{Y}_{q_2}^{\mu(\tau)\circ})\Big)^{1/2}\\&=\frac{1}{\Im z^2}\Big(2\sum_{\mu=1}^m\sum_{\substack{p_1\neq p_2\\q_1\neq q_2}}B^2_{\mathbf{q},\mathbf{p}}(Y_{p_1}^\mu Y_{p_2}^\mu\bar{Y}_{q_1}^\mu\bar{Y}_{q_2}^\mu-Y_{p_1}^{\mu(\tau)\circ}Y_{p_2}^{\mu(\tau)\circ}\bar{Y}_{q_1}^\mu\bar{Y}_{q_2}^\mu\\&-Y_{p_1}^\mu Y_{p_2}^\mu\bar{Y}_{q_1}^{\mu(\tau)\circ}\bar{Y}_{q_2}^{\mu(\tau)\circ}+Y_{p_1}^{\mu(\tau)\circ}Y_{p_2}^{\mu(\tau)\circ}\bar{Y}_{q_1}^{\mu(\tau)\circ}\bar{Y}_{q_2}^{\mu(\tau)\circ})\Big)^{1/2}.\end{aligned}$$

Notice that in view of (3.5) and (2.5), the entries where one of the indexes $\{p_1,p_2,q_1,q_2\}$ differs from others are equal to zero. Thus,

$$\begin{aligned}|\mathrm{Tr}(\widetilde{G}-\widetilde{G}^\tau)|&\le\frac{1}{\Im z^2}\Big(2\sum_{\mu=1}^m\sum_{\substack{\mathbf{p}=\mathbf{q}\\\bar{\mathbf{p}}=\mathbf{q}}}B^2_{\mathbf{q},\mathbf{p}}(Y_{p_1}^\mu Y_{p_2}^\mu\bar{Y}_{q_1}^\mu\bar{Y}_{q_2}^\mu-Y_{p_1}^{\mu(\tau)\circ}Y_{p_2}^{\mu(\tau)\circ}\bar{Y}_{q_1}^\mu\bar{Y}_{q_2}^\mu\\&-Y_{p_1}^\mu Y_{p_2}^\mu\bar{Y}_{q_1}^{\mu(\tau)\circ}\bar{Y}_{q_2}^{\mu(\tau)\circ}+Y_{p_1}^{\mu(\tau)\circ}Y_{p_2}^{\mu(\tau)\circ}\bar{Y}_{q_1}^{\mu(\tau)\circ}\bar{Y}_{q_2}^{\mu(\tau)\circ})\Big)^{1/2}.\end{aligned}$$

Relations (3.5) and (2.5) imply

$$\begin{aligned}&\mathbf{E}\{|Y_{p_1}^\mu|^2|Y_{p_2}^\mu|^2-Y_{p_1}^{\mu(\tau)\circ}Y_{p_2}^{\mu(\tau)\circ}\bar{Y}_{p_1}^\mu\bar{Y}_{p_2}^\mu-Y_{p_1}^\mu Y_{p_2}^\mu\bar{Y}_{p_1}^{\mu(\tau)\circ}\bar{Y}_{p_2}^{\mu(\tau)\circ}+|Y_{p_1}^{\mu(\tau)\circ}|^2|Y_{p_2}^{\mu(\tau)\circ}|^2\}\\&=1-(1+o(1))-(1+o(1))+(1+o(1))=o(1).\end{aligned}$$

Combining all above, we get

$$\frac{1}{n^2}|\mathbf{E}\{\mathrm{Tr}(\widetilde{G}-\widetilde{G}^{\tau})\}| < \frac{(2m\mathrm{Tr}(JB)^2 o(1))^{1/2}}{N\Im z^2} = \frac{\sqrt{2m}}{n\Im z^2}o(1).$$

Finally, in view of (3.10), we have

$$\frac{1}{n^2}|\mathbf{E}\{\mathrm{Tr}(G(z)^{(1)}-G^{\tau}(z))\}| < \frac{\sqrt{m}}{\sqrt{2}n|\Im z|}o(1) = o(1).$$

■

It follows from Lemma 1 that for our purposes it suffices to prove Theorem 1 for the matrix M_n^{τ}. Hence below we will assume that M_n is replaced by M_n^{τ}. To simplify the notations, we drop the index τ and denote

$$G(z) = (M_n - z)^{-1},\ G^{\mu}(z) = G\mid_{X^{\mu}=0},\ N = n^2.$$

In the proof of the main theorem we need some results

Lemma 2. *If F is a non-random $N \times N$ matrix such that $|F| \le c$, then* (*i*)

$$\begin{aligned} &\mathbf{E}\{(FG^{\mu}X^{\mu}, X^{\mu})\} = \mathrm{Tr}(FG^{\mu}BJB), \\ &\mathbf{Var}\{N^{-1}(FG^{\mu}X^{\mu}, X^{\mu})\} = o(1),\ n \to +\infty; \end{aligned} \tag{3.11}$$

(*ii*)

$$\frac{1}{N}|\mathrm{Tr}F(G-G^{\mu})| = O(N^{-1}); \tag{3.12}$$

(*iii*)

$$\mathbf{Var}\{N^{-1}\mathrm{Tr}(FG)\} \le \frac{c}{N}. \tag{3.13}$$

The proof of the lemma is given in Section 4..

According to (3.4), we have

$$G_{\mathbf{i},\mathbf{j}} = G^{\mu}_{\mathbf{i},\mathbf{j}} - N^{-1}\frac{(G^{\mu}X^{\mu})_{\mathbf{i}}(G^{\mu}\bar{X}^{\mu})_{\mathbf{j}}}{1+N^{-1}(G^{\mu}X^{\mu},X^{\mu})}.$$

Hence,

$$(GX^{\mu})_{\mathbf{i}} = \frac{(G^{\mu}X^{\mu})_{\mathbf{i}}}{1+N^{-1}(G^{\mu}X^{\mu},X^{\mu})}.$$

Take any $N \times N$ bounded matrix K. Then

$$\frac{1}{N}\mathrm{Tr}(KGM) = \frac{1}{N^2}\sum_{\mu=1}^{m}\sum_{\mathbf{i},\mathbf{j}} K_{\mathbf{j},\mathbf{i}}(GX^{\mu})_{\mathbf{i}}\bar{X}^{\mu}_{\mathbf{j}}$$

$$= \frac{1}{N^2}\sum_{\mu=1}^{m}\sum_{\mathbf{j}} \frac{(KG^{\mu}X^{\mu})_{\mathbf{j}}\bar{X}^{\mu}_{\mathbf{j}}}{1+N^{-1}(G^{\mu}X^{\mu},X^{\mu})} = \frac{1}{N^2}\sum_{\mu=1}^{m}\frac{(KG^{\mu}X^{\mu},X^{\mu})}{1+N^{-1}(G^{\mu}X^{\mu},X^{\mu})}. \qquad (3.14)$$

To analyze the r.h.s. of (3.14), let us show first that if $\mathcal{C}$ and $\mathcal{D}$ are random variables such that $\mathbf{E}\{|\mathcal{C}|^2+|\mathcal{D}|^2\} < c$ and

$$\bar{\mathcal{C}} = \mathbf{E}\{\mathcal{C}\}, \quad \mathcal{C}^{\circ} = \mathcal{C} - \bar{\mathcal{C}}, \quad \bar{\mathcal{D}} = \mathbf{E}\{\mathcal{D}\}, \quad \mathcal{D}^{\circ} = \mathcal{D} - \bar{\mathcal{D}},$$

then

$$\mathbf{E}\left\{\frac{\mathcal{C}}{\mathcal{D}}\right\} = \frac{\bar{\mathcal{C}}}{\bar{\mathcal{D}}} + O\left(\mathbf{E}\left\{\frac{|\mathcal{C}^{\circ}|^2}{|\bar{\mathcal{D}}|^2} + \frac{|\mathcal{D}^{\circ}|^2}{|\bar{\mathcal{D}}|^2}\right\}\right). \qquad (3.15)$$

Indeed,

$$\frac{\mathcal{C}}{\mathcal{D}} = \frac{\bar{\mathcal{C}}+\mathcal{C}^{\circ}}{\bar{\mathcal{D}}} - \frac{(\bar{\mathcal{C}}+\mathcal{C}^{\circ})\mathcal{D}^{\circ}}{\bar{\mathcal{D}}^2} + O\left(\left(\frac{\mathcal{D}^{\circ}}{\bar{\mathcal{D}}}\right)^3\right).$$

Thus,

$$\mathbf{E}\left\{\frac{\mathcal{C}}{\mathcal{D}}\right\} = \frac{\bar{\mathcal{C}}}{\bar{\mathcal{D}}} + \mathbf{E}\left\{\frac{\mathcal{C}^{\circ}\mathcal{D}^{\circ}}{\bar{\mathcal{D}}^2}\right\} + O\left(\frac{|\mathcal{D}^{\circ}|^3}{\bar{\mathcal{D}}^3}\right) \le \frac{\bar{\mathcal{C}}}{\bar{\mathcal{D}}} + \mathbf{E}\left\{\frac{|\mathcal{C}^{\circ}|^2}{|\bar{\mathcal{D}}|^2} + c\frac{|\mathcal{D}^{\circ}|^2}{|\bar{\mathcal{D}}|^2}\right\}.$$

The last inequality implies (3.15).

Let $\mathcal{C} = N^{-1}(KG^{\mu}X^{\mu},X^{\mu})$, $\mathcal{D} = 1+2N^{-1}(G^{\mu}X^{\mu},X^{\mu})$. Since the matrix K is bounded, it follows from (3.11) that

$$\mathbf{E}_{\mu}\{|\mathcal{C}^{\circ}|^2\} = \mathbf{E}_{\mu}\{|\mathcal{D}^{\circ}|^2\} = o(1),\ n \to +\infty.$$

This, (3.14) and (3.15) imply

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}(KGM)\} = \frac{1}{N}\sum_{\mu=1}^{m}\Big(\mathbf{E}\Big\{\frac{N^{-1}\mathrm{Tr}(KG^{\mu}BJB)}{1+N^{-1}\mathrm{Tr}(G^{\mu}BJB)}\Big\} + o(1)\Big). \qquad (3.16)$$

In the r.h.s. of (3.16) result (3.12) allows us to replace G^{μ} with G,

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}(KGM)\} = \mathbf{E}\Big\{\frac{c_n N^{-1}\mathrm{Tr}(KGBJB)}{1+N^{-1}\mathrm{Tr}(GBJB)} + o(1))\Big\}. \qquad (3.17)$$

The last step is to replace $N^{-1}\mathrm{Tr}(KGBJB)$ and $N^{-1}\mathrm{Tr}(GBJB)$ in (3.17) with their expectations. We use again (3.15) with $\mathcal{C} = N^{-1}\mathrm{Tr}(KGBJB)$, $\mathcal{D} = 1 + N^{-1}\mathrm{Tr}(GBJB)$. It follows from (3.17) and (3.13) that

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}(KGM)\} = \frac{c_n N^{-1}\mathbf{E}\{\mathrm{Tr}(KGBJB)\}}{1 + N^{-1}\mathbf{E}\{\mathrm{Tr}(GBJB)\}} + o(1). \tag{3.18}$$

Notice that

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}(KGM)\} = \frac{1}{N}\mathbf{E}\{\mathrm{Tr}(K(G(M-z) + Gz))\} = \frac{1}{N}\mathbf{E}\{\mathrm{Tr}K\} + \frac{z}{N}\mathbf{E}\{\mathrm{Tr}(KG)\}.$$

This and (3.18) imply that for any bounded matrix K

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}K\} = \frac{1}{N}\mathbf{E}\{\mathrm{Tr}(KG(c_n b_n^{-1}BJB - z))\} + o(1), \tag{3.19}$$

where

$$b_n = 1 + N^{-1}\mathbf{E}\{\mathrm{Tr}(GBJB)\}. \tag{3.20}$$

Taking $K = (c_n b_n^{-1}BJB - z)^{-1}$, we obtain

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}(c_n b_n^{-1}BJB - z)^{-1}\} = f_n(z) + o(1), \tag{3.21}$$

where

$$g_n(z) = \frac{1}{N}\mathrm{Tr}(G(z)), \quad f_n(z) = \mathbf{E}\{g_n(z)\}.$$

It follows from (3.19) with $K = I$ that

$$\frac{1}{N}\mathbf{E}\{\mathrm{Tr}(I + zG)\} = \frac{c_n}{b_n}(b_n - 1) + o(1).$$

Then we get

$$1 + zf_n(z) = c_n(1 - \frac{1}{b_n}) + o(1).$$

Now we can find b_n:

$$b_n = \frac{c_n}{c_n - zf_n(z) - 1 + o(1)}. \tag{3.22}$$

This and (3.21) yield

$$f_n(z) = f_n^{(0)}\left(\frac{z}{c_n - zf_n(z) - 1}\right)(c_n - zf_n(z) - 1)^{-1} + o(1), \tag{3.23}$$

where

$$f_n^{(0)}(z) = \frac{1}{N}\mathbf{E}\{\mathrm{Tr}(BJB - z)^{-1}\}.$$

The sequence $\{f_n\}$ consists of the functions that are analytic and uniformly bounded in n and z. Hence, there exists an analytic in $\mathbb{C}\backslash\mathbb{R}$ function f and a subsequence $\{f_{n_j}\}$ that converges to f uniformly on any compact set of $\mathbb{C}\backslash\mathbb{R}$. In addition, we have

$$\Im f_n(z)\Im z > 0,\ \Im z \neq 0,$$

and thus $\Im f(z)\Im z \geq 0,\ \Im z \neq 0$. By Proposition 1(vi) and the hypothesis of the theorem on the weak convergence of the sequence σ_n to σ, the sequence $f_n^{(0)}$ of their Stieltjes transforms consists of analytic in $\mathbb{C}\backslash\mathbb{R}$ functions that converge uniformly on a compact set of $\mathbb{C}\backslash\mathbb{R}$ to the Stieltjes transform $f^{(0)}$ of the limiting counting measure σ of matrices BJB. This allows us to pass to the limit $n \to +\infty$ in (3.23) and to obtain that the limit f of any converging subsequence of the sequence f_n satisfies the functional equation

$$f(z) = f^{(0)}\left(\frac{z}{c - zf(z) - 1}\right)\Big(c - zf(z) - 1\Big)^{-1}, \tag{3.24}$$

and $\Im f(z)\Im z \geq 0,\ \Im z \neq 0$. The proof of the uniqueness of solution of the equation in the class of functions, analytic for $\Im z \neq 0$ and such that $\Im f(z)\Im z \geq 0,\ \Im z \neq 0$, is analogues to that from [9]. Hence, the whole sequence f_n converges uniformly on a compact set of $\mathbb{C}\backslash\mathbb{R}$ to the unique solution f of the equation. Let us show that the solution possesses the properties $\Im f(z)\Im z \geq 0,\ \Im z \neq 0$ and $\lim_{\eta\to+\infty} \eta|f(i\eta)| = 1$. Assume that $\Im f(z_0) = 0,\ \Im z_0 \neq 0$. Then (3.24) implies that

$$\Im \int \frac{d\sigma(\lambda)}{(c-1)\lambda - z_0(f(z_0) - 1)} = C\Im f^{(0)}(\tilde{z}) = 0,$$

where C is some real constant and $\Im\tilde{z} \neq 0$. This is impossible because, according to Proposition 1(ii), $\Im f^{(0)}(z)$ is strictly positive for any nonreal z. Since $|f(i\eta)| < \eta^{-1}$, we have

$$\lim_{\eta\to+\infty} \eta|f(i\eta)| = \lim_{\eta\to+\infty} \int \frac{\eta d\sigma(\lambda)}{(c-1)\lambda - i\eta - i\eta f(i\eta)} = 1.$$

This and Proposition 1(iv) imply that f is the Stieltjes transform of a probability measure. ∎

4. Proofs of Lemma 2

(i) It follows from (2.5) that

$$\mathbf{E}_\mu\{(FG^\mu X^\mu, X^\mu)\} = \mathrm{Tr}(FG^\mu BJB).$$

Denote

$$r_n^\mu = (FG^\mu X^\mu, X^\mu) - \mathrm{Tr}(FG^\mu BJB).$$

We need to show that $\mathbf{E}_\mu\{(N^{-1}r^\mu)^2\} = o(1)$, $n \to +\infty$. Rewrite

$$\begin{aligned}
r_n^\mu &= \sum_{\mathbf{i},\mathbf{j},\mathbf{p},\mathbf{q}} (FG^\mu)_{\mathbf{i},\mathbf{j}} B_{\mathbf{j},\mathbf{p}} B_{\mathbf{q},\mathbf{i}} (Y^\mu_{p_1} Y^\mu_{p_2} \bar{Y}^\mu_{q_1} \bar{Y}^\mu_{q_2} - J_{\mathbf{p},\mathbf{q}}) \\
&= \sum_{\mathbf{i},\mathbf{j}} (FG^\mu)_{\mathbf{i},\mathbf{j}} \Big(\sum_{\mathbf{p}} B_{\mathbf{j},\mathbf{p}} B_{\mathbf{p},\mathbf{i}} \Big(|Y^\mu_{p_1}|^2 |Y^\mu_{p_2}|^2 - 1 \Big) \\
&\qquad + \sum_{\mathbf{p}} B_{\mathbf{j},\mathbf{p}} B_{\bar{\mathbf{p}},\mathbf{i}} \Big(|Y^\mu_{p_1}|^2 |Y^\mu_{p_2}|^2 - 1 \Big) + \sum_{\substack{\mathbf{p}\neq\mathbf{q} \\ \bar{\mathbf{p}}\neq\mathbf{q}}} B_{\mathbf{j},\mathbf{p}} Y^\mu_{p_1} Y^\mu_{p_2} B_{\mathbf{q},\mathbf{i}} \bar{Y}^\mu_{q_1} \bar{Y}^\mu_{q_2} \Big) \\
&= \sum_{\mathbf{i},\mathbf{j}} (FG^\mu)_{\mathbf{i},\mathbf{j}} \Big(\sum_{\mathbf{p}} B_{\mathbf{j},\mathbf{p}} (JB)_{\mathbf{p},\mathbf{i}} \Big(|Y^\mu_{p_1}|^2 |Y^\mu_{p_2}|^2 - 1 \Big) \\
&\qquad + \sum_{\substack{\mathbf{p}\neq\mathbf{q} \\ \bar{\mathbf{p}}\neq\mathbf{q}}} B_{\mathbf{j},\mathbf{p}} Y^\mu_{p_1} Y^\mu_{p_2} B_{\mathbf{q},\mathbf{i}} \bar{Y}^\mu_{q_1} \bar{Y}^\mu_{q_2} \Big).
\end{aligned}$$

Since G^μ is independent of Y^μ, we obtain

$$\begin{aligned}
\mathbf{E}_\mu\{(N^{-1}r^\mu)^2\} &= \frac{1}{N^2} \mathbf{E}_\mu \Big\{ \Big(\sum_{\mathbf{i},\mathbf{j}} (FG^\mu)_{\mathbf{i},\mathbf{j}} \Big)^2 \Big(\sum_{\mathbf{p}} B_{\mathbf{j},\mathbf{p}} (JB)_{\mathbf{p},\mathbf{i}} \Big(|Y^\mu_{p_1}|^2 |Y^\mu_{p_2}|^2 - 1 \Big) \\
&\qquad + \sum_{\substack{\mathbf{p}\neq\mathbf{q} \\ \bar{\mathbf{p}}\neq\mathbf{q}}} B_{\mathbf{j},\mathbf{p}} Y^\mu_{p_1} Y^\mu_{p_2} B_{\mathbf{q},\mathbf{i}} \bar{Y}^\mu_{q_1} \bar{Y}^\mu_{q_2} \Big)^2 \Big\} \\
&= \frac{1}{N^2} \mathbf{E}_\mu \Big\{ \sum_{\mathbf{i},\mathbf{j}} \sum_{\mathbf{i}',\mathbf{j}'} (FG^\mu)_{\mathbf{i},\mathbf{j}} (\bar{F}\bar{G}^\mu)_{\mathbf{i}',\mathbf{j}'} \\
&\qquad \times \Big(\sum_{\substack{\mathbf{p}\neq\mathbf{q} \\ \bar{\mathbf{p}}\neq\mathbf{q}}} \sum_{\substack{\mathbf{p}'\neq\mathbf{q}' \\ \bar{\mathbf{p}}'\neq\mathbf{q}'}} B_{\mathbf{j},\mathbf{p}} Y^\mu_{p_1} Y^\mu_{p_2} B_{\mathbf{q},\mathbf{i}} \bar{Y}^\mu_{q_1} \bar{Y}^\mu_{q_2} \bar{B}_{\mathbf{j}',\mathbf{p}'} \bar{Y}^\mu_{p'_1} \bar{Y}^\mu_{p'_2} \bar{B}_{\mathbf{q}',\mathbf{i}'} Y^\mu_{q'_1} Y^\mu_{q'_2} \Big\} \\
&\qquad + \frac{1}{N^2} \mathbf{E}_\mu \Big\{ \sum_{\mathbf{i},\mathbf{j}} \sum_{\mathbf{i}',\mathbf{j}'} (FG^\mu)_{\mathbf{i},\mathbf{j}} (F\bar{G}^\mu)_{\mathbf{i}',\mathbf{j}'}
\end{aligned}$$

$$\times \sum_{\mathbf{p}} \sum_{\mathbf{p}'} B_{\mathbf{j},\mathbf{p}}(JB)_{\mathbf{p},\mathbf{i}} \bar{B}_{\mathbf{j}',\mathbf{p}'}(J\bar{B})_{\mathbf{p}',\mathbf{i}'} \Big(|Y^\mu_{p_1}|^2|Y^\mu_{p_2}|^2 - 1\Big)\Big(|Y^\mu_{p'_1}|^2|Y^\mu_{p'_2}|^2 - 1\Big)\Big\}$$

$$+ \frac{2}{N^2}\mathbf{E}_\mu\Big\{ \sum_{\mathbf{i},\mathbf{j}} \sum_{\mathbf{i}',\mathbf{j}'} (FG^\mu)_{\mathbf{i},\mathbf{j}} (\bar{F}\bar{G}^\mu)_{\mathbf{i}',\mathbf{j}'}$$

$$\times \sum_{\mathbf{p}} \sum_{\substack{\mathbf{p}'\neq\mathbf{q}' \\ \bar{\mathbf{p}}'\neq\mathbf{q}'}} B_{\mathbf{j},\mathbf{p}}(JB)_{\mathbf{p},\mathbf{i}} \Big(|Y^\mu_{p_1}|^2|Y^\mu_{p_2}|^2 - 1\Big) \bar{B}_{\mathbf{j}',\mathbf{p}'} \bar{Y}^\mu_{p'_1} \bar{Y}^\mu_{p'_2} \bar{B}_{\mathbf{q}',\mathbf{i}'} Y^\mu_{q'_1} Y^\mu_{q'_2}\Big)\Big\}$$

$$=: \frac{1}{N^2}(R_1 + R_2 + R_3).$$

Denote

$$H = BFG^\mu B,$$

and introduce an $N \times N$ matrix Δ such that

$$\Delta_{\mathbf{i},\mathbf{j}} = \delta_{i_1 j_2}\delta_{i_2 j_1}.$$

It is easy to check that for any $N \times N$ matrix A

$$\begin{aligned} A_{i_2 i_1, j_1 j_2} &= (\Delta A)_{\mathbf{i},\mathbf{j}}, \\ A_{i_1 i_2, j_2 j_1} &= (A\Delta)_{\mathbf{i},\mathbf{j}}. \end{aligned} \tag{4.25}$$

Let us define the set $E = \{p_1, p_2, q_1, q_2, p'_1, p'_2, q'_1, q'_2\}$. Notice that if in the set E there are more than 4 different numbers, then

$$\mathbf{E}_\mu\{Y^\mu_{p_1} Y^\mu_{p_2} \bar{Y}^\mu_{q_1} \bar{Y}^\mu_{q_2} \bar{Y}^\mu_{p'_1} \bar{Y}^\mu_{p'_2} Y^\mu_{q'_1} Y^\mu_{q'_2}\} = 0.$$

Hence we need to consider the sets I_1, I_2, I_3 and I_4 of all multi-indexes $\{\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}'\}$ of the special form:

$$I_1 = \Big\{\{\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}'\} = \{(a,b),(a,c),(d,b),(d,c)\}\Big\},$$
$$I_2 = \Big\{\{\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}'\} = \{(a,b),(c,d),(a,b),(c,d)\}\Big\},$$

where the numbers a, b, c and d are all pairwise different,

$$I_3 = \Big\{\{\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}'\} : \text{there are 3 different numbers (i, j, k) in the set } E\Big\},$$
$$I_4 = \Big\{\{\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}'\} : \text{there are 2 different numbers (i, j) in the set } E\Big\}$$

or any inversion in the multi-indexes of the same form. Since B, F, Δ and G^μ (in view of (3.2)) are bounded, then there exists a constant c such that $|H| < c$.

Hence, in view of (4.25) and (3.6),

$$R_1 \le \mathbf{E}_\mu\Big\{\sum_{I_1} H_{\mathbf{p},\mathbf{q}}\bar{H}_{\mathbf{p}',\mathbf{q}'}|Y_a^\mu|^2|Y_b^\mu|^2|Y_c^\mu|^2|Y_d^\mu|^2 + \sum_{I_2} H_{\mathbf{p},\mathbf{q}}\bar{H}_{\mathbf{p}',\mathbf{q}'}|Y_a^\mu|^2|Y_b^\mu|^2|Y_c^\mu|^2|Y_d^\mu|^2$$
$$+ \sum_{I_3} H_{\mathbf{p},\mathbf{q}}\bar{H}_{\mathbf{p}',\mathbf{q}'}(|Y_i^\mu|^4|Y_j^\mu|^2|Y_k^\mu|^2 + Y_i^\mu|^3|Y_j^\mu|^3|Y_k^\mu|^2)$$
$$+ \sum_{I_4} H_{\mathbf{p},\mathbf{q}}\bar{H}_{\mathbf{p}',\mathbf{q}'}(|Y_i^\mu|^4|Y_j^\mu|^4 + |Y_i^\mu|^6|Y_j^\mu|^2 + |Y_i^\mu|^5|Y_j^\mu|^3)\Big\}$$
$$\le \tilde{c}\Big(\sum_{p_1,p_1',p_2,q_2}(H + \Delta H + H\Delta + \Delta H\Delta)_{p_1p_2,q_1q_2}(\bar{H} + \Delta\bar{H} + \bar{H}\Delta + \Delta\bar{H}\Delta)_{p_1'p_2,p_1'q_2}$$
$$+ \mathrm{Tr}(H + \Delta H + H\Delta + \Delta H\Delta)(H + \Delta H + H\Delta + \Delta H\Delta)^*$$
$$+ |I_3|c^2n\tau^2 + |I_4|c^2n^2\tau^4\Big).$$

Since $\Delta^2 = I$ and $|I_3| = c_1n^3$, $|I_2| = c_2n^2$, we have

$$R_1 \le \tilde{c}\Big(\sum_{p_1,p_1',p_2,q_2} C_{p_1p_2,q_1q_2}C^*_{p_1'p_2,p_1'q_2} + \mathrm{Tr}HH^* + \mathrm{Tr}\Delta HH^* + cn^4\tau\Big),$$

where

$$C = H + \Delta H + H\Delta + \Delta H\Delta.$$

Denote by $\tilde{C}$ an $n \times n$ matrix with the coordinates

$$\tilde{C}_{p_2q_2} = \sum_{p_1=1}^n C_{p_1p_2,p_1q_2}.$$

Then

$$R_1 \le c\Big(\mathrm{Tr}\tilde{C}\tilde{C}^* + \mathrm{Tr}HH^* + \mathrm{Tr}\Delta HH^* + cn^4\tau\Big).$$

It is easy to see that $|\tilde{C}| < n|H| < nc$, hence

$$R_1 \le c(n^3 + n^2 + n^4\tau).$$

Divide the set $\{(\mathbf{p},\mathbf{p}')\}$ of all possible indexes into four sets $\{I_i\}_{i=1}^4$ such that $(\mathbf{p},\mathbf{p}') \in I_i$ if there are exactly i different numbers in the set (p_1,p_2,p_1',p_2'). The matrices H and J are bounded and, in view of (3.5) and (3.6), we have

$$R_2 \le c\mathbf{E}\Big\{\sum_{I_1}|Y_1^\mu|^8 + \sum_{I_2}(|Y_1^\mu|^4|Y_2^\mu|^4 + |Y_1^\mu|^6|Y_2^\mu|^2) + \sum_{I_3}|Y_1^\mu|^4|Y_2^\mu|^2|Y_3^\mu|^2$$
$$+ \sum_{I_4}(|Y_1^\mu|^2|Y_2^\mu|^2 - 1)(|Y_3^\mu|^2|Y_4^\mu|^2 - 1)\Big\}$$
$$= c(|I_1|n^3\tau^6 + |I_2|n^2\tau^4 + |I_3|n\tau^2 + |I_4|o(1)) = cn^4(\tau + o(1)).$$

Notice that if the set of indexes $\{p_1, p_2, p'_1, p'_2, q'_1, q'_2\}$ has more than 3 or less than 2 different numbers, then

$$\mathbf{E}\Big\{\Big(|Y^\mu_{p_1}|^2|Y^\mu_{p_2}|^2 - 1\Big)\bar{Y}^\mu_{p'_1}\bar{Y}^\mu_{p'_2}Y^\mu_{q'_1}Y^\mu_{q'_2}\Big\} = 0.$$

Other terms are divided into the sets I_1 (3 different numbers) and I_2 (2 different numbers). Similarly to the previous case,

$$R_3 \le c\Big(\sum_{I_1} n\tau^2 + \sum_{I_2} n^2\tau^4\Big) = cn^4\tau.$$

Finally, we get

$$\mathbf{E}_\mu\{(N^{-1}r^\mu)^2\} \le o(1) + c\tau.$$

Since this inequality is true for every τ, we have

$$\mathbf{E}_\mu\{(N^{-1}r^\mu)^2\} = o(1).$$

(ii) According to (3.4),

$$(F(G-G^\mu))_{\mathbf{i},\mathbf{j}} = -\frac{N^{-1}(FG^\mu X^\mu)_{\mathbf{i}}\overline{(G^\mu X^\mu)}_{\mathbf{j}}}{1+N^{-1}(G^\mu X^\mu, X^\mu)}.$$

Hence,

$$|\mathrm{Tr}(F(G-G^\mu))| = \left|\frac{N^{-1}(FG^\mu X^\mu, G^\mu X^\mu)}{1+N^{-1}(G^\mu X^\mu, X^\mu)}\right| \le \frac{|F|\,|((G^\mu)^*G^\mu X^\mu, X^\mu)|}{|\Im(G^\mu X^\mu, X^\mu)|}.$$

On the other hand, by the spectral theorem,

$$(G^\mu X^\mu, X^\mu) = \sum_{k=1}^{m-1}\frac{(v^k, X^\mu)^2}{\lambda_k - z},$$

where $\{\lambda_k\}$ are the eigenvalues of G^μ and $\{v^k\}$ are the eigenvectors of G^μ. Then

$$|\Im(G^\mu X^\mu, X^\mu)| = |\Im z|\sum_{k=1}^{m-1}\frac{|(v^k, X^\mu)|^2}{(\lambda_k - z)(\lambda_k^* - z)}.$$

Besides,

$$((G^\mu)^*G^\mu X^\mu, X^\mu) = \sum_{k=1}^{m-1}\frac{|(v^k, X^\mu)|^2}{(\lambda_k - z)(\lambda_k^* - z)}.$$

Finally, we get

$$\frac{1}{N}\mathrm{Tr}F(G-G^{\mu}) \le \frac{|F|}{N|\Im z|} = O(N).$$

(iii) To prove the lemma, we need the following statement of martingale bounds (see [6] for results and references):

Lemma 3. *Let $\{Y^{\mu}\}_{\mu=1}^{m}$ be a sequence of i.i.d random vectors of $\mathbb{R}^n(\mathbb{C}^n)$. Assume that the function $\phi : \mathbb{R}^{nm}(\mathbb{C}^{nm}) \to \mathbb{C}$ is a bounded Boreal function such that*

$$\sup_{X^1,\dots,X^{\mu}\in\mathbb{R}^n(\mathbb{C}^n)} |\phi-\phi^{\mu}| \le c,$$

where $\phi^{\mu} = \phi\,|_{X^{\mu}=0}$. Then

$$\mathbf{Var}\{\phi(Y^1,\dots,Y^{\mu})\} \le 4c^2 m.$$

Take $\phi = \mathrm{Tr}(FG)$. Then, using representation (3.4), we obtain

$$|\phi-\phi^{\mu}| = |\mathrm{Tr}G - \mathrm{Tr}G^{\mu}| = \left|\frac{N^{-1}(G^{\mu}FG^{\mu}X^{\mu},X^{\mu})}{1+N^{-1}(G^{\mu}X^{\mu},X^{\mu})}\right|.$$

Similarly to the proof of the previous result, we have

$$\left|\frac{N^{-1}(G^{\mu}FG^{\mu}X^{\mu},X^{\mu})}{1+N^{-1}(G^{\mu}X^{\mu},X^{\mu})}\right| \le c|\Im z|^{-1}.$$

Thus,

$$|\phi-\phi^{\mu}| \le c|\Im z|^{-1}.$$

So, according to Lemma 3,

$$\mathbf{Var}\{g_n\} \le 4c^2 c_n/N.$$

■

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References

[1] *N.I. Akhiezer and I.M. Glazman*, Theory of Linear Operators in Hilbert Space. Dover, New York, 1993.

[2] *G. Akemann, J. Baik, and P. Di Francesco*, The Oxford Handbook of Random Matrix Theory. Oxford Univ. Press, Oxford, 2011.

[3] *Z.D. Bai and J.W. Silverstein*, Spectral Analysis of Large Dimensional Random Matrices. Springer, New York, 2010.

[4] *P.J. Forrester*, Log-Gases and Random Matrices. Princeton Univ. Press, Princeton, New York, 2010.

[5] *J.S. Geronimo and T.P. Hill*, Necessary and Suffcient Condition that the Limit of Stieltjes Transforms is the Stieltjes Transform. J. Approx. Theory, 2003.

[6] *V.L. Girko*, Theory of Stochastic Canonical Equations. Vols. I, II. Kluwer, Dordrecht, 2001.

[7] *T. Kato*, Perturbation Theory for Linear Operators. Springer-Verlag, Berlin, 1976.

[8] *A. Lytova and L. Pastur*, Central Limit Theorem for Linear Eigenvalue Statistics of Random Matrices with Independent Entries. — *Annals of Probability* **37** (2009), No. 5, 1778–1840 .

[9] *V. Marchenko and L. Pastur*, The Eigenvalue of Distribution in Some Ensembles of Random Matrices. — *Math. USSR Sbornik* **1** (1967).

[10] *L. Pastur and M. Shcherbina*, Eigenvalue Distribution of Large Random Matrices. Mathematical Survives and Monographs. Vol. 171. AMS Providence, RI, 2011.

[11] *M. Shcherbina*, Central Limit Theorem for Linear Eigenvalue Statistics of the Wigner and Sample Covariance Random Matrices. — *J. Math. Phys., Anal., Geom.* **7** (2011), No. 2, 176–192.