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Maxwell–Bloch Equations without Spectral Broadening: Gauge Equivalence, Transformation Operators and Matrix Riemann–Hilbert Problems

M.S. Filipkovska, V.P. Kotlyarov,
and E.A. Melamedova (Moskovchenko)

B. Verkin Institute for Low Temperature Physics and Engineering
of the National Academy of Sciences of Ukraine
47 Nauky Ave., Kharkiv, 61103, Ukraine

E-mail: filipkovskaya@ilt.kharkov.ua
kotlyarov@ilt.kharkov ua
melamedova@ilt.kharkov.ua

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A mixed initial-boundary value problem for nonlinear Maxwell–Bloch (MB) equations without spectral broadening is studied by using the inverse scattering transform in the form of the matrix Riemann–Hilbert (RH) problem. We use transformation operators whose existence is closely related with the Goursat problems with nontrivial characteristics. We also use a gauge transformation which allows us to obtain Goursat problems of the canonical type with rectilinear characteristics, the solvability of which is known. The transformation operators and a gauge transformation are used to obtain the Jost type solutions of the Ablowitz–Kaup–Newel–Segur equations with well-controlled asymptotic behavior by the spectral parameter near singular points. A well posed regular matrix RH problem in the sense of the feasibility of the Schwartz symmetry principle is obtained. The matrix RH problem generates the solution of the mixed problem for MB equations.

Key words: Maxwell–Bloch equations, gauge equivalence, transformation operators, matrix Riemann–Hilbert problems.

Mathematics Subject Classification 2010: 34L25, 34M50, 35F31, 35Q15, 35Q51.

The paper is dedicated to the 95th anniversary of Vladimir Aleksandrovich Marchenko

1. Introduction

Special integral operators that transform solutions of differential equations with constant coefficients into solutions of equations with variable coefficients are a distinctive feature of the Marchenko's papers. In this paper, we propose a systematic use of the transformation operators for the construction of matrix Riemann–Hilbert problems which lead to the solution of the initial-boundary value problem for nonlinear Maxwell–Bloch equations without spectral broadening. The transformation operators proposed are closely related to the Goursat problems with nontrivial characteristics. The use of a gauge transformation allows one to obtain the Goursat problems of the canonical type with rectilinear characteristics as well as their solvability. We use the same transformation to establish a gauge equivalence between two pairs of the Ablowitz–Kaup–Newel–Segur (AKNS) equations to construct their Jost type solutions with the well-controlled asymptotic behavior by a spectral parameter on the complex plane near the singular points. As a result, the well-posed regular matrix RH problem, which generates the solution of the mixed problem for MB equations, is obtained.

The Maxwell–Bloch equations in the integrable case have the following form (sf. [16]):

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{E}}{\partial x} = \langle \rho \rangle, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + 2\mathrm{i}\lambda\rho = \mathcal{N}\mathcal{E}, \tag{2}$$

$$\frac{\partial \mathcal{N}}{\partial t} = -\frac{1}{2}\left(\overline{\mathcal{E}}\rho + \mathcal{E}\overline{\rho}\right). \tag{3}$$

Here the symbol $^{-}$ denotes a complex conjugation, $\mathcal{E} = \mathcal{E}(t,x)$ is a complex valued function of the space variable x and the time t, $\rho = \rho(t,x,\lambda)$, and $\mathcal{N}(t,x,\lambda)$ are the complex valued and real functions of t, x and a spectral parameter λ. The angular brackets $\langle\rangle$ mean the averaging by λ with the given "weight" function $n(\lambda)$,

$$\langle \rho \rangle = \int_{-\infty}^{\infty} \rho(t,x,\lambda) n(\lambda) d\lambda, \qquad \int_{-\infty}^{\infty} n(\lambda) d\lambda = \pm 1. \tag{4}$$

If $n(\lambda) > 0$, then an unstable medium is considered (the so-called two-level laser amplifier). If $n(\lambda) < 0$, then a stable medium is considered (the so-called attenuator).

Equations (1)–(4) have appeared in many physical models. However, first they were studied in [22–25]. The most important is a model of the propagation of electromagnetic waves in a medium with distributed two-level atoms. For example, there are models of self-induced transparency [1,2,14–17], and two-level laser amplifiers [14–16, 27–30]. For these models, $\mathcal{E}(t,x)$ is the complex valued envelope of electromagnetic wave of fixed polarization, $\mathcal{N}(t,x,\lambda)$ and $\rho(t,x,\lambda)$ are the entries of a density matrix of the atomic subsystem

$$\hat{\rho}(t,x,\lambda) = \begin{pmatrix} \mathcal{N}(t,x,\lambda) & \rho(t,x,\lambda) \\ \overline{\rho}(t,x,\lambda) & -\mathcal{N}(t,x,\lambda) \end{pmatrix}.$$

The parameter λ denotes a deviation of the transition frequency from its mean value. The weight function $n(\lambda)$ characterizes the inhomogeneous broadening which is the difference between the initial population of the upper and lower levels. Short reviews on the Maxwell–Bloch equations and applying to them of the inverse scattering transform (IST) method can be found in [1, 2, 16].

We restrict our study to the case where $n(\lambda) = \delta(\lambda)$, i.e., without spectral broadening. Then $\langle\rho\rangle = \rho$ and the system (1)–(4) is written as

$$\frac{\partial\mathcal{E}}{\partial t} + \frac{\partial\mathcal{E}}{\partial x} = \rho, \qquad \frac{\partial\rho}{\partial t} = \mathcal{N}\mathcal{E}, \qquad \frac{\partial\mathcal{N}}{\partial t} = -\frac{1}{2}\left(\overline{\mathcal{E}}\rho + \mathcal{E}\overline{\rho}\right). \tag{5}$$

These equations are simpler than (1)–(4). However, applying of the IST method to (5) is somewhat complicated. The matrix Riemann–Hilbert problem for MB equations (1)–(4) was studied in [19]. The main goal of this paper is to study a mixed problem for the Maxwell–Bloch equations which is defined by the initial and boundary conditions:

$$\mathcal{E}(0,x) = \mathcal{E}_0(x), \quad \rho(0,x) = \rho_0(x), \quad \mathcal{N}(0,x) = \mathcal{N}_0(x), \quad \mathcal{E}(t,0) = \mathcal{E}_1(t), \tag{6}$$

where $x \in (0,l)$ $(l \le \infty)$ and $t \in \mathbb{R}_+$. The function $\mathcal{E}_1(t)$ is a Schwartz-type function (smooth and fast decreasing at infinity). The functions $\mathcal{E}_0(x)$, $\rho_0(x)$, $1 - \mathcal{N}_0(x)$ are smooth if $x \in [0,l]$ or of Schwartz type if $x \in \mathbb{R}_+$.

The functions $\rho(t,x)$, $\mathcal{N}(t,x)$ are not independent. Indeed, equations (2), (3) give $\frac{\partial}{\partial t}\left(|\rho(t,x)|^2 + \mathcal{N}(t,x)^2\right) = 0$, and we put

$$|\rho(t,x)|^2 + \mathcal{N}(t,x)^2 \equiv 1.$$

If we define $\rho(0,x)$, then

$$\mathcal{N}(0,x) = \mp\sqrt{1 - |\rho(0,x)|^2}.$$

If one chooses the sign "minus", then the problem is considered in a stable medium, in the so-called attenuator (for example, a model of self-induced transparency). The matrix Riemann–Hilbert problems were studied for this case in [18, 20]. If the sign "plus" is chosen, then the problem is considered in an unstable medium (for example, a model of a two-level laser amplifier), which is the subject of our study.

We suppose that the functions $\mathcal{E}(t,x)$, $\rho(t,x)$ and $\mathcal{N}(t,x)$ satisfy the MB equations (5) in the domain $x,t \in (0,l)\times(0,\infty)$. We develop the IST method in the form of the matrix Riemann–Hilbert problem in a complex z-plane. The method is based on using the transformation operators for constructing the Jost type solutions of the AKNS equations. The RH problem is defined by spectral functions which, in turn, are defined through the given initial and boundary conditions for the MB equations. The RH problem is meromorphic and simple in some sense: it is deduced by using standard approaches to the inverse scattering transform for the quarter of the xt-plane. Unfortunately, this RH problem has an essential deficiency because it may have multiple eigenvalues and spectral singularities. Therefore we deduce a new regular matrix RH problem, free from the mentioned deficiency, which has the unique solution. Then we prove that this regular RH problem generates a system of compatible differential equations, which is the AKNS system of linear equations [2] for the MB equations without broadening. Thus the RH problem generates a solution to the MB equations.

Our approach differs from those considered for the mixed problem to the MB equations given in [1, 16, 18, 27, 28]. We develop an approach to the simultaneous spectral analysis proposed in [10–13] and in [3–7, 21] for other nonlinear equations and prove that the mixed problem for the MB equations is completely linearizable by the appropriate matrix RH problem.

2. Gauge Transformation of the Ablowitz–Kaup–Newel–Segur Equations

The Ablowitz–Kaup–Newel–Segur equations for the Maxwell–Bloch equations without spectral broadening have the form:

$$\Phi_t = U(t,x,\lambda)\Phi, \qquad U(t,x,\lambda) = -(\mathrm{i}\lambda\sigma_3 + H(t,x)), \tag{7}$$

$$\Phi_x = V(t,x,\lambda)\Phi, \qquad V(t,x,\lambda) = \mathrm{i}\lambda\sigma_3 + H(t,x) + \frac{\mathrm{i}F(t,x)}{4\lambda}, \tag{8}$$

where $H(t,x) = \frac{1}{2}\begin{pmatrix} 0 & \mathcal{E}(t,x) \\ -\overline{\mathcal{E}}(t,x) & 0 \end{pmatrix}$, $F(t,x) = \begin{pmatrix} \mathcal{N}(t,x) & \rho(t,x) \\ \overline{\rho}(t,x) & -\mathcal{N}(t,x) \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix. It is well known [2] that the over-determined system of differential equations (7), (8) is compatible if and only if the compatibility condition

$$U_x - V_t + [U,V] = 0 \tag{9}$$

holds. Equation (9) is equivalent to the system of nonlinear equations

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} = \frac{1}{4}[\sigma_3, F], \qquad \frac{\partial F}{\partial t} = [F,H], \tag{10}$$

which are the matrix form of MB equations (5).

Two systems of the AKNS equations,

$$\Phi_t = U(t,x,\lambda)\Phi,$$
$$\Phi_x = V(t,x,\lambda)\Phi$$

and

$$\Psi_t = \hat{U}(t,x,\lambda)\Psi, \tag{11}$$
$$\Psi_x = \hat{V}(t,x,\lambda)\Psi, \tag{12}$$

are called gauge equivalent [9] if their compatible solutions are related by

$$\Phi(t,x,\lambda) = g(t,x)\Psi(t,x,\lambda),$$

where the unitary matrix $g(t,x)$ does not depend on λ. Obviously, the matrices $(\hat{U},\,\hat{V})$ and $(U,\,V)$ are connected by the relations:

$$\hat{U} = g^{-1}Ug - g^{-1}g_t, \tag{13}$$
$$\hat{V} = g^{-1}Vg - g^{-1}g_x. \tag{14}$$

The nonlinear equations, defined by the corresponding compatibility conditions

$$U_x - V_t + [U,V] = 0, \qquad \hat{U}_x - \hat{V}_t + [\hat{U},\hat{V}] = 0,$$

are also called gauge equivalent [9].

Among all gauge transformations we are interested only in those which make the matrix $F(t,x)$ be diagonal, i.e.,

$$F(t,x) = g(t,x)\sigma_3 g^{-1}(t,x), \qquad g^{-1}(t,x) = g^*(t,x), \tag{15}$$

where $*$ means the Hermitian conjugation. This equality does not define the unitary matrix $g(t,x)$ uniquely. It is defined up to the unitary diagonal matrix

$$g(t,x) = D(t,x)\mathrm{e}^{\chi(t,x)\sigma_3}, \tag{16}$$

where $\chi = \chi(t,x)$ is an imaginary scalar function. It is convenient to chose the matrix $D(t,x)$ in the form

$$D(t,x) = \frac{1}{\sqrt{2(1+N(t,x))}}\begin{pmatrix} 1+N(t,x) & -\rho(t,x) \\ \bar{\rho}(t,x) & 1+N(t,x)\end{pmatrix}, \tag{17}$$

where the root is its arithmetic value. It is easy to verify that $\det D(t,x) \equiv 1$ and $D^{-1}(t,x) = D^*(t,x)$. Then for the matrix $\hat{U}$ (13) we obtain

$$\hat{U} = -\mathrm{i}\lambda S(t,x) + \hat{H}(t,x),$$

where

$$\begin{aligned} S(t,x) :=& g^{-1}(t,x)\sigma_3 g(t,x), \\ \hat{H}(t,x) :=& g^{-1}(t,x)H(t,x)g(t,x) + g^{-1}(t,x)\dot{g}_t(t,x). \end{aligned}$$

The matrix $\hat{H}(t,x) = \mathrm{e}^{-\chi(t,x)\sigma_3}(D^{-1}HD + D^{-1}\dot{D}_t + \dot{\chi}_t\sigma_3)\mathrm{e}^{\chi(t,x)\sigma_3}$. It will be identically equal to zero if the matrix $\hat{D} = D^{-1}HD + D^{-1}\dot{D}_t$ commutes with σ_3. Indeed, in this case, the matrix $\hat{D}$ is proportional to σ_3,

$$\hat{D} = f(t,x)\sigma_3. \tag{18}$$

Then putting $\dot{\chi}_t = -f(t,x)$, one finds that $\hat{H}(t,x) \equiv 0$. In order to prove (18), we use the second equation from (10). Since $F = g\sigma_3 g^{-1} = D\mathrm{e}^{\chi\sigma_3}\sigma_3\mathrm{e}^{-\chi\sigma_3}D^{-1} = D\sigma_3 D^{-1}$, then $\dot{F}_t - [F,H] = D[D^{-1}\dot{D}_t, \sigma_3]D^{-1} - [D\sigma_3 D^{-1}, H] =$

$$= D[D^{-1}\dot{D}_t + D^{-1}HD, \sigma_3]D^{-1} = D[\hat{D}, \sigma_3]D^{-1} = 0$$

It means that $[\hat{D}, \sigma_3] = 0$ and hence $\hat{D} = f(t,x)\sigma_3 + \beta(t,x)I$, where $f = f(t,x)$ and $\beta = \beta(t,x)$ are arbitrary scalars. Further, since $\operatorname{tr}\hat{D} = \operatorname{tr} H + \operatorname{tr} D^{-1}\dot{D}_t = \operatorname{tr} D^{-1}\dot{D}_t = 2N\dot{N}_t + \dot{\rho}_t\bar{\rho} + \dot{\bar{\rho}}\rho_t \equiv 0$, one finds $\beta = 0$. Finally, in view of

$$\dot{\chi}_t = -f(t,x), \qquad f(t,x) = (D^{-1}HD + D^{-1}\dot{D}_t)_{11},$$

where $(\cdot)_{11}$ means (11) element of the matrix, we have that $\hat{H}(t,x) \equiv 0$, and the matrix $\hat{U}$ is equal to

$$\hat{U}(t,x,\lambda) = -\mathrm{i}\lambda S(t,x), \qquad S(t,x) = \begin{pmatrix} \nu(t,x) & p(t,x) \\ \bar{p}(t,x) & -\nu(t,x) \end{pmatrix},$$

where $S(t,x) = g^{-1}(t,x)\sigma_3 g(t,x) = S^*(t,x)$ and $S^2 \equiv I$.

For the matrix $\hat{V} = \hat{V}(t,x,\lambda)$, we have

$$\hat{V} = g^{-1}Vg - g^{-1}g'_x = \mathrm{i}\lambda S(t,x) + R(t,x) + \frac{\mathrm{i}\sigma_3}{4\lambda},$$

where $R = g^{-1}Hg - g^{-1}g'_x = \mathrm{e}^{-\chi\sigma_3}(D^{-1}HD - D^{-1}D'_x - \chi'_x\sigma_3)\mathrm{e}^{\chi\sigma_3}$. Since $\operatorname{tr} R = 0$, then the matrix R takes the form

$$R = (h(t,x) - \chi'_x)\sigma_3 + \frac{\sigma_3}{2}[\sigma_3, g^{-1}Hg - g^{-1}g'_x],$$

where $h(t,x) = (D^{-1}HD - D^{-1}D'_x)_{11}$. By putting $\chi'_x = h(t,x)$, we find that

$$R(t,x) = \frac{\sigma_3}{2}[\sigma_3, \mathrm{e}^{-\chi\sigma_3}(D^{-1}HD - D^{-1}D'_x)\mathrm{e}^{\chi\sigma_3}] = \begin{pmatrix} 0 & r(t,x) \\ -\bar{r}(t,x) & 0 \end{pmatrix}.$$

All written above is true if the equations

$$\dot\chi_t = -f(t,x), \quad f(t,x) = \frac{\bar\rho}{4}\left(\mathcal{E} - \frac{\dot\rho_t}{1+N}\right) - \frac{\rho}{4}\left(\bar{\mathcal{E}} - \frac{\bar{\dot\rho}_t}{1+N}\right),$$
$$\dot\chi_x = h(t,x), \quad h(t,x) = \frac{\bar\rho}{4}\left(\mathcal{E} + \frac{\rho'_x}{1+N}\right) - \frac{\rho}{4}\left(\bar{\mathcal{E}} + \frac{\bar{\rho}'_x}{1+N}\right)$$

are compatible. Indeed, these simple equations are compatible if and only if

$$\frac{\partial h}{\partial t} + \frac{\partial f}{\partial x} = 0.$$

By using the Maxwell–Bloch equations (5), this condition can be verified by routine calculations. Then $\chi(t,x)$ is defined as an integral

$$\chi(t,x) = \int_{(0,0)}^{(t,x)} h(t,s)ds - f(\tau,x)d\tau + \chi(0,0),$$

which does not depend on a path of integration. The free parameter $\chi(0,0)$ will be used in Sec. 5.

The gauge Eqs. (11) and (12) with

$$\hat U = -\mathrm{i}\lambda S(t,x), \qquad \hat V = \mathrm{i}\lambda S(t,x) + R(t,x) + \frac{\mathrm{i}\sigma_3}{4\lambda}$$

are also compatible

$$\hat U_x - \hat V_t + [\hat U, \hat V] = 0.$$

The last equation is equivalent to the system of nonlinear equation

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} = [R,S], \qquad R_t = \frac{1}{4}[S,\sigma_3].$$

Thus we have proved the theorem below.

Theorem 1. *The Maxwell–Bloch Eqs.* (5) *are gauge equivalent to the equations*

$$\frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial x} = p\bar r + \bar p r, \quad \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} = -2\nu r, \quad \frac{\partial r}{\partial t} = -\frac{1}{2}p,$$

where $\nu = \nu(t,x)$ is real, and $p = p(t,x)$, $r = r(t,x)$ are complex valued functions which constitute the matrices $S = \begin{pmatrix} \nu(t,x) & p(t,x) \\ \bar p(t,x) & -\nu(t,x)\end{pmatrix}$ and $R = \begin{pmatrix} 0 & r(t,x) \\ \bar r(t,x) & 0\end{pmatrix}$.

R e m a r k 1. The second matrix equation $R_t = \frac{1}{4}[S, \sigma_3]$ gives

$$S = \nu\sigma_3 + [R_t, \sigma_3] = \nu\sigma_3 + 2R_t\sigma_3 = \begin{pmatrix} \nu & -2r_t \\ -2\bar{r}_t & -\nu \end{pmatrix}.$$

Then the first matrix equation $\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} = [R, S]$ is read as

$$\frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial x} + 2\frac{\partial |r|^2}{\partial t} = 0, \qquad \frac{\partial^2 r}{\partial t^2} + \frac{\partial^2 r}{\partial x \partial t} = \nu r.$$

The gauge AKNS equations (11) and (12) will be used below for the transformation of some Goursat problems and also for the construction of compatible solutions of the AKNS equations (7) and (8) which have a well-controlled asymptotic behavior as $z \to 0$.

3. Basic Solutions of the Ablowitz–Kaup–Newel–Segur Linear Equations

We suppose here that the solution $(\mathcal{E}(t,x),\ \mathcal{N}(t,x),\ \rho(t,x))$ of the mixed problem (5), (6) for the Maxwell–Bloch equations in the domain $t \in \mathbb{R}_+$, $0 \le x \le l \le \infty$ exists, and it is unique and smooth. Then the AKNS linear equations (7) and (8) are compatible. To construct their solutions we use the following lemma.

Lemma 1. *Let Eqs.* (7) *and* (8) *be compatible for all $t, x, \lambda \in \mathbb{R}$. Let $\Phi(t,x,\lambda)$ be a matrix satisfying the t-equation* (7) *for all x (the x-equation* (8) *for all t). Assume that $\Phi(t_0,x,\lambda)$ satisfies the x-equation* (8) *for some $t = t_0 \le \le \infty$ (the t-equation* (7) *for some $x = x_0 \le \infty$). Then $\Phi(t,x,\lambda)$ satisfies the x-equation* (8) *for all t (satisfies the t-equation* (7) *for all x).*

P r o o f. The proof can be found in [3] (Lemma 2.1). ■

Let $Y(t,x,\lambda)$ be a product of the matrices

$$Y(t,x,\lambda) = W(t,x,\lambda)\Phi(t,\lambda), \tag{19}$$

where $W(t,x,\lambda)$ satisfies the x-equation (8) for all t and $W(t,0,\lambda) = I$, and $\Phi(t,\lambda)$ satisfies the t-equation (7) for $x = 0$ under the initial condition

$$\lim_{t\to\infty} \Phi(t,\lambda) e^{\mathrm{i}\lambda t \sigma_3} = I.$$

Let $Z(t,x,\lambda)$ be a product of the matrices

$$Z(t,x,\lambda) = \Psi(t,x,\lambda) w(x,\lambda), \tag{20}$$

where $\Psi(t,x,\lambda)$ satisfies the t-equation (7) for all x and $\Psi(0,x,\lambda) = I$, and $w(x,\lambda)$ satisfies the x-equation (8) for $t=0$ under the initial condition $w(l,\lambda) = e^{\mathrm{i}l\mu(\lambda)\sigma_3}$, where $\mu(\lambda) = \lambda + \frac{1}{4\lambda}$. If $l=\infty$, the initial condition takes the form

$$\lim_{x\to\infty} w(x,\lambda)e^{-\mathrm{i}x\mu(\lambda)\sigma_3} = I.$$

It is easy to see that due to Lemma 1, the matrices $Y(t,x,\lambda)$ and $Z(t,x,\lambda)$ are compatible solutions of the AKNS system of Eqs. (7), (8).

Lemma 2. *Let $\mathcal{E}(t,0) = \mathcal{E}_1(t)$ be smooth and fast decreasing as $t\to\infty$. Then for $\operatorname{Im}\lambda = 0$ there exists the Jost solution $\Phi(t,\lambda)$ of the t-equation* (7) *with $x=0$ represented by the transformation operator*

$$\Phi(t,\lambda) = e^{-\mathrm{i}\lambda t\sigma_3} + \int_t^\infty K(t,\tau)e^{-\mathrm{i}\lambda\tau\sigma_3}d\tau, \qquad \operatorname{Im}\lambda = 0. \tag{21}$$

The kernel $K(t,\tau)$ satisfies the symmetry condition $\overline{K}(t,\tau) = \Lambda K(t,\tau)\Lambda^{-1}$ with matrix $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and it is defined by the Goursat problem:

$$\begin{aligned} \sigma_3\frac{\partial K(t,\tau)}{\partial t} + \frac{\partial K(t,\tau)}{\partial \tau}\sigma_3 &= H(t,0)\sigma_3 K(t,\tau), \\ \sigma_3 K(t,t) - K(t,t)\sigma_3 &= \sigma_3 H(t,0), \\ \lim_{t+\tau\to+\infty} K(t,\tau) &= 0. \end{aligned}$$

The kernel $K(t,\tau)$ is smooth and fast decreasing as $t+\tau\to\infty$.

The proof uses the Goursat problem and the corresponding integral equations which allow one to prove their unique solvability and thus to prove the integral representation (21) (sf. [9]). Due to (21), the vector columns $\Phi[1](t,\lambda)$ and $\Phi[2](t,\lambda)$ of the matrix $\Phi(t,\lambda) = (\Phi[1](t,\lambda), \Phi[2](t,\lambda))$ have analytic continuations $\Phi[1](t,z)$ and $\Phi[2](t,z)$ to the lower half-plane $\mathbb{C}_-$ and the upper half-plane $\mathbb{C}_+$ of the complex z-plane, respectively. Thus the vector columns $\Phi[1](t,z)$, $\Phi[2](t,z)$ are analytic in $\mathbb{C}_-$, $\mathbb{C}_+$, respectively, continuous in $\mathbb{C}_-\cup\mathbb{R}$, $\mathbb{C}_+\cup\mathbb{R}$ and have the following asymptotics:

$$\begin{aligned} \Phi[1](t,z)e^{\mathrm{i}zt} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}(z^{-1}), & \operatorname{Im} z \le 0,\ z\to\infty; \\ \Phi[2](t,z)e^{-\mathrm{i}zt} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{O}(z^{-1}), & \operatorname{Im} z \ge 0,\ z\to\infty. \end{aligned}$$

The symbol O(.) means a matrix whose entries have the indicated order.

Lemma 3. *Let $\mathcal{E}(t,x)$, $\mathcal{N}(t,x)$, $\rho(t,x)$ be smooth. Then for any x and $\operatorname{Im}\lambda = 0$ there exists the solution $\Psi(t,x,\lambda)$ of the t-equation (7) represented by the transformation operator*

$$\Psi(t,x,\lambda) = e^{-\mathrm{i}\lambda t\sigma_3} + \int_{-t}^{t} K_0(t,\tau,x)e^{-\mathrm{i}\lambda\tau\sigma_3}d\tau, \qquad \operatorname{Im}\lambda = 0. \tag{22}$$

The kernel $K_0(t,\tau,x)$ is smooth, it satisfies the symmetry condition $\overline{K_0(t,\tau,x)} = \Lambda K_0(t,\tau,x)\Lambda^{-1}$ with matrix $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and is defined by the Goursat problem:

$$\begin{aligned} \sigma_3\frac{\partial K_0(t,\tau,x)}{\partial t} + \frac{\partial K_0(t,\tau,x)}{\partial \tau}\sigma_3 = H(t,x)\sigma_3 K_0(t,\tau,x), \\ \sigma_3 K_0(t,t,x) - K_0(t,t,x)\sigma_3 = H(t,x)\sigma_3, \\ \sigma_3 K_0(t,-t,x) + K_0(t,-t,x)\sigma_3 = 0. \end{aligned}$$

The proof of the lemma can be found in [3].

The integral representation (22) gives the analyticity of the solution $\Psi(t,x,z)$ for $z \in \mathbb{C}$ and its asymptotic behavior as $z \to \infty$:

$$\Psi(t,x,z)e^{\mathrm{i}zt} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\mathrm{i}zt} \end{pmatrix} + \mathrm{O}(z^{-1}) + \mathrm{O}(e^{2\mathrm{i}zt}z^{-1}), \quad z \to \infty;$$

$$\Psi(t,x,z)e^{-\mathrm{i}zt} = \begin{pmatrix} e^{-2\mathrm{i}zt} & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{O}(z^{-1}) + \mathrm{O}(e^{-2\mathrm{i}zt}z^{-1}), \quad z \to \infty.$$

Since t is positive, these asymptotics mean that $\Psi(t,x,z)e^{\mathrm{i}zt}$ is bounded in $\overline{\mathbb{C}_+}$, and $\Psi(t,x,z)e^{-\mathrm{i}zt}$ is bounded in $\overline{\mathbb{C}_-}$ as $z \to \infty$. For any fixed t and x, they are bounded on any compact set of the complex plane.

Now we pass to the construction of the solutions of the x-equation (8) which has two singular points ∞ and 0 (t-equation (7) had the singular point at ∞ only). First, we consider the Jost solution of (8) which has a good behavior as $z \to \infty$. With this purpose, we represent the x-equation (8) in the form

$$W_x = \left[\mathrm{i}\mu(\lambda)\sigma_3 + H(t,x) + \frac{\mathrm{i}}{4\lambda}(F(t,x) - \sigma_3)\right]W, \qquad \mu(\lambda) = \lambda + \frac{1}{4\lambda}. \tag{23}$$

If $H(t,x) \equiv 0$ and $F(t,x) \equiv \sigma_3$, the x-equation has the exact solution $e^{\mathrm{i}x\mu(\lambda)\sigma_3}$.

Lemma 4. *Let $\mathcal{E}(t,x)$, $\mathcal{N}(t,x)$, $\rho(t,x)$ be the smooth functions for $0 \le x \le l$, and*

$$\mathcal{N}^2(t,x) + |\rho(t,x)|^2 \equiv 1.$$

Then for any t and $\operatorname{Im}\mu(\lambda) = 0$ there exists the solution $W(t,x,\lambda)$ of the x-equation (23) represented by the transformation operators

$$W(t,x,\lambda) = e^{\mathrm{i}x\mu(\lambda)\sigma_3} + \int_{-x}^{x} L(x,y,t)e^{\mathrm{i}y\mu(\lambda)\sigma_3}dy + \frac{1}{\mathrm{i}\lambda}\int_{-x}^{x} M(x,y,t)e^{\mathrm{i}y\mu(\lambda)\sigma_3}dy. \quad (24)$$

The kernels $L(x,y,t)$ and $M(x,y,t)$ are smooth, they satisfy the symmetry condition $\overline{L}(x,y,t) = \Lambda L(x,y,t)\Lambda^{-1}$, $\overline{M}(x,y.t) = \Lambda M(x,y,t)\Lambda^{-1}$ with matrix $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and are defined by the formulas $L(x,y,t) = \hat{L}(x,y,t)$ and $M(x,y,t) = g(t,x)\hat{M}(x,y,t)$, where the unitary matrix $g(t,x)$ is the same as in Section 2. The matrices $\hat{L}(x,y,t)$ and $\hat{M}(x,y,t)$ are the unique solution of the Goursat problem:

$$\begin{aligned}
\sigma_3\frac{\partial \hat{L}}{\partial x} + \frac{\partial \hat{L}}{\partial y}\sigma_3 =& \sigma_3 H(t,x)\hat{L} + \sigma_3[\sigma_3, g(t,x)]\hat{M}, \\
\sigma_3\frac{\partial \hat{M}}{\partial x} + \frac{\partial \hat{M}}{\partial y}\sigma_3 =& \sigma_3 R(t,x)\hat{M} + \frac{\sigma_3}{4}[g^{-1}(t,x), \sigma_3]\hat{L}, \\
\sigma_3\hat{L}(x,x,t) - \hat{L}(x,x,t)\sigma_3 =& \sigma_3 H(t,x), \\
\sigma_3\hat{M}(x,x,t) - \hat{M}(x,x,t)\sigma_3 =& \frac{1}{4}[\sigma_3, g^{-1}(t,x)]\sigma_3, \\
\sigma_3\hat{L}(x,-x,t) + \hat{L}(x,-x,t)\sigma_3 =& 0, \\
\sigma_3\hat{M}(x,-x,t) + \hat{M}(x,-x,t)\sigma_3 =& 0,
\end{aligned} \quad (25)$$

where $R(t,x) = g^{-1}(t,x)H(t,x)g(t,x) - g^{-1}(t,x)g'_x(t,x)$.

P r o o f. Substituting (24) into equation (23) and integrating by parts, we get the Goursat problem $(-x < y < x)$:

$$\begin{aligned}
\frac{\partial L}{\partial x} + \sigma_3\frac{\partial L}{\partial y}\sigma_3 =& H(t,x)L + (\sigma_3 - F(t,x))M, \\
\frac{\partial M}{\partial x} + F(t,x)\frac{\partial M}{\partial y}\sigma_3 =& H(t,x)M + \frac{1}{4}\left(\sigma_3 - F(t,x)\right)M,
\end{aligned}$$

$$\begin{aligned}
\sigma_3 L(x,x,t) - L(x,x,t)\sigma_3 =& \sigma_3 H(t,x), \\
F(t,x)M(x,x,t) - M(x,x,t)\sigma_3 =& -\frac{1}{4}\left(\sigma_3 - F(t,x)\right)\sigma_3, \\
\sigma_3 L(x,-x,t) + L(x,-x,t)\sigma_3 =& 0, \\
F(t,x)M(x,-x,t) + M(x,-x,t)\sigma_3 =& 0.
\end{aligned}$$

To integrate the summand with the factor $\frac{1}{\lambda^2}$ we have to use the identity $\frac{1}{\lambda^2} = 4\frac{\mu}{\lambda} - 4$.

We have obtained the Goursat problem with variable coefficients at derivatives. To reduce this problem to the standard form with constant matrices at derivatives, we use the same gauge transformation as in the previous section. We put $F(t,x) = g(t,x)\sigma_3 g^{-1}(t,x)$ where the unitary matrix $g(t,x)$ is the same as in Section 2. Further we put $L(x,y,t) = \hat{L}(x,y,t)$ and $M(x,y,t) = g(t,x)\hat{M}(x,y,t)$. Taking into account that $\frac{\partial M}{\partial x} = g(t,x)\frac{\partial \hat{M}}{\partial x} + g'_x(t,x)\hat{M}$ and $\frac{\partial M}{\partial y} = g(t,x)\frac{\partial \hat{M}}{\partial y}$, we find that the Goursat problem reduces to the form:

$$\begin{aligned}
\frac{\partial \hat{L}}{\partial x} + \sigma_3 \frac{\partial \hat{L}}{\partial y}\sigma_3 =& H(t,x)\hat{L} + [\sigma_3, g(t,x)]\hat{M},\\
\frac{\partial \hat{M}}{\partial x} + \sigma_3 \frac{\partial \hat{M}}{\partial y}\sigma_3 =& R(t,x)\hat{M} + \frac{1}{4}[g^{-1}(t,x), \sigma_3]\hat{L},\\
\sigma_3\hat{L}(x,x,t) - \hat{L}(x,x,t)\sigma_3 =& \sigma_3 H(t,x),\\
\sigma_3\hat{M}(x,x,t) - \hat{M}(x,x,t)\sigma_3 =& \frac{1}{4}[\sigma_3, g^{-1}(t,x)]\sigma_3,\\
\sigma_3\hat{L}(x,-x,t) + \hat{L}(x,-x,t)\sigma_3 =& 0,\\
\sigma_3\hat{M}(x,-x,t) + \hat{M}(x,-x,t)\sigma_3 =& 0.
\end{aligned}$$

This problem coincides with (25). Thus we obtain the classical Goursat problem which in turn gives the existence of representation (24). ■

The integral representation (24) gives the analyticity of the solution $W(t,x,z)$ for $z \in \mathbb{C} \setminus \{0\}$ and its asymptotic behavior as $z \to \infty$ and $z \to 0$:

$$W(t,x,z)e^{-\mathrm{i}x\mu(z)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\mathrm{i}x\mu(z)} \end{pmatrix} + \mathrm{O}(z^{-1}) + \mathrm{O}(e^{-2\mathrm{i}x\mu(z)}z^{-1}), & z \to \infty; \\ \mathrm{O}(1) + \mathrm{O}(e^{-2\mathrm{i}x\mu(z)}), & z \to 0; \end{cases}$$

$$W(t,x,z)e^{\mathrm{i}x\mu(z)} = \begin{cases} \begin{pmatrix} e^{2\mathrm{i}x\mu(z)} & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{O}(z^{-1}) + \mathrm{O}(e^{2\mathrm{i}x\mu(z)}z^{-1}), & z \to \infty; \\ \mathrm{O}(1) + \mathrm{O}(e^{2\mathrm{i}x\mu(z)}), & z \to 0. \end{cases}$$

Taking into account that $\operatorname{Im}\mu(z) = (1 - \frac{1}{4|z|^2})\operatorname{Im} z$, and since $x > 0$, the above asymptotics mean that $W(t,x,z)e^{-\mathrm{i}x\mu(z)}$ is bounded in the domain $\{z \in \mathbb{C} : \operatorname{Im}\mu(z) \le 0\}$, and $W(t,x,z)e^{\mathrm{i}x\mu(z)}$ is bounded in the domain $\{z \in \mathbb{C} : \operatorname{Im}\mu(z) \ge 0\}$.

Lemma 5. *Let the initial functions $\mathcal{E}_0(x)$, $\rho_0(x)$, $\mathcal{N}_0(x)$ from* (6) *be smooth or of Schwartz type if $x \in \mathbb{R}_+$, i.e., $l = \infty$, and*

$$\mathcal{N}_0^2(x) + |\rho_0(x)|^2 \equiv 1.$$

Then the Jost solution $w(x,\lambda)$ can be represented in the form ($\operatorname{Im}\mu(\lambda) = 0$*),*

$$w(x,\lambda) = e^{\mathrm{i}x\mu(\lambda)\sigma_3} + \int\limits_x^{2l-x} L(x,y)e^{\mathrm{i}y\mu(\lambda)\sigma_3}dy + \frac{1}{\mathrm{i}\lambda}\int\limits_x^{2l-x} M(x,y)e^{\mathrm{i}y\mu(\lambda)\sigma_3}dy. \quad (26)$$

The kernels $L(x,y)$ and $M(x,y)$ satisfy the symmetry conditions

$$\overline{L}(x,y) = \Lambda L(x,y)\Lambda^{-1}, \qquad \overline{M}(x,y) = \Lambda M(x,y)\Lambda^{-1}$$

with matrix $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and they are defined by the formulas $L(x,y) = \hat{L}(x,y)$ and $M(x,y) = g(0,x)\hat{M}(x,y)$, where the unitary matrix $g(0,x)$ is the same as in Section 2 with $t = 0$. The matrices $\hat{L}(x,y)$ and $\hat{M}(x,y)$ are the unique solution of the Goursat problem:

$$\begin{aligned}
\sigma_3\frac{\partial \hat{L}}{\partial x} + \frac{\partial \hat{L}}{\partial y}\sigma_3 =& \sigma_3 H(0,x)\hat{L} + \sigma_3[\sigma_3, g(0,x)]\hat{M},\\
\sigma_3\frac{\partial \hat{M}}{\partial x} + \frac{\partial \hat{M}}{\partial y}\sigma_3 =& \sigma_3 R(0,x)\hat{M} + \frac{\sigma_3}{4}[g^{-1}(0,x), \sigma_3]\hat{L},\\
\sigma_3\hat{L}(x,x) - \hat{L}(x,x)\sigma_3 =& H(0,x)\sigma_3,\\
\sigma_3\hat{M}(x,x) - \hat{M}(x,x)\sigma_3 =& \frac{1}{4}[g^{-1}(0,x), \sigma_3]\sigma_3,\\
\sigma_3\hat{L}(x,2l-x) + \hat{L}(x,2l-x)\sigma_3 =& 0,\\
\sigma_3\hat{M}(x,2l-x) + \hat{M}(x,2l-x)\sigma_3 =& 0,
\end{aligned}$$

or

$$\lim_{x+y\to+\infty}\hat{L}(x,y) = \lim_{x+y\to+\infty}\hat{M}(x,y) = 0, \quad \text{if} \quad l = \infty,$$

where $R(0,x) = g^{-1}(0,x)H(0,x)g(0,x) - g^{-1}(0,x)g'_x(0,x)$. The kernels $L(x,y)$ and $M(x,y)$ are smooth and fast decreasing as $x + y \to \infty$ if $l = \infty$.

P r o o f. By substituting (26) into equation (23) and integrating by parts, we get the Goursat problem ($x < y < 2l - x$):

$$\frac{\partial L}{\partial x} + \sigma_3\frac{\partial L}{\partial y}\sigma_3 = H(0,x)L + (\sigma_3 - F(0,x))M,$$

$$
\begin{aligned}
\frac{\partial M}{\partial x} + F(0,x)\frac{\partial M}{\partial y}\sigma_3 =& H(0,x)M + \frac{1}{4}\left(\sigma_3 - F(0,x)\right)M,\\
\sigma_3 L(x,x) - L(x,x)\sigma_3 =& H(0,x)\sigma_3,\\
F(0,x)M(x,x) - M(x,x)\sigma_3 =& \frac{1}{4}\left(\sigma_3 - F(0,x)\right)\sigma_3,\\
\sigma_3 L(x,2l-x) + L(x,2l-x)\sigma_3 =& 0,\\
F(0,x)M(x,2l-x) + M(x,2l-x)\sigma_3 =& 0.
\end{aligned}
$$

If $l = \infty$, then the last two conditions (for $y = 2l - x$) are changed with

$$
\lim_{x+y\to+\infty} L(x,y) = \lim_{x+y\to+\infty} M(x,y) = 0, \quad \text{if} \quad l = \infty.
$$

Further we put $L(x,y) = \hat{L}(x,y)$ and $M(x,y) = g(0,x)\hat{M}(x,y)$, and we finish the proof in the same way as in Lemma (4). ■

Introduce the notations: $\Omega_\pm = \{z \in \mathbb{C}_\pm \,\big|\, |z| > \frac{1}{2}\}$, $D_\pm = \{z \in \mathbb{C}_\pm \,\big|\, |z| < \frac{1}{2}\}$, $\Sigma = \mathbb{R} \cup C_{up} \cup C_{low}$, where the semicircles C_{up} and C_{low} are: $C_{up} = \{z \in \mathbb{C} \,\big|\, |z| = \frac{1}{2}, \arg z \in (0,\pi)\}$ and $C_{low} = \{z \in \mathbb{C} \,\big|\, |z| = \frac{1}{2}, \arg z \in (\pi, 2\pi)\}$. Let $\overline{\Omega}_\pm$ and $\overline{D}_\pm$ be the closures of the domains $\Omega_\pm$ and $D_\pm$, respectively. The contour Σ is the set where $\operatorname{Im}\mu(\lambda) = 0$:

$$
\Sigma = \{\lambda \in \mathbb{C} : \operatorname{Im}\left(\lambda + \frac{1}{4\lambda}\right) = 0\} = \mathbb{R} \cup C_{up} \cup C_{low}.
$$

The orientation on Σ is depicted in Figure (1).

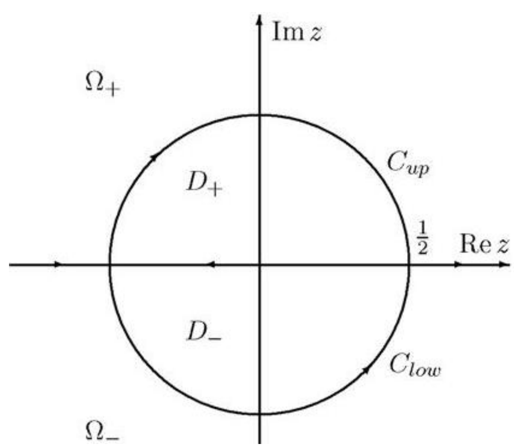

Fig. 1. The domains $\Omega_\pm$, $D_\pm$ and the oriented contour Σ.

The vector columns of the matrix $w(x,\lambda)$ have analytic continuations $w[1](x,z)$ and $w[2](x,z)$ in the domains $\Omega_+\cup D_-$ and $\Omega_-\cup D_+$, respectively. These vectors have the asymptotics:

$$w[1](x,z)e^{-\mathrm{i}x\mu(z)}=\begin{cases}\begin{pmatrix}1\\0\end{pmatrix}+\mathrm{O}(z^{-1}), & z\in\overline{\Omega}_+,\ z\to\infty,\\ \mathrm{O}(1), & z\in\overline{D}_-\setminus\{0\},\ z\to 0.\end{cases}\tag{27}$$

$$w[2](x,z)e^{\mathrm{i}x\mu(z)}=\begin{cases}\begin{pmatrix}0\\1\end{pmatrix}+\mathrm{O}(z^{-1}), & z\in\overline{\Omega}_-,\ z\to\infty,\\ \mathrm{O}(1), & z\in\overline{D}_+\setminus\{0\},\ z\to 0.\end{cases}\tag{28}$$

Formula (19), Lemmas 2, 4 and Eqs. (27), (28) imply the following properties of $Y(t,x,\lambda)=(Y[1](t,x,\lambda)\quad Y[2](t,x,\lambda))$:

1) $Y(t,x,\lambda)$ $(\lambda\neq 0)$ satisfies the $t-$ and $x-$equations (7), (8);

2) $Y(t,x,\lambda)=\Lambda\overline{Y}(t,x,\lambda)\Lambda^{-1},\ \ \lambda\in\mathbb{R}\setminus\{0\}$, where $\Lambda=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$;

3) $\det Y(t,x,\lambda)\equiv 1,\quad \lambda\in\mathbb{R}\setminus\{0\}$;

4) the map $(x,t)\longmapsto Y(t,x,\lambda)$ $(\lambda\neq 0)$ is smooth in t and x;

5) the vector functions $Y[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}\mu(z)x}$, $Y[1](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}\mu(z)x}$ and $Y[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}\mu(z)x}$, $Y[2](t,x,z)e^{\mathrm{i}zt-\mathrm{i}\mu(z)x}$ are analytic in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively, continuous up to the boundary with exception of $\lambda=0$ and have the following asymptotic behavior:

$$\begin{aligned}Y[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}\mu(z)x}&=\begin{pmatrix}1\\0\end{pmatrix}+\mathrm{O}(z^{-1}), && z\in\overline{\Omega}_-,\ z\to\infty,\\ Y[1](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}\mu(z)x}&=\mathrm{O}(1), && z\in\overline{D}_-\setminus\{0\},\ z\to 0,\end{aligned}\tag{29}$$

$$\begin{aligned}Y[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}\mu(z)x}&=\begin{pmatrix}0\\1\end{pmatrix}+\mathrm{O}(z^{-1}), && z\in\overline{\Omega}_+,\ z\to\infty,\\ Y[2](t,x,z)e^{\mathrm{i}zt-\mathrm{i}\mu(z)x}&=\mathrm{O}(1), && z\in\overline{D}_+\setminus\{0\},\ z\to 0.\end{aligned}\tag{30}$$

Formula (20) and Lemmas 3, 5 imply the following properties of
$Z(t,x,\lambda)=(Z[1](t,x,\lambda)\quad Z[2](t,x,\lambda))$:

1) $Z(t,x,\lambda)$ $(\lambda\neq 0)$ satisfies the t- and x-equations (7), (8);

2) $Z(t,x,\lambda)=\Lambda\overline{Z}(t,x,\lambda)\Lambda^{-1},\ \ \lambda\in\mathbb{R}\setminus\{0\}$;

3) $\det Z(t,x,\lambda)\equiv 1,\ \lambda\in\mathbb{R}\setminus\{0\}$;

4) the map $(x,t)\longmapsto Z(t,x,\lambda)$ $(\lambda\neq 0)$ is smooth in t and x;

5) the maps $z\longmapsto Z[1](t,x,z)$ and $z\longmapsto Z[2](t,x,z)$ are analytic in $\Omega_+\cup D_-$ and $\Omega_-\cup D_+$, respectively, and the asymptotic behavior of $Z[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}x\mu(z)}$,

$Z[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)}$ is as follows:

$$Z[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}x\mu(z)} = \begin{pmatrix}1\\0\end{pmatrix} + \mathrm{O}(z^{-1}), \qquad z\in\overline{\Omega}_+,\ z\to\infty, \tag{31}$$
$$Z[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}x\mu(z)} = \mathrm{O}(1), \qquad z\in\overline{D}_-\setminus\{0\}\ z\to 0,$$

$$Z[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)} = \begin{pmatrix}0\\1\end{pmatrix} + \mathrm{O}(z^{-1}), \qquad z\in\overline{\Omega}_-,\ z\to\infty, \tag{32}$$
$$Z[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)} = \mathrm{O}(1), \qquad z\in\overline{D}_+\setminus\{0\},\ z\to 0.$$

Since the matrices $Y(t,x,\lambda)$ and $Z(t,x,\lambda)$ are the solutions of the $t-$ and $x-$equations (7), (8), they are linear dependent. Consequently, there exists a transition matrix $T(\lambda)$, independent of x and t, such that

$$Y(t,x,\lambda) = Z(t,x,\lambda)T(\lambda). \tag{33}$$

The transition matrix is equal to

$$T(\lambda) = Z^{-1}(0,0,\lambda)Y(0,0,\lambda) = w^{-1}(0,\lambda)\Phi(0,\lambda),$$

and hence $T(\lambda) = \Lambda\overline{T}(\lambda)\Lambda^{-1}$, $\lambda\in\mathbb{R}\setminus\{0\}$, i.e., $T(\lambda)$ has the form

$$T(\lambda) = \begin{pmatrix}\overline{a}(\lambda) & b(\lambda)\\ -\overline{b}(\lambda) & a(\lambda)\end{pmatrix}. \tag{34}$$

The scattering relation (33) can be written in the form

$$Y[1](t,x,\lambda) = \overline{a}(\lambda)Z[1](t,x,\lambda) - \overline{b}(\lambda)Z[2](t,x,\lambda), \quad \lambda\in\mathbb{R}\setminus\{0\}, \tag{35}$$
$$Y[2](t,x,\lambda) = a(\lambda)Z[2](t,x,\lambda) + b(\lambda)Z[1](t,x,\lambda), \quad \lambda\in\mathbb{R}\setminus\{0\}. \tag{36}$$

Relations (35), (36) give

$$a(\lambda) = \det(Z[1](t,x,\lambda), Y[2](t,x,\lambda)), \quad \overline{a}(\lambda) = \det(Y[1](t,x,\lambda), Z[2](t,x,\lambda)),$$
$$b(\lambda) = \det(Y[2](t,x,\lambda), Z[2](t,x,\lambda)), \quad \overline{b}(\lambda) = -\det(Z[1](t,x,\lambda), Y[1](t,x,\lambda)).$$

It is easy to see that the matrix

$$w(0,\lambda) = \begin{pmatrix}\alpha(\lambda) & -\overline{\beta}(\lambda)\\ \beta(\lambda) & \overline{\alpha}(\lambda)\end{pmatrix} \tag{37}$$

is the spectral function of the x-equation for $t=0$, which is uniquely defined by the given initial functions $\mathcal{E}(0,x)$, $\rho(0,x)$ and $\mathcal{N}(0,x)$, and the matrix

$$\Phi(0,\lambda) = \begin{pmatrix}\overline{A}(\lambda) & B(\lambda)\\ -\overline{B}(\lambda) & A(\lambda)\end{pmatrix} \tag{38}$$

is the spectral function of the t-equation for $x = 0$, which is uniquely defined by the boundary condition $\mathcal{E}(t, 0)$.

The functions $\alpha(\lambda)$, $\beta(\lambda)$ and $\overline{\alpha}(\lambda)$, $\overline{\beta}(\lambda)$ have analytic continuations in $\Omega_+ \cup \cup D_-$ and $\Omega_- \cup D_+$, respectively, the functions $A(\lambda)$, $B(\lambda)$ and $\overline{A}(\lambda)$, $\overline{B}(\lambda)$ have analytic continuations in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

Let $f^\star(z) := \overline{f}(\overline{z})$ denote the Schwartz conjugate of a function $f(z)$. Then analytic continuations are denoted as ($\alpha(z)$, $\beta(z)$, $\alpha^\star(z)$, $\beta^\star(z)$, $A(z)$, $B(z)$, $A^\star(z)$, $B^\star(z)$) for z in the domains of their analyticity. They have the following asymptotic behavior:

$$\begin{aligned}
\alpha(z) &= 1 + \mathrm{O}(z^{-1}), & \beta(z) &= \mathrm{O}(z^{-1}), & z &\to \infty, & z &\in \overline{\Omega}_+;\\
\alpha(z) &= \mathrm{O}(1), & \beta(z) &= \mathrm{O}(1), & z &\to 0, & z &\in \overline{D}_-;\\
\alpha^\star(z) &= 1 + \mathrm{O}(z^{-1}), & \beta^\star(z) &= \mathrm{O}(z^{-1}), & z &\to \infty, & z &\in \overline{\Omega}_-;\\
\alpha^\star(z) &= \mathrm{O}(1), & \beta^\star(z) &= \mathrm{O}(1), & z &\to 0, & z &\in \overline{D}_+;
\end{aligned}$$

$$\begin{aligned}
A(z) &= 1 + \mathrm{O}(z^{-1}), & B(z) &= \mathrm{O}(z^{-1}), & z &\to \infty, & z &\in \mathbb{C}_+;\\
A^\star(z) &= 1 + \mathrm{O}(z^{-1}), & B^\star(z) &= \mathrm{O}(z^{-1}), & z &\to \infty, & z &\in \mathbb{C}_-;\\
A(z) &= \mathrm{O}(1), & B(z) &= \mathrm{O}(1), & z &\to 0, & z &\in \mathbb{C}_+;\\
A^\star(z) &= \mathrm{O}(1), & B^\star(z) &= \mathrm{O}(1), & z &\to 0, & z &\in \mathbb{C}_-.
\end{aligned}$$

The entries of the transition matrix $T(\lambda)$ in the domains of their analyticity are equal to

$$\begin{aligned}
a(z) &= \alpha(z)A(z) - \beta(z)B(z), & z &\in \Omega_+;\\
b(z) &= \alpha^\star(z)B(z) + \beta^\star(z)A(z), & z &\in D_+;\\
a^\star(z) &= \alpha^\star(z)A^\star(z) - \beta^\star(z)B^\star(z), & z &\in \Omega_-;\\
b^\star(z) &= \alpha(z)B^\star(z) + \beta(z)A^\star(z), & z &\in D_-.
\end{aligned}$$

The spectral functions $a(z)$ and $b(z)$ are defined and smooth for $z \in \Sigma \setminus \{0\}$. The determinant of $T(z) \equiv 1$ and, hence, $a(z)a^\star(z) + b(z)b^\star(z) \equiv 1$ for $z \in \Sigma \setminus \{0\}$. The spectral functions have the asymptotics:

$$\begin{aligned}
a(z) &= 1 + \mathrm{O}(z^{-1}), & z &\to \infty, & z &\in \overline{\Omega}_+; \qquad (39)\\
b(z) &= \mathrm{O}(1), & z &\to 0, & z &\in \overline{D}_+;\\
a^\star(z) &= 1 + \mathrm{O}(z^{-1}), & z &\to \infty, & z &\in \overline{\Omega}_-; \qquad (40)\\
b^\star(z) &= \mathrm{O}(1), & z &\to 0, & z &\in \overline{D}_-.
\end{aligned}$$

If the function $a(z)$ has zeroes $z_j \in \Omega_+$, $j = \overline{1, n}$, then

$$a(z_j) = \det(Z[1](t, x, z_j), Y[2](t, x, z_j)) = 0.$$

Therefore, the vector columns are linear dependent:

$$Y[2](t,x,z_j)=\gamma_j Z[1](t,x,z_j),\quad \gamma_j=\frac{B(z_j)}{\alpha(z_j)}=\frac{A(z_j)}{\beta(z_j)},\quad j=\overline{1,n}. \tag{41}$$

At the conjugate points $\overline{z}_j\in\Omega_-$, $j=\overline{1,n}$,

$$a^\star(\overline{z}_j)=\det(Y[1](t,x,\overline{z}_j),Z[2](t,x,\overline{z}_j))=0.$$

Consequently,

$$Y[1](t,x,\overline{z}_j)=\overline{\gamma}_j Z[2](t,x,\overline{z}_j),\quad \overline{\gamma}_j=\frac{B^\star(\overline{z}_j)}{\alpha^\star(\overline{z}_j)}=\frac{A^\star(\overline{z}_j)}{\beta^\star(\overline{z}_j)},\quad j=\overline{1,n}. \tag{42}$$

If the function $b(z)$ has zeroes $\zeta_k\in D_+$, $k=\overline{1,m}$, then

$$b(\zeta_k)=\det(Y[2](t,x,\zeta_k),Z[2](t,x,\zeta_k))=0.$$

Therefore,

$$Z[2](t,x,\zeta_k)=\eta_k Y[2](t,x,\zeta_k),\quad \eta_k=\frac{B(\zeta_k)}{\beta^\star(\zeta_k)}=-\frac{A(\zeta_k)}{\alpha^\star(\zeta_k)},\quad k=\overline{1,m}. \tag{43}$$

At the conjugate points $\overline{\zeta}_k\in D_-$, $k=\overline{1,m}$,

$$b^\star(\overline{\zeta}_k)=-\det(Z[1](t,x,\overline{\zeta}_k),Y[1](t,x,\overline{\zeta}_k))=0.$$

Hence,

$$Z[1](t,x,\overline{\zeta}_k)=-\overline{\eta}_k Y[1](t,x,\overline{\zeta}_k),\quad \overline{\eta}_k=\frac{B^\star(\overline{\zeta}_k)}{\beta(\overline{\zeta}_k)}=-\frac{A^\star(\overline{\zeta}_k)}{\alpha(\overline{\zeta}_k)},\quad k=\overline{1,m}. \tag{44}$$

Let us define the matrix

$$M(t,x,z)=\begin{cases}\begin{pmatrix} Z[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}x\mu(z)} & \dfrac{Y[2](t,x,z)}{a(z)}e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)}\end{pmatrix}, & z\in\Omega_+,\\ \begin{pmatrix} \dfrac{Y[1](t,x,z)}{a^\star(z)}e^{\mathrm{i}zt-\mathrm{i}x\mu(z)} & Z[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)}\end{pmatrix}, & z\in\Omega_-,\\ \begin{pmatrix} \dfrac{Y[2](t,x,z)}{b(z)}e^{\mathrm{i}zt-\mathrm{i}x\mu(z)} & Z[2](t,x,z)e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)}\end{pmatrix}, & z\in D_+,\\ \begin{pmatrix} Z[1](t,x,z)e^{\mathrm{i}zt-\mathrm{i}x\mu(z)} & -\dfrac{Y[1](t,x,z)}{b^\star(z)}e^{-\mathrm{i}zt+\mathrm{i}x\mu(z)}\end{pmatrix}, & z\in D_-.\end{cases} \tag{45}$$

The matrix M is analytic for $z\in\mathbb{C}\setminus\Sigma$ if $a(z)\neq 0$ and $b(z)\neq 0$. It is meromorphic for $z\in\mathbb{C}\setminus\Sigma$ and has poles at $z_j\in\Omega_+$ and $\overline{z}_j\in\Omega_-$, where $a(z_j)=$

0 and $a^\star(\overline{z}_j)=0$ $j=\overline{1,n}$. The matrix M also has poles at $\zeta_k\in D_+$ and $\overline{\zeta}_k\in D_-$, where $b(\zeta_k)=0$ and $b^\star(\overline{\zeta}_k)=0$, $k=\overline{1,m}$. It has the asymptotics $M(t,x,z)=I+\mathrm{O}(z^{-1})$ as $z\to\infty$.

Let the contour Σ have the orientation shown in Fig. 1. Then the matrix (45) has jumps over the contour Σ,

$$M(t,x,\lambda)_-=M_+(t,x,\lambda)J(t,x,\lambda),\qquad \lambda\in\Sigma\setminus\{0\},$$

where

$$J(t,x,\lambda)=\begin{pmatrix}1+|r(\lambda)|^2 & -r(\lambda)e^{-2\mathrm{i}\theta(t,x,\lambda)}\\ -\overline{r}(\lambda)e^{2\mathrm{i}\theta(t,x,\lambda)} & 1\end{pmatrix},\quad \lambda\in\mathbb{R},\ |z|>\frac{1}{2},$$
$$=\begin{pmatrix}1 & -r^{-1}(\lambda)e^{-2\mathrm{i}\theta(t,x,\lambda)}\\ -\overline{r}^{-1}(\lambda)e^{2\mathrm{i}\theta(t,x,\lambda)} & 1+|r(\lambda)|^{-2}\end{pmatrix},\quad \lambda\in\mathbb{R},\ |z|<\frac{1}{2},\ \lambda\neq0;\tag{46}$$

$$=\begin{pmatrix}0 & -r(\lambda)e^{-2\mathrm{i}\theta(t,x,\lambda)}\\ r^{-1}(\lambda)e^{2\mathrm{i}\theta(t,x,\lambda)} & 1\end{pmatrix},\quad \lambda\in C_{up},$$
$$=\begin{pmatrix}0 & r^\star(\lambda)e^{-2\mathrm{i}\theta(t,x,\lambda)}\\ -r^{\star-1}(\lambda)e^{2\mathrm{i}\theta(t,x,\lambda)} & 1\end{pmatrix},\quad \lambda\in C_{low},\tag{47}$$

where $r(\lambda):=b(\lambda)/a(\lambda)$ is defined on $\mathbb{R}\cup C_{up}$, $r^\star(\lambda):=b^\star(\lambda)/a^\star(\lambda)$ is defined on $\mathbb{R}\cup C_{low}$, and $\theta(t,x,\lambda)=\lambda t-x\mu(\lambda)$, $\lambda\in\Sigma\setminus\{0\}$. It is easy to see that $\det J(t,x,\lambda)\equiv1$, $\lambda\in\Sigma\setminus\{0\}$.

Suppose the number of zeroes is finite and these zeroes are simple poles of $M(t,x,z)$, i.e., $a(z_j)=0$, $a'(z_j)=\frac{da(z)}{dz}\Big|_{z=z_j}\neq0$, $z_j\in\Omega_+$, $j=\overline{1,n}$, and $b(\zeta_k)=0$, $b'(\zeta_k)=\frac{db(z)}{dz}\Big|_{z=\zeta_k}\neq0$, $\zeta_k\in D_+$, $k=\overline{1,m}$. Then

$$\operatorname*{res}_{z=z_j}M[2](t,x,z)=\frac{\gamma_j}{a'(z_j)}e^{-2\mathrm{i}\theta(t,x,z_j)}M[1](t,x,z_j),\tag{48}$$

$$\operatorname*{res}_{z=\overline{z}_j}M[1](t,x,z)=\frac{\overline{\gamma}_j}{a^{\star\prime}(\overline{z}_j)}e^{2\mathrm{i}\theta(t,x,\overline{z}_j)}M[2](t,x,\overline{z}_j),\tag{49}$$

$$\operatorname*{res}_{z=\zeta_k}M[2](t,x,z)=\frac{\eta_k}{b'(\zeta_k)}e^{-2\mathrm{i}\theta(t,x,\zeta_k)}M[1](t,x,\zeta_k),\tag{50}$$

$$\operatorname*{res}_{z=\overline{\zeta}_k}M[1](t,x,z)=\frac{\overline{\eta}_k}{b^{\star\prime}(\overline{\zeta}_k)}e^{2\mathrm{i}\theta(t,x,\overline{\zeta}_k)}M[2](t,x,\overline{\zeta}_k),\tag{51}$$

where γ_j, $\overline{\gamma}_j$ are defined in (41), (42), and η_k, $\overline{\eta}_k$ are defined in (43), (44).

4. Matrix Riemann–Hilbert Problems

In this section we give a reconstruction of the solution to the MB equations in terms of the spectral functions $a(\lambda), b(\lambda)$, which are defined through the spectral functions $A(\lambda), B(\lambda)$, defined by input pulse $\mathcal{E}_1(t)$, and $\alpha(\lambda), \beta(\lambda)$ defined by initial data $\mathcal{E}_0(x)$, $\rho_0(x)$, $\mathcal{N}_0(t)$. In the previous section we proved that the matrices (45) are the solutions of the following matrix RH problem:
Find a 2×2 matrix $M(t,x,z)$ such that

- $M(t,x,z)$ is meromorphic in $z \in \mathbb{C} \setminus \Sigma$ (or analytic if $a(z) \neq 0$ and $b(z) \neq 0$) and continuous up to the set $\Sigma \setminus \{-\frac{1}{2}, 0, \frac{1}{2}\}$; $RH1$
- If $a(z_j) = a^\star(\overline{z}_j) = 0, j = 1,2,\dots,n$ and all these zeroes are simple, then $M(x,t,z)$ has poles at the points $z = z_j$, $z = \overline{z}_j$, $j = 1,2,\dots,n$, and the corresponding residues satisfy relations (48) and (49); $RH2$
- If $b(\zeta_j) = b^\star(\overline{\zeta}_j) = 0, j = 1,2,\dots,m$ and all these zeroes are simple, then $M(x,t,z)$ has poles at the points $z = \zeta_j$, $z = \overline{\zeta}_j$, $j = 1,2,\dots,m$, and the corresponding residues satisfy relations (50) and (51); $RH3$
- $M_-(t,x,\lambda) = M_+(t,x,\lambda)J(t,x,\lambda), \qquad \lambda \in \Sigma \setminus \{-\frac{1}{2}, 0, \frac{1}{2}\}$, where $J(t,x,\lambda)$ is defined in (46) and (47); $RH4$
- $M(t,x,z)$ is bounded in the neighborhoods of the points $\{-\frac{1}{2}, 0, \frac{1}{2}\}$;
- $M(t,x,z) = I + O(z^{-1}), \quad |z| \to \infty.$ $RH5$

Theorem 2. *Let the functions $\mathcal{E}(t,x)$, $\mathcal{N}(t,x)$ and $\rho(t,x)$ be the solution of the mixed problem (5)–(6) for the Maxwell–Bloch equations. Let the corresponding spectral functions $a(\lambda)$ and $b(\lambda)$ have finite and simple zeroes in the domains of their analyticity. Then there exists the matrix $M(t,x,z)$ which is the solution of the Riemann–Hilbert problem $RH1$–$RH5$, and the complex electric field envelope $\mathcal{E}(t,x)$ is defined by the relation*

$$\mathcal{E}(t,x) = -\lim_{z\to\infty} 4\mathrm{i}zM_{12}(t,x,z). \tag{52}$$

P r o o f. The existence of the matrix $M(t,x,z)$ follows from the above considerations. We only need to prove equation (52). The matrix $M(t,x,z)$ defines the solution $\Phi(t,x,z)$ of the AKNS equations (7) and (8) by the formula

$$\Phi(t,x,z) = M(t,x,z)e^{-\mathrm{i}\theta(t,x,z)\sigma_3}.$$

Formula (52) follows from (7) and ($RH5$). Indeed, by substituting the last formula into equation (7), we find

$$M_t + \mathrm{i}z[\sigma_3, M] + HM = 0. \tag{53}$$

Using $(RH5)$, we put

$$M(t,x,z) = I + \frac{m(t,x)}{z} + \mathrm{o}(z^{-1}),$$

where

$$m(t,x) = \lim_{z\to\infty} z(M(t,x,z) - I).$$

This asymptotics and equation (53) give

$$H(t,x) = -\mathrm{i}[\sigma_3, m(t,x)],$$

and hence

$$\mathcal{E}(t,x) = -4\mathrm{i}m_{12} = -\lim_{z\to\infty} 4\mathrm{i}zM_{12}(t,x,z).$$

■

Taking into account the well-known fact that $a(z)$ can have multiple zeroes or infinitely many zeroes with limit points on the contour Σ or zeroes on Σ (the so-called spectral singularities), to avoid the complexities, we propose below another formulation of the matrix RH problem. We use the ideas proposed in [6, 7, 21] and introduce another set of solutions of the AKNS equations with suitable asymptotic behavior of the solutions in the vicinity of the origin ($z = 0$).

5. Basic Solutions with Well-Controlled Asymptotic Behavior at $z = 0$

We will use here the gauge AKNS equations defined by (11), (12) and (13), (14). Let $\hat{X}(t,x,\lambda)$ be a product of the matrices

$$\hat{X}(t,x,\lambda) = \hat{W}(t,x,\lambda)\hat{\Phi}(t,\lambda), \tag{54}$$

where $\hat{W}(t,x,\lambda)$ satisfies the x-equation (12) for all t and $\hat{W}(t,0,\lambda) = I$, and $\hat{\Phi}(t,\lambda)$ satisfies the t-equation (11) for $x = 0$ under the initial condition $\hat{\Phi}(0,\lambda) = I$. By the same way as in Lemmas 3.2 and 3.3, we prove the existence of representations

$$\hat{\Phi}(t,\lambda) = \mathrm{e}^{-\mathrm{i}t\lambda\sigma_3} + \mathrm{i}\lambda \int_{-t}^{t} \hat{K}_0(t,\tau)\mathrm{e}^{-\mathrm{i}\tau\lambda\sigma_3} d\tau \tag{55}$$

and

$$\hat{W}(t,x,\lambda) = \mathrm{e}^{\mathrm{i}x\mu(\lambda)\sigma_3} + \int_{-x}^{x} \hat{L}_0(x,y,t)\mathrm{e}^{\mathrm{i}y\mu(\lambda|\sigma_3} dy + \mathrm{i}\lambda \int_{-x}^{x} \hat{M}_0(x,y,t)\mathrm{e}^{\mathrm{i}y\mu(\lambda|\sigma_3} dy \tag{56}$$

with some matrix functions $\hat{K}_0(t,\tau)$, $\hat{L}_0(x,y,t)$, $\hat{M}_0(x,y,t)$ which have the same symmetry properties as in Lemmas 2–5: $\overline{\hat{L}_0(x,y,t)} = \Lambda \hat{L}_0(x,y,t)\Lambda^{-1}$, etc. These representations guarantee the analyticity of $\hat{X}$ in $z \in \mathbb{C}\setminus\{0\}$ and the following asymptotics as $z \to 0$:

$$\hat{X}[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} = \begin{pmatrix}1\\0\end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im} z \geq 0, \quad z \to 0;$$

$$\hat{X}[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} = \begin{pmatrix}0\\1\end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im} z \leq 0, \quad z \to 0.$$

Due to Lemma 1, the matrix (54) $\hat{X}(t,x,\lambda)$ is a compatible solution of the gauge AKNS Eqs. (11), (12). Then, due to Sec. 2, the matrix

$$X(t,x,\lambda) = g(t,x)\hat{X}(t,x,\lambda) \tag{57}$$

with g defined by (15)–(17) is a solution of the AKNS system of Eqs. (7), (8). The matrix X is analytic in $z \in \mathbb{C}\setminus\{0\}$ and its columns have the asymptotics:

$$X[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} = g(t,x)\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im} z \geq 0, \quad z \to 0; \tag{58}$$

$$X[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} = g(t,x)\begin{pmatrix}0\\1\end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im} z \leq 0, \quad z \to 0. \tag{59}$$

Finally, X is bounded in z for any compact set of the complex plane except the vicinity of the origin and for any fixed t and x.

Let $\hat{Z}(t,x,\lambda)$ be a product of the matrices

$$\hat{Z}(t,x,\lambda) = \hat{\Psi}(t,x,\lambda)\hat{w}(x,\lambda), \tag{60}$$

where $\hat{\Psi}(t,x,\lambda)$ satisfies the t-equation (11) for all x and $\hat{\Psi}(0,x,\lambda) = I$, and $\hat{w}(x,\lambda)$ satisfies the x-equation (12) with $t = 0$ under the initial condition $\hat{w}(l,\lambda) = e^{\mathrm{i}l\mu(\lambda)\sigma_3}$, where $\mu(\lambda) = \lambda + \frac{1}{4\lambda}$. If $l = \infty$, then the initial condition takes the form

$$\lim_{x\to\infty} \hat{w}(x,\lambda)e^{-\mathrm{i}x\mu(\lambda)\sigma_3} = I.$$

It is easy to see that due to Lemma 1, the matrix $\hat{Z}(t,x,\lambda)$ is a compatible solution of the gauge AKNS Eqs. (11), (12).

Again we prove the existence of representations

$$\hat{\Psi}(t,x,\lambda) = \mathrm{e}^{-\mathrm{i}t\lambda\sigma_3} + \mathrm{i}\lambda\int_{-t}^{t} \hat{K}_\Psi(t,\tau,x)\mathrm{e}^{-\mathrm{i}\tau\lambda\sigma_3}d\tau$$

and

$$\hat{w}(x,\lambda) = \mathrm{e}^{\mathrm{i}x\mu(\lambda)\sigma_3} + \int\limits_x^{2l-x} \hat{L}_w(x,y)\mathrm{e}^{\mathrm{i}y\mu(\lambda|\sigma_3}dy + \mathrm{i}\lambda \int\limits_x^{2l-x} \hat{M}_w(x,y)\mathrm{e}^{\mathrm{i}y\mu(\lambda|\sigma_3}dy$$

with some matrix functions $\hat{K}_\Psi(t,\tau,x)$, $\hat{L}_w(x,y)$, $\hat{M}_w(x,y)$ which have the mentioned above symmetry properties. These representations guarantee the analyticity of the columns $\hat{Z}[1](t,x,z)$ and $\hat{Z}[2](t,x,z)$ of the matrix (60) in the domains $\operatorname{Im}\mu(z) \geq 0$ and $\operatorname{Im}\mu(z) \leq 0$, respectively, and give the following asymptotics as $z \to 0$:

$$\hat{Z}[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im}\mu(z) \geq 0, \quad z \to 0; \tag{61}$$

$$\hat{Z}[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im}\mu(z) \leq 0, \quad z \to 0. \tag{62}$$

Due to Section 2, the matrix

$$\widetilde{Z}(t,x,\lambda) = g(t,x)\hat{Z}(t,x,\lambda) \tag{63}$$

with g defined by (15), (16), (17) is a solution of the AKNS equations (7), (8). The columns $\widetilde{Z}[1](t,x,z)$ and $\widetilde{Z}[2](t,x,z)$ of the matrix $\widetilde{Z}$ are analytic in the domains $\operatorname{Im}\mu(z) \geq 0$ and $\operatorname{Im}\mu(z) \leq 0$, respectively, and these columns have the asymptotics:

$$\widetilde{Z}[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} = g(t,x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im} z \geq 0, \quad z \to 0; \tag{64}$$

$$\widetilde{Z}[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} = g(t,x)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{O}(z), \qquad \operatorname{Im} z \leq 0, \quad z \to 0. \tag{65}$$

Thus the construction of the compatible solutions (57) and (63) with well-controlled behavior (58), (59) and (64), (65) as $z \to 0$ is finished.

Introduce the corresponding spectral functions. They are generated by the linear dependence of the above solutions. Thus, for $\operatorname{Im}\mu(\lambda) = 0$, the matrices $\widetilde{Z}(t,x,\lambda)$ and $X(t,x,\lambda)$ are linear dependent:

$$\widetilde{Z}(t,x,\lambda) = X(t,x,\lambda)T_{\widetilde{Z}}(\lambda),\ T_{\widetilde{Z}}(\lambda) = \widetilde{Z}(0,0,\lambda) = \hat{w}(0.\lambda) = \begin{pmatrix} \alpha_0^\star(\lambda) & \beta_0(\lambda) \\ -\beta_0^\star(\lambda) & \alpha_0(\lambda) \end{pmatrix}. \tag{66}$$

The spectral functions $\alpha_0(\lambda)$, $\alpha_0^\star(\lambda) = \overline{\alpha}_0(\overline{\lambda})$ and $\beta_0(\lambda)$, $\beta_0^\star(\lambda) = \overline{\beta}_0(\overline{\lambda})$ are defined by the gauge x-equation (12) with $t = 0$ which, in turn, is determined by the function $g(0,x)$ and the initial functions $\mathcal{E}_0(x)$, $\rho_0(x)$ and $\mathcal{N}_0(x)$. These spectral functions are analytic in $z \in D_+$ and, due to (61), (62), have the asymptotic behavior:

$$\alpha_0(z) = 1 + \mathrm{O}(z), \qquad \beta_0(z) = \mathrm{O}(z), \quad z \to 0, \quad z \in \overline{D_+} \setminus \{0\}. \tag{67}$$

The functions $\alpha_0^\star(z)$ and $\beta_0^\star(z)$ are analytic in $z \in D_-$ and have the asymptotics:

$$\alpha_0^\star(z) = 1 + \mathrm{O}(z), \qquad \beta_0^\star(z) = \mathrm{O}(z), \quad z \to 0, \quad z \in \overline{D_-} \setminus \{0\}. \tag{68}$$

Formulas (67), (68) yield that there exists $\varepsilon > 0$ and a set $O_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$ such that

$$\alpha_0(z) \neq 0, \quad z \in \overline{D_+} \cap O_\varepsilon \setminus \{0\}; \qquad \alpha_0^\star(z) \neq 0, \quad z \in \overline{D_-} \cap O_\varepsilon \setminus \{0\}.$$

The relation

$$Z(t,x,\lambda) = X(t,x,\lambda)T_Z(\lambda), \qquad \operatorname{Im}\mu(\lambda) = 0,$$

where $T_Z(\lambda) = X^{-1}(0,0,\lambda)Z(0,0,\lambda) = g^{-1}(0,0)w(0,\lambda)$, generates the spectral functions

$$T_Z(\lambda) = g^{-1}(0,0)\begin{pmatrix} \alpha(\lambda) & -\beta^\star(\lambda) \\ \beta(\lambda) & \alpha^\star(\lambda) \end{pmatrix} =: \begin{pmatrix} \alpha_g(\lambda) & -\beta_g^\star(\lambda) \\ \beta_g(\lambda) & \alpha_g^\star(\lambda) \end{pmatrix}, \tag{69}$$

where $g^{-1}(0,0)$ is the unitary matrix, and the entries of the matrix $w(0,\lambda)$ were defined in (37).

Finally, the relation

$$Y(t,x,\lambda) = X(t,x,\lambda)T_Y(\lambda), \qquad \lambda \in \mathbb{R} \setminus \{0\},$$

defines the transition matrix $T_Y(\lambda) = X^{-1}(0,0,\lambda)Y(0,0,\lambda) = g^{-1}(0,0)\Phi(0,\lambda)$, which has the form

$$T_Y(\lambda) = g^{-1}(0,0)\begin{pmatrix} \overline{A}(\lambda) & B(\lambda) \\ -\overline{B}(\lambda) & A(\lambda) \end{pmatrix} =: \begin{pmatrix} \overline{A}_g(\lambda) & B_g(\lambda) \\ -\overline{B}_g(\lambda) & A_g(\lambda) \end{pmatrix}, \quad \operatorname{Im}\lambda = 0, \tag{70}$$

where the entries of the matrix $\Phi(0,\lambda)$ were defined in (38).

The spectral functions defined by $\hat{w}(0,\lambda)$ and $w(0,\lambda)$ are related each other by the formula

$$\hat{w}(0,\lambda) = g^{-1}(0,0)w(0,\lambda), \qquad \operatorname{Im}\mu(\lambda) = 0. \tag{71}$$

Indeed, the matrices $w(x,\lambda)$ and $g(0,x)\hat{w}(x,\lambda)$ are the solutions of (8). Therefore they are linear dependent, i.e., $w(x,\lambda) = g(0,x)\hat{w}(x,\lambda)C(\lambda)$, where $C(\lambda) = \mathrm{e}^{-\mathrm{i}l\mu(\lambda)\sigma_3}g^{-1}(0,l)\mathrm{e}^{\mathrm{i}l\mu(\lambda)\sigma_3}$. Without loss of generality, we put $\mathcal{N}_0(l) =$

1. Then $\rho_0(l) = 0$, $g^{-1}(0,l) = \mathrm{e}^{-\chi(0,l)\sigma_3}$ where $\chi(0,l) = \int\limits_0^l h(0,x)dx + \chi(0,0)$. Since $\chi(0,0)$ is a free parameter, we can put $\chi(0,l) = 0$. Then $\hat{w}(x,\lambda) = g^{-1}(0,x)w(x,\lambda)$, and hence (71) is valid. Finally, from (66) and (69), we find

$$\begin{pmatrix} \alpha_0^\star(\lambda) & \beta_0(\lambda) \\ -\beta_0^\star(\lambda) & \alpha_0(\lambda) \end{pmatrix} = \begin{pmatrix} \alpha_g(\lambda) & -\beta_g^\star(\lambda) \\ \beta_g(\lambda) & \alpha_g^\star(\lambda) \end{pmatrix}. \tag{72}$$

All the introduced compatible solutions X, Y, Z, $\widetilde{Z}$ of the AKNS equations (7), (8) and the spectral functions defined by $T(\lambda)$ (34), $T_{\widetilde{Z}}(\lambda)$ (66), $T_Z(\lambda)$ (69), $T_Y(\lambda)$ (70) allow us to formulate a new matrix RH problem which is regular, i.e, without poles. Moreover, this formulation does not require any additional assumptions on the absence of spectral singularities.

6. Regular Matrix Riemann–Hilbert Problem and a Solution of the Mixed Problem for the Maxwell–Bloch Equations

Due to (39), $a(z) = 1 + \mathrm{O}(z^{-1})$ and $a^\star(z) = 1 + \mathrm{O}(z^{-1})$ as $z \to \infty$.

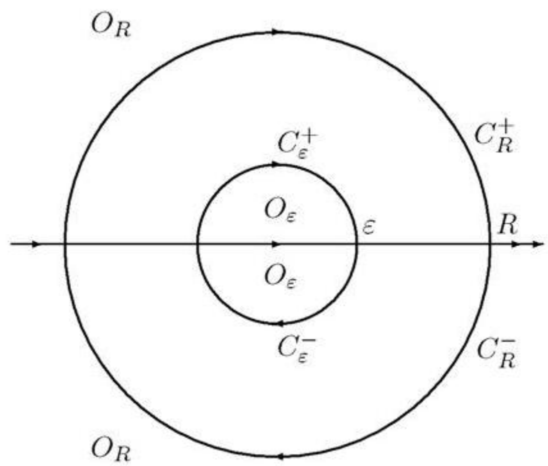

Fig. 2. The domains O_R, O_ε and the oriented contour Γ. $O_R = \{z \in \mathbb{C} : |z| > R\}$, $O_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$, $C_R^+ = \{z \in \mathbb{C} \,\big|\, |z| = R, \arg z \in (0,\pi)\}$, $C_R^- = \{z \in \mathbb{C} \,\big|\, |z| = R, \arg z \in (\pi, 2\pi)\}$, $C_\varepsilon^+ = \{z \in \mathbb{C} \,\big|\, |z| = \varepsilon, \arg z \in (0,\pi)\}$, $C_\varepsilon^- = \{z \in \mathbb{C} \,\big|\, |z| = \varepsilon, \arg z \in (\pi, 2\pi)\}$, $\Gamma := \mathbb{R} \cup C_\varepsilon \cup C_R = \mathbb{R} \cup C_\varepsilon^+ \cup C_\varepsilon^- \cup C_R^+ \cup C_R^-$.

Hence there exists $R > 0$ and a set $O_R = \{z \in \mathbb{C} : |z| > R\}$ such that

$$a(z) \neq 0, \quad z \in \overline{\Omega_+} \cap O_R; \qquad a^\star(z) \neq 0, \quad z \in \overline{\Omega_-} \cap O_R.$$

Define the new contour Γ which contains the real axis $\mathbb{R}$ orientated from left to right and two circles of the radii ε and R oriented clockwise, i.e., $\Gamma := \mathbb{R} \cup C_\varepsilon \cup C_R$ (Fig. 2).

Now, using the vectors $Y[1]$, $Y[2]$, $Z[1]$, $Z[2]$ from Sec. 3 and the vectors (58), (59), (64), (65), we can define a new matrix

$$M^{reg}(t,x,z) = \begin{pmatrix} Z[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & \dfrac{Y[2](t,x,z)}{a(z)}\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}, \quad z \in \Omega_+ \cap O_R, \tag{73}$$

$$= \begin{pmatrix} X[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & \dfrac{\widetilde{Z}[2](t,x,z)}{\alpha_g^\star(z)}\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}, \quad z \in D_+ \cap O_\varepsilon, \tag{74}$$

$$= \begin{pmatrix} X[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & X[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}, \quad \varepsilon < |z| < R, \tag{75}$$

$$= \begin{pmatrix} \dfrac{\widetilde{Z}[1](t,x,z)}{\alpha_g(z)}\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & X[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}, \quad z \in D_- \cap O_\varepsilon, \tag{76}$$

$$= \begin{pmatrix} \dfrac{Y[1](t,x,z)}{a^\star(z)}\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & Z[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}, \quad z \in \Omega_- \cap O_R, \tag{77}$$

which satisfies the following properties:

- $M^{reg}(t,x,z)$ is analytic in $z \in \mathbb{C} \setminus \Gamma$ and continuous up to the set $\mathbb{R} \setminus \{-R, -\varepsilon, \varepsilon, R\}$; *RRH*1
- $M_-^{reg}(t,x,\lambda) = M_+^{reg}(t,x,\lambda) J^{reg}(t,x,\lambda), \quad \lambda \in \Gamma \setminus \{-R,-\varepsilon,0,\varepsilon,R\}$ *RRH*2
- $M^{reg}(t,x,z)$ is bounded in the vicinity of the points $\{-R,-\varepsilon,0,\varepsilon,R\}$; *RRH*3
- $M^{reg}(t,x,z) = M_0(t,x) + O(z), \quad |z| \to 0.$ *RRH*4
- $M^{reg}(t,x,z) = I + O(z^{-1}), \quad |z| \to \infty.$ *RRH*5

Formulas (73)–(77) give the jump matrices on the circles:

$$J^{reg}(t,x,\lambda) = \begin{pmatrix} \dfrac{A_g(\lambda)}{a(\lambda)} & -\dfrac{B_g(\lambda)}{a(\lambda)}\mathrm{e}^{-2\mathrm{i}\theta(t,x,\lambda)} \\ -\beta_g(\lambda)\mathrm{e}^{2\mathrm{i}\theta(t,x,\lambda)} & \alpha_g(\lambda) \end{pmatrix}, \quad \lambda \in C_R^+; \tag{78}$$

$$= \begin{pmatrix} 1 & -r_g^\star(\lambda)\mathrm{e}^{-2\mathrm{i}\theta(t,x,\lambda)} \\ 0 & 1 \end{pmatrix}, \quad \lambda \in C_\varepsilon^+; \tag{79}$$

$$J^{reg}(t,x,\lambda) = \begin{pmatrix} 1 & 0 \\ r_g(\lambda)\mathrm{e}^{2\mathrm{i}\theta(t,x,\lambda)} & 1 \end{pmatrix}, \qquad \lambda \in C_\varepsilon^-; \tag{80}$$

$$= \begin{pmatrix} \alpha_g^\star(\lambda) & \beta_g^\star(\lambda)\mathrm{e}^{-2\mathrm{i}\theta(t,x,\lambda)} \\ \dfrac{B_g^\star(\lambda)}{a^\star(\lambda)}\mathrm{e}^{2\mathrm{i}\theta(t,x,\lambda)} & \dfrac{A_g^\star(\lambda)}{a^\star(\lambda)} \end{pmatrix}, \qquad \lambda \in C_R^- \tag{81}$$

and on the real line:

$$J^{reg}(t,x,\lambda) = \begin{pmatrix} \dfrac{1}{|\alpha_g^\star(\lambda)|^2} & r_g^\star(\lambda)\mathrm{e}^{-2\mathrm{i}\theta(t,x,\lambda)} \\ r_g(\lambda)\mathrm{e}^{2\mathrm{i}\theta(t,x,\lambda)} & 1 \end{pmatrix}, \qquad \lambda \in (-\varepsilon,\varepsilon) \setminus \{0\}; \tag{82}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon < |\lambda| < R \quad \operatorname{Im}\lambda = 0; \tag{83}$$

$$= \begin{pmatrix} \dfrac{1}{|a(\lambda)|^2} & -r(\lambda)\mathrm{e}^{-2\mathrm{i}\theta(t,x,\lambda)} \\ -\overline{r}(\lambda)\mathrm{e}^{2\mathrm{i}\theta(t,x,z)} & 1 \end{pmatrix}, \quad \lambda \in (-\infty,-R) \cup (R,\infty), \tag{84}$$

where

$$r_g(\lambda) = \frac{\beta_g(\lambda)}{\alpha_g(\lambda)}, \quad r_g^\star(\lambda) = \frac{\beta_g^\star(\lambda)}{\alpha_g^\star(\lambda)}, \qquad r(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad \overline{r}(\lambda) = \frac{\overline{b}(\lambda)}{\overline{a}(\lambda)}.$$

The matrix $M^{reg}(t,x,z)$ is bounded in the vicinity of the points $\{-R,-\varepsilon,0,\varepsilon,R\}$ by the construction. Further, at the point $z=0$, there exist nontangential limits

$$\lim_{z\to 0\pm \mathrm{i}0} M^{reg}(t,x,z) = M_0(t,x),$$

where $M_0(t,x) = g(t,x)$. Hence, $M^{reg}(t,x,z)$ has a continuation up to a continuous function at the point $z=0$. Indeed, taking into account (58), (59), (61), (62), (67), (68) and (72), we have that

$$\lim_{z\to 0+\mathrm{i}0} M^{reg}(t,x,z) = \lim_{z\to 0+\mathrm{i}0} \begin{pmatrix} X[1](t,x,z)\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & \dfrac{\widetilde{Z}[2](t,x,z)}{\alpha_0(z)}\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}$$
$$= g(t,x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

and

$$\lim_{z\to 0-\mathrm{i}0} M^{reg}(t,x,z) = \lim_{z\to 0-\mathrm{i}0} \begin{pmatrix} \dfrac{\widetilde{Z}[1](t,x,z)}{\overline{\alpha}_0(z)}\mathrm{e}^{\mathrm{i}\theta(t,x,z)} & X[2](t,x,z)\mathrm{e}^{-\mathrm{i}\theta(t,x,z)} \end{pmatrix}$$

$$= g(t,x)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Asymptotics at the infinity is evident in view of (29), (30), (31), (32), (39), (40).

Theorem 3. *Let the functions $\mathcal{E}(t,x)$, $\mathcal{N}(t,x)$ and $\rho(t,x)$ be a solution of the mixed problem (5)–(6) to the Maxwell–Bloch equations. Then there exists the matrix $M(t,x,z)$ which solves the regular Riemann–Hilbert problem $RRH1$–$RRH5$. The matrix $M(t,x,z)$ gives the field $\mathcal{E}(t,x)$ by the relation*

$$\mathcal{E}(t,x) = -\lim_{z\to\infty} 4\mathrm{i}zM_{12}(t,x,z). \tag{85}$$

The entries $\mathcal{N}(t,x)$ and $\rho(t,x)$ of the density matrix $F(t,x)$ are defined as follows:

$$F(t,x) = M_0(t,x)\sigma_3 M_0^{-1}(t,x), \qquad M_0(t,x) = \lim_{z\to 0\pm \mathrm{i}0} M(t,x,z). \tag{86}$$

P r o o f. The existence of the matrix $M(t,x,z)$ follows from the above considerations. We only need to prove equations (85) and (86). The matrix $M(t,x,z)$ defines the solution $\Phi(t,x,z)$ of the AKNS equations (7) and (8) by the formula

$$\Phi(t,x,z) = M(t,x,z)e^{-\mathrm{i}\theta(t,x,z)\sigma_3}.$$

Formulas (85) follow from (7) and ($RRH5$). Indeed, substituting the last formula into Eq. (7), we can find

$$M_t + \mathrm{i}z[\sigma_3, M] + HM = 0. \tag{87}$$

Using ($RRH5$), we put

$$M(t,x,z) = I + \frac{m(t,x)}{z} + \mathrm{o}(z^{-1}),$$

where

$$m(t,x) = \lim_{z\to\infty} z(M(t,x,z) - I).$$

This asymptotics and Eq. (87) give

$$H(t,x) = -\mathrm{i}[\sigma_3, m(t,x)],$$

and hence

$$\mathcal{E}(t,x) = -4\mathrm{i}m_{12} = -\lim_{z\to\infty} 4\mathrm{i}zM_{12}(t,x,z).$$

Further, since $\lim\limits_{z\to 0\pm\mathrm{i}0} M^{reg}(t,x,z) = M_0(t,x)$, then the x-equation for $M(t,x,z)$,

$$M_x + \frac{\mathrm{i}}{4z}M\sigma_3 = \mathrm{i}z[\sigma_3, M] + HM + \frac{\mathrm{i}F}{4z}M,$$

gives $F(t,x) = M_0(t,x)\sigma_3 M_0^{-1}(t,x)$. ■

Thus the mixed initial boundary value problem in the quarter xt-plane to the Maxwell–Bloch equations without spectral broadening is completely linearizable.

7. The Study of the Matrix Riemann–Hilbert Problem

It remains to consider the matrix Riemann–Hilbert problem independently of its origin. Namely, let the conjugation contour $\Gamma = \mathbb{R} \cup C_\varepsilon \cup C_R$ and all spectral functions be given. Let us consider the following problem:
Find a 2×2 matrix $M(t,x,z)$ such that the conditions $RRH1$–$RRH5$ are satisfied by the jump matrices given by (78)–(84).

Let t and x be fixed. We look for the solution $M(t,x,z)$ of the RH problem in the form

$$M(t,x,z) = I + \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} \frac{P(t,x,s)[I - J(t,x,s)]}{s-z} ds, \qquad z \notin \Gamma. \tag{88}$$

The Cauchy integral (88) provides all properties of the RH problem (cf. [8]) if and only if the matrix $Q(t,x,\lambda) := P(t,x,\lambda) - I$ satisfies the singular integral equation

$$Q(t,x,z) - \mathcal{K}[Q](t,x,z) = R(t,x,z), \qquad z \in \Gamma. \tag{89}$$

The singular integral operator $\mathcal{K}$ and the right-hand side $R(t,x,z)$ are as follows:

$$\mathcal{K}[Q](t,x,z) := \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} \frac{Q(t,x,s)[I - J(t,x,s)]}{s - z_+} ds,$$
$$R(t,x,z) := \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} \frac{I - J(t,x,s)}{s - z_+} ds.$$

We consider this integral equation in the space $L^2(\Gamma)$ of the 2×2 matrix complex valued functions $Q(z) := Q(t,x,z)$, $z \in \Gamma$. The norm of $Q \in L^2(\Gamma)$ is given by

$$||Q||_{L^2(\Gamma)} = \left(\int\limits_{\Gamma} \mathrm{tr}(\mathrm{Q}^*(\mathrm{z})\mathrm{Q}(\mathrm{z}))|\mathrm{dz}| \right)^{1/2} = \left(\sum_{j,l=1}^{2} \int\limits_{\Gamma} |Q_{jl}(z))|^2 |dz| \right)^{1/2}.$$

The operator $\mathcal{K}$ is defined by the jump matrix $J(t,x,z)$ and the generalized function

$$\frac{1}{s - z_+} = \lim_{z' \to z, z' \in side+} \frac{1}{s - z'}.$$

For the unique solvability of this integral equation the conjugation contour and the jump matrices on its non real part should be Schwartz symmetric. The contour Γ is symmetric with respect to the real axis, and the jump matrices satisfy the relations:

$$J(t,x,\lambda)|_{\lambda \in C_\varepsilon^+} = J^{*-1}(t,x,\lambda)|_{\lambda \in C_\varepsilon^-}, \qquad J(t,x,\lambda)|_{\lambda \in C_R^+} = J^{*-1}(t,x,\lambda)|_{\lambda \in C_R^-},$$

which can be easily verified under the clockwise orientation of the contours $C_\varepsilon^\pm$ and $C_R^\pm$. We recall that the asterisk is the Hermitian conjugation. It means that the Schwartz symmetry principle is performed. Further, the matrix

$$J(t,x,\lambda)+J^*(t,x,\lambda)$$

is positive definite on the real axis ($\lambda\in\mathbb{R}$). Then Theorem 9.3 from [31] (p. 984) guarantees the L^2 invertibility of the operator $Id-\mathcal{K}$ (Id is the identical operator). The function $R(t,x,z)$ belongs to $L^2(\Gamma)$ because $I-J(t,x,z)\in L^2(\Gamma)$ when $z\in\Gamma$, and the Cauchy operator

$$C_+[f](z):=\frac{1}{2\pi\mathrm{i}}\int\limits_\Gamma\frac{f(s)}{s-z_+}ds=\frac{f(z)}{2}+\text{p.v.}\frac{1}{2\pi\mathrm{i}}\int\limits_\Gamma\frac{f(s)}{s-z}ds$$

is bounded in the space $L^2(\Gamma)$ [26]. Therefore, the singular integral equation (89) has a unique solution $Q(t,x,z)\in L^2(\Gamma)$ for any fixed $x,t\in\mathbb{R}$, and formula (88) gives the solution of the above RH problem.

The uniqueness can be proved in the same way as in [8] (p. 194–198). Thus the next theorem is valid.

Theorem 4. *Let a contour Γ and a jump matrix $J(t,x,z)$ (78)–(84) satisfy the Schwartz symmetry principle. Let $I-J(t,x,.)\in L^2(\Gamma)\cap L^\infty(\Gamma)$. Then for any fixed and real t,x the regular RH problem $RRH1$–$RRH5$ has the unique solution $M(t,x,z)$ given by (88).*

Theorem 5. *Let $M(t,x,z)$ be the solution of the RH problem $RRH1$–$RRH5$ given by Theorem 4. If $M(t,x,z)$ is absolutely continuous (smooth) in t and x, then $\Phi(t,x,z)$ ($z\in\mathbb{C}\setminus\Gamma$) satisfies the AKNS equations*

$$\Phi_t=-\left(\mathrm{i}z\sigma_3+H(t,x)\right)\Phi, \tag{90}$$

$$\Phi_x=\left(\mathrm{i}z\sigma_3+H(t,x)+\mathrm{i}\frac{F(t,x)}{4z}\right)\Phi \tag{91}$$

almost everywhere (point-wise) with respect to t and x. The matrix $H(t,x)$ is given by

$$H(t,x)=-\mathrm{i}[\sigma_3,m(t,x)],\qquad m(t,x)=\frac{1}{\pi}\int\limits_\Gamma(I+Q(t,x,\lambda))(J(t,x,\lambda)-I)d\lambda, \tag{92}$$

where $Q(t,x,\lambda)$ is the unique solution of the singular integral equation (89). The matrix $F(t,x)=M(t,x,0)\sigma_3M^{-1}(t,x,0)$ is Hermitian and it has the structure

$$F(t,x)=\begin{pmatrix}\mathcal{N}(t,x)&\rho(t,x)\\\rho^*(t,x)&-\mathcal{N}(t,x)\end{pmatrix}.$$

P r o o f. The matrix $\Phi(t,x,z) := M(t,x,z)\mathrm{e}^{(-\mathrm{i}zt+\mathrm{i}\eta(z)x)\sigma_3}$ is analytic in $z \in \mathbb{C}\setminus\Gamma$ and has a jump across the contour Γ,

$$\Phi_-(t,x,\lambda) = \Phi_+(t,x,\lambda)J_0(\lambda), \qquad \lambda\in\Gamma,$$

where $J_0(\lambda) := \mathrm{e}^{(\mathrm{i}\lambda t-\mathrm{i}\eta(\lambda)x)\sigma_3}J^{reg}(t,x.\lambda)\mathrm{e}^{(-\mathrm{i}\lambda t+\mathrm{i}\eta(\lambda)x)\sigma_3}$ is independent on t and x. This relation implies:

$$\frac{d\Phi_-(t,x,\lambda)}{dt}\Phi_-^{-1}(t,x,\lambda) = \frac{d\Phi_+(t,x,\lambda)}{dt}\Phi_+^{-1}(t,x,\lambda),$$
$$\frac{d\Phi_-(t,x,\lambda)}{dx}\Phi_-^{-1}(t,x,\lambda) = \frac{d\Phi_+(t,x,\lambda)}{dx}\Phi_+^{-1}(t,x,\lambda)$$

for $\lambda\in\Gamma$. The relations obtained mean that the matrix logarithmic derivatives $\Phi_t(t,x,z)\Phi^{-1}(t,x,z)$ and $\Phi_x(t,x,z)\Phi^{-1}(t,x,z)$ are analytic in $z\in\mathbb{C}\setminus\{0\}$ with exception of self-intersection points of the contour Γ. The matrix $M(t,x,z)$ and its derivative $M_t(t,x,z)$ are analytic in $z\in\mathbb{C}\setminus\Gamma$, and the Cauchy integral (88) gives the asymptotic formula

$$M(t,x,z) = I + \frac{m_\pm(t,x)}{z} + \mathrm{O}(z^{-2}), \qquad z\to\infty, \quad z\in\mathbb{C}_\pm.$$

Hence,

$$\begin{aligned}\Phi_t(t,x,z)\Phi^{-1}(t,x,z) &= -\mathrm{i}z\sigma_3 + \mathrm{i}[\sigma_3, m_+(t,x)] + \mathrm{O}(z^{-1})\\ &= -\mathrm{i}z\sigma_3 + \mathrm{i}[\sigma_3, m_-(t,x)] + \mathrm{O}(z^{-1}), \qquad z\to\infty,\end{aligned}$$

where

$$m_-(t,x) = m_+(t,x) = m(t,x) = \frac{\mathrm{i}}{2\pi}\int\limits_\Gamma P(t,x,z)[I-J(t,x,z)]dz.$$

Since $M(t,x,z)$ is bounded up to the boundary, then $z=0$, the end points and self-intersection points are removable singularities for $\Phi_t(t,x,z)\Phi^{-1}(t,x,z)$. Therefore, by Liouville's theorem, this derivative is a polynomial

$$U(z) := \Phi_t(t,x,z)\Phi^{-1}(t,x,z) = -\mathrm{i}z\sigma_3 - H(t,x),$$

where $H(t,x) := -\mathrm{i}[\sigma_3, m(t,x)] = \begin{pmatrix}0 & q(t,x)\\ p(t,x) & 0\end{pmatrix}$. Using the Schwartz symmetries of the jump matrix $J(t,x,z)$, we show that $U(z) = \sigma_2\overline{U}(\overline{z})\sigma_2$, where $\sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$. These reductions imply $H(t,x) = -H^*(x,t)$, i.e., $q(t,x) = -\overline{p}(t,x)$, and we put $q(t,x) := \mathcal{E}(t,x)/2$. Thus, $\Phi(t,x,z)$ satisfies equation (90), and a scalar function $\mathcal{E}(t,x)$ is defined by (92). The function $\mathcal{E}(t,x)$ is smooth in t and

x, because the matrix $M(x,t,z)$ and hence $m(t,x)$ are smooth in t and x by supposition. In the same way as above, we find that $\Phi_x(x,t,\lambda)\Phi^{-1}(x,t,\lambda)$ is a rational matrix function

$$V(z) := \Phi_x(x,t,\lambda)\Phi^{-1}(x,t,\lambda) = \mathrm{i}z\sigma_3 + H(t,x) + \frac{\mathrm{i}F(t,x)}{4z}$$

because the two asymptotics are true:

$$\Phi_x(t,x,z)\Phi^{-1}(t,x,z) = \mathrm{i}z\sigma_3 + H(t,x) + \mathrm{O}(z^{-1}), \qquad z \to \infty$$

and

$$\Phi_x(t,x,z)\Phi^{-1}(t,x,z) = \frac{\mathrm{i}F(t,x)}{4z} + F_0(t,x) + \mathrm{O}(z), \qquad z \to 0,$$

where $F(t,x) = M(t,x,0)\sigma_3 M^{-1}(t,x,0)$ and $F_0(t,x)$ is some matrix. Moreover, the previous relations give that $F_0(t,x) \equiv H(t,x)$. Thus we can see that the matrix $\Phi(x,t,z)$ satisfies two differential equations (90) and (91). Their compatibility ($\Phi_{xt}(x,t,z) = \Phi_{tx}(x,t,z)$) gives the identity in z,

$$U_x(z) - V_t(z) + [U(z), V(z)] = 0, \quad U = -\mathrm{i}z\sigma_3 - H, \quad V = \mathrm{i}z\sigma_3 + H + \frac{\mathrm{i}F}{4z},$$

i.e.,

$$H_t(t,x) + H_x(t,x) + \left[\mathrm{i}z\sigma_3 + H(t,x), \mathrm{i}z\sigma_3 + H(t,x) + \frac{\mathrm{i}F(t,x)}{4z}\right] = 0.$$

This identity is equivalent to the system of matrix equations:

$$H_t(t,x) + H_x(t,x) = \frac{1}{4}[\sigma_3, F(t,x)], \tag{93}$$

$$F_t(t,x) = [F(t,x), H(t,x)]. \tag{94}$$

Using the Schwartz symmetries of the jump matrix $J(t,x,z)$, we can find that $F(t,x)$ is a Hermitian and traceless matrix, and we put

$$F(t,x) = \begin{pmatrix} \mathcal{N}(t,x) & \rho(t,x) \\ \rho^*(t,x) & -\mathcal{N}(t,x) \end{pmatrix}.$$

Matrix equations (93) and (94) are equivalent to the scalar equations (5). Thus, we have proved that the matrix $\Phi(t,x,z)$ satisfies Eqs. (90) and (91), which coincide with the AKNS Eqs. (7) and (8), and the scalar functions $\mathcal{E}(t,x)$, $\mathcal{N}(t,x)$, $\rho(t,x)$ satisfy the MB equations (5) due to the compatibility of Eqs. (90) and (91). ■

The differentiability of the matrix $M(t,x,z)$ (88) with respect to t and x can be proved if we assume that the boundary and initial conditions exponentially vanish at infinity or have a finite support. Then the introduced spectral functions will be analytic in some strip along the real axis of the complex plane. Further, using a factorization of the jump matrices into upper / lower triangular matrices, one can deform the contour into a complex plane in the neighborhoods of zero and infinity in such a way that on the deformed contour the new jump matrices will be exponentially close to the identity matrices when $z \to 0$ and $z \to \infty$. Outside these points, the new contour coincides with Γ. This enables us to differentiate (many times) the singular integral Eq. (89) with respect to the parameters t and x under the integral sign, and thereby to prove the smoothness of the matrices $Q(t,x,\lambda) = P(t,x,\lambda) - I$ and $M(t,x,z)$ with respect to t and x. As a consequence, we obtain the smoothness of the matrix $F(t,x) = M(t,x,0)\sigma_3 M^{-1}(t,x,0)$ and, due to (92), the smoothness of the matrix $H(t,x)$, i.e., the smoothness of the solution $\mathcal{E}(t,x)$, $\mathcal{N}(t,x)$, $\rho(t,x)$ of the Maxwell–Bloch equations.

It remains only to show that these functions give exactly the solution of the mixed problem. In proving, the deformations of the matrix RH problem $RRH1$–$RRH5$ into Rieman–Hilbert problems can be used. One of the problems should be in the one-to-one correspondence with the initial functions while another RH problem should be in the one-to-one correspondence with the boundary condition. These deformations can be done in the same way as in [21]. The so-called global relation (sf. [10,12,13]) does not appear in our consideration because the Maxwell–Bloch equations are the PDEs of the first order.

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