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On Lin's Condition for Products of Random Variables

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The paper presents an elaboration of some results on Lin's conditions. A new proof is given to the fact that if densities of independent random variables ξ_1 and ξ_2 satisfy Lin's condition, then the same is true for their product. Also, it is shown that without the condition of independence, the statement is no longer valid.

Key words: random variable, absolutely continuous distribution, Lin's condition.

Mathematical Subject Classification 2010: 60E05.

1. Introduction

Lin's condition plays a significant role in establishing "checkable" conditions for the moment (in)determinacy of probability distributions. This condition expresses certain regularity in the behaviour of probability densities. Given a probability density f, the tool used here is a function L_f which was brought into consideration by G.D. Lin [3] and called Lin's function in subsequent researches starting from [8]. The function is defined as follows.

Definition 1.1. Let f be a probability density continuously differentiable on $(0, \infty)$. The function

$$L_f(x) := -\frac{xf'(x)}{f(x)} \tag{1.1}$$

is called *Lin's function* of f.

Clearly, Lin's function of f is defined only at the points where f does not vanish. In this work, we deal only with probability densities of positive random variables whose Lin's functions are defined for all $x > 0$. In particular, it is assumed that all densities do not vanish for all $x > 0$, that is, Lin's function for them is well-defined and, in addition, only continuously-differentiable densities are considered. For such densities, the following condition was first considered by G.D. Lin in [3] with regard to the problem of moments.

Definition 1.2. Let $f \in C^1(0,\infty)$ be a probability density of a positive random variable. It is said that f satisfies *Lin's condition* on (x_0,∞) if $L_f(x)$ is monotone increasing on (x_0,∞) and $\lim_{x\to+\infty} L_f(x) = +\infty$.

As this condition is used widely to study the moment determinacy of absolutely continuous probability distributions (see, [4, 6] and references therein), a natural question to ask is which operations on random variables preserve Lin's condition. In [1], Kopanov and Stoyanov established that if a density $f \in C^1(0,+\infty)$ of a random variable X satisfies Lin's condition, then the densities of X^r, $r > 0$ and $\ln X$ also satisfy Lin's condition. Further, if $L_f(x)/x \to +\infty$ as $x \to +\infty$, then the density of e^X also satisfies Lin's condition. In the same article [1], it was stated that if X_1 and X_2 are *independent* positive random variables whose densities satisfy Lin's condition, then the density of their product also satisfies Lin's condition. The approach suggested in [1] is based on the application of the mean value theorem for integrals. In this work, a different approach is proposed, which may be used in other problems such as estimation of moments. Furthermore, it is shown that the condition of independence is crucial here. In general, the statement is not true for the product of dependent random variables whose densities satisfy Lin's condition.

2. Statement of the results

The first result of this work was presented in [1], and its proof based on the application of the mean value theorem was given in [9]. In the present paper, an alternative proof is provided which uses the technique of [5] and also fills some gaps in the previously available proof.

Theorem 2.1. *If ξ_1 and ξ_2 are positive independent random variables whose densities f_1 and f_2 satisfy Lin's condition on $(0,+\infty)$, then the density g of their product satisfies Lin's condition on $(0,+\infty)$.*

Obviously, the result can be extended by induction on the product of n independent random variables.

The next theorem shows that the condition of ξ_1 and ξ_2 being independent is crucial for the validity of the statement and, in general, it cannot be left out whatever the densities of ξ_1 and ξ_2 are.

Theorem 2.2. *Let f_1 and f_2 be two densities of the positive random variables satisfying Lin's condition on $(0,+\infty)$. Then there exists a random vector (ξ_1,ξ_2) with absolutely continuous distribution such that the coordinates ξ_1 and ξ_2 have densities f_1 and f_2 respectively, the density g of the product $\xi_1\xi_2$ is continuously differentiable on $(0,+\infty)$ and the following relations are valid:*

$$\limsup_{x\to+\infty} L_g(x) = +\infty, \qquad \liminf_{x\to+\infty} L_g(x) = -\infty. \tag{2.1}$$

Obviously, equalities (2.1) imply that g does not satisfy Lin's condition on any interval (x_0,∞).

3. Some auxiliary results

To begin with, let us recall that if $f(x,y)$ is a joint probability density of positive random variables ξ_2 and ξ_2, then the density g of their product is given by

$$g(x) = \int_0^{+\infty} f\left(t, \frac{x}{t}\right) \frac{dt}{t}. \tag{3.1}$$

See, for example [2, p. 618, formula (18.5-17)]. With the help of (3.1), the next useful outcome can be derived.

Lemma 3.1 ([9]). *Let ξ_1 and ξ_2 be independent random variables whose densities f_1 and f_2 possess Lin's functions. If g is the (continuous) density of the product $\xi_1 \cdot \xi_2$, then*

$$\begin{aligned} L_g(x) &= \frac{1}{g(x)} \int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t) L_{f_2}(t) \frac{dt}{t} \\ &= \frac{1}{g(x)} \int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t) L_{f_1}\left(\frac{x}{t}\right) \frac{dt}{t}. \end{aligned} \tag{3.2}$$

Lemma 3.2. *Let f be a probability density such that $L_f(x)$ is monotone increasing for all $x > 0$. Then, for every $0 < a < b$, the function*

$$\tau(x) := \frac{f(ax)}{f(bx)} \tag{3.3}$$

is monotone increasing in x.

Proof. Indeed,

$$\tau'(x) = \frac{f(ax)}{xf(bx)} \left[L_f(bx) - L_f(ax)\right] > 0 \quad x > 0. \qquad \square$$

4. Proofs of the theorems

Proof of Theorem 2.1. **1.** First, we are going to prove that $L_g(x)$ is monotone increasing on $(0, +\infty)$. Chose $0 < x < y$ and consider $L_g(y) - L_g(x)$. By virtue of (3.1) and (3.2), one has

$$\begin{aligned} &L_g(y) - L_g(x) \\ &\quad = \int_0^{+\infty} f_1(y/v) f_2(v) L_{f_2}(v) \frac{dv}{v} \Big/ \int_0^{+\infty} f_1(y/v) f_2(v) \frac{dv}{v} \\ &\qquad - \int_0^{+\infty} f_1(x/u) f_2(u) L_{f_2}(u) \frac{du}{u} \Big/ \int_0^{+\infty} f_1(x/u) f_2(u) \frac{du}{u} \\ &\quad = \frac{1}{g(x)g(y)} \int_0^{+\infty} \int_0^{+\infty} f_1(y/v) f_2(v) f_1(x/u) f_2(u) \left[L_{f_2}(v) - L_{f_2}(u)\right] \frac{dudv}{uv} \\ &\quad = \frac{1}{g(x)g(y)} \left[\iint_{A_1} + \iint_{A_2}\right], \end{aligned}$$

where $A_1 = \{(u,v) : u > v\}$ and $A_2 = \{(u,v) : u < v\}$. Now, interchanging u and v in $\iint_{A_2}$, one derives

$$\iint_{A_2} = -\iint_{A_1} f_1(y/u)f_1(x/v)f_2(u)f_2(v)\left[L_{f_2}(v) - L_{f_2}(u)\right]\frac{dudv}{uv}.$$

Therefore,

$$L_g(y) - L_g(x) = \iint_{A_1} \frac{f_2(u)f_2(v)}{uv}\left[L_{f_2}(v) - L_{f_2}(u)\right] \times \left[f_1(y/v)f_1(x/u) - f_1(y/u)f_1(x/v)\right] dudv. \quad (4.1)$$

Now, consider the expressions in the both brackets. Since $u > v$ in A_1 and L_{f_2} is strictly increasing, it follows that the first one is negative everywhere in A_1. The second one can be rewritten as follows:

$$f_1(y/v)f_1(x/u) - f_1(y/u)f_1(x/v) = f_1(y/v)f_1(x/u)\left[1 - \frac{\tau(y)}{\tau(x)}\right],$$

where $\tau(x)$ is defined by (3.3) with $a = 1/u < 1/v = b$. Lemma 3.2 implies that $\tau(y) > \tau(x)$ and, as a result, $f_1(y/v)f_1(x/u) - f_1(y/u)f_1(x/v) < 0$ in A_1. To summarize, the integrand in (4.1) is positive, whence $L_g(y) > L_g(x)$ whenever $y > x$, as stated.

2. At this stage, we are going to prove that $L_g(x) \to +\infty$ as $x \to +\infty$. By virtue of Lemma 3.1, formula (3.2), one has

$$g(x)L_g(x) = \int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t)L_{f_2}(t)\frac{dt}{t} = \int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t)L_{f_1}\left(\frac{x}{t}\right)\frac{dt}{t}$$

which, after substitution $t \mapsto \sqrt{x}t$, leads to

$$2g(x)L_g(x) = \int_0^\infty f_1\left(\frac{\sqrt{x}}{t}\right) f_2(\sqrt{x}t)\left[L_{f_1}\left(\frac{\sqrt{x}}{t}\right) + L_{f_2}\left(\sqrt{x}t\right)\right]\frac{dt}{t}.$$

Now, since both L_{f_1} and L_{f_2} are increasing in their arguments, it follows that for every $t > 0$,

$$L_{f_1}\left(\frac{\sqrt{x}}{t}\right) + L_{f_2}\left(\sqrt{x}t\right) \geq \min\left\{L_{f_1}\left(\sqrt{x}\right), L_{f_2}\left(\sqrt{x}\right)\right\} =: \widetilde{L}\left(\sqrt{x}\right).$$

Therefore,

$$2g(x)L_g(x) \geq \widetilde{L}\left(\sqrt{x}\right)\int_0^\infty f_1\left(\frac{\sqrt{x}}{t}\right) f_2\left(\sqrt{x}t\right)\frac{dt}{t} = \widetilde{L}\left(\sqrt{x}\right)g(x),$$

implying

$$L_g(x) \geq \frac{1}{2}\widetilde{L}\left(\sqrt{x}\right).$$

The statement now follows. □

Corollary 4.1. *If $f_1 = f_2$, then*

$$L_g(x) \geq \frac{1}{2} L_f(\sqrt{x}).$$

Proof of Theorem 2.2. Let us denote by $C(a, b; r) := \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 = r^2\}$, $D(a, b; r) := \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 \leq r^2\}$ the circumference and the disc with the center (a, b) and radius r, respectively. For any $0 < a < v$, consider the square

$$K = \{(x, y) : v - a \leq x, y \leq v + a\} \subset \mathbb{R}^2.$$

Fix $0 < r < a/4$ and consider a function $\rho(x, y) \in C^\infty(\mathbb{R}^2)$ such that:

(i) $\rho(x, y) = 1$ when $(x, y) \in D(0, 0; r/2)$;

(ii) $\rho(x, y) = 0$ when $(x, y) \notin D(0, 0; r)$;

(iii) $0 \leq \rho(x, y) \leq 1$ for all $(x, y) \in \mathbb{R}^2$.

Such a function can be constructed, for a example, in the following way. Starting with $q(t) \in C^\infty[0, \infty)$ satisfying the conditions $q(t) = 1$ for $t \in [0, r^2/4]$, $q(t) = 0$ for $t > r^2$, and $q(t)$ being monotone decreasing on $(r^2/4, r^2)$, we set

$$\rho(x, y) := q(x^2 + y^2) \tag{4.2}$$

which is a desired function. Now, using the function ρ, put

$$\varphi(x, y) = \beta \sin(\nu x y) \rho(x - v - a/2, y - v - a/2), \tag{4.3}$$

where β is a fixed number such that

$$0 < \beta < \min\{f_1(x) f_2(y) : (x, y) \in K\}$$

and $\nu > 0$ is a parameter whose value will be determined later. Obviously, $\varphi(x, y) \in C^\infty\left(\mathbb{R}^2\right)$, $\varphi(x, y) \geq 0$,

$$\varphi(x, y) = \begin{cases} \beta \sin(\nu x y), & \text{for } (x, y) \in D(v + a/2, v + a/2; r/2), \\ 0, & \text{for } (x, y) \notin D(v + a/2, v + a/2; r). \end{cases}$$

Now, define

$$f(x, y) := f_1(x) f_2(y) - \varphi(x, y) + \varphi(x, y + a) - \varphi(x + a, y + a) + \varphi(x + a, y).$$

Obviously, $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$ and $f(x, y) = 0$ outside of the first quadrant. What is more, $f(x, y)$ is a joint probability density of some positive random variables, say, ξ_1 and ξ_2, whose marginal distributions have given densities f_1 and f_2 and, as such, the densities of ξ_1 and ξ_2 satisfy Lin's condition on $(0, +\infty)$. What about the density g of their product $\xi_1 \xi_2$?

To derive the conclusion of this Theorem, notice that for each

$$z \in \left((v + a/2)^2 - r^2/10, (v + a/2)^2 + r^2/10\right),$$

hyperbola $\Gamma_z := \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ : xy = z\}$ intersects both of the circumferences $S(v+a/2, v+a/2; r)$ and $S(v+a/2, v+a/2; r/2)$ at two distinct points. Denote the abscissas of these points by $x_1 < x_2 < x_3 < x_4$. By formula (3.1), the density

$$\begin{aligned} g(z) = &\int_0^\infty f\left(x, \frac{z}{x}\right)\frac{dx}{x} - \int_{x_1}^{x_2} \varphi\left(x, \frac{z}{x}\right)\frac{dx}{x} \\ &- \int_{x_2}^{x_3} \varphi\left(x, \frac{z}{x}\right)\frac{dx}{x} - \int_{x_3}^{x_4} \varphi\left(x, \frac{z}{x}\right)\frac{dx}{x} \\ &=: p(z) - I_1(z) - I_2(z) - I_3(z) =: p(z) - I(z). \end{aligned}$$

Here, $p(z)$ is the (continuous) density of the product of independent random variables with densities f_1 and f_2. By (4.3), one has

$$I_2(z) = \int_{x_2}^{x_3} \beta \sin(\nu z)\frac{dx}{x} = \beta \sin(\nu z) \log \frac{x_3}{x_2}.$$

We notice that $\log(x_3/x_2) \geq c = c_{v,a,r}$. As for $I_1(z)$ and $I_2(z)$, it can be observed that they have the same sign as $\sin(\nu z)$ whenever $\sin(\nu z) \neq 0$. Let z_1, z_2, and z_3 be successive extreme points of $\sin(\nu z)$ falling into interval $\big((v+a/2)^2 - r^2/10, (v+a/2)^2 + r^2/10\big)$. To be specific, opt for

$$\sin(\nu z_1) = 1, \quad \sin(\nu z_2) = -1, \quad \text{and} \quad \sin(\nu z_3) = 1.$$

This can be achieved by taking ν large enough for the extreme points to become very close. Since for all these values of z, one has $\log(x_3/x_2) \geq c > 0$, it follows that

$$\begin{aligned} I(z_1) &> I_2(z_1) \geq \beta \sin(\nu z_1)c = c\beta, \\ I(z_2) &< I_2(z_2) \leq \beta \sin(\nu z_2)c = -c\beta, \end{aligned}$$

implying $I(z_1) - I(z_2) > 2c\beta$. Correspondingly, there exists $z_* \in (z_1, z_2)$ such that

$$I'(z_*) < -\frac{2c\beta}{\pi/\nu} = -\frac{2c\beta\nu}{\pi},$$

which can achieve arbitrarily large negative values for sufficiently large ν. Likewise, adding z_3, one obtains: $I(z_3) \geq c\beta$, whence

$$I(z_3) - I(z_2) \geq 2c\beta$$

and, consequently,

$$I'(z_{**}) > \frac{2c\beta\nu}{\pi} \quad \text{for some } z_{**} \in (z_2, z_3).$$

Since $g'(z) = p'(z) - I'(z)$ and $p'(z)$ is bounded on $\big[(v-a)^2, (v+a)^2\big]$ by a constant independent from ν, it follows that for ν large enough, there exist points

$$z_*, z_{**} \in \big((v+a/2)^2 - r^2/10, (v+a/2)^2 + r^2/10\big)$$

such that $g'(z_*) \geq A$ and $g'(z_{**}) \leq -B$ for any prescribed $A, B > 0$.

Applying the same procedure to an infinite sequence of disjoint squares

$$K_n = \{(x,y) : v_n - a_n \leq x, y \leq v_n + a_n, \}, \quad n \in \mathbb{N}$$

one derives the statement of Theorem 2.2. □

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Щодо умови Ліна для добутку випадкових величин

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Робота містить уточнення наявних результатів стосовно умови Ліна. Представлено нове доведення того факту, що якщо щільності розподілів випадкових величин ξ_1 та ξ_2 задовольняють умову Ліна, те ж саме виконується для їхнього добутку. Також показано, що, коли умову незалежності відкинуто, твердження перестає бути вірним.

Ключові слова: випадкова величина, абсолютно неперервний розподіл, умова Ліна.