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On Dynamical Behavior of the p-adic λ-Ising Model on Cayley Tree

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In the present paper, we continue to study some features of the mixed type p-adic λ-Ising model which was studied in [3]. In that study, the existence of the p-adic Gibbs measures and phase transitions were investigated for the model on the Cayley tree of order two. In the current paper, we study the dynamical behavior of the fixed points which have been found in [3]. As the main result, we proved that the fixed point u_0 is an attractor and the other fixed points $u_{1,2}$ are repellent fixed points for the mixed type p-adic λ-Ising model. In addition, the size of basin of attractor for the fixed point u_0 is described.

Key words: p-adic numbers, p-adic quasi Gibbs measure, dynamical systems, Cayley tree.

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1. Introduction

The p-adic numbers were firstly described by German mathematician K. Hensel, and they have been attracting interest of scientists since then. By now, many theoretical and practical applications have being studied in the p-adic field. Also some features of dynamical behavior of a dynamical system were studied in [2, 7, 10, 13, 17].

A number of scientists applied renormalization techniques in statistical mechanics, and they got lots of interesting results. As consequences of such results about phase transitions of spin models on hierarchical lattices represented that they make the exact calculation of multifarious physical quantities [1, 5]. One of these useful hierarchical lattices is a Cayley tree or a Bethe lattice (see [19]). The lattice is not real but it provides to do some certain calculations of many physical quantities. And it helps to construct corresponding dynamical systems for many complicated models [21].

Renormalization methods were widely applied to study the Ising model [1]. At the same time, one of the generalizations of the Ising model is the so-called λ-model on the Cayley tree (see [11, 20]). In [3], we combined these two most studied models and called "p-adic λ-Ising model". In the same work we proved the existence and uniqueness of this kind of p-adic quasi Gibbs measures. Moreover, we proved the occurrence of the phase transition according to the p-adic λ-Ising model.

Currently, we continue studying different aspects of this model. In [3], we obtained a dynamical function and got three fixed points $u_0, u_{1,2}$. In the present paper, we will explore dynamical behavior of these fixed points.

As the main result of this paper, we have proved that u_0 is an attractor and $u_{1,2}$ are repellent fixed points for the mixed type p-adic λ-Ising model on the Cayley tree of order two.

The result is obtained for the p-adic case. However, it is not valid for the real case.

2. Preliminaries

2.1. p-adic numbers. Suppose that p is a fixed prime number. The set $\mathbb{Q}_p$ is defined as a completion of the rational numbers $\mathbb{Q}$ with the norm $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by

$$|x|_p = \begin{cases} p^{-r}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

where $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z}$, $n \in \mathbb{N}$, $(m,p) = (n,p) = 1$. The absolute value $|\cdot|_p$ is called non-Archimedean norm, and it satisfies the *strong triangle inequality*

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

This is the most crucial property of the norm, i.e., if $|x|_p > |y|_p$, then $|x+y|_p = |x|_p$. Notice that this very useful feature can be employed only in the non-Archimedean norm.

Any p-adic number $x \in \mathbb{Q}_p$, $x \neq 0$ is uniquely represented in the form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots), \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0, 1, 2, \ldots$, in the case $|x|_p = p^{-\gamma(x)}$.

We recall that the p-adic integers set is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ |x|_p \leq 1\}.$$

And a p-adic exponential function is defined by

$$\exp_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

and is convergent for every $x \in B(0, p^{-1/(p-1)})$. It is known [9] that for any $x \in B(0, p^{-1/(p-1)})$ one has

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1.$$

Let

$$\mathcal{E}_p = \{x \in \mathbb{Q}_p : |x-1|_p < p^{-1/(p-1)}\}. \tag{2.2}$$

Lemma 2.1 ([18]). *The set $\mathcal{E}_p$ has the following properties:*

1. *$\mathcal{E}_p$ is a group under multiplication.*

2. *$|a-b|_p < 1$ for all $a, b \in \mathcal{E}_p$.*

3. *If $a, b \in \mathcal{E}_p$, then it holds*

$$|a+b|_p = \begin{cases} \frac{1}{2} & \text{if } p = 2 \\ 1 & \text{if } p \neq 2 \end{cases}.$$

4. *If $a \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $a = \exp_p(h)$.*

5. *Let $x, y \in \mathbb{Q}_p$. If $|xy|_p = 1$ and $|x+y|_p < 1$, then $|x|_p = |y|_p = 1$.*

2.2. p-adic measure. Assume that $(X, \mathcal{B})$ is a measurable space, where $\mathcal{B}$ is an algebra of subsets of X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is called a *p-adic measure* if the equality

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j)$$

holds for any $A_1, \ldots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$.

A p-adic measure is called a *probability measure* whenever $\mu(X) = 1$.

2.3. Cayley tree. Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root $x^{(0)}$. Here, the set of all vertices is V and the set of all edges is L. A collection of the pairs $\langle x, x_1 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a *path* from the vertex x to the vertex y. The distance $d(x, y), x, y \in V$, on the Cayley tree is the length of the shortest path from x to y. The vertices x and y are the *nearest neighbors* denoted by $l = \langle x, y \rangle$ if $d(x, y) = 1$. Two vertices $x, y \in V$ are considered as the *next-nearest neighbors* if $d(x, y) = 2$. The next-nearest-neighbors vertices x and y are the *prolonged next-nearest neighbors* if $x \in W_{n-2}$ and $y \in W_n$ for some $n \geq 1$, which are shown by $\rangle x, y \langle$, and called *one-level next-nearest-neighbors* if $x, y \in W_n$ for some n and shown by $\rangle \overline{x, y} \langle$.

$$W_n = \left\{ x \in V : \ d(x, x^{(0)}) = n \right\}, \qquad V_n = \bigcup_{m=0}^{n} W_m,$$
$$L_n = \left\{ l = \langle x, y \rangle \in L : \ x, y \in V_n \right\}.$$

The direct successors set of x is defined by

$$S(x) = \left\{ y \in W_{n+1} : \ d(x, y) = 1 \right\}, \quad x \in W_n.$$

2.4. Dynamical systems in $\mathbb{Q}_p$. In this subsection, we remind some standard notions of dynamical systems. Let $r, s > 0$ s.t. $r < s$, and $a \in \mathbb{Q}_p$. Then we define the p-adic balls and a p-adic sphere as follows:

$$B_r(a) = \{x \in \mathbb{Q}_p : \ |x-a|_p < r\}, \qquad \bar{B}_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}. \quad (2.3)$$
$$B_r(a) = \{x \in \mathbb{Q}_p : \ r < |x-a|_p < s\}, \quad S_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}. \quad (2.4)$$

It is obvious that $\bar{B}_r(a) = B_r(a) \cup S_r(a)$.

If any function $f : B_r(a) \to \mathbb{Q}_p$, which converges uniformly on the ball $B_r(a)$, can be represented by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad f \in \mathbb{Q}_p,$$

then it is said to be *analytic*.

Letting a dynamical system (f, B) in $\mathbb{Q}_p$, where $f : x \in B \to f(x) \in B$, be an analytic function and $B = B_r(a)$ or $\mathbb{Q}_p$. Denote $x^{(n)} = f^n(x^{(0)})$, where $x^0 \in B$ and $f^n(x) = \underbrace{f \circ \cdots \circ f(x)}_{n}$. If $f(x^{(0)}) = x^{(0)}$, then $x^{(0)}$ is said to be a *fixed point*.

A fixed point $x^{(0)}$ is called an *attractor* if there exists a neighborhood $U(x^{(0)})(\subset B)$ of $x^{(0)}$ such that for all points $y \in U(x^{(0)})$ it holds $\lim_{n\to\infty} y^n = x^{(0)}$ where $y^n = f^n(y)$. If $x^{(0)}$ is an attractor, then its *basin of attraction* is

$$A(x^{(0)}) = \{y \in \mathbb{Q}_p : \ y^n \to x^{(0)}, \text{ as } n \to \infty\}.$$

A fixed point $x^{(0)}$ is said to be a *repellent* if there exists a neighborhood $U(x^{(0)})$ of $x^{(0)}$ such that $|f(x) - x^{(0)}|_p > |x - x^{(0)}|_p$ for $x \in U(x^{(0)})$, $x \neq x^{(0)}$.

For a fixed point $x^{(0)}$ of a function $f(x)$ a ball $B_r(x^{(0)})$ (contained in B) is said to be a *Siegel disc* if each sphere $S_\rho(x^{(0)})$, $\rho < r$, is an invariant sphere of $f(x)$, i.e., if $x \in S_\rho(x^{(0)})$, then all iterated points $x^{(n)} \in S_\rho(x^{(0)})$ for all $n = 1, 2 \ldots$. The union of all Siegel discs with the center at $x^{(0)}$ is said to be *a maximum Siegel disc* and it is denoted by $SI(x^{(0)})$.

In other words, let $x^{(0)}$ be a fixed point of an analytic function $f(x)$. Then

$$\lambda = \frac{d}{dx} f(x^{(0)}).$$

It is clear that λ is a usual derivative of f at $x^{(0)}$. Therefore, the point $x^{(0)}$ is *attractive* whenever $0 \leq |\lambda|_p < 1$, *neutral* whenever $|\lambda|_p = 1$, and *repellent* whenever $|\lambda|_p > 1$.

3. Construction of p-adic Gibbs measure and dynamical function

In this section, we deal with the mixed type p-adic λ-Ising model where spins take values $\{\pm 1\}$, and these values are assigned to the vertices of the Cayley tree $\Gamma^k_+ = (V, L)$. A configuration σ on V is defined as a function of $\sigma : x \in V \to$

$\sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and ω on V_n and W_n, respectively. The set of all configurations on V (respectively V_n, W_n) is the same with $\Omega = \Phi^V$ (respectively $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$) and it is easy to see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Due to the given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$, their concatenation is defined by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), & \text{whenever } x \in V_{n-1} \\ \omega(x), & \text{whenever } x \in W_n \end{cases}.$$

It is obvious that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

$\lambda : \Phi \times \Phi \to \mathbb{Q}_p$ is a function defined on each edge $\langle x, y\rangle \in L$. However, the Hamiltonian of the p-adic λ-Ising model can be defined by

$$H_n(\sigma) = \sum_{\langle x,y\rangle \in L_n} \lambda(\sigma(x), \sigma(y)) + J \sum_{\rangle x,y\langle} \sigma(x)\sigma(y), \tag{3.1}$$

where $H_n : \Omega_{V_n} \to \mathbb{Q}_p$.

In what follows, we suppose that $|\lambda(x, y)|_p \leq 1/p$ and $|J|_p \leq 1/p$ ($p \geq 3$). These conditions guarantee the existence of $\exp_p(H_n(\sigma))$ for all $n \in \mathbb{N}$ and σ. $\mathbf{h} : x \in V \setminus \{x^{(0)}\} \to h_x \in \mathbb{Q}_p$ is a mapping. For any $n \in \mathbb{N}$, a p-adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on the entire configuration Ω_{V_n} is

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_{n,h}^{(\mathbf{h})}} \exp_p(H_n(\sigma)) \prod_{x \in W_n} (h_x)^{\sigma(x)}. \tag{3.2}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_{n,h}^{(\mathbf{h})}$ is the related function called a *partition function* and given by

$$Z_{n,h}^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \exp_p(H_n(\sigma)) \prod_{x \in W_n} (h_x)^{\sigma(x)}. \tag{3.3}$$

Remember [12] that one of the main results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions, which is a well-known *Kolmogorov's Extension Theorem* [23]. Recall that a p-adic probability measure μ on Ω is *compatible* if one holds

$$\mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n) \quad \text{for all } \sigma_n \in \Omega_{V_n},\ n \in \mathbb{N}. \tag{3.4}$$

The existence of the measure μ is guaranteed by Kolmogorov's theorem [4,8]. Namely, if the measures $\mu_{\mathbf{h}}^{(n)}$, $n \geq 1$ satisfy the *compatibility condition*, i.e.,

$$\sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}) \tag{3.5}$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$, then there is a unique p-adic Gibbs measure μ on Ω with (3.4).

Now, following [12], if the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition for a function $\mathbf{h}$, then there is a unique p-adic probability measure, which is denoted by $\mu_{\mathbf{h}}$. The measure $\mu_{\mathbf{h}}$ is said to be a *p-adic quasi Gibbs measure* corresponding to the p-adic λ-Ising model. By $Q\mathcal{G}(H)$, we denote the set of all p-adic quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x,\ x \in V\}$.

Theorem 3.1. *The measure $\mu_h^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, satisfies Kolmogorov's consistency condition (3.5) if and only if the following equations hold for any $x \in V$:*

$$h_x = \sum_{y \in S(x)} F_{x,y}(h_y, \lambda, J), \tag{3.6}$$

and

$$h_x^2 = \frac{a^2\theta^2 h_y^2 h_z^2 + ab(h_y^2 + h_z^2) + b^2\theta^2}{c^2\theta^2 h_y^2 h_z^2 + cd(h_y^2 + h_z^2) + d^2\theta^2}, \tag{3.7}$$

where $a = \exp_p(\lambda(1,1))$, $b = \exp_p(\lambda(1,-1))$, $c = \exp_p(\lambda(-1,1))$, $d = \exp_p(\lambda(-1,-1))$, $\theta = \exp_p(J)$.

Proof. Necessity. Assume that (3.5) holds; we want to obtain (3.7). We substitute (3.2) in (3.5) and hence, for any configuration $\sigma_{n-1} \in \Omega_{V_n}$, we get

$$Z_n^{-1} \sum_{\sigma^{(n)}} \exp_p \left[\sum_{\langle x,y \rangle \in L_n} \lambda_{x,y}(\sigma(x), \sigma(y)) + J \sum_{\substack{\rangle x,y \langle \\ x,y \in W_n}} \sigma(x)\sigma(y) + \sum_{x \in W_n} h_x \sigma(x) \right]$$
$$= Z_{n-1}^{-1} \sum_{\sigma^{(n)}} \exp_p \left[\sum_{\langle x,y \rangle \in L_{n-1}} \lambda_{x,y}(\sigma(x), \sigma(y)) \right.$$
$$\left. + J \sum_{\substack{\rangle x,y \langle \\ x,y \in W_{n-1}}} \sigma(x)\sigma(y) + \sum_{x \in W_{n-1}} h_x \sigma(x) \right].$$

It yields that

$$\exp_p(H_n(\sigma_{n-1})) \prod_{x \in W_{n-2}} \prod_{y \in S(x)} \left(h_{xy,\sigma(x)\sigma(y)} \right)^{\sigma(x)\sigma(y)}$$
$$= L_n \sum_{\eta \in \Omega_{W_n}} \exp_p \left[H_n(\sigma_{n-1}) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \lambda_{x,y}(\sigma(x), \sigma(y)) \right.$$
$$\left. + J \sum_{\substack{y,z \in S(x) \\ y \neq z}} \sigma(y)\sigma(z) \right] \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \left(h_{xy,\sigma(x)\eta(y)} \right)^{\sigma(x)\eta(y)}.$$

From here we get

$$\frac{Z_{n-1}}{Z_n} \sum_{\sigma(n)} \exp_p \left[\sum_{x \in W_{n-1}} \sum_{y \in S(x)} \lambda_{x,y}(\sigma(x), \sigma(y)) + J \sum_{\substack{y,z \in S(x) \\ y \neq z}} \sigma(y)\sigma(z) \right.$$

$$+ \sum_{y\in S(x)} h_y\sigma(y)\Bigg] = \prod_{x\in W_{n-1}} \exp_p(h_x\sigma(x)).$$

Then, we find

$$\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{\sigma(y)\in\pm 1}\exp_p\Bigg[\sum_{x\in W_{n-1}}\lambda_{x,y}(\sigma(x),\sigma(y))$$
$$+ J\sum_{\substack{y,z\in S(x)\\ y\neq z}}\sigma(y)\sigma(z) + h_y\sigma(y)\Bigg] = \prod_{x\in W_{n-1}}\exp_p(h_x\sigma(x)).$$

In the equation above, if we substitute $\sigma(x) = 1$ and $\sigma(x) = -1$ respectively, then, after dividing the corresponding equations by each other, we get equation (3.7), which means that (3.6) holds.

Sufficiency. Assume that (3.6) is valid. Then it implies the existence of $a(x) \in \mathbb{Q}_p$ such that

$$\prod_{y\in S(x)}\sum_{\tilde\sigma(y)\in\{\pm 1\}}\exp_p\Bigg[\lambda(\sigma(x),\tilde\sigma(y))$$
$$+ J\sum_{\substack{z\in S(x)\\ z\neq y\\ \eta(z)\in\Phi}}\tilde\sigma(y)\eta(z) + \tilde\sigma(y)h_y\Bigg] = a(x)\exp_p(\sigma(x)h_x), \tag{3.8}$$

where $\sigma(x) \in \{\pm 1\}$. From the last equality, one gets

$$\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{\tilde\sigma(y)\in\{\pm 1\}}\exp_p\Bigg[\lambda(\sigma(x),\tilde\sigma(y))$$
$$+ J\sum_{\substack{z\in S(x)\\ z\neq y\\ \eta(z)\in\Phi}}\tilde\sigma(y)\eta(z) + \tilde\sigma(y)h_y\Bigg] = \prod_{x\in W_{n-1}} a(x)\exp_p(\sigma(x)h_x). \tag{3.9}$$

Now, multiplying both sides of (3.9) by $\exp_p(H_{n-1}(\sigma))$ and denoting

$$A_n(x) = \prod_{x\in W_n} a(x), \tag{3.10}$$

we get

$$U_n\exp_p(H_{n-1}(\sigma))\prod_{x\in W_{n-2}}\prod_{y\in S(x)}\left(h_{xy,\sigma(x)\sigma(y)}\right)^{\sigma(x)\sigma(y)}$$
$$= \exp_p(H_{n-1}(\sigma))\prod_{x\in W_{n-2}}\prod_{y\in S(x)}\prod_{z\in S(y)}\sum_{\eta(z)\in\{\pm 1\}}\exp_p\Bigg[\lambda(\sigma(x),\tilde\sigma(y))$$

$$+ J \sum_{\substack{z \in S(x) \\ z \neq y \\ \eta(z) \in \Phi}} \tilde{\sigma}(y)\eta(z) + \tilde{\sigma}(y)h_y \Bigg].$$

From (3.2) it follows that

$$U_{n-1} Z_{n-1}^{(\mathbf{h})} \mu_{\mathbf{h}}^{(n-1)}(\sigma) = Z_n^{(\mathbf{h})} \sum_{\eta} \mu_{\mathbf{h}}^{(n)}(\sigma \vee \eta). \tag{3.11}$$

Since the measure $\mu_{\mathbf{h}}^{(n)}, (n \geq 1)$ is a probability measure for each $n \in \mathbb{N}$, i.e.,

$$\sum_{\sigma \in \Omega_{V_{n-1}}} \mu_{\mathbf{h}}^{(n-1)}(\sigma) = \sum_{\sigma \in \Omega_{V_{n-1}}} \sum_{\eta \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma \vee \eta) = 1.$$

Hence, from (3.11) we obtain

$$Z_n^{(\mathbf{h})} = U_{n-1} Z_{n-1}^{(\mathbf{h})}. \tag{3.12}$$

(3.11) and (3.12) imply (3.5). The proof is completed. $\square$

This proof yields the uniqueness of the p-adic quasi Gibbs measures for our model.

In this study, our main goal is to analyze the p-adic dynamical behavior of the fixed points of the dynamic function (3.7). And the existence of three non-trivial fixed points belonging to the model was proved in [3].

Recall that a function $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V \setminus \{x^0\}}$ is called *translation-invariant* if $\mathbf{h}_x = \mathbf{h}_y$ for all $x, y \in V$. A p-adic Gibbs measure $\mu_{\mathbf{h}}$, corresponding to a translation-invariant function $\mathbf{h}$, is called a *translation-invariant p-adic quasi Gibbs measure*.

By Theorem 3.1, we reduced the existence of translation-invariant p-adic quasi Gibbs measures to the solutions of equation (3.7). In generally, solutions of such kind of equations are very complicated. Therefore, firstly we need to reduce the equation to the simplest form possible. Then we suppose that $h_x = h_y = h_z = h$ since h is translation invariant for all $x, y, z \in V$. Therefore we rewrite equation (3.7) as following:

$$h^2 = \frac{a^2\theta^2 h^4 + 2abh^2 + b^2\theta^2}{c^2\theta^2 h^4 + 2cdh^2 + d^2\theta^2}. \tag{3.13}$$

For more simplicity of (3.13), let $h^2 = u = f(u)$. Then the last equation is reduced to the following function, and we call it a dynamical function:

$$f(u) = \frac{a^2\theta^2 u^2 + 2abu + b^2\theta^2}{c^2\theta^2 u^2 + 2cdu + d^2\theta^2}, \tag{3.14}$$

where $a, b, c, d, \theta \in \mathcal{E}_p$. Now we need to analyze fixed points of (3.14).

Lemma 3.2. *Let $p \geq 3$, then for any $a, b \in \mathcal{E}_p$ one has $|a+1|_p = 1$, $|a+b|_p = 1$.*

The proof is obvious from strong triangle inequality.

3.1. Uniqueness of p-adic quasi Gibbs measures. Due to [6], the following lemma guarantees the existence of at least one solution of the dynamical function (3.14) for the mixed type λ-Ising model.

Lemma 3.3 ([3])**.** *Let $p \geq 3$, $a, b, c, d, \theta \in \mathcal{E}_p$, and f be as given in* (3.14)*.*

1. *The function f has a unique fixed point $u_0 \in \mathcal{E}_p$.*

2. *Then $f(\mathcal{E}_p) \subset \mathcal{E}_p$ and $|f(u) - f(v)|_p \leq \frac{1}{p}|u - v|_p$ for all $u, v \in \mathcal{E}_p$.*

For the proof, we suggest you to refer to [3].

Hence the function f satisfies the Banach contraction principle. This proof guarantees the existence of a unique fixed point u_0 of f in $\mathcal{E}_p$. By using the results of [14, 15], we can state the following theorem.

Theorem 3.4. *If $p \geq 3$ and $a, b, c, d, \theta \in \mathcal{E}_p$, then there exists a unique translational invariant p-adic quasi Gibbs measure μ_0 associated to the unique fixed point u_0 for the model* (3.1) *on Cayley tree of order two.*

Now we need to study other fixed points of (3.14) if they exist. For simplicity, let us assume that $b = \lambda(1, -1) = \lambda(-1, 1) = c$ and let us take it as b. Then we rewrite (3.14) as follows:

$$b^2\theta^2u^3 + (2bd - a^2\theta^2)u^2 + (d^2\theta^2 - 2ab)u - b^2\theta^2 = 0, \tag{3.15}$$

where $a, b, d, \theta \in \mathcal{E}_p$. Moreover, by Lemma 3.3, there exists at least one fixed point $u_0 \in \mathcal{E}_p$. Therefore we can write (3.15) in the form

$$(u - u_0)(b^2\theta^2u^2 + Au + B) = 0, \tag{3.16}$$

where $A = 2bd + b^2\theta^2u_0 - a^2\theta^2$ and $B = \frac{b^2\theta^2}{u_0}$ since u_0 is a fixed point of (3.14).

To find other fixed points of (3.16), which are different from u_0, we need to analyze

$$u^2 + \frac{2bd + b^2\theta^2u_0 - a^2\theta^2}{b^2\theta^2}u + \frac{1}{u_0} = 0, \tag{3.17}$$

where $a, b, d, \theta, u_0 \in \mathcal{E}_p$. To find out all fixed points of (3.16), let us perform the results developed in [22]. Therefore, we rewrite (3.17) in the form

$$u^2 + eu = f, \tag{3.18}$$

where $e = \frac{2bd+b^2\theta^2u_0-a^2\theta^2}{b^2\theta^2}$ and $f = -\frac{1}{u_0}$, $a, b, d, \theta, u_0 \in \mathcal{E}_p$. And the discriminant of (3.18) is $\Delta = e^2 + 4f$.

To find other fixed points of (3.18) in $\mathbb{Q}_p$, we will need the proposition below without the proof.

Proposition 3.5 ([3])**.** *Let $p > 3$ and assume that $\sqrt{\Delta} = \sqrt{e^2 + 4f}$ exists, where $e = \frac{2bd+b^2\theta^2u_0-a^2\theta^2}{b^2\theta^2}$, $f = -\frac{1}{u_0}$, and $a, b, d, \theta, u_0 \in \mathcal{E}_p$. Then equation* (3.18) *has two fixed points $u_{1,2} \in \mathbb{Q}_p$ which are different from $u_0 \in \mathcal{E}_p$.*

Due to [3], we get easily two fixed points $u_{1,2} \in \mathbb{Q}_p$ which are different from $u_0 \in \mathcal{E}_p$,

$$u_1 = \frac{1}{2}(-e + \sqrt{\Delta}), \quad u_2 = \frac{1}{2}(-e - \sqrt{\Delta}). \tag{3.19}$$

And the conditions $|u_{1,2}|_p = 1$, $\sqrt{|\Delta|_p} < 1$ and $|u_{1,2} - 1|_p = |e \pm \sqrt{\Delta} + 2|_p = 1$ hold.

Now, we are ready to study the dynamical behavior of the fixed points of (3.14).

4. Dynamical behavior of the fixed points

In this section, we are going to investigate the p-adic norm of the dynamic system at the fixed points u_0 and $u_{1,2}$. Due to the notion of dynamical system in $\mathbb{Q}_p$, the point $x^{(0)}$ is called *attractive* if $0 \leq |\lambda|_p < 1$, *neutral* if $|\lambda|_p = 1$, and *repellent* if $|\lambda|_p > 1$, where λ is a usual derivative of the dynamical function.

To study the dynamical behavior of the fixed points $u_0, u_{1,2}$ under the dynamic function (3.14), we need to find the derivative of (3.14). After that we are going to check the boundedness of the derivative of (3.14) at the fixed points u_0, $u_{1,2}$.

Firstly, before stating the main theorem, we need the following lemma:

Lemma 4.1. *Let $p > 3$, $a, b, c, d, \theta \in \mathcal{E}_p$ and $u_{1,2} \in \mathbb{Q}_p - \mathcal{E}_p$. Then $|u_{1,2} + 1|_p < 1$ and $|u_1|_p = |u_2|_p = 1$.*

Proof. From Proposition 3.5 and the strong triangle inequality, we get

$$\begin{aligned} |u_{1,2} + 1|_p &= \left|\frac{-2bd - b^2\theta^2 u_0 + a^2\theta^2 \pm b^2\theta^2\sqrt{\Delta} + 2b^2\theta^2}{2b^2\theta^2}\right|_p \\ &= \frac{|-2(bd-1) - (b^2\theta^2 u_0 - 1) + (a^2\theta^2 - 1)|_p}{|2b^2\theta^2|_p} \\ &\pm \frac{|b^2\theta^2\sqrt{\Delta} + 2(b^2\theta^2 - 1)|_p}{|2b^2\theta^2|_p} < \frac{1}{p}, \end{aligned}$$

and from (3.19), we get $|u_1|_p = |u_2|_p = 1$. □

Now we are ready to state the main theorem.

Theorem 4.2. *Let $p > 3$ and $a, b, c, d, \theta \in \mathcal{E}_p$. Then the p-adic dynamical function (3.14) has three fixed points which are $u_0 \in \mathcal{E}_p$ and $u_{1,2} \in \mathbb{Q}_p - \mathcal{E}_p$ under the condition $b = exp_p(\lambda(1,-1)) = exp_p(\lambda(-1,1)) = c$, and*

(i) *u_0 is an attractor;*

(ii) *$u_{1,2}$ are repellent.*

Proof. From Lemma 3.3 and Proposition 3.5, the dynamical function (3.14) has three fixed points which are $u_0 \in \mathcal{E}_p$ and $u_{1,2} \in \mathbb{Q}_p - \mathcal{E}_p$. In here, to prove (i) and (ii), let

$$f(u) = \frac{a^2\theta^2 u^2 + 2abu + b^2\theta^2}{b^2\theta^2 u^2 + 2bdu + d^2\theta^2}, \tag{4.1}$$

where $a, b, d, \theta \in \mathcal{E}_p$. If we find $\lambda = \frac{df}{du}$, then we get

$$f^{'}(u) = \frac{2[(a^2bd\theta^2 - ab^2\theta^2)u^2 + (a^2d^2\theta^2 - b^4\theta^4)u + abd^2\theta^2 - b^3d\theta^2]}{[b^2\theta^2u^2 + 2bdu + d^2\theta^2]^2}. \tag{4.2}$$

Hereafter, to prove (i) and (ii), we are going to check the p-adic norm of $\lambda = f^{'}(u)$ at $u_0, u_{1,2}$.

(i) Let us plug u_0 in (4.2). Then we get

$$\begin{aligned}
\lambda|_{u_0} &= f^{'}(u_0)\\
&= 2\frac{(a^2bd\theta^2 - ab^2\theta^2)u_0^2 + (a^2d^2\theta^2 - b^4\theta^4)u_0 + abd^2\theta^2 - b^3d\theta^2}{[b^2\theta^2u_0^2 + 2bdu_0 + d^2\theta^2]^2}\\
&= 2\frac{(a^2bd\theta^2 - 1 - (ab^2\theta^2 - 1))u_0^2 + (a^2d^2\theta^2 - 1 - (b^4\theta^4 - 1))u_0}{[(b^2\theta^2 - 1)u_0^2 + u_0^2 + 2(bd - 1)u_0 + u_0 + d^2\theta^2 - 1 + 1]^2}\\
&\quad + 2\frac{abd^2\theta^2 - 1 - (b^3d\theta^2 - 1)}{[(b^2\theta^2 - 1)u_0^2 + u_0^2 + 2(bd - 1)u_0 + u_0 + d^2\theta^2 - 1 + 1]^2}.
\end{aligned} \tag{4.3}$$

From here, taking the p-adic norm of (4.3), we obtain

$$|f^{'}(u_0)|_p \le \frac{1}{p} < 1$$

since $|2|_p = 1$ and $|u_0 + 1|_p = 1$.

Then u_0 is an attractor.

(ii) Let us plug $u_{1,2}$ in (4.2). Then we get

$$\begin{aligned}
\lambda|_{u_{1,2}} &= f^{'}(u_{1,2})\\
&= 2\frac{(a^2bd\theta^2 - ab^2\theta^2)u_{1,2}^2 + (a^2d^2\theta^2 - b^4\theta^4)u_{1,2} + abd^2\theta^2 - b^3d\theta^2}{[b^2\theta^2u_{1,2}^2 + 2bdu_{1,2} + d^2\theta^2]^2}\\
&= 2\frac{(a^2bd\theta^2 - 1 - (ab^2\theta^2 - 1))u_{1,2}^2 + (a^2d^2\theta^2 - 1 - (b^4\theta^4 - 1))u_{1,2}}{[(b^2\theta^2 - 1)u_{1,2}^2 + u_{1,2}^2 + 2(bd - 1)u_{1,2} + u_{1,2} + d^2\theta^2 - 1 + 1]^2}\\
&\quad + 2\frac{abd^2\theta^2 - 1 - (b^3d\theta^2 - 1)}{[(b^2\theta^2 - 1)u_{1,2}^2 + u_{1,2}^2 + 2(bd - 1)u_{1,2} + u_{1,2} + d^2\theta^2 - 1 + 1]^2}.
\end{aligned} \tag{4.4}$$

Since

$$|(b^2\theta^2 - 1)u_{1,2}^2 + u_{1,2}^2 + 2(bd - 1)u_{1,2} + u_{1,2} + d^2\theta^2 - 1 + 1|_p = |u_{1,2} + 1|_p,$$

taking the p-adic norm of (4.4), we obtain

$$|f^{'}(u_{1,2})|_p \ge p^3 > 1$$

from Lemma 4.1 and the strong triangle inequality.

In this case, we conclude that $u_{1,2}$ are repellent. □

Then, as a conclusion, we state that the fixed point u_0 is an attractor and the other fixed points $u_{1,2}$ are repellent.

Now we are going to describe the basin of attraction of the fixed point u_0,

$$A(u_0) = \{u \in \mathbb{Q}_p : \ f^n(u) \to u_0\}.$$

Let

$$\mathcal{B} = \{u \in S_1(u_0) : \ \exists n_0 \in \mathbb{N} \ f^{n_0}(x) \in K\},$$

where $K = \{u \in S_1(u_0) : |u_0 - 1|_p < 1\}$.

One can see that

$$\mathcal{B} = \left(\bigcup_{n\geq 1} f^{-n}(K)\right) \bigcap S_1(u_0). \tag{4.5}$$

Now we are going to describe the size of an attractor of the dynamic function as in [16].

Theorem 4.3. *Let $a, b, d, \theta \in \mathcal{E}_p$. Then the following statements hold:*

(i) *If $p > 3$, $\sqrt{\Delta} = \sqrt{(\frac{2bd+b^2\theta^2u_0-a^2\theta^2}{b^2\theta^2})^2 - 4\frac{1}{u_0}}$, then one has*

$$A(u_0) = \mathcal{B} \cup (\mathbb{Q}_p \setminus S_1(u_0)) \cup B_1(0).$$

(ii) *Otherwise, $A(u_0) = \mathbb{Q}_p$.*

Proof. (i) Due to Lemma 3.3, for any $u \in \mathcal{E}_p$, we infer that $u \in A(u_0)$, which means that $\mathcal{E}_p \subset A(u_0)$. We notice that $\mathcal{E}_p = B_1(u_0)$.

Consider several cases.

(I) Assume that $|u|_p < 1$, i.e., $u \in B_1(0)$. Now, let us examine $| \ f(u) \ |_p$ and $| \ f(u) - 1 \ |_p$. One can see that

$$\begin{aligned} | \ f(u) \ |_p &= \frac{|(a^2\theta^2-1)u^2+u^2+2(ab-1)u+2u+(b^2\theta^2-1)+1|_p}{|(b^2\theta^2-1)u^2+u^2+2(bd-1)u+2u+(d^2\theta^2-1)+1|_p} \\ &= \frac{|u+1|_p^2}{|u+1|_p^2} = 1, \end{aligned}$$

and

$$| \ f(u) - 1 \ |_p = \frac{|(a^2\theta^2-b^2\theta^2)u^2+2(ab-bd)u+b^2\theta^2-d^2\theta^2|_p}{|u+1|_p^2} < 1,$$

which imply $f(u) \in \mathcal{E}_p$. Hence, $u \in A(u_0)$, which means that $B_1(0) \subset A(u_0)$. So,

$$B_1(0) \cup \mathcal{E}_p \subset A(u_0).$$

(II) Now assume that $|u|_p > 1$ taking into account that $u \notin S_1(u_0)$. Then we have

$$|f(u)|_p = \frac{|u+1|_p^2}{|u+1|_p^2} = 1,$$

and since

$$\max\{|(a^2\theta^2-1)u^2|_p, |(b^2\theta^2-1)u^2|_p\} < |u|_p^2 \quad \text{and} \quad |u+1|_p^2 = |u|_p^2,$$

we obtain

$$\begin{aligned} |\ f(u)-1\ |_p &= \frac{|(a^2\theta^2-b^2\theta^2)u^2+2(ab-1)u-(bd-1)u+b^2\theta^2-d^2\theta^2|_p}{|u+1|_p^2} \\ &= \frac{|(a^2\theta^2-1)u^2-(b^2\theta^2)u^2+u^2+2(ab-bd)u+b^2\theta^2-d^2\theta^2|_p}{|u+1|_p^2} \\ &= \frac{\max\{|(a^2\theta^2-1)u^2|_p, |(b^2\theta^2-1)u^2|_p\}}{|u|_p^2} < 1, \end{aligned}$$

which means $f(u) \in \mathcal{E}_p$. So, $u \in A(u_0)$.

Consequently, if $|u|_p > 1$, then $u \in A(u_0)$.

(III) Let us consider $|u|_p = 1$. Then

$$|f(u)|_p = \frac{|u+1|_p^2}{|u+1|_p^2} = 1,$$

and since

$$\max\{|(a^2\theta^2-1)u^2|_p, |(b^2\theta^2-1)u^2|_p\} < |u|_p^2 = 1 \quad \text{and} \quad |u+1|_p^2 = |u|_p^2 = 1,$$

$$\begin{aligned} |f(u)-1|_p &= \frac{|(a^2\theta^2-b^2\theta^2)u^2+2(ab-1)u-(bd-1)u+b^2\theta^2-d^2\theta^2|_p}{|u+1|_p^2} \\ &= \frac{|(a^2\theta^2-1)u^2-(b^2\theta^2)u^2+u^2+2(ab-bd)u+b^2\theta^2-d^2\theta^2|_p}{|u+1|_p^2} \\ &= \frac{\max\{|(a^2\theta^2-1)u^2|_p, |(b^2\theta^2-1)u^2|_p\}}{|u|_p^2} < 1. \end{aligned}$$

Hence one gets $f(u) \in \mathcal{E}_p$, so $u \in A(u_0)$.

Therefore, we have

$$\mathcal{B} \cup (\mathbb{Q}_p \setminus S_1(x_0)) \cup B_1(0) \subset A(x_0).$$

Now assume that $u \notin \mathcal{B} \cup (\mathbb{Q}_p \setminus S_1(u_0)) \cup B_1(0)$. Due to (4.5), it yields that $f^n(u) \notin K$ for all $n \in \mathbb{N}$. This means that

$$|f^n(u)-1|_p \not< 1, \quad n \in \mathbb{N}. \tag{4.6}$$

If $f^n(u) \to u_0$ as $n \to \infty$, then from (4.6) one finds $|f^n(u)-1|_p < 1$, which is a contradiction. Therefore, we infer that $u \notin A(u_0)$. So,

$$A(u_0) \subset \mathcal{B} \cup (\mathbb{Q}_p \setminus S_1(u_0)) \cup B_1(0).$$

This completes the part (i).

(ii) The proof of this part immediately follows from (i). □

5. Conclusions

In [3], it is proven that the p-adic λ-Ising model has three fixed points, which means that in the model there exist three transition invariant p-adic quasi Gibbs measures on the Cayley tree of order two. In this work, we continued the investigation of the dynamical properties of these fixed points.

As a main result of this paper, we proved that the p-adic norm of the derivative of the dynamical function (3.14) are $|f^{'}(u_0)|_p < 1$ and $|f^{'}(u_{1,2})|_p > 1$ at the fixed points. Hence, u_0 is an attractor and $u_{1,2}$ are repellent fixed points. Moreover, we described the size of the basin attractor of the fixed point, u_0.

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Про динамічну поведінку p-адичної λ-ізінгової моделі на дереві Кейлі

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У даній статті ми продовжуємо вивчення деяких властивостей p-адичної λ-ізінгової моделі змішаного типу, що вивчалася в [3]. У тій роботі існування p-адичних мір Гібса і фазових переходів досліджувалося в моделі на дереві Кейлі другого порядка. У даній статті ми вивчаємо динамічну поведінку нерухомих точок, які було знайдено в [3]. Основним результатом є те, що ми довели, що нерухома точка u_0 є атрактором, а інші нерухомі точки $u_{1,2}$ репелентними нерухомими точками для p-адичної λ-ізінгової моделі змішаного типу. На додаток описано розмір басейну атрактора для нерухомої точки u_0.

Ключові слова: p-адичні числа, p-адична квазіміра Гіббса, динамічні системи, дерево Кейлі.