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Simple Morse Functions on an Oriented Surface with Boundary

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In the paper, smooth functions with non-degenerate critical points on a smooth compact surface with boundary are considered. Firstly, it is shown that these functions are topologically equivalent to m-functions. The equipped Reeb graph is used to describe their topological structure. Secondly, the authors characterize the topological structure of all simple functions with at most 5 critical points. And finally, a formula for the genus of the surface based on the equipped Reeb graph is obtained.

Key words: topological classification, non-degenerate critical point, equipped Reeb graph.

Mathematical Subject Classification 2010: 57R45; 57R70.

1. Introduction

Smooth functions with non-degenerate singularities and their classification are the main topics of research in many fields of mathematics. There is a number of papers devoted to the functions with non-degenerate critical points on closed 2-dimensional manifolds [1,9,11,17]. Furthermore, some significant results on the surfaces with boundary were obtained in [2–4,6,7]. For instance, M. Morse [11] and others [2,3] gave a canonical representation of a function in the neighborhood of its non-degenerate critical point in the form of a second degree polynomial. In [1]), O.V. Bolsinov and A.T. Fomenko introduced the definition of atom and f-atom using fiberwise and fiberwise frame equivalences respectively. Other denominations of these equivalences are layer and layer equipped equivalences used in [4].

Let us recall that two given smooth functions f and g, defined on the smooth compact surfaces M and N respectively, are called *layer (layer equipped) equivalent* if there exists a homeomorphism $\lambda : M \to N$, which maps the components of the level sets of f onto the components of the level sets of g (and preserves growing direction of the functions). Thus an atom (f-atom) is a class of the layer (layer equipped) equivalence of the function f restricted to the set $f^{-1}(c-\varepsilon, c+\varepsilon)$, where c is a critical value of f, for small enough ε, such that the line segment $[c-\varepsilon, c+]$ does not include critical values with the exception of c. If additionally the homeomorphism λ, which defines the layer equipped equivalence of the functions

f and g on the oriented surfaces M and N respectively, preserves orientation, then these functions are called *$\mathcal{O}$-equivalent*. In this manner, the $\mathcal{O}$-equivalence class of the pair $(U, f|_U)$ is called an *$\mathcal{O}$-atom* for an oriented surface.

A *level line* $c \in \mathbb{R}$ of the function f is the set $L_c = \{f^{-1}(c)\} := \{p \in M | f(p) = c\}$. A level c is *critical* if the corresponding L_c includes the critical point and *the regular value* in the opposite case. A function $f : M \to \mathbb{R}$ which has at most one critical point at each level line is *simple* (see [4]). In this paper, we will consider only simple functions. Analogically, an atom ($\mathcal{O}$-atom) is called *simple* (see [1]) if it includes one critical point. Thus, every atom ($\mathcal{O}$-atom) considered in this paper is simple since every considered function is simple.

The smooth functions f and g, defined on a smooth compact surface M, are called *topologically equivalent* if there exist homeomorphisms $h_1 : M \to M$, $h_2 : \mathbb{R} \to \mathbb{R}$ such that $h_2 \circ f = g \circ h_1$ and h_2 preserves the orientation of $\mathbb{R}$. Topologically equivalent functions f and g, defined on an oriented surface M, are called *topologically $\mathcal{O}$-equivalent* if (as denoted above) the homeomorphism h_1 preserves the orientation of M.

Topological equivalence and $\mathcal{O}$-equivalence are equivalence relations. The set of functions splits into equivalent classes. The investigation of these classes is the main aim of topological and $\mathcal{O}$-classifications. In particular, these classifications study invariant, which allows us to describe all topological properties of functions. Many studies have been carried out on topological classification of functions with non-degenerate critical points on surfaces without boundary such as [5,12,13,18]. Simple functions are strongly connected with the Reeb graph introduced by H. Reeb [17] and A.S. Kronrod [9]. In this paper, the definition of the Reeb graph is generalized to the case of a surface with boundary. A Morse function $f : M \to \mathbb{R}$ is said to be an *m-function* if all its critical points are interior, the restriction f_∂ to its boundary is also a Morse function and the critical levels of f include exactly one critical point of f and do not include a critical point of f_∂. The mentioned above equivalences of m-functions on a surface with boundary were studied in [10,14,16].

This paper is focused on the topological properties of functions with non-degenerate critical points on the boundary of a surface. These functions are interesting, because they are Lyapunov functions of flows of general position on surfaces with boundary and their application is one of the main research methods in dynamical systems.

2. Atoms of simple functions on a surface with boundary

Let $f : M \to \mathbb{R}$ be simple smooth functions on an oriented smooth compact connected surface M with the boundary ∂M such that:

(i) if a critical point p_0 of f does not belong to the boundary ∂M, then it is non-degenerate critical point of f;

(ii) if a critical point p_0 of f belongs to the boundary ∂M, then it is non-degenerate critical point of f and of its restriction to the boundary $f|_{\partial M}$;

(iii) if p_0 is a critical point of the function $f|_{\partial M}$, then it is also a critical point of the function f.

The class of these functions will be denoted by $\Omega(M)$. Thus, $\Omega(M) = \{f : M \to \mathbb{R} | CP(f) = NDCP(f) \supset CP(f|_{\partial M}) = NDCP(f|_{\partial M})\}$, where $CP(f)$ ($NDCP(f)$) is the set of (non-degenerate) critical points of f.

Theorem 2.1. *Let M be an oriented connected smooth compact surface. Then the following statements hold true:*

1) *for any arbitrary function $f \in \Omega(M)$ there exists an m-function $g : M \to \mathbb{R}$ which is topologically equivalent to f;*

2) *for any arbitrary m-function $g : M \to \mathbb{R}$ there exists a function $f \in \Omega(M)$ such that f and g are topologically equivalent.*

Proof. Note that the second statement follows from the first one and vice versa because of the symmetry of topological equivalence. For this reason, we will prove only the first statement.

Let $f \in \Omega(M)$. Then f has the representation $f(x, y) = x + C$ for some constant $C = f(0, 0)$ in a neighborhood of a regular point [3]. M-functions have the same representations in the regular neighborhood. Then the function g can be constructed as $g \equiv f$ in small enough regular neighborhoods.

In the neighborhoods of interior critical points, the function f is an m-function(which follows directly from the definition) and the m-function$g$ can be chosen as $g \equiv f$.

The last possible situation is when the critical points belong to the boundary of the surface. Note that the function f has the local representation $f(x, y) = \pm x^2 \pm y^2, y \geq 0$ [2, 3] in a neighborhood of a critical point on the boundary and the m-function$g(x, y) = \pm x^2 \pm y, y \geq 0$ [6] in a neighborhood of a critical point of the restriction to the boundary ∂M.

Let $p_1, \dots, p_l \in \partial M$ be critical points of the function f. Consider the rectangular neighborhoods $-2\varepsilon_i \leq x \leq 2\varepsilon_i, 0 \leq y \leq \varepsilon_i$, $i = \overline{1, l}$ for some small enough ε_i, such that these neighborhoods do not include other critical points of f with the exception of $p_i, i = \overline{1, l}$. We denote $\varepsilon = min_{i=\overline{1,l}} \varepsilon_i$ and restrict the neighborhoods described above to $V_i = \{-2\varepsilon \leq x \leq 2\varepsilon,\ 0 \leq y \leq \varepsilon\}$, $i = \overline{1, l}$. Then consider the δ-neighborhoods of the boundaries of V_i, $i = \overline{1, l}$, which have the form $U(V_i) = \{-2\varepsilon - \delta \leq x \leq -2\varepsilon + \delta,\ 0 \leq y \leq \varepsilon + \delta\} \bigcup \{2\varepsilon - \delta \leq x \leq 2\varepsilon + \delta,\ 0 \leq y \leq \varepsilon + \delta\} \bigcup \{-2\varepsilon \leq x \leq 2\varepsilon,\ \varepsilon - \delta \leq y \leq \varepsilon + \delta\}$, $i = \overline{1, l}$ for some small enough δ (such that $U(V_i)$ includes only the regular values of f). Then on the sets $V_i \backslash U(V_i)$, $i = \overline{1, l}$, the m-functionis defined as $g = \pm x^2 \pm y$, $y \geq 0$, and the topological equivalence can be defined via the homeomorphisms $h_1(x, y) = (x, y^2)$, $h_2(z) = z$. In the regular neighborhoods outside of V_i, we have that $g \equiv f$. And finally, after smoothing of the function g on the sets $U(V_i)$ (for instance, as in the paper [8]), $i = \overline{1, l}$, we get an m-function(see Fig. 2.1). □

Let $f \in \Omega(M)$. The components of the level lines of the function f are called layers. These layers are homeomorphic to the circle or to the line segment for

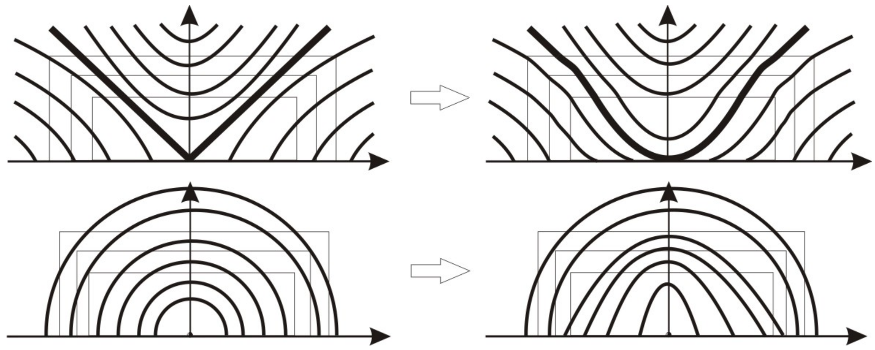

Fig. 2.1

regular values of the function. Then the surface M can be considered as a union of layers and we get a foliation with singularities. A layer is of the first (second) type if it corresponds to the component homeomorphic to the line segment (circle). Let us consider the equivalence relation on M such that points are equivalent if and only if they belong to the same layer. Thus, after examining the natural factor-topology, we get the graph Γ_f, whose edges are drawn by solid or dashed-dotted lines depending on whether they correspond to the layers of the first or second type. In this way, we get the *edges* of *the first* and *second types*, and this classification of edges is said to be *edges division* of graph Γ_f.

Definition 2.2. The vertices of graph Γ_f of the function f with degrees 3 and 4 incident to the first type edges are said to be Y- and X-*vertices*, respectively.

We denote the X-vertex as in Fig. 2.2.

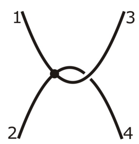

Fig. 2.2

For an arbitrary Y-vertex of graph Γ_f, we fix the cycle order for the edges incident to this vertex. In the figure, this order is defined by passing the edges counterclockwise. The same can be applied to X-vertices.

Next, we consider the $\mathcal{O}$-equivalence of f and g. Let us fix the orientation of edges of the graphs Γ_f and Γ_g from the lower vertex to the upper one. This orientation is not shown on the graph, as it is defined for each graph, however we suppose that the graph is oriented.

Definition 2.3. *The Equipped Reeb graph* of a function $f \in \Omega(M)$ is the graph Γ_f equipped with edges division, orientation and cycle order at Y- and X-vertices.

There are possibly 7 (simple) atoms and 13 (simple) $\mathcal{O}$-atoms, the classification of which depends on the index of the critical point and its belonging to the boundary ∂M. The functions of the class $\Omega(M)$ are topologically equivalent to m-functions (Theorem 2.1), and atoms can be considered as atoms of the height function of m-function. Thus, we have the following atoms and $\mathcal{O}$-atoms:

(i) 3 atoms if $p_0 \in \partial M$: A, B, C, and 6 corresponding $\mathcal{O}$-atoms: A_1, A_2, B_1, B_2 and C_1, C_2 (2.3);

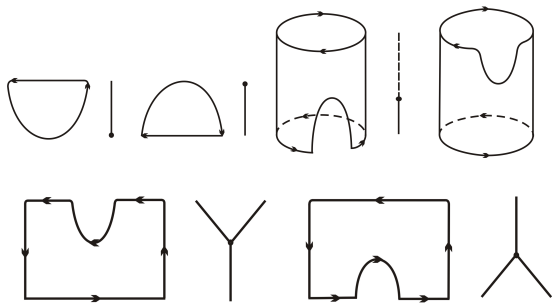

Fig. 2.3

(ii) 2 atoms if $p_0 \notin \partial M$ and the atoms have empty intersections with the boundary: D and E, and 4 corresponding $\mathcal{O}$-atoms: D_1, D_2, E_1, E_2 (see Fig. 2.4);

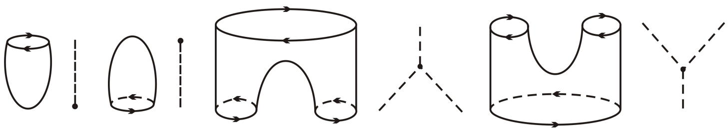

Fig. 2.4

(iii) 2 atoms if $p_0 \notin \partial M$ and the atoms have non-empty intersections with the boundary: F and G, and 3 corresponding $\mathcal{O}$-atoms: F_1, F_2 and $G = G_1 = G_2$ (see Fig. 2.5).

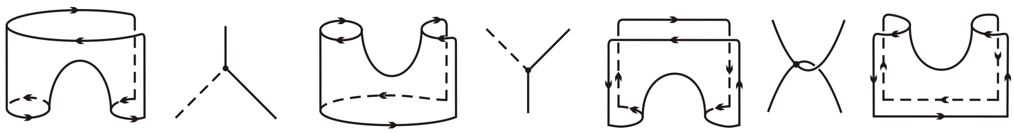

Fig. 2.5

The orientation of atoms, which can be embedded into the plane, is defined by the orientation of the plane (see Figs. 2.3 $(1, 2, 5, 6)$, and 2.5 (3)). Otherwise (see Figs. 2.3 $(3, 4)$, 2.4 $(1, 2, 3, 4)$, and 2.5 $(1, 2)$), we fix the orientation in the following way: counterclockwise on the lower circles (parts of circles) and clockwise on the upper circles (parts of circles). The corresponding equipped Reeb graphs are also shown in Figs. 2.3, 2.4, and 2.5.

Theorem 2.4. *Each simple $\mathcal{O}$-atom of a function from the class $\Omega(M)$ coincides with one of the $\mathcal{O}$-atoms A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2 and G.*

The proof of Theorem 2.4 follows from [16, Theorem 1] and Theorem 2.1.

Recall that a critical point which is not local extremum is called a *saddle critical point* [3].

Some cases of degree $\deg(v)$ of a vertex v of the graph Γ_f is shown in Figs. 2.3, 2.4, and 2.5:

(i) if $\deg(v) = 1$, then v corresponds to the minimum or maximum point;

(ii) if $\deg(v) = 2$, then v corresponds to the saddle critical point, which belongs to the boundary of the surface;

(iii) if $\deg(v) = 3$, then v corresponds to the saddle critical point, which either is interior or belongs to the boundary (Y-vertex);

(iv) if $\deg(v) = 4$, then v corresponds to the interior saddle critical point (X-vertex).

Definition 2.5. The equipped Reeb graphs Γ_f and Γ_g of the functions $f, g \in \Omega(M)$ are said to be *equivalent* by means of the isomorphism $\varphi : \Gamma_f \to \Gamma_g$ and denoted by $\Gamma_f \sim \Gamma_g$ or $\Gamma_f \sim_\varphi \Gamma_g$ if φ :

(i) preserves the edges division;

(ii) preserves the cycle order of the edges at each X- and Y-vertex;

(iii) preserves the edges orientation.

The above-described relation $\sim$ is an equivalence relation.

3. Equipped Reeb graph of simple functions defined on surfaces with boundaries

Let $f \in \Omega(M)$ be a function which has n critical points $p_1, p_2, \ldots, p_n$ with the corresponding critical values $c_1 < c_2 < \ldots < c_n$ and $M_t = \{p \in M | f(p) \leq t\}$, $L_t = \{p \in M | f(p) = t\}$. In other words, M_t is the *subsurface* of M consisting of all points at which the function f takes values less than or equal to t, and L_t is the set of points where the value of f is exactly t (level line). We consider an oriented connected surface M, so the topological type of M can be defined from the information about the genus g and the number of components of the boundary ∂ of the surface (see [14]). As the parameter t changes, we can observe the changes of invariants of the subsurface M_t and of the number of connected components. Recall that the genus of changes of a non-connected surface is defined as the sum of the genera of all connected components.

Let us consider the changing of the topological type of the subsurface M_t depending on the gluing of the corresponding $\mathcal{O}$-atoms (here $\triangle g$, $\triangle\partial$, $\triangle c$ are the changes of the genus, the number of boundary components and the number of connected components of M_t). The function f takes its maximum value c_n and minimum value c_1. Since there is no point p with $f(p) < a$, we have $M_t = \emptyset$ for $t < c_1$. Also we have $f(p) \leq c_n$ at any point $p \in M$, that is, if $t \geq c_n$, then $M_t = M$. Thus, the fundamental idea is to trace the change of shapes of M_t as the parameter t starts from a value greater than or equal to c_1 and changes up to c_n. If f does not have critical values on the interval $[t_1, t_2]$, then the subsurfaces M_{t_1} and M_{t_2} are diffeomorphic (see [2]). In other words, the topological type of M_t does not change at the regular value t.

Thus, important is the change of M_t when t passes through a critical value c_i, $i = \overline{1,n}$. If $t_i = c_i$ and p_i is a local minimum or maximum, then, for some small enough $\varepsilon_i > 0$, the subsurface $M_{t_i+\varepsilon_i}$ can be represented as a disjoint union $M_{t_i-\varepsilon_i} \coprod D_1$ [19] ($p_i \notin \partial M$) or $M_{t_i-\varepsilon_i} \coprod A_1$ [3] ($p_i \in \partial M$). In particular, if it is an $\mathcal{O}$-atom A_1, then $\triangle\partial = +1$ and $\triangle c = +1$, and if it is an $\mathcal{O}$-atom, then D_1 $-$ $\triangle g = 0$, $\triangle\partial = +1$, $\triangle c = +1$. The gluing of these $\mathcal{O}$-atoms causes an increase of the number of connected components $\triangle c$.

In case p_i is a local maximum, either a half of a disk D^2_+ (if additionally $p_i \in \partial M$) or a disk D^2 (if additionally $p_i \notin \partial M$) is glued to the subsurface $M_{t_i-\varepsilon_i}$, which is homeomorphic to $\mathcal{O}$-atoms A_2 and D_2 (see [3, 19]). The gluing of $\mathcal{O}$-atoms D_2 causes a decrease of the number of boundary components by 1 ($\triangle\partial = -1$) while other invariants do not change.

In case p_i is a saddle critical point, the rectangle $[0,1] \times [0,1]$ is glued to the subsurface $M_{t_i-\varepsilon_i}$ such that the two opposite sides are glued to the boundary of the level line $L_{t_i-\varepsilon_i}$ and the other two sides, to the boundary ∂M (see [3, 15]). As a result, we get the gluing of the atoms: E, B, C, F, and G. The gluing of the $\mathcal{O}$-atoms B_2 and C_2 does not change the topological type of the subsurface (see Fig. 2.3 (4, 6)), and the gluing of the $\mathcal{O}$-atoms B_1, E_2 and F_2 increases the number of boundary components, $\triangle\partial = +1$, since we get a new component of the boundary of $M_{t_i+\varepsilon_i}$ (see Figs. 2.3 (3), 2.4 (4), and 2.5 (2)). For other $\mathcal{O}$-atoms (C_1, E_1, F_1, and G), the cases $P(i:j)$, $\{i,j\} \subset \{1,2\}$ should be considered, when the atom is glued to the i connected components and j components of the boundary of the level line L_{t_i}. The case $P(2:1)$ is impossible. As shown in Figs. 2.3 (5) and 2.5 (3), the gluing of the $\mathcal{O}$-atom C_1 or G to one component of the boundary of L_{t_i} ($P(1:1)$) increases the number of boundary components, i.e., $\triangle\partial = +1$. Moreover, the case $P(1:1)$ is impossible for the $\mathcal{O}$-atoms E_1 and F_1 (see Figs. 2.4 (3) and 2.5 (1)). In the case of $P(1:2)$, the $\mathcal{O}$-atoms C_1 (see Fig. 2.3 (1)), E_1 and G (see Figs. 2.4 (3) and 2.5 (3)) increase the genus of the surface ($\triangle g = +1$) and decrease the number of boundary components ($\triangle\partial = -1$). And finally, in the case of $P(2:2)$, we get one of the $\mathcal{O}$-atoms C_1, E_1, F_1, and G, whose gluing causes a decrease of the number of boundary components and connected components by 1 (see Figs. 2.3 (5), 2.4 (3), and 2.5 (1, 3)).

All the described calculations can be summed up in Table 3.1, where $P1$ is the case $P(1:1)$, $P2$ is the case $P(1:2)$ and $P3$ is the case $P(2:2)$.

Remark 3.1. Let $f \in \Omega(M)$ be a function which has n critical points with corresponding critical values $t_1, \ldots, t_n$. Then the topological invariants of the

	A_1	A_2	B_1	B_2	C_1	C_1	C_1	C_2	D_1	D_2
P	1	1	1	1	1	2	3	1	1	1
$\triangle g$	0	0	0	0	0	$+1$	0	0	0	0
$\triangle\partial$	$+1$	0	$+1$	0	$+1$	-1	-1	0	$+1$	-1
$\triangle c$	$+1$	0	0	0	0	0	-1	0	$+1$	0
	E_1	E_1	E_2	F_1	F_1	F_2	G	G	G	
P	2	3	1	2	3	1	1	2	3	
$\triangle g$	$+1$	0	0	$+1$	0	0	0	$+1$	0	
$\triangle\partial$	-1	-1	$+1$	-1	-1	$+1$	$+1$	-1	-1	
$\triangle c$	0	-1	0	0	-1	0	0	0	-1	

Table 3.1

surface M can be calculated using the following formulas:

$$g = \triangle g_1 + \ldots + \triangle g_n,$$
$$\partial = \triangle\partial_1 + \ldots + \triangle\partial_n,$$
$$(c = \triangle c_1 + \ldots + \triangle c_n = 1),$$

where $\triangle g_i$, $\triangle\partial_i$, $\triangle c_i$ are the changes of the genus, the number of boundary components and the number of connected components of the surface M_{t_i}, $i = \overline{1, n}$.

The definition of the $\mathcal{O}$-equivalence can be locally represented as follows. The functions $f \in \Omega(M)$ and $g \in \Omega(N)$ defined on the oriented smooth compact surfaces M and N are *$\mathcal{O}$-equivalent in some neighborhoods of their critical levels* $f^{-1}(c_1)$, $g^{-1}(c_2)$ if there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and a homeomorphism $\lambda : f^{-1}(c_1 - \varepsilon_1, c_1 + \varepsilon_1) \to g^{-1}(c_2 - \varepsilon_2, c_2 + \varepsilon_2)$ which maps the level lines of the function f onto the level lines of the function g and preserves the growing direction of functions and the orientation of surfaces.

Theorem 3.2. *Let M, N be smooth compact surfaces (with boundaries) such that $f \in \Omega(M)$, $g \in \Omega(N)$. Then f and g are $\mathcal{O}$-equivalent if and only if their equipped Reeb graphs Γ_f and Γ_g are equivalent.*

The statement of Theorem 3.2 follows from [16, Theorem 2] and Theorem 2.1.

Let us enumerate the vertices of the equipped Reeb graph of the functions $f, g \in \Omega(M)$ according to ordinal numbers (by functions growing) of critical points. Then the equivalent graphs Γ_f and Γ_g are said to be *orderly equivalent* if the isomorphism φ additionally preserves the numeration of graphs' vertices. Note that in the case of orderly equivalent graphs, in the definition of equivalent graphs condition (iii) is redundant.

Corollary 3.3. *Let M, N be smooth compact surfaces (with boundaries) such that $f \in \Omega(M)$, $g \in \Omega(N)$. Then f and g are topologically $\mathcal{O}$-equivalent if and only if their equipped Reeb graphs Γ_f and Γ_g are orderly equivalent.*

4. Examples of calculations

According to Theorem 3.2, the $\mathcal{O}$-equivalence of the functions of class $\Omega(M)$ is an equivalence relation, because the equivalence of equipped Reeb graphs is an equivalence relation. Thus, $\Omega(M)$ can be considered as a set of equivalence classes, which we denote for the function $f \in \Omega(M)$ by $[f]$.

Let i, j be an in-degree and out-degree of a vertex of an oriented graph γ. The pair of numbers (i, j) is said to be a *vertex index*. Let us denote by $\Gamma_n^{\leq 4}$ a set of all connected oriented graphs γ with n vertices, the indexes of which are $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ and at least one vertex with index $(0, 1)$ and one with index $(1, 0)$ belong to each of these graphs. If the number of vertices is not important to us, then we denote the set by $\Gamma^{\leq 4}$.

Let us consider the operations with the graphs from $\Gamma^{\leq 4}$:

(a1) addition of vertices and incident edges (see Fig. 4.1 (1));

(a2) division of one of the edges by the interior point, which is said to be a new vertex of the graph (see Fig. 4.1 (2));

(a3) addition of edges without new vertices (see Fig. 4.1 (3)).

Next, we consider operations (a1)–(a3) for obtaining a graph from the set $\Gamma^{\leq 4}$.

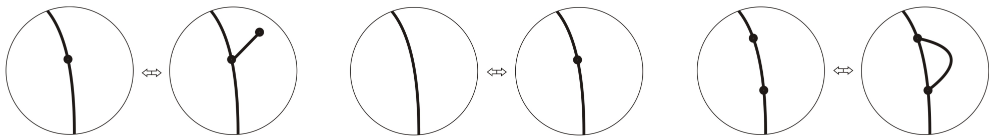

Fig. 4.1

Definition 4.1. *An operation of increasing (decreasing) the number of vertices of the graph $\gamma \in \Gamma_n^{\leq 4}$* is one of the following actions $(a1)$, $(a2)$, $(a3)$ ($(a1)^{-1}$, $(a2)^{-1}$, $(a3)^{-1}$) or their finite sequence.

Lemma 4.2. *Each graph $\gamma \in \Gamma_n^{\leq 4}$ can be obtained from the graph $\widetilde{\gamma} \in \Gamma_2^{\leq 4}$ (see Fig. 4.2) after some operations of increasing the number of vertices of the graph $\widetilde{\gamma}$.*

Fig. 4.2

Proof. It is enough to show that the graph $\gamma \in \Gamma_n^{\leq 4}$ can be deformed into the graph depicted in Fig. 4.2 by the operation of decreasing the number of vertices.

The graph γ is a tree that includes at least one simple cycle (without repeating vertices). If γ includes a simple cycle, then, after some operations $(a3)^{-1}$, it can be represented as a graph without cycles. Thus, we can suppose that γ is a tree and hence it is planar and has an edges orientation which forms an acute angle with the positive direction of the second line of coordinate system (after embedding of γ into the plane). We fix a vertex v_0 with index $(0,1)$ and consider the relation $\ll$ on the set of vertices V of the graph γ as follows: $v_i, v_j \in V$: $v_i \ll v_j \Leftrightarrow l(v_i) \leq l(v_j) = j$, where $l(v_s)$ is a non-oriented distance (the least number of edges of all possible non-oriented paths) from v_s to v_0. Thus, the set V can be represented as a disjoint union of classes $V = V^1 \bigcup V^2 \bigcup \ldots \bigcup V^l$ for some natural l, where $V^j = \{v_1^j, v_2^j, \ldots, v_{i_j}^j\} \subset V$, $j = \overline{1,l}$ are such that $l(v_0, v_p^j) = l(v_0, v_q^j) = j$, $p, q \in \{1, 2, \ldots, i_j\}$, $j = \overline{1,l}$. Note that $\forall j = \overline{1,l} : V^j \neq \emptyset$, and V^l consists of the vertices with index $(1,0)$, and $|V^l| \geq 1$. Let $v_1^l \in V^l$, and let $v_1^{l-1} \in V^{l-1}$ be a vertex connected with the previous one by an edge. Next, we consider the operation of decreasing the number of v_1^{l-1} depending on the vertex index:

(i) $(0,1)$: γ is connected and it coincides with the graph in Fig. 4.2;

(ii) $(1,1)$: by operation $(a2)^{-1}$ we contract the vertex v_1^{l-1}, then $v_1^l \in V^{l-1}$ (it is equivalent to the contraction of the vertex v_1^l and the corresponding edge);

(iii) $(1,2)$: in this case, there exists $v_2^l \in V^l$ and by applying operation $(a1)^{-1}$ twice we contract the vertices v_1^l and v_2^l;

(iv) $(2,1)$: there exists the vertex $v_2^l \in V^l$ which is connected with v_1^{l-1} by some edge, therefore, by applying operation $(a1)^{-1}$ to the vertex v_2^l, we obtain the case $(1,1)$;

(v) $(2,2)$: we have $|V^l| \geq 3$, in other words, there exist two vertices v_2^l, v_3^l connected with v_1^{l-1}, and we obtain case $(1,2)$ or $(2,1)$ (operation $(a1)^{-1}$ at v_2^l or at v_3^l).

Thus, the graph γ can be represented in the form of $V^l = \emptyset$ if index of v_1^{l-1} equals $(1,1)$ and of $V^l = \{v_1^l\}$ (i.e., $|V^l| = 1$) otherwise. In the same way, we assume that the classes $V^1, V^2, \ldots, V^{l-1}$ do not have any vertex with degree 1. The last statement means that the whole graph is the path between v_0 and v_1^l. And finally, we get the graph by using operation $(a1)^{-1}$, which is depicted in Fig. 4.2. □

Theorem 4.3. *For any arbitrary function $f \in \Omega(M)$ with at most 5 critical pints, the equipped Reeb graph of its equivalence class $[f]$ has one of the representations from Fig. 4.3.*

Proof. Each function $f \in \Omega(M)$, except $f = const$, takes its maximum and minimum values on a compact set and corresponding points are extremum points of the function. Thus, each equipped Reeb graph has at least two vertices. There exist exactly 2 equipped Reeb graphs with two vertices, because there are two possibilities for determining the edge of the first or second type (see Fig. 4.3 $(1,2)$). An equipped Reeb graph with three vertices has one of the forms depicted in Fig. 4.3 $(3,4)$. It follows from the statement that the edges incident with the vertices with degree 2 should be of different types (see Fig. 2.2 $(3,4)$).

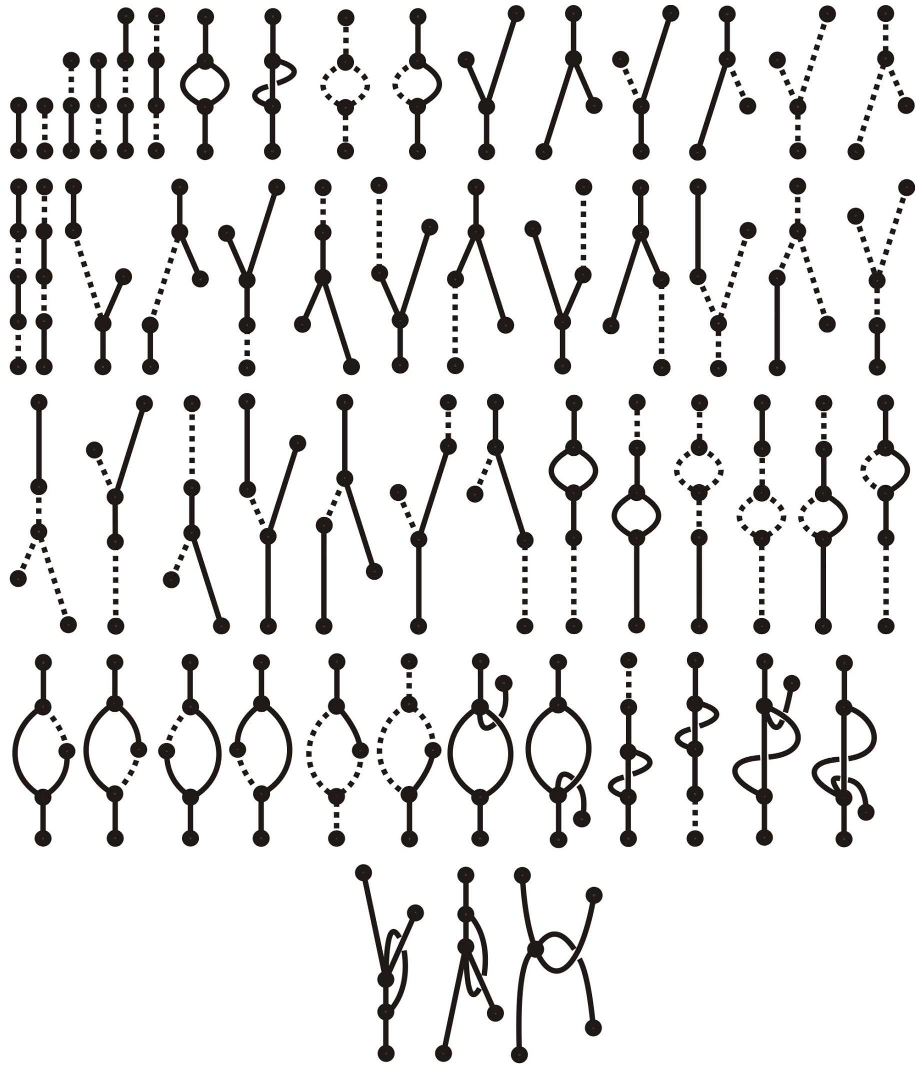

Fig. 4.3

Each graph with the vertices as in Figs. 2.3, 2.4, and 2.5, which has n vertices, can be obtained from the graph with the less number of vertices and edges after the procedure of increasing the number of vertices of the graph. Thus, firstly we consider the graphs of the functions from $[f]$ with three vertices and without edges division. Let us consider all above-described operations with these graphs, after which we get the graphs with four and five vertices with at most $deg = 4$. We fix the edges division such that some neighborhood of each of the vertex has the form depicted in Figs. 2.3, 2.4, and 2.5. Hence, for the function from $[f]$ we get one of the graphs from Fig. 4.3. □

Corollary 4.4. *There exist* 57 *topologically* $\mathcal{O}$*-inequivalent* m*-functions, the equipped Reeb graphs of which are depicted in Fig.* 4.3.

Remark 4.5. In [16], the oriented layer equipped classification of m-functions with minimal number and at most 6 critical points is obtained. Thus, the graphs 1, 2, 7–10 in Fig. 4.3 correspond to the minimal m-functions.

5. Topological type of a surface

Let M be an oriented smooth compact connected surface with boundary, and $f \in \Omega(M)$.

Definition 5.1. A vertex with degree 2 (3) of the graph Γ_f of the function f, which is incident with the edges of both types, is said to be a T-*vertex* (D-*vertex*).

As it was mentioned earlier, the topological type of the surface can be defined basing on the information given by the genus g and the number of boundary components ∂.

Let us consider the definition of a boundary cycle. We consider the first type edge and an arbitrary moving direction along this edge. When moving along this direction, we reach the vertex. The final move is defined by the orientation (cycle order) in Y- and X-vertices along the other edge of the first type in the T-vertex or along the same edge in the opposite direction in the D-vertex and on the vertices with degree 1. After passing the edge, the next edge is passed in the same way. We continue this procedure until we reach the initial vertex. As a result, we obtain a cycle with the initial moving direction which is said to be a *boundary cycle*. From the structure of this cycle it follows that the number of boundary cycles is equal to the number of boundary components ∂.

The first and the second type edges of the graph Γ_f are called I- and O-edges, respectively.

To define the genus of the surface, we consider the following designation: let E_{I} (E_{O}) be the number of I-edges (O-edges) and V_{I} (V_{O}) be the number of the vertices incident only with I-edges (O-edges).

Theorem 5.2. *Let a graph Γ_f of a function $f \in \Omega(M)$ include either O-edges or I-edges. Then the genus of the surface can be computed using formulas (5.1) and (5.2), respectively, where*

$$g_{\mathsf{O}} = E_{\mathsf{O}} - V_{\mathsf{O}} + 1, \tag{5.1}$$

$$g_{\mathsf{I}} = \frac{E_{\mathsf{I}} - V_{\mathsf{I}} + 2 - \partial}{2}. \tag{5.2}$$

Proof. If the graph Γ_f includes only O-edges, then the surface M can not include the boundary, i.e., $\partial = 0$. If Γ_f is a tree, then the genus equals 0, and adding the edges with new vertices at interior vertices of the connected graph increases the genus by 1. Thus, (5.1) follows from the last statement.

If Γ_f is a tree and includes only I-edges, then the surface is a 2-dimensional disk. The gluing of an edge to the inner vertices is equivalent to the gluing of a rectangle to the surface M. If the rectangle is glued to one boundary component, then the number of boundary components increases by 1 while the genus does not change. Otherwise, the number of boundary components decreases by 1 while the genus increases by 1. Thus, we obtain (5.2). □

Theorem 5.3. *The genus of a surface can be computed using the formula*

$$g = g_{\mathsf{O}} + g_{\mathsf{I}} + V_{\mathsf{D}} + V_{\mathsf{T}} - c_{\mathsf{O}} - c_{\mathsf{I}} + 1, \tag{5.3}$$

where g_{O} is a summary genus of the subgraph, which consists only of the edge of the second type, such that the genus of each graph component is defined by formula (5.1), g_{I} is a summary genus of the subgraph, which consists only of the edge of the first type, such that the genus of each graph component is defined by formula (5.2), V_{D} is the number of D-vertices and V_{T} is the number of T-vertices, c_{O} is the number of connected components of the subgraph, which consists only of the edge of the second type, and c_{I} is the number of connected components of the subgraph, which consists only of the edge of the first type.

Proof. Let us cut the graph in D- and T-vertices. The union of the second type edges with D- and T-vertices forms a subgraph of the second type. The genus of each component of this subgraph is defined by (5.1) and the genus g_{O} of the whole subgraph is defined as the sum of these genera. In the same way, we determine a subgraph of the first type and its genus g_{I}. The graph G can be obtained as a disjoint union of graphs of the first and second types by gluing the corresponding edges of the second type to the edges of the first type in D- and T-vertices. Then, either the number of connected components decreases by 1 while the genus does not change, or the number of connected components does not change while the genus increases by 1. □

The next statement follows from 2.1 :

Corollary 5.4. *Let f be an m-function. Then formulas (5.1), (5.2) and (5.3) hold true.*

6. Conclusion

We proved the topological equivalence of functions of class $\Omega(M)$ and m-functions. We obtained the classification of functions of class $\Omega(M)$ up to the $\mathcal{O}$-equivalence by means of equipped Reeb graphs. We computed the number of functions from class $\Omega(M)$ with at most five critical points. Moreover, the problem of topological type of the equipped Reeb graph was solved.

The results obtained in the paper can be generalized to the case of non-oriented surfaces with boundary.

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Прості функції Морса на орієнтованій поверхні з межею

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У даній роботі розглядаються гладкі функції з невиродженими критичними точками на гладкій компактній орієнтованій поверхні з межею. Спочатку показано, що такі функції топологічно еквівалентні m-функціям. Для опису їх топологічної структури використовується оснащений граф Ріба. Потім автори характеризують топологічну структуру всіх простих функцій з не більш ніж 5-ма критичними точками. Нарешті, виводиться формула для обчислення роду поверхні, яка базується на оснащеному графі Ріба.

Ключові слова: топологічна класифікація, невироджена критична точка, оснащений граф Ріба.