Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 3, pp. 379–394
doi: https://doi.org/10.15407/mag15.03.379

On Einstein Sequential Warped Product Spaces

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In this paper, Einstein sequential warped product spaces are studied. Here we prove that if M is an Einstein sequential warped product space with negative scalar curvature, then the warping functions are constants. We find out some obstructions for the existence of such Einstein sequential warped product spaces. We also show that if $\bar{M} = (M_1 \times_f I_{M_2}) \times_{\bar{f}} I_{M_3}$ is a sequential warped product of a complete connected $(n-2)$-dimensional Riemannian manifold M_1 and one-dimensional Riemannian manifolds I_{M_2} and I_{M_3} with some certain conditions, then (M_1, g_1) becomes a $(n-2)$-dimensional sphere of radius $\rho = \frac{n-2}{\sqrt{r^1+\alpha}}$. Some examples of the Einstein sequential warped product space are given in Section 3.

Key words: warped product, sequential warped product, Einstein manifold.

Mathematical Subject Classification 2010: 53C21, 53C25, 53C50.

1. Introduction

A Riemannian manifold $(M^n, g), n \geq 2$, is said to be an Einstein manifold if for every vector field $X, Y \in \chi(M)$ there exists a real constant λ such that $\mathrm{Ric} = \lambda g$, where g is called Einstein metric and Ric denotes the Ricci tensor of M. It is obvious that $\lambda = \frac{r}{n}$, where $r(= \mathrm{tr}(\mathrm{Ric}))$ is the scalar curvature of M and n is the dimension of M.

The notion of a warped product was introduced by R. Bishop and B. O'Nill [1] for studying manifolds of negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with $\dim B = m > 0$, $\dim F = k > 0$ and $f : B \to (0, \infty)$, $f \in C^\infty(B)$. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with Riemannian structure such that $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2$ for any vector field X on M. Thus, we have that $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M, and F is the fiber. The function f is called the warping function of the warped product [9].

Now, we can generalize warped products to multiply warped products. A multiply warped product is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \oplus \cdots \oplus b_m^2 g_{F_m}$, where for each

$i \in \{1, 2, \ldots, m\}$, $b_i : B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. In particular, when $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold. We call M the multiply generalized Robertson–Walker space-time.

A multiply twisted product (M, g) is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \oplus \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, 2, \ldots, m\}$, the warping functions $b_i : B \times F_i \to (0, \infty)$ are smooth.

In 2015, S. Shenawy introduced a new type of warped product manifolds, namely, a sequential warped product [14]. Let (M_i, g_j), $i = 1, 2, 3$, be three Riemannian manifolds. Let $f : M_1 \to (0, \infty)$ and $\bar{f} : M_1 \times M_2 \to (0, \infty)$ be two smooth positive functions on M_1 and $M_1 \times M_2$ respectively. Then the sequential warped product manifold, denoted by $(M_1 \times_f M_2) \times_{\bar{f}} M_3$, is the triple product manifold $(M_1 \times M_2) \times M_3$, with the metric tensor $\bar{g} = (g_1 \oplus f^2 g_2) \oplus \bar{f}^2 g_3$. The functions f and $\bar{f}$ are called warping functions.

Let (M_i, g_i), $i = 1, 2$ be two n_i-dimensional Riemannian manifolds. Let $\bar{f} : M_1 \times M_2 \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ be two smooth positive functions. Then $(n_1 + n_2 + 1)$-dimensional product manifold $I \times_{\bar{f}} (M_1 \times_f M_2)$, with the metric tensor $\bar{g} = -\bar{f}^2 dt^2 \oplus (g_1 \oplus f^2 g_2)$, is a standard static space-time, where I is an open, connected subinterval of $\mathbb{R}$, and dt^2 is the Euclidean metric tensor on I. Also, the $(n_1 + n_2 + 1)$-dimensional product manifold $I_{\bar{f}} \times (M_1 \times_f M_2)$, with the metric tensor $\bar{g} = -dt^2 \oplus \bar{f}^2 (g_1 \oplus f^2 g_2)$, is a generalized Robertson–Walker space-time, where I is an open, connected subinterval of $\mathbb{R}$, $\bar{f} : I \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ are smooth functions, and dt^2 is the Euclidean metric tensor on I.

Many authors studied Einstein warped product spaces. In 2002, D.S. Kim established that there does not exist a compact Einstein warped product space with nonconstant warping function [6]. In [8], S. Kim constructed compact base manifolds with positive scalar curvature, which do not admit any non-trivial Ricci-flat Einstein warped product, and noncompact complete base manifolds. In 2011, M. Rimoldi [15] proved a result for Einstein warped products that is the extension of a theorem from [7] to the case of noncompact bases. A.S. Diallo obtained recent results on the existence of compact Einstein warped product Riemannian manifolds in [3]. In [4], D. Dumitru gave some obstructions to the existence of compact Einstein warped products. Also Q. Qu, Y. Wang [12], S. Pahan, B. Pal and A. Bhattacharyya [10], [11] etc. studied Einstein warped product and multiply warped product with affine connections.

In this paper, we study Einstein sequential warped product spaces. First, we prove that if $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ has negative scalar curvature, then the warping functions f and $\bar{f}$ are constants. Next, in Theorem 2.7, we show some obstructions to the existence of such spaces. Then, in Theorem 2.8, we show that an Einstein sequential warped product space with the complete connected $(n-2)$-dimensional base is isometric to an $(n-2)$-dimensional sphere. Later we prove a result in the static space-time with some conditions. In the last section, we give an example of the Einstein sequential warped space.

For more convenience, we give a summary of indices used in the paper as follows:

(M_i, g_i):	Riemannian manifolds;
n_i:	dimensions of M_i;
λ, μ :	scalar functions;
Ric:	the Ricci tensor of a Riemannian manifold;
r_i:	scalar curvature of Riemannian manifolds M_i;
$f, \bar{f}$:	the warping functions defined on M_1 and $M_1 \times M_2$;
∇f:	gradient of a smooth function f;
H^f:	the Hessian of a smooth function f is defined as its second covariant differential $H^f = \nabla\nabla f$, where ∇ is the Levi-Civita connection on the Riemannian manifold M_1;
Δf:	Laplacian of a smooth function f is the divergence of its gradient;
$\chi(M)$:	the set of all vector fields on M.

2. Einstein sequential warped product spaces

Now we consider the following propositions from [14], which will be helpful in proving the main results of this section.

Proposition 2.1. *Let $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ be a sequential warped product with metric $g = (g_1 \oplus f^2 g_2) \oplus \bar{f}^2 g_3$ and also let $X_i, Y_i, Z_i \in \chi(M_i)$. Then*

1) $\overline{\mathrm{Ric}}(X_1, Y_1) = \mathrm{Ric}^1(X_1, Y_1) - \dfrac{n_2}{f} H_1^f(X_1, Y_1) - \dfrac{n_3}{\bar{f}} H^{\bar{f}}(X_1, Y_1),$

2) $\overline{\mathrm{Ric}}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - f^2 g_2(X_2, Y_2) f^* - \dfrac{n_3}{\bar{f}} H^{\bar{f}}(X_2, Y_2),$

3) $\overline{\mathrm{Ric}}(X_3, Y_3) = \mathrm{Ric}^3(X_3, Y_3) - \bar{f}^2 g_3(X_3, Y_3) \bar{f}^*,$

4) $\overline{\mathrm{Ric}}(X_i, Y_j) = 0,$ *$i \neq j$, where* $f^* = \dfrac{\Delta^1 f}{f} + (n_2 - 1)\dfrac{|\nabla^1 f|^2}{f^2}$ *and* $\bar{f}^* = \dfrac{\Delta \bar{f}}{\bar{f}} + (n_1 + n_2 - 1)\dfrac{|\nabla \bar{f}|^2}{\bar{f}^2}.$

Proposition 2.2. *The sequential warped product $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ ($\dim M_1 = n_1$, $\dim M_2 = n_2$, $\dim M_3 = n_3$), is Einstein with $\overline{\mathrm{Ric}} = \lambda \bar{g}$ if and only if*

1) $\mathrm{Ric}^1 = \lambda g_1 + \dfrac{n_2}{f} H_1^f + \dfrac{n_3}{\bar{f}} H^{\bar{f}},$

2) $\mathrm{Ric}^2 = \omega g_2 + \dfrac{n_3}{\bar{f}} H^{\bar{f}},$ *where* $\omega = f^2 \left(\lambda + \dfrac{\Delta^1 f}{f} + (n_2 - 1)\dfrac{|\nabla^1 f|^2}{f^2}\right),$

3) *M_3 is Einstein with* $\mathrm{Ric}^3 = \mu g_3,$

4) $\mu = \bar{f}^2 \left(\lambda + \dfrac{\Delta \bar{f}}{\bar{f}} + (n_2 - 1)\dfrac{|\nabla \bar{f}|^2}{\bar{f}^2}\right).$

Now, we state a lemma whose detailed proof is given in [6].

Lemma 2.3. *Let f be a smooth function on a Riemannian manifold M_1. Then for any vector X, the divergence of the Hessian tensor H^f satisfies*

$$\operatorname{div}\left(H^f\right)(X) = \operatorname{Ric}(\nabla f, X) - \Delta(df)(X),$$

where $\Delta = d\delta + \delta d$ denotes the Laplacian on M_1 acting on differential forms.

Now we prove the following propositions for later use.

Proposition 2.4. *Let (M_1, g_1) be a compact Riemannian manifold of dimension $n_1 \geq 2$. Suppose that f is a nonconstant smooth function on M_1 satisfying*

$$\operatorname{Ric}^1 = \lambda g_1 + \frac{n_2}{f} H_1^f + \frac{n_3}{\bar{f}} H^{\bar{f}},$$

for a constant $\lambda \in R$ and $n_2 \geq 2$, and if the condition

$$\frac{n_2 n_3}{f\bar{f}} H^{\bar{f}}(X, \nabla^1 f) + \operatorname{div}\left(\frac{n_3}{\bar{f}} H^{\bar{f}}\right) = \frac{2n_2}{f} d(\Delta^1 f) + \frac{n_3}{2} d\left(\frac{\Delta \bar{f}}{\bar{f}}\right)$$

holds, then f satisfies

$$\omega = f^2\left(\lambda + \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}\right)$$

for a constant $\omega \in R$.

Proof. By taking trace of both sides of $\operatorname{Ric}^1 = \lambda g_1 + \frac{n_2}{f} H_1^f + \frac{n_3}{\bar{f}} H^{\bar{f}}$, we have

$$r_1 = \lambda n_1 + \frac{n_2}{f}\Delta^1 f + \frac{n_3}{\bar{f}}\Delta \bar{f}, \tag{2.1}$$

where r_1 denotes the scalar curvature of M_1 given by $\operatorname{tr}(\operatorname{Ric}^1)$. From [9], the second Bianchi identity implies that

$$dr_1 = 2 \operatorname{div} \operatorname{Ric}^1. \tag{2.2}$$

From equations (2.1) and (2.2), we obtain

$$\operatorname{div}\operatorname{Ric}^1(X) = \frac{n_2}{2}\left[-\frac{\Delta^1 f}{f^2} df + \frac{1}{f} d(\Delta^1 f)\right] + \frac{n_3}{2}\left[-\frac{\Delta \bar{f}}{\bar{f}^2} d\bar{f} + \frac{1}{\bar{f}} d(\Delta \bar{f})\right].$$

Also, by the definition, we have

$$\operatorname{div}\left(\frac{1}{f} H_1^f\right)(X) = \sum_i \left(D_{e_i}\left(\frac{1}{f} H^f\right)\right)(e_i, X) = -\frac{1}{f^2} H_1^f(\nabla f, X) + \frac{1}{f} \operatorname{div} H_1^f(X)$$

for any vector field X and an orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ of M_1. Since $H_1^f(\nabla f, X) = (D_X df)(\nabla f) = \frac{1}{2} d(|\nabla^1 f|^2)(X)$, the last equation becomes

$$\operatorname{div}\left(\frac{1}{f} H_1^f\right)(X) = -\frac{1}{2f^2} d\left(|\nabla^1 f|^2\right)(X) + \frac{1}{f} \operatorname{div} H_1^f(X)$$

for a vector field X on M_1. From Lemma 2.3, it follows that

$$\operatorname{div}\left(\frac{1}{f}H_1^f\right)(X)=\frac{1}{2f^2}\left[d(|\nabla^1 f|^2)(X)+2\lambda f(df)(X)-2fd(\Delta^1 f)\right] + \frac{n_3}{f\bar{f}}H^{\bar{f}}(X,\nabla^1 f).$$

But Proposition 2.2 gives

$$\operatorname{div}\mathrm{Ric}^1=\operatorname{div}\left(\frac{n_2}{f}H_1^f\right)+\operatorname{div}\left(\frac{n_3}{\bar{f}}H^{\bar{f}}\right).$$

Therefore, using the condition

$$\frac{n_2n_3}{f\bar{f}}H^{\bar{f}}\left(X,\nabla^1 f\right)+\operatorname{div}\left(\frac{n_3}{\bar{f}}H^{\bar{f}}\right)=\frac{2n_2}{f}d(\Delta^1 f)+\frac{n_3}{2}d\left(\frac{\Delta\bar{f}}{\bar{f}}\right),$$

we obtain

$$\frac{n_2}{2f^2}\left[2f\lambda(df)(X)+fd(\Delta^1 f)+\Delta^1 f df+(n_2-1)d(|\nabla^1 f|^2)(X)\right]=0,$$

that is,

$$f\Delta^1 f+(n_2-1)|\nabla^1 f|^2+\lambda f^2=\omega$$

for some constant ω. Thus the proposition is proved. □

Now, in a similar way, we will consider the following lemma.

Lemma 2.5. *Let $\bar{f}$ be a smooth function on a Riemannian manifold $M_1\times M_2$. Then, for any vector X, the divergence of the Hessian tensor $H^{\bar{f}}$ satisfies*

$$\operatorname{div}\left(H^{\bar{f}}\right)(X)=\mathrm{Ric}\left(\nabla\bar{f},X\right)-\Delta\left(d\bar{f}\right)(X),$$

where $\Delta=d\delta+\delta d$ denotes the Laplacian on $M_1\times M_2$ acting on differential forms.

Now we prove another following proposition that will be also helpful in proving the next theorem.

Proposition 2.6. *Let (M_1,g_1) and (M_2,g_2) be two compact Riemannian manifolds of dimension $n_1\geq 2$ and $n_2\geq 2$. Suppose that $\bar{f}$ is a nonconstant smooth function on $M_1\times M_2$ satisfying $\mathrm{Ric}^2=\omega g_2+\frac{n_3}{\bar{f}}H^{\bar{f}}$ for a constant $\omega\in R$ and if the condition*

$$(n_3-n_2)d\left(|\nabla\bar{f}|^2\right)(X)+2\bar{f}(\omega-\lambda)\left(d\bar{f}\right)(X)=4\bar{f}d\left(\Delta\bar{f}\right)(X)$$

holds, then $\bar{f}$ satisfies

$$\mu=\bar{f}^2\left(\lambda+\frac{\Delta\bar{f}}{\bar{f}}+(n_2-1)\frac{|\nabla\bar{f}|^2}{\bar{f}^2}\right)$$

for a constant $\mu\in R$. Hence, for a compact Einstein space (M_3,g_3) of dimension $n_3\geq 2$ with $\mathrm{Ric}^3=\mu g_3$, we get a constant Einstein sequential warped product space $\bar{M}=(M_1\times_f M_2)\times_{\bar{f}} M_3$ with $\overline{\mathrm{Ric}}=\lambda\bar{g}$.

Proof. By taking trace of both sides of $\mathrm{Ric}^2 = \omega g_2 + \frac{n_3}{\bar{f}} H^{\bar{f}}$, we have

$$r_2 = \omega n_2 + n_3 \frac{\Delta \bar{f}}{\bar{f}},$$

where r_2 denotes scalar curvature of M_2 given by $\mathrm{tr}(\mathrm{Ric}^2)$. As in the proof of Proposition 2.4, we can show that $\mu = \bar{f}^2\Big(\lambda + \frac{\Delta \bar{f}}{\bar{f}} + (n_2 - 1)\frac{|\nabla \bar{f}|^2}{\bar{f}^2}\Big)$. For a compact Einstein space (M_3, g_3) of dimension $n_3 \geq 2$ with $\mathrm{Ric}^3 = \mu g_3$, we make a constant Einstein sequential warped product space $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ with $\overline{\mathrm{Ric}} = \lambda \bar{g}$ by the sufficiencies of Proposition 2.2. □

Now we prove the following theorem.

Theorem 2.7. *Let $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ be an Einstein sequential warped product space, where M_1 and M_2 are compact spaces, and M_3 is a compact Einstein space. If $\bar{M}$ has the negative scalar curvature, then the warping functions f and $\bar{f}$ are constants.*

Proof. We have $\mathrm{Ric}^2 = \omega g_2 + \frac{n_3}{\bar{f}} H^{\bar{f}}$, where

$$\omega = f^2 \left(\lambda + \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2} \right),$$

and M_3 is Einstein with $\mathrm{Ric}^3 = \mu g_3$, where

$$\mu = \bar{f}^2 \left(\lambda + \frac{\Delta \bar{f}}{\bar{f}} + (n_2 - 1)\frac{|\nabla \bar{f}|^2}{\bar{f}^2} \right).$$

We see that ω and μ are constants. Let $p, q \in M_1$ be two points, where f attains its maximum and minimum in M_1. Then $\nabla^1 f(p) = 0 = \nabla^1 f(q)$ and also $\Delta^1 f(p) \leq 0 \leq \Delta^1 f(q)$. Since $\bar{M}$ has negative scalar curvature, $\lambda < 0$. Now we also have $f > 0$. Hence we obtain $-\lambda f^2(p) > -\lambda f^2(q)$. We also have $\omega = f^2\Big(\lambda + \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}\Big)$. Therefore, we get

$$0 \geq f(p)\Delta^1 f(p) = \omega - \lambda f^2(p) > \omega - \lambda f^2(q) = f(q)\Delta^1 f(q) \geq 0.$$

So, we can write

$$\omega - \lambda f^2(p) = \omega - \lambda f^2(q).$$

Thus, $\lambda < 0$ implies that $f(p) = f(q)$, i.e., f is constant.

Similarly, let $(p_1, q_1), (p_2, q_2) \in M_1 \times M_2$ be two points, where $\bar{f}$ attains its maximum and minimum in $M_1 \times M_2$. Then $\nabla \bar{f}(p_1, q_1) = 0 = \nabla \bar{f}(p_2, q_2)$ and also $\Delta \bar{f}(p_1, q_1) \leq 0 \leq \Delta \bar{f}(p_2, q_2)$. Since M has negative scalar curvature, $\lambda < 0$. Now we also have $f > 0$. Hence we obtain $-\lambda \bar{f}^2(p_1, q_1) > -\lambda \bar{f}^2(p_2, q_2)$. We also have $\mu = \bar{f}^2(\lambda + \frac{\Delta \bar{f}}{\bar{f}} + (n_2 - 1)\frac{|\nabla \bar{f}|^2}{\bar{f}^2})$. Therefore, we get

$$0 \geq \bar{f}(p_1, q_1)\Delta \bar{f}(p_1, q_1) = \mu - \lambda \bar{f}^2(p_2, q_2)$$

$$> \mu - \lambda \bar{f}^2(p_2, q_2) = \bar{f}(p_2, q_2)\Delta \bar{f}(p_2, q_2) \geq 0.$$

So, we can write

$$\mu - \lambda \bar{f}^2(p_1, q_1) = \mu - \lambda \bar{f}^2(p_2, q_2).$$

Thus, $\lambda < 0$ implies that $\bar{f}(p_1, q_1) = \bar{f}(p_2, q_2)$, i.e., $\bar{f}$ is constant. This completes the proof of the theorem. □

Theorem 2.8. *Let $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ be a sequential warped product space, where M_1 and M_2 are compact spaces with $\dim M_1 = n_1$, $\dim M_2 = n_2$, and M_3 is an Einstein space with $\dim M_3 = n_3$, $\mathrm{Ric}_{M_3} = \lambda g_{M_3}$. Then the following conditions hold:*

a) *If $r_3 \leq 0$ and $\lambda > 0$, then $\bar{f}$ is constant.*

b) *If $n_2 = 1$ and $\omega >$ (or $<$)$f^2\lambda$, then f is constant. Hence $\bar{f}$ is constant when $\lambda > 0$.*

c) *If $|\nabla^1 f| \geq \sqrt{\frac{\omega}{n_2-1}}$, $|\nabla \bar{f}| \geq \sqrt{\frac{\mu}{n_2-1}}$ and also $\lambda < 0$, then f and $\bar{f}$ are constant.*

Proof. Taking trace of Proposition 2.2, we have

$$\begin{aligned} r_1 &= \lambda n_1 + \frac{n_2}{f}\Delta^1 f + \frac{n_3}{\bar{f}}\Delta \bar{f}, \\ r_2 &= \omega n_2 + \frac{n_3}{\bar{f}}\Delta \bar{f}, \\ r_3 &= \mu n_3. \end{aligned}$$

a) If $r_3 < 0$, then $\mu \leq 0$. From Proposition 2.2, we have

$$\mu = \bar{f}^2\lambda + \bar{f}\Delta\bar{f} + (n_2 - 1)|\nabla \bar{f}|^2.$$

So, we can write

$$\bar{f}^2\lambda + \bar{f}\Delta\bar{f} = \mu - (n_2 - 1)|\nabla \bar{f}|^2 \leq 0.$$

Hence,

$$\bar{f}\Delta\bar{f} \leq -\bar{f}^2\lambda < 0.$$

Therefore $\bar{f}$ is constant.

b) From Proposition 2.2, we have

$$\omega = f^2 \left[\lambda + \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2} \right].$$

Using the conditions, we can easily say that f is constant.

As f is constant, $\omega = f^2\lambda$. Therefore we obtain $f^2\lambda + \frac{n_3}{f}\Delta\bar{f} = 0$. Using the condition $\lambda > 0$, $\Delta\bar{f} > 0$. Thus $\bar{f}$ is constant.

c) We know

$$\omega = f^2\lambda + f\Delta^1 f + (n_2 - 1)|\nabla^1 f|^2.$$

Then

$$f^2\lambda + f\Delta^1 f = \omega - (n_2 - 1)|\nabla^1 f|^2.$$

Using the condition $|\nabla^1 f| \geq \sqrt{\frac{\omega}{n_2-1}}$ and $\lambda < 0$, we get

$$f\Delta^1 f \geq -f^2\lambda > 0.$$

Thus f is constant. Similarly, from Proposition 2.2 and using the conditions $|\nabla \bar{f}| \geq \sqrt{\frac{\mu}{n_2-1}}$ and $\lambda < 0$, we can say that $\bar{f}$ is constant. □

Theorem 2.9. *Let $\bar{M} = (M_1 \times_f I_{M_2}) \times_{\bar{f}} I_{M_3}$ be a sequential warped product of a complete connected $(n-2)$-dimensional Riemannian manifold M_1 and the one-dimensional Riemannian manifolds I_{M_2} and I_{M_3}. If $(\bar{M}, g)$ is an Einstein manifold with constant associated scalars α, $U \in \chi(\bar{M})$ and the Hessian of f or $\bar{f}$ is proportional to the metric tensor g_1 on M_1, then (M_1, g_1) is an $(n-2)$-dimensional sphere of radius $\rho = \frac{n-2}{\sqrt{r^1+\alpha}}$.*

Proof. Let $\bar{M}$ be a connected sequential warped product manifold. Then from [14] we have

$$\mathrm{Ric}^1(X,Y) = \overline{\mathrm{Ric}}(X_1,Y_1) + \frac{1}{f}H_1^f(X,Y) + \frac{1}{\bar{f}}H^{\bar{f}}(X,Y). \tag{2.3}$$

Since $(\bar{M}, g)$ is an Einstein manifold with constant associated scalars α, then we have

$$\overline{\mathrm{Ric}}(X,Y) = \alpha g(X,Y). \tag{2.4}$$

Decomposing the vector field U uniquely into its components U_{M_1}, $U_{I_{M_2}}$ $U_{I_{M_3}}$ on M_1, I_{M_2} and I_{M_3}, respectively, we have

$$U = U_{M_1} + U_{I_{M_2}} + U_{I_{M_3}}. \tag{2.5}$$

Putting the value of (2.4) and (2.5) in (2.3), we get

$$\mathrm{Ric}^1(X,Y) = \alpha g(X,Y) + \frac{1}{f}H_1^f(X,Y) + \frac{1}{\bar{f}}H^{\bar{f}}(X,Y).$$

By the contraction over X and Y, we get

$$r_1 = \bar{r} - 2\alpha + \frac{\Delta^1 f}{f} + \frac{\Delta \bar{f}}{\bar{f}}. \tag{2.6}$$

From [14], we obtain

$$\bar{r} = -n\frac{\Delta^1 f}{f} - \frac{\Delta \bar{f}}{\bar{f}}. \tag{2.7}$$

From equations (2.6) and (2.7), it follows that

$$r_1 + 2\alpha = (1-n)\frac{\Delta^1 f}{f}.$$

Since the Hessian of f is proportional to the metric tensor g_1, then we have

$$H^f(X,Y) = \frac{1}{(n-2)}\left[(n-1)\frac{\Delta^1 f}{(n-2)}\right] g_1(X,Y).$$

From the above equation, we obtain

$$H^f(X,Y) + \frac{(r^1+2\alpha)f}{(n-2)^2} g_1(X,Y) = 0.$$

So, M_1 is isometric to the $(n-2)$-dimensional sphere of radius $\frac{(n-2)}{\sqrt{r^1+2\alpha}}$ [13].

Again from [14], we obtain

$$\bar{r} = -n\frac{\Delta \bar{f}}{\bar{f}} - n(n-2)\frac{|\nabla \bar{f}|^2}{\bar{f}^2}. \tag{2.8}$$

From equations (2.6) and (2.8), we get

$$r^1 + 2\alpha = -(n-1)\frac{\Delta \bar{f}}{\bar{f}} - n(n-2)\frac{|\nabla \bar{f}|^2}{\bar{f}^2} + \frac{\Delta^1 f}{f}. \tag{2.9}$$

Since the Hessian of f is proportional to the metric tensor g_1, then we obtain

$$H^{\bar{f}}(X,Y) = \frac{1}{(n-2)}\left[(n-1)\frac{\Delta \bar{f}}{(n-2)} + n\frac{|\nabla \bar{f}|^2}{\bar{f}} - \frac{\bar{f}\Delta^1 f}{(n-2)f}\right] g_1(X,Y).$$

Hence, from equation (2.9), we have

$$H^{\bar{f}}(X,Y) + \frac{(r^1+2\alpha)\bar{f}}{(n-2)^2} g_1(X,Y) = 0.$$

Thus M_1 is isometric to the $(n-2)$-dimensional sphere of radius $\frac{(n-2)}{\sqrt{r^1+2\alpha}}$ [13]. This completes the proof. □

Let (M_i, g_i), $i = 1, 2$ be two n_i-dimensional Riemannian manifolds. Let $\bar{f} : M_1 \times M_2 \to (0,\infty)$ and $f : M_1 \to (0,\infty)$ be two smooth positive functions. Then (n_1+n_2+1)-dimensional product manifold $I \times_{\bar{f}} (M_1 \times_f M_2)$, with the metric tensor $\bar{g} = -\bar{f}^2 dt^2 \oplus (g_1 \oplus f^2 g_2)$, is a standard static space-time, where I is an open, connected subinterval of $\mathbb{R}$, and dt^2 is the Euclidean metric tensor on I.

Theorem 2.10. *Let $\bar{M} = I \times_{\bar{f}} (M_2 \times_f M_3)$ be a sequential warped product with the metric tensor $-\bar{f}^2 dt^2 \oplus (g_2 \oplus f^2 g_3)$ and $\dim M_2 = n_2$, $\dim M_3 = n_3$. Then $(\bar{M}, g)$ is an Einstein manifold with constant associated scalar λ if and only if the following conditions are satisfied:*

i) *(M_2, g_2) is an Einstein manifold with scalar λ_2 when Hessian tensor $H^{\bar{f}}$ is proportional to the metric tensor g on M_2,*

ii) $\lambda = \frac{n_2}{f} f'' + \frac{n_3}{\bar{f}} \bar{f}''$,

iii) $\lambda_2 - \lambda f^2 + f f'' + (n_2-1)(f')^2 - n_3 f^2 = 0$,

iv) *(M_3, g_3) is an Einstein manifold with scalar λ_3,*

v) $\lambda_3 - \lambda \bar{f}^2 + \bar{f}\bar{f}'' + n_2(\bar{f}')^2 = 0.$

Proof. From [14], we have

$$\overline{\mathrm{Ric}}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=-\frac{n_2}{f}H_1^f\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)-\frac{n_3}{\bar f}H^{\bar f}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right), \tag{2.10}$$

$$\overline{\mathrm{Ric}}(X_2,Y_2)=\mathrm{Ric}^2(X_2,Y_2)+f^2\left[\frac{f''}{f}+(n_2-1)\frac{(f')^2}{f^2}\right]g_2(X_2,Y_2)$$
$$-\frac{n_3}{\bar f}H^{\bar f}(X_2,Y_2), \tag{2.11}$$

$$\overline{\mathrm{Ric}}(X_3,Y_3)=\mathrm{Ric}^3(X_3,Y_3)+\bar f^2\left[\frac{\bar f''}{\bar f}+n_2\frac{(\bar f')^2}{\bar f^2}\right]g_3(X_3,Y_3). \tag{2.12}$$

Since $\bar M$ is an Einstein manifold, we have

$$\overline{\mathrm{Ric}}=\lambda g.$$

Now,

$$\overline{\mathrm{Ric}}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\lambda g_1\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right).$$

From equation (2.10), we obtain

$$\lambda=\frac{n_2}{f}f''+\frac{n_3}{\bar f}\bar f''.$$

Since $H^{\bar f}$ is proportional to g on M_2, we can write $H^{\bar f}(X_2,Y_2)=\bar f f^2 g_2(X_2,Y_2)$.

Therefore, from equation (2.11), we get

$$\mathrm{Ric}^2(X_2,Y_2)=\left[\lambda f^2-ff''-(n_2-1)(f')^2+n_3f^2\right]g_2(X_2,Y_2). \tag{2.13}$$

Hence M_2 is an Einstein manifold with λ_2.

From equation (2.13), it follows that

$$\lambda_2-\lambda f^2+ff''+(n_2-1)\{(f')\}^2-n_3f^2=0.$$

Again from equation (2.12), we have

$$\mathrm{Ric}^3(X_3,Y_3)=\left[\lambda\bar f^2-\bar f\bar f''-n_2(\bar f')^2\right]g_3(X_3,Y_3).$$

Hence we can say that M_3 is an Einstein manifold with λ_3.

From the above equation we can easily see that

$$\lambda_3-\lambda\bar f^2+\bar f\bar f''+n_2(\bar f')^2=0. \qquad \square$$

Remark 2.11. From above discussions, we can also prove similar results and theorems for more general metrics given by (2.11) of [2] and Section 3 of [5].

3. Example of Einstein sequential warped product spaces

Example 3.1. Let us consider a five-dimensional pseudo-Riemannian manifold M^5 endowed with the metric given by

$$ds^2 = -dt^2 + (e^t)^2(dx^2 + dy^2 + dz^2) + (e^t)^2 d\psi^2,$$

the fifth coordinate is taken to be space-like unlike in [16].

Then, in a local coordinate, the only non-vanishing components of the Christoffel symbols are

$$\begin{aligned} \Gamma^1_{22} &= \Gamma^1_{33} = \Gamma^1_{44} = \Gamma^1_{55} = (e^t)^2, \\ \Gamma^2_{21} &= \Gamma^3_{31} = \Gamma^4_{41} = \Gamma^5_{51} = 1. \end{aligned}$$

The non-vanishing curvature tensors and the Ricci tensors are

$$\begin{aligned} R_{1221} &= R_{1331} = R_{1441} = R_{1551} = (e^t)^2, \\ R_{2332} &= R_{2442} = R_{3443} = -(e^t)^4, \\ R_{2552} &= R_{3553} = R_{4554} = -(e^t)^4, \end{aligned}$$

and

$$R_{11} = 4, \quad R_{55} = -4(e^t)^2, \quad R_{22} = R_{33} = R_{44} = -4(e^t)^2.$$

Therefore, we can say that

$$R_{ij} = \alpha g_{ij},$$

where $i, j = \{1, 2, 3, 4, 5\}$ and $\alpha = -4$.

Hence this space-time is an Einstein space with scalar curvature $\alpha = -4$.

Now we rewrite the metric in the following way:

$$ds^2 = -dt^2 + (e^t)^2 \left[(dx^2 + dy^2 + dz^2) + d\psi^2\right],$$

i.e., in the form of $(n_1 + n_2 + 1)$-dimensional product manifold $I \times_{\bar{f}} (M_1 \times_f M_2)$ with the metric tensor $\bar{g} = -dt^2 \oplus \bar{f}^2(g_1 \oplus f^2 g_2)$. In this case, I is any open set, say, $(a, b) \subset \mathbb{R}$, $M_1 = \mathbb{R}^3$ and $M_2 = \mathbb{R}$, and $\bar{f} : (a, b) \to (0, \infty)$ is smooth and given by $\bar{f} = (e^t)^2 > 0$ and $f : \mathbb{R}^3 \to (0, \infty)$ is smooth and given by $f = 1 > 0$. Therefore, the above manifold is an example of the Einstein sequential warped product space-time.

Example 3.2. Next we consider a spherically symmetric solution given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{V(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where $V(r)$ is a positive smooth function.

Now we know that (M_i, g_i), $i = 1, 2$, are two n_i-dimensional Riemannian manifolds. Let $\bar{f} : M_1 \times M_2 \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ be two smooth positive functions. Then $(n_1 + n_2 + 1)$-dimensional product manifold $I \times_{\bar{f}} (M_1 \times_f$

M_2), with metric tensor $\bar{g} = -\bar{f}^2 dt^2 \oplus (g_1 \oplus f^2 g_2)$, is a standard static space-time, where I is an open, connected subinterval of $\mathbb{R}$, and dt^2 is the Euclidean metric tensor on I.

To define a sequential warped product for this case, we consider the warping function $\psi : R \to (0, \infty)$ by $\psi = r\sqrt{V(r)}$ and observe that ψ is a smooth function. $\bar{f} : R \times R^3 \to (0, \infty)$ is given by $\bar{f} = \sqrt{f(r)}$ which is smooth.

Therefore the metric can be expressed as

$$ds^2 = -f(r)dt^2 + \frac{1}{V(r)}(dr^2 + r^2 V(r)[d\theta^2 + \sin^2\theta d\phi^2]),$$

which is an example of the Einstein sequential warped product space, because we know that any spherically symmetric solution is Einstein.

4. Appendix

Here we rewrite some basic formulas in a tensor language, which is more common in physical community. We choose local coordinates $(x^1, x^2, \ldots, x^n)$ of any point $p \in M$. Consider $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ and $g = \det(g_{ij})$. Any two smooth vector fields X, Y on M can be written as $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$. We also know that $\mathrm{Ric}(X,Y) = R_{ij}$, $r = R^i_i$, that is, $r = \mathrm{tr}(\mathrm{Ric})$, $\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}$, $\operatorname{div} X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}(X^j \sqrt{g})$, $\mathrm{Hess}(f) = \nabla(\nabla f) = \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}$, the Laplacian of $f = \Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j}\right)$. These are the tensor forms used in this paper.

First, let $\bar{M} = (M_1 \times_f M_2) \times_{\bar{f}} M_3$ be a sequential warped product with metric $g = (g_1 \oplus f^2 g_2) \oplus \bar{f}^2 g_3$, and let $X_i, Y_i, Z_i \in \chi(M_i)$. Here we can derive Proposition 2.1 with tensor approach in the following way:

1) In terms of local coordinate system, we suppose

$$X_1 = \frac{\partial}{\partial x^\alpha} \in \chi(M_1), \quad Y_1 = \frac{\partial}{\partial x^\beta} \in \chi(M_1),$$

where $\alpha, \beta \in \{1, 2, \ldots, n_1\}$. The Riemannian metric corresponding to the smooth manifold M_1 is g_1, and the component of the metric is denoted by $g_{\alpha\beta}$. Then we obtain

$$\bar{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{n_2}{f}\left[\frac{\partial}{\partial x^\alpha}\frac{\partial f}{\partial x^\beta} - \Gamma^k_{\alpha\beta}\frac{\partial f}{\partial x^k}\right] - \frac{n_3}{\bar{f}}\left[\frac{\partial}{\partial x^\alpha}\frac{\partial \bar{f}}{\partial x^\beta} - \Gamma^m_{\alpha\beta}\frac{\partial \bar{f}}{\partial x^m}\right].$$

2) In terms of local coordinate system, we consider

$$X_2 = \frac{\partial}{\partial x^\gamma} \in \chi(M_2), \quad Y_2 = \frac{\partial}{\partial x^\delta} \in \chi(M_2),$$

where $\gamma, \delta \in \{1, 2, \ldots, n_2\}$. The Riemannian metric corresponding to the smooth manifold M_2 is g_2, and the component of the metric is denoted by $g_{\gamma\delta}$. Then we have

$$\bar{R}_{\gamma\delta} = R_{\gamma\delta} - f^2 g_{\gamma\delta}\left[\frac{1}{f}\frac{1}{\sqrt{(\det(g_{\alpha\beta})}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\det(g_{\alpha\beta})} g^{\alpha\beta}\frac{\partial f}{\partial x^\beta}\right)\right.$$

$$+\frac{n_2-1}{f^2}\left|g^{\alpha\beta}\frac{\partial f}{\partial x^\beta}\frac{\partial}{\partial u^\alpha}\right|^2\Bigg]-\frac{n_3}{\bar{f}}\left[\frac{\partial}{\partial x^\gamma}\left(\frac{\partial \bar{f}}{\partial x^\delta}\right)-\Gamma^p_{\gamma\delta}\frac{\partial \bar{f}}{\partial x^p}\right],$$

where f is a smooth function on M_1 and the components of the Riemannian metric g_1 corresponding to the smooth manifold M_1 are $g_{\alpha\beta}$, $\alpha,\beta \in \{1,2,\ldots,n_1\}$.

3) In terms of local coordinate system, we choose

$$X_3=\frac{\partial}{\partial x^\eta}\in\chi(M_3),\quad Y_3=\frac{\partial}{\partial x^\kappa}\in\chi(M_3),$$

where $\eta,\kappa \in \{1,2,\ldots,n_3\}$. The Riemannian metric corresponding to the smooth manifold M_3 is g_3, and the component of the metric is denoted by $g_{\eta\kappa}$. Then we write

$$\bar{R}_{\eta\kappa}=R_{\eta\kappa}-\bar{f}^2 g_{\eta\kappa}\Bigg[\frac{1}{\bar{f}}\frac{1}{\sqrt{\det(g_{\sigma\varsigma})}}\frac{\partial}{\partial x^\sigma}\left(\sqrt{\det(g_{\sigma\varsigma})}g^{\sigma\varsigma}\frac{\partial \bar{f}}{\partial x^\varsigma}\right)$$
$$+\frac{n_1+n_2-1}{\bar{f}^2}\left|g^{\sigma\varsigma}\frac{\partial \bar{f}}{\partial x^\varsigma}\frac{\partial}{\partial u^\sigma}\right|^2\Bigg],$$

where $\bar{f}$ is a smooth function on $M_1\times_f M_2=N$ (say), dimension of $N=n_1+n_2$, the components of the Riemmanian metric corresponding to the smooth manifold N are $g_{\sigma\varsigma}$, $\sigma,\varsigma\in\{1,2,\ldots,n_1+n_2\}$.

4) $\bar{R}_{ij}=0$, $i\in\{\alpha,\gamma,\eta\}$, $j\in\{\beta,\delta,\kappa\}$ and when $i=\alpha, j\neq\beta$; $i=\gamma, j\neq\delta$; $i=\eta, j\neq\kappa$.

Now we can rewrite Proposition 2.2 with tensor approach in the following way. In terms of local coordinate system, we suppose

$$X_1=\frac{\partial}{\partial x^\alpha}\in\chi(M_1),\qquad Y_1=\frac{\partial}{\partial x^\beta}\in\chi(M_1),$$
$$X_2=\frac{\partial}{\partial x^\gamma}\in\chi(M_2),\qquad Y_2=\frac{\partial}{\partial x^\delta}\in\chi(M_2),$$
$$X_3=\frac{\partial}{\partial x^\eta}\in\chi(M_3),\qquad Y_3=\frac{\partial}{\partial x^\kappa}\in\chi(M_3),$$

where $\alpha,\beta\in\{1,2,\ldots,n_1\}$, $\gamma,\delta\in\{1,2,\ldots,n_2\}$ and $\eta,\kappa\in\{1,2,\ldots,n_3\}$. The Riemannian metrics corresponding to the smooth manifolds M_1, M_2 and M_3 are g_1, g_2 and g_3, respectively, and the components of the metrics are denoted by $g_{\alpha\beta}$, $g_{\gamma\delta}$ and $g_{\eta\kappa}$ respectively.

The sequential warped product $\bar{M}=(M_1\times_f M_2)\times_{\bar{f}} M_3$ (dim $M_1=n_1$, dim $M_2=n_2$, dim $M_3=n_3$,) is Einstein with $\bar{R}_{ij}=\lambda\bar{g}_{ij}$, $i,j\in\{1,2,\ldots,n_1+n_2+n_3\}$ if and only if

1) $R_{\alpha\beta}=\lambda g_{\alpha\beta}+\frac{n_2}{f}\left[\frac{\partial}{\partial x^\alpha}\left(\frac{\partial f}{\partial x^\beta}\right)-\Gamma^k_{\alpha\beta}\frac{\partial f}{\partial x^k}\right]+\frac{n_3}{\bar{f}}\left[\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \bar{f}}{\partial x^\beta}\right)-\Gamma^m_{\alpha\beta}\frac{\partial \bar{f}}{\partial x^m}\right],$

2) $R_{\gamma\delta} = \omega g_{\gamma\delta} + \frac{n_3}{\bar{f}}\left[\frac{\partial}{\partial x^\gamma}\left(\frac{\partial \bar{f}}{\partial x^\delta}\right) - \Gamma^p_{\gamma\delta}\frac{\partial \bar{f}}{\partial x^p}\right]$, where

$$\omega = f^2\left(\lambda + \frac{1}{f}\frac{1}{\sqrt{(\det(g_{\alpha\beta})}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\det(g_{\alpha\beta})}g^{\alpha\beta}\frac{\partial f}{\partial x^\beta}\right) + \frac{n_2-1}{f^2}\left|g^{\alpha\beta}\frac{\partial f}{\partial x^\beta}\frac{\partial}{\partial u^\alpha}\right|^2\right),$$

3) M_3 is Einstein with $R_{\eta\kappa} = \mu g_{\eta\kappa}$,

4) $\mu = \bar{f}^2\left(\lambda + \frac{1}{\bar{f}}\frac{1}{\sqrt{\det(g_{\sigma\varsigma})}}\frac{\partial}{\partial x^\sigma}\left(\sqrt{\det(g_{\sigma\varsigma})}g^{\sigma\varsigma}\frac{\partial \bar{f}}{\partial x^\varsigma}\right) + \frac{n_2-1}{\bar{f}^2}\left|g^{\sigma\varsigma}\frac{\partial \bar{f}}{\partial x^\varsigma}\frac{\partial}{\partial u^\sigma}\right|^2\right)$, where f is a smooth function on M_1, the components of the Riemmanian metric g_1 corresponding to the smooth manifold M_1 are $g_{\alpha\beta}$, $\alpha, \beta \in \{1, 2, \dots, n_1\}$, and $\bar{f}$ is a smooth function on $M_1 \times_f M_2 = N$. The dimension of N is $n_1 + n_2$, the components of the Riemmanian metric corresponding to the smooth manifold N are $g_{\sigma\varsigma}$, $\sigma, \varsigma \in \{1, 2, \dots, n_1 + n_2\}$.

5. Conclusions

An Einstein manifold is a Riemannian or pseudo-Riemannian differentiable manifold whose Ricci tensor is proportional to the metric tensor. A warped product manifold is a Riemannian or pseudo-Riemannian manifold which plays very important role not only in geometry but also in mathematical physics, especially in general relativity. We know that the Einstein equations are fundamental in the construction of cosmological models. The physical motivation for studying various types of space-time models in cosmology is to obtain the information about the evolution of the universe. The study of Einstein sequential warped product spaces is important because such space-time represents different phases in the evolution of the universe. Consequently, the investigations of Einstein sequential warped product spaces help us to have a deeper understanding of the global character of the universe.

Acknowledgments. The authors wish to express their sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.

Supports. The first author is supported by UGC-DSKPDF of India No. F.4-2/2006(BSR)/MA/18-19/0007.

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Received January 5, 2018, revised June 26, 2018.

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Про секвенціально викривлені добутки, що є просторами Ейнштейна

Sampa Pahan and Buddhadev Pal

У роботі вивчаються секвенціально викривлені добутки, що є просторами Ейнштейна. Доведено, якщо M — секвенціально викривлений добуток, що є простором Ейнштейна з від'ємною скалярною кривизною, то функції викривлення є константами. З'ясовано деякі перешкоди для існування таких секвенціально викривлених добутків, що є просторами Ейнштейна. Також показано, що коли $\bar{M} = (M_1 \times_f I_{M_2}) \times_{\bar{f}} I_{M_3}$ є секвенціально викривленим добутком повного зв'язного $(n-2)$-вимірного многовида Римана M_1 та одновимірних многовидів Римана I_{M_2} і I_{M_3}, то за певних умов (M_1, g_1) стає $(n-2)$-вимірною сферою з радіусом $\rho = \frac{n-2}{\sqrt{r^1+\alpha}}$. Приклади секвенціально викривлених добутків, що є просторами Ейнштейна, наведено в Розділі 3.

Ключові слова: викривлений добуток, секвенціально викривлений добуток, многовид Ейнштейна.