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Notes on the Asymptotic Properties of Some Class of Unbounded Strongly Continuous Semigroups

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The abstract Cauchy problem in the Banach and Hilbert space setting is considered and the asymptotic behavior of individual orbits of corresponding C_0-semigroup is studied. The possibility to find uniformly stable dense subset of initial states in the case of unstable semigroups (so-called polynomial stability) is discussed. Also, the existence of the fastest growing orbit (so-called maximal asymptotics) for certain class of semigroups is studied.

Key words: linear differential equations, asymptotic behavior of solutions, maximal asymptotics, asymptotic stability, polynomial stability.

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1. Introduction

We consider the abstract Cauchy problem given in a Banach space X, namely

$$\begin{cases} \dot{x}(t) = Ax(t), \quad x(t) \in X, \\ x(0) = x_0 \in X. \end{cases} \tag{1.1}$$

The above problem is well-posed, i.e., it has a unique classical solution for any initial condition $x_0 \in D(A)$ or a unique mild solution for any $x_0 \in X$ if the operator A is the generator of a C_0-semigroup (strongly continuous semigroup) of the operators $\{T(t)\}_{t\geq 0}$. The characterization of the generators of C_0-semigroups is given by the well-known Hille–Yosida theorem (for more details see, e.g., [6,7]). In particular, a closed densely defined operator $A : D(A) \subset X \to X$ is the generator of the C_0-semigroup if and only if it satisfies some resolvent estimates and has nonempty resolvent set. Further, a semigroup $\{T(t)\}_{t\geq 0}$ provides a solution $x(t)$ of system (1.1) given by the formula

$$x(t) = T(t)x_0, \quad t \geq 0.$$

The important and desirable property of dynamical systems is their stability. We call the semigroup $T(t)$ strongly asymptotically stable if for any initial condition

$x_0 \in X$ the orbit $x(t) = T(t)x_0$ tends to zero when $t \to +\infty$. In the theory of asymptotic behavior of C_0-semigroups the so-called growth bound ω_0 given by

$$\omega_0 := \omega_0(T) := \lim_{t\to+\infty} \frac{\ln \|T(t)\|}{t}$$

or, equivalently, by

$$\omega_0 := \inf\{\omega : \|T(t)\| \le M_\omega e^{\omega t}, t \ge 0\} \tag{1.2}$$

plays an essential role. For any C_0-semigroup the inequality $\omega_0 < +\infty$ holds, however, it may happen that $\omega_0 = -\infty$. From (1.2), we can see that in the case when $\omega_0 < 0$ the norm of any solution $T(t)x_0$ satisfies

$$\|T(t)x_0\| \le M_\varepsilon e^{(\omega_0+\varepsilon)t}\|x_0\|$$

for any $\varepsilon > 0$ and some positive constant M_ε. If we choose ε small enough to have $\omega_0 + \varepsilon < 0$, then we see that all orbits tend to zero uniformly with exponential rate. In the case $\omega_0 > 0$, the uniform boundedness principle implies that at least one orbit $T(t)x_0$ is unbounded, so the semigroup can not be asymptotically stable. For the critical case $\omega_0 = 0$, the norm of the semigroup can not tend to zero with any rate, and the asymptotic stability may occur (but does not have to) only if the semigroup is bounded (i.e., there exists a constant $M > 0$ such that $\|T(t)\| \le M, t \ge 0$). The following theorem, due to Sklyar–Shirman [17] (the case of bounded generator A) and Arendt–Batty [1], Lyubich–Phong [9] (general case), is the criterion of asymptotic stability

Theorem 1.1. *Let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\ge0}$ on a Banach space X and let $\sigma(A) \cap (i\mathbb{R})$ be at most countable. Then the semigroup $\{T(t)\}_{t\ge0}$ is strongly asymptotically stable if and only if the adjoint operator A^* has no pure imaginary eigenvalues.*

Even if the semigroup is asymptotically stable, its norm may not tend to zero as $t \to +\infty$ (e.g., when $\sigma(A) \cap (i\mathbb{R}) \ne \emptyset$). In this case there does not exist any function $g(t) \to 0$ such that

$$\|T(t)x\| \le g(t)M_x, \quad t \ge 0,\ x \in X,$$

where $M_x > 0$ depends on the choice of x. Indeed, if the function $g(t)$ existed, the uniform boundedness principle would imply $\|T(t)\| \le Mg(t)$, and thus $\|T(t)\| \to 0$, which is a contradiction.

This implies that the orbits $T(t)x$ tend to zero arbitrarily slow, i.e., there is no minimal rate of decay of these orbits. In particular, this means the system has no maximal asymptotics [13,14] (for more details see Section 3).

On the other hand, in [3], Batty proved the following.

Theorem 1.2. *Let $\{T(t)\}_{t\ge0}$ be a bounded C_0-semigroup on a Banach space X with the generator A. If, in addition,*

$$\sigma(A) \cap (i\mathbb{R}) = \emptyset, \tag{1.3}$$

then

$$\|T(t)A^{-1}\| \to 0, \quad t \to +\infty. \tag{1.4}$$

Remark 1.3. Condition (1.4) is equivalent to the existence of the function $g(t) \to 0$, namely $g(t) := \|T(t)A^{-1}\|$, such that

$$\|T(t)x\| \le g(t)\|x\|_{D(A)}, \quad t \ge 0,\ x \in D(A), \tag{1.5}$$

where $\|\cdot\|_{D(A)}$ is a graph norm induced by the operator A, i.e., $\|x\|_{D(A)} = \|x\| + \|Ax\|$.

Inequality (1.5) shows that the orbits starting in the domain of A (a dense subset of X) are uniformly stable on the unit ball in the space $(D(A), \|\cdot\|_{D(A)})$ and they tend to zero at least as the function $g(t)$. Further, Batty and Duyckaerts proved (see [4]) that condition (1.3) is also necessary for the bounded semigroup to satisfy (1.4). Later on, in [15], it was shown that the relations between the location of the spectrum of generator and the stability in the domain (1.4) can be derived from Theorem 1.1. This approach does not exploit the assumption of boundedness of the semigroup, which allows us to prove the necessity of condition (1.3) for unbounded semigroups as well. Namely, we have

Theorem 1.4. *Let $\{T(t)\}_{t\ge 0}$ be a C_0-semigroup on a Banach space X with the generator A. Then for any $\lambda \notin \sigma(A)$,*

a) $\|T(t)(A-\lambda I)^{-1}\| \to 0 \quad \Rightarrow \quad \sigma(A) \subset \{\operatorname{Re}\lambda < 0\}$;

if, in addition, the semigroup $T(t)$ is bounded, then

b) $\|T(t)(A-\lambda I)^{-1}\| \to 0 \quad \Leftarrow \quad \sigma(A) \subset \{\operatorname{Re}\lambda < 0\}$,

where the above limits are taken as $t \to +\infty$.

Of course, the condition on the location of the spectrum can not be, in general, sufficient for asymptotic stability in the domain. The first argument for that is the fact that we can easily build an example of the semigroup such that the spectral bound of the generator (i.e., $s(A) := \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$) is strictly less than the growth bound of the semigroup $\omega_0(T)$ (see Theorem 3.1 (iii) in [18]). However, even in the case of $s(A) = \omega_0(T) = 0$, the semigroup can be unbounded and possess unbounded orbits in the domain of the generator.

One of the purposes of this paper is to show the directions for Theorem 1.1 and Theorem 1.2 to be developed in the case of unbounded semigroups. Looking at Theorem 1.2, the following important question arises, whether the orbits starting in the domain of the generator can be uniformly asymptotically stable for an unbounded semigroup, i.e., whether they tend to zero at some minimal rate. This problem is not solved completely in general setting, but in Section 2 we present new results concerning some special class of operators. Section 3 is devoted to the development of the concept of maximal asymptotics [13]. It is shown there that Theorem 1.1 can be interpreted in a more general setting, namely as a condition of nonexistence of the fastest growing trajectory (maximal asymptotics) for a given semigroup. Now we give a new result on this subject (Theorem 3.6). We consider a class of semigroups whose generators have the spectrum splitted into a countable number of separated bounded sets. We show that the existence of maximal asymptotics of the semigroup is possible only if it is achieved on one of the Riesz projections corresponding to the bounded parts of the spectrum.

2. Polynomial stability

Despite the fact that C_0-semigroup is unbounded, some of the orbits (those that start in some dense set) may still tend to zero with the same rate. This fact is the base of the following definition (taken from [2])

Definition 2.1. We call the semigroup (or corresponding equation) polynomially stable if there exist the constants $M, \alpha, \beta > 0$, such that

$$\|T(t)x\| \leq Mt^{-\beta}\|x\|_{D(A^\alpha)}, \quad t \geq 0,\ x \in D(A^\alpha)$$

or, equivalently,

$$\|T(t)A^{-\alpha}\| \leq Mt^{-\beta}, \quad t \geq 0.$$

For fractional powers of closed operators we refer the reader to [11]. Note that the definition is incorrect if $0 \in \sigma(A)$, but in this case we can replace the operator A by $A - \mu I$ for any $\mu \notin \sigma(A)$ (the domain remains the same and the corresponding norms in the domain subspace are equivalent). Thus, for convenience, we assume that $0 \notin \sigma(A)$.

The characterization of polynomial stability only by the location of the spectrum is impossible, however Borichev and Tomilov [5] obtained some characterization for bounded semigroups acting on Hilbert space via the behavior of the resolvent operator on the imaginary axis. In particular, they obtained

Theorem 2.2. *Let $\{T(t)\}_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with the generator A. If $i\mathbb{R} \subset \rho(A)$, then for any $\alpha > 0$ the following conditions are equivalent:*

$$\begin{aligned} &\|T(t)A^{-1}\| = O\left(t^{-\frac{1}{\alpha}}\right), && t \to +\infty, \\ &\|T(t)A^{-\alpha}\| = O(t^{-1}), && t \to +\infty, \\ &\|R(A, is)\| = O(|s|^\alpha), && s \to \pm\infty. \end{aligned} \tag{2.1}$$

The above theorem shows that a slower rate of growth of the norm of resolvent on the imaginary axis provide faster vanishing of the orbits of semigroup starting in the domain of the generator. However, in general, an explicit form of the resolvent operator can be difficult or even impossible to find. On the other hand, Bátkai, Engel, Prüss and Schnaubelt in [2] connected the behavior of the resolvent operator with the rate of approach of the spectrum of the generator to the imaginary axis, namely they obtained

Theorem 2.3. *Let A be the generator of a bounded C_0-semigroup on a Banach space X, and let $\sigma(A) \subset \mathbb{C}_-$. If*

$$\|R(A, is)A^{-\alpha}\| \leq C, \quad s \in \mathbb{R}, \tag{2.2}$$

then there exists $\delta > 0$ such that

$$|\operatorname{Im}\lambda| \geq C|\operatorname{Re}\lambda|^{-\frac{1}{\alpha}} \tag{2.3}$$

for $\lambda \in \sigma(A) : |\operatorname{Re}\lambda| < \delta$.

Remark 2.4. In [8], Latushkin and Shvydkoy showed that conditions (2.1) and (2.2) on the behavior of the resolvent are equivalent. Moreover, in the case of a bounded semigroup on a Hilbert space, condition (2.3) is necessary for the polynomial stability as a consequence of Theorems 2.2 and 2.3.

Note that the knowledge about the location of the spectrum of generator is not sufficient for determining the behavior of the semigroup or the resolvent, at least the rank of the particular eigenvalues plays an important role. To obtain more specific criterion for polynomial stability we have to narrow the class of generators under considerations. We focus on the class of operators satisfying the following assumptions:

(A1) $A : D(A) \subset H \to H$ generates the C_0-group in the Hilbert space H;

(A2) $\sigma(A) = \bigcup_{k\in\mathbb{Z}} \sigma_k$ such that

- (a) $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$,
- (b) $\#\sigma_k \leq N, \quad k \in \mathbb{Z}$,
- (c) $\inf\{|\lambda - \mu| : \lambda \in \sigma_i, \mu \in \sigma_j, i \neq j\} = d > 0$;

(A3) linear span of the generalized eigenvectors of the operator A is dense in the space H.

Assumption (A2) means that the spectrum of the generator A can be splitted into a countable number of finite families σ_k such that the number of elements (counted with multiplicities) in each family does not exceed the common number N, and the families are separated, i.e., the distance between any two of them is greater than the constant d. Let us denote by $\mathcal{A}$ a class of operators satisfying assumptions (A1)–(A3). In [19], Zwart proved that operators $A \in \mathcal{A}$ always possess a Riesz basis of the finite-dimensional A-invariant subspaces. Namely, the images of the Riesz projections $P_k : H \to H, k \in \mathbb{Z}$,

$$P_k x := \frac{1}{2\pi i} \oint_{\Gamma_k} R(A, \lambda)x \, d\lambda,$$

V_k, say, are finite-dimensional ($\dim V_k \leq N$) almost orthogonal and constitute the basis of subspaces, i.e., for any $x \in H$ there exists a unique decomposition $x = \sum_k P_k x$ and there exist positive constants m, M such that

$$m\|x\|^2 \leq \sum_{k\in\mathbb{Z}} \|P_k x\|^2 \leq M\|x\|^2, \quad x \in H. \tag{2.4}$$

The Riesz basis of finite-dimensional subspaces is the key tool for proving the following theorem (see [16] for the proof).

Theorem 2.5. *Let an operator $A \in \mathcal{A}$ have the spectrum $\sigma(A)$ contained in the open left half-plane. Suppose that there exist constants $C, \alpha > 0$, such that*

$$|\operatorname{Im}\lambda| \geq C|\operatorname{Re}\lambda|^{-\frac{1}{\alpha}}$$

holds for every $\lambda \in \sigma(A)$. *Then*

$$\begin{aligned} \|T(t)A^{-N\alpha}\| &= O(\frac{1}{t}), && t>0, \\ \|T(t)A^{-n}\| &= O(t^{N-1-\frac{n}{\alpha}}), && t>0, \qquad n\in\mathbb{N}, \\ \|R(A,is)\| &= O(|s|^{N\alpha}), && s\to\pm\infty. \end{aligned}$$

The above theorem provides the strictly spectral condition for the polynomial stability of a certain class of generators. In particular, taking initial conditions from the domain of sufficiently large power of the generator (sufficiently smooth initial states), we obtain the uniform stability of orbits on the unit ball in this domain, with arbitrary polynomial rate of decay.

3. Maximal asymptotics

In this section, we focus on the behavior of individual orbits $T(t)x$ without any restrictions on the initial condition x. As we mentioned before, the rate of growth/decay of the norm of semigroup $\|T(t)\|$ may not be achieved by any orbit $\|T(t)x\|$. In the case of a bounded semigroup, this means that the orbits should tend to zero. However, this property is worth studying even out of the context of stability. In [13], Sklyar gave the following definition of maximal asymptotics of C_0-semigroup

Definition 3.1. We say that the equation $\dot{x} = Ax$ (or the corresponding semigroup $\{T(t),\ t\geq 0\}$) possesses a maximal asymptotics if there exists a real positive function $f(t), t \geq 0$, such that

(i) for any initial vector $x \in X$ the function $\frac{\|T(t)x\|}{f(t)}$ is bounded on $[0, +\infty)$,

(ii) there exists at least one $x_0 \in X$ such that

$$\lim_{t\to+\infty} \frac{\|T(t)x_0\|}{f(t)} = 1. \tag{3.1}$$

If instead of (ii) there exists at least one $x_0 \in X$ such that start

(i′) there exists at least one $x_0 \in X$ such that

$$\limsup_{t\to+\infty} \frac{\|T(t)x_0\|}{f(t)} = 1,$$

then we call $f(t)$ a quasi-maximal asymptotics.

If for the semigroup $T(t)$ there exists a maximal asymptotics $f(t)$, then the function $\|T(t)\|$ is at least a quasi-maximal asymptotics. Moreover, any maximal asymptotics is commensurate with $\|T(t)\|$, i.e., for any maximal asymptotics $f(t)$ there exist positive constants m, M, t_0 such that

$$m\|T(t)\| \leq f(t) \leq M\|T(t)\|, \quad t > t_0.$$

With the notion of maximal asymptotics, Theorem 1.1 can be generalized to any type of C_0-semigroup, namely

Theorem 3.2. *Assume that*

(i) $\sigma(A) \cap \{\lambda : Re\,\lambda = \omega_0\}$ *is at most countable;*

(ii) *the operator* A^* *does not possess eigenvalues with the real part* ω_0.

Then the equation $\dot{x} = Ax$ *(the semigroup* $\{T(t),\ t \geq 0\}$*) does not have any maximal asymptotics.*

Corollary 3.3. *Let the assumptions of Theorem 3.2 be satisfied and let* $f(t)$ *be a positive function such that*

(a) $\log f(t)$ *is concave,*

(b) *for any* $x \in X$ *the function* $\|T(t)x\|/f(t)$ *is bounded.*

Then

$$\lim_{t\to+\infty} \frac{\|T(t)x\|}{f(t)} = 0, \quad x \in X.$$

Remark 3.4. Corollary 3.3 is not a direct conclusion from Theorem 3.2, some technical effort is required to prove it. The proofs of them both can be found in [14]. Furthermore, it is clear that Theorem 3.2 is a generalization of Theorem 1.1.

The proof of Theorem 3.2 is based on a special construction of the quotient semigroup. We will use a similar technique to prove Theorem 3.6 on nonexistence of maximal asymptotics for a certain class of semigroups. First, let us note that for the generators $A \in \mathcal{A}$ (defined in Section 2) the existence of the Riesz basis of A-invariant subspaces allows us to prove easily that if in any basis subspaces (say, $H_k, k \in \mathbb{N}$) for every initial condition $x_k \in H_k$ the rate of growth of $\|T(t)x_k\|$ is slower than $\|T(t)\|$, then there is no initial state $x \in H$ such that the corresponding orbit grows like $\|T(t)\|$. Shortly, the group $T(t)$ does not possess any maximal asymptotics. Namely, we have

Theorem 3.5. *Let* $T(t)$ *be a* C_0*-group generated by the operator* $A \in \mathcal{A}$ *such that* $\omega_0(T) \neq -\infty$*, and let* $\{H_k\}_{k\in\mathbb{N}}$ *be the Riesz basis of* H*, where each* H_k *is an image of the Riesz projection* P_k*. Assume*

$$\lim_{t\to+\infty} \frac{\|T(t)x_k\|}{\|T(t)\|} = 0, \quad x_k \in H_k, \quad k \in \mathbb{N}. \tag{3.2}$$

Then the C_0*-group* $T(t)$ *does not possess any maximal asymptotics.*

Proof. Assume the opposite, i.e., $f(t)$ is the maximal asymptotics and $x_0 \in H$ is the initial state satisfying (3.1). From the fact that $f(t)$ and $\|T(t)\|$ must be commensurate we have that

$$\frac{\|T(t)x_k\|}{f(t)} = \frac{\|T(t)x_k\|}{\|T(t)\|}\frac{\|T(t)\|}{f(t)} \to 0.$$

Thus, also for any finite number $N \in \mathbb{N}$,

$$\lim_{t\to+\infty} \sum_{k=1}^{N} \frac{\|T(t)x_k\|^2}{f^2(t)} \to 0.$$

Let $x \in H, \varepsilon > 0$ be arbitrary. Choose N large enough to have

$$\sum_{k=N}^{+\infty} \|x_k\|^2 \leq \frac{\varepsilon}{2M} \inf_{t\geq 0} \frac{f^2(t)}{\|T(t)\|^2}, \tag{3.3}$$

where $x_k = P_k x$ is the projection of x on the subspace H_k and the constant M satisfies (2.4). Then choose t_0 large enough to have

$$\sum_{k=1}^{N} \frac{\|T(t)x_k\|^2}{f^2(t)} \leq \frac{\varepsilon}{2M}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\frac{\|T(t)x\|^2}{f^2(t)} \leq M \sum_{k=1}^{+\infty} \frac{\|T(t)x_k\|^2}{f^2(t)} \leq M \sum_{k=1}^{N} \frac{\|T(t)x_k\|^2}{f^2(t)} + M \frac{\|T(t)\|^2}{f^2(t)} \sum_{k=N}^{+\infty} \|x_k\|^2 \leq \varepsilon.$$

The above means that $\frac{\|T(t)x\|}{f(t)} \to 0$ contradicts the assumption that x_0 satisfies (3.1). □

It can be proved that for any C_0-semigroup acting on a Banach space X such that $\omega_0(T) = -\infty$, the semigroup does not possess any maximal asymptotics. The key assumption in the above theorem was that the subspaces H_k with no maximal asymptotics constitute the Riesz basis of subspaces. In the next theorem we show that this assumption can be weakened.

Theorem 3.6. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of operators acting on a Banach space X with the generator A, and $\omega_0(T) = 0$. Assume*

(A) *for any $\lambda \in \sigma(A) \cap (i\mathbb{R})$ there exists a closed and bounded component of $\sigma(A)$, say σ_λ, containing λ (i.e., $\lambda \in \sigma_\lambda \subset \sigma(A)$) and a regular bounded curve Γ_λ enclosing σ_λ such that $\Gamma_\lambda \cap \sigma(A) = \emptyset$;*

(B) *for any $\lambda \in \sigma(A) \cap (i\mathbb{R})$ and $x \in X_\lambda$,*

$$\lim_{t\to+\infty} \frac{\|T(t)x\|}{\|T(t)\|} \to 0,$$

where X_λ is an image of the Riesz projection corresponding to the curve Γ_λ.

Then the semigroup $T(t)$ has no maximal asymptotics.

Remark 3.7. If the semigroup $T(t)$ does not possess any maximal asymptotics, then also the function $\|T(t)\|$ can not be the one. Hence, for all orbits $T(t)x$, the limit

$$\lim_{t\to+\infty} \frac{\|T(t)x\|}{\|T(t)\|}$$

is either zero or it does not exist. This means that any orbit grows much slower than the function $\|T(t)\|$ or it is incomparable to it. If the function $\|T(t)\|$ is sufficiently regular, that is $\ln \|T(t)\|$ is concave, then the last limit is always 0 (see Theorem 9 in [13]).

Remark 3.8. Theorem 3.6 holds in the general case for any $\omega_0(T)$. For the proof one can use rescaling argument defining a new semigroup $\{e^{-\omega_0 t}T(t)\}$. The case $\omega_0 = 0$ is the most important in the context of stability.

Remark 3.9. Note that the spectrum of the generator A on the line $\Re\lambda = \omega_0$ does not have to be countable. Note also that assumption (A) implies the part of spectrum $\sigma(A)$ close to the line $\Re\lambda = \omega_0$ can be splitted into (at most) countably many disjoint closed components. However, the infimum of distances between these components can be zero. In this case, the images X_λ of the Riesz projections P_Γ may not constitute the Riesz basis of subspaces X_λ.

Proof of Theorem 3.6. The proof is based on the idea used in the proof of Theorem 5 of [13] and some of constructions are taken from there. Suppose, by contradiction, that $f(t)$ is a maximal asymptotics of the semigroup $T(t)$. This means that there exist constants $m, M > 0$ such that

$$m\|T(t)\| \leq f(t) \leq M\|T(t)\|, \quad t > t_0, \tag{3.5}$$

and the initial state $x_0 \in X$ satisfying (3.1). It is proven in [13, Proof of Theorem 5] that there exists nonnegative function $\overline{f}(t) : [0; +\infty) \to \mathbb{R}$ such that the function $\ln \overline{f}(t)$ is concave, $\overline{f}(t) \geq f(t)$, $t \geq 0$, and there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \to +\infty$, such that $\overline{f}(t_n) = f(t_n)$, $n \in \mathbb{N}$. This means the function $\overline{f}(t)$ may not be a maximal asymptotics, but it is at least a quasi-maximal asymptotics, i.e., it satisfies

$$\limsup_{t\to+\infty} \frac{\|T(t)x_0\|}{\overline{f}(t)} = c, \tag{3.6}$$

some $c > 0$ instead of (3.1) for the same x_0. Moreover, [13, Lemma 6] states that for any $\Delta > 0$,

$$\lim_{t\to+\infty} \frac{\overline{f}(t+\Delta)}{\overline{f}(t)} = 1. \tag{3.7}$$

The regularity of the function $\overline{f}(t)$ will be crucial for the existence of some limit at the end of the proof. Now we would like to "cut of" a part of the initial states for which limit (3.6) is zero. To this end, we define the seminorm $\ell(\cdot)$ on X by the formula

$$\ell(x) := \limsup_{t\to+\infty} \frac{\|T(t)x\|}{\overline{f}(t)}$$

and the closed subspace $L := \{x : \ell(x) = 0\}$. Further, $L \neq X$ because $\ell(x_0) \neq 0$, so we can define the quotient space $X/L := \{\hat{x} = x + L : x \in X\}$ and equip it with the norm $\|\hat{x}\|' := \hat{\ell}(\hat{x}) := \ell(x)$ instead of the natural one $\|\hat{x}\|^\sim := \inf\{\|x\| : x \in \hat{x}\}$. Note that for any $x \in X$,

$$\ell(x) = \limsup_{t\to+\infty} \frac{\|T(t)x\|}{\overline{f}(t)} \leq \frac{1}{m}\|x\|$$

holds, which means $\|\hat{x}\|' \leq \frac{1}{m}\|\hat{x}\|^\sim$ for any $\hat{x} \in X/L$. Because of that the space $(X/L, \|\cdot\|')$ can not be complete. We denote its complement in the norm $\|\cdot\|'$

by $\hat{X}$. Consider the family of operators $\{\hat{T}(t)\}$ acting on $\hat{X}$ induced by the semigroup $\{T(t)\}$ by the formula $\hat{T}(t)\hat{x} = T(t)x + L$, for $\hat{x} \in X/L$. It can be checked directly that also $\hat{A}\hat{x} = Ax + L$ and $\hat{R}(\hat{A},\lambda)\hat{x} = R(A,\lambda)x + L$, for $\hat{x} \in X/L$, where $\hat{A}, A$ are the generators of $\hat{T}(t), T(t)$ respectively, and $\hat{R}, R$ are their resolvents. It is easy to see that $\hat{T}(t)$ is an isometry on X/L for each $t \geq 0$, and it can be extended on the whole $\hat{X}$ continuously. Indeed, for any $x \in \hat{x}$ we have

$$\|\hat{T}(t)\hat{x}\|' = \limsup_{s\to+\infty} \frac{\|T(t+s)x\|}{\overline{f}(t+s)} \frac{\overline{f}(t+s)}{\overline{f}(s)} = \|\hat{x}\|',$$

where (3.7) is used.

Next, the semigroup $\hat{T}(t)$ can be extended to a C_0-group of isometries and $\sigma(\hat{A}) \subset \sigma(A) \cap (i\mathbb{R})$ (see [13] for more details). The set $\sigma(\hat{A})$ is contained in the imaginary axis and is nonempty as a spectrum of the generator of a C_0-group of isometries (see [10]). Let $\lambda \in \sigma(\hat{A}) \subset \sigma(A)$. By the assumption, the point λ can be encircled by a bounded curve $\Gamma : \Gamma \cap \sigma(\hat{A}) = \Gamma \cap \sigma(A) = \emptyset$. Consider the Riesz projections $P_\Gamma, \hat{P}_\Gamma$ corresponding to the curve Γ in the spaces X and $\hat{X}$ respectively. It is easy to see that $\hat{P}_\Gamma\hat{x} = \widehat{P_\Gamma x} := P_\Gamma x + L$ for any $\hat{x} \in X/L$. Moreover, the curve Γ indicates a division of the space $\hat{X}$ into direct sum $\hat{Y} + \hat{Z}$ and analogically $X = Y + Z$, where $\hat{Y}, Y$ are the images of $\hat{P}_\Gamma, P_\Gamma$ and $\hat{Z}, Z$ are the images of $\hat{I} - \hat{P}_\Gamma, I - P_\Gamma$, and all the subspaces are $A, T(t)$ (or $\hat{A}, \hat{T}(t)$) invariant. Then the spectra of restricted operators A_Y, A_Z are the intersections of the spectrum $\sigma(A)$ with the interior and the exterior of the area bounded by the curve Γ respectively.

The set $\sigma(\hat{A})$ consists of approximate eigenvalues only because it is its own boundary (see [13, Lemma 7]). Thus there exists a sequence $\{\hat{x}_k\} : \|\hat{x}_k\|' = 1$ such that

$$\|\hat{A}\hat{x}_k - \lambda\hat{x}_k\|' \to 0 \text{ for } k \to +\infty. \tag{3.8}$$

Let us split the sequence $\hat{x}_k = \hat{y}_k + \hat{z}_k$ such that $\hat{y}_k \in \hat{Y}$ and $\hat{z}_k \in \hat{Z}$. Then (3.8) implies that both sequences $\|\hat{A}\hat{y}_k - \lambda\hat{y}_k\|' and \|\hat{A}\hat{z}_k - \lambda\hat{z}_k\|'$ tend to zero. However, $\hat{z}_k \to 0$, because otherwise λ would be the approximate eigenvalue of the operator $\hat{A}$ restricted to the subspace $\hat{Z}$, which is a contradiction. Thus, $\hat{y}_k$ must be approximate eigenvector of $\hat{A}$, moreover $\hat{y}_k$ can be chosen from $\hat{P}_\Gamma(X/L) \subset \hat{Y}$ (it follows by the density of X/L in $\hat{X}$ and the boundedness of $\hat{A}$ restricted to $\hat{Y}$). Then we have

$$\hat{y}_k = \hat{P}_\Gamma(y_k + L) = P_\Gamma y_k + L$$

for some sequence $y_k \in X$. Thus, for the norm we have

$$1 \sim \|\hat{y}_k\|' = \|P_\Gamma y_k + L\|' = \ell(P_\Gamma y_k).$$

But the above contradicts

$$\ell(P_\Gamma y_k) = \limsup_{t\to+\infty} \left(\frac{\|T(t)P_\Gamma y_k\|}{\|T(t)\|} \frac{\|T(t)\|}{\overline{f}(t)} \right) \leq \gamma \limsup_{t\to+\infty} \frac{\|T(t)P_\Gamma y_k\|}{\|T(t)\|} = 0,$$

where $\gamma = \limsup_{t\to+\infty} \frac{\|T(t)\|}{\overline{f}(t)} \leq \frac{1}{m}$ is a finite positive constant. □

In the end, we would like to analyze the existence of maximal asymptotics for special generator A that does not posses the eigenbasis. In the following example we present the construction of such a generator. The example is based on [12].

Example. We consider a complex space $(\ell_2, \|\cdot\|)$ with the canonical basis $\{e_n\}_{n\in\mathbb{N}}$ and the right shift operator $S : H \to H$ defined by $Se_n = e_{n+1}, n \in \mathbb{N}$. Given a sequence $\{c^{(k)}\}_{k=1}^{\infty} \in \ell_2$. Replacing the norm by

$$\|\{c^{(k)}\}\|_1 := \|\{(I-S)c^{(k)}\}\| = \|\{c^{(k+1)} - c^{(k)}\}\| = \left(\sum_{k=0}^{\infty} \left|c^{(k+1)} - c^{(k)}\right|^2\right)^{\frac{1}{2}},$$

where $c^{(0)} := 0$, we obtain a new normed vector space ℓ_2^0. Note that $\ell_2^0 \supset \ell_2$ and it is not a complete space. We denote the completion of $(\ell_2^0, \|\cdot\|_1)$ by $(\ell_2^1, \|\cdot\|_1)$. It turns out (see [12, Proposition 3]) that ℓ_2^1 is a separable Hilbert space, the sequence $\{e_n\}_{n\in\mathbb{N}}$ is still complete in the space ℓ_2^1 but it is no longer the Schauder basis. Now we are going to construct a C_0-semigroup acting on ℓ_2^1. Let us define the family of operators $T(t) : \ell_2^1 \to \ell_2^1$, $t \geq 0$, by

$$T(t)e_n = e^{\lambda_n t} e_n, \quad \lambda_n := i \log n, \quad n \in \mathbb{N}.$$

It can be verified (see [12]) that $\{T(t)\}_{t\geq 0}$ is the C_0-semigroup with the generator $A : D(A) \subset \ell_2^1 \to \ell_2^1$ given by

$$Ae_n = \lambda_n e_n = i \log n \cdot e_n, \quad n \in \mathbb{N},$$

where $D(A) = \{\{c^{(k)}\} \in \ell_2^1 : \{\lambda_k c^{(k)}\} \in \ell_2^1\}$.

Remark 3.10. The above semigroup $T(t)$ is unbounded. In fact, one can show $\|T(t)\| \sim t$. On the other hand, it is easy to see that asymptotics t can not be achieved on any eigenspace span$\{e_n\}$. Therefore Theorem 3.6 yields that the semigroup $T(t)$ does not have any maximal asymptotics.

The nonexistence of maximal asymptotics (especially for unbounded semigroups) opens the way to look for a dense subsets of initial states that has stable orbits, in particular the problem of polynomial stability of such semigroups could be very interesting.

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Нотатки про асимптотичні властивості деякого класу необмежених сильно неперервних напівгруп

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Розглянуто постановку абстрактної задачі Коші в банаховому та гільбертовому просторах та вивчено асимптотичне поводження окремих орбіт відповідних C_0-напівгруп. Обговорено можливість знаходження стійкої щільної множини початкових станів у випадку нестійкої напівгрупи (так звану поліноміальну стійкість). Вивчено також існування орбіти найшвидшого росту (так званої максимальної асимптотики) для деякого класу напівгруп.

Ключові слова: лінійні диференціальні рівняння, асимптотичне поводження розв'язків, максимальна асимптотика, асимптотична стійкість, поліноміальна стійкість.