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Some Non-Trivial and Non-Gradient Closed Pseudo-Riemannian Steady Ricci Solitons

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In this paper, we study the Ricci soliton equation on compact indecomposable Lorentzian 3-manifolds that admit a parallel light-like vector field with closed orbits. These compact structures that are geodesically complete, admit non-trivial, i.e., non-Einstein and non-gradient steady Lorentzian Ricci solitons with zero scalar curvature which show the difference between closed Riemannian and pseudo-Riemannian Ricci solitons. The associated potential vector field of a Ricci soliton structure in all the cases that we construct on these manifolds is a space-like vector field. However, we show that there are examples of closed pseudo-Riemannian steady Ricci solitons in the neutral signature $(2, 2)$ with zero scalar curvature such that the associated potential vector field can be time-like or null. These compact manifolds are also geodesically complete and they cannot admit a conformal-Killing vector field.

Key words: Ricci solitons, closed pseudo-Riemannian manifolds, parallel light-like vector field.

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1. Introduction

Let (M, g) be a pseudo-Riemannian manifold and X be a smooth vector field on M. We say that the triple (M, g, X) is a pseudo-Riemannian Ricci soliton if the equation

$$L_X(g) + \mathrm{Ric}(g) = \lambda g \tag{1.1}$$

is satisfied, where L_X is the Lie-derivative with respect to X, Ric is the Ricci tensor and λ is a real number. A Ricci soliton is called shrinking, steady or expanding according to whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. If for a smooth function f on a manifold (M, g), $X = 1/2\nabla f$, where ∇f is the gradient of f, then equation (1.1) leads to

$$\mathrm{Hess}_f(g) + \mathrm{Ric}(g) = \lambda g, \tag{1.2}$$

where Hess_f denotes the Hessian of the function f. In this case, the soliton is called the gradient Ricci soliton and f is called the potential function. In what follows we let (M, g, f) be a gradient Ricci soliton.

Ricci solitons are natural generalizations of Einstein manifolds. If X is a Killing vector field in the Ricci soliton equation (1.1) or f is a constant function in the gradient Ricci soliton equation (1.2), then we obtain the Einstein equation $\mathrm{Ric}(g) = \lambda g$ and the soliton is an Einstein manifold or, in the steady case, it is a Ricci-flat manifold. The concept of Ricci solitons was first introduced in [8] by Hamilton as a self-similar solution of Hamilton's Ricci flow, $\partial_t g(t) = -2Ric(g(t))$, on Riemannian metrics. Ricci flow is an evolutionary intrinsic geometric flow introduced in 1982 by Hamilton on Riemannian metrics for studying the topology of 3-dimensional manifolds [9].

A soliton for the Ricci flow is a metric that changes only by rescaling and by a pullback of a one-parameter family of diffeomorphisms as it evolves under the Ricci flow. If we have a Riemannian or pseudo-Riemannian metric g, a complete vector field X and a real number λ (all independent of time) that satisfy the Ricci soliton equation (1.1), then $g(t) = \sigma(t)\phi_t^*(g)$ is a solution of the Ricci flow, where $\sigma(t) := 1 - 2\lambda t$ and ϕ_t is a family of diffeomorphisms generated by the t-dependent vector field $\sigma(t)^{-1}X$. See [7] for more details.

Geometry of Riemannian Ricci solitons has been studied widely because of the role of the Ricci flow in solving the Poincaré conjecture and Thurston's geometrization conjecture that were finally proved by Perelman [12]. Ricci solitons often arise as limits of dilations of singularities in the Ricci flow [4]. The geometric structure of Ricci solitons in pseudo-Riemannian setting has been studied by a number of authors. For some recent results and further references on pseudo-Riemannian Ricci solitons, we may refer to [3] and references therein. Also, solutions of Euclidean signature Einstein gravity coupled to a free massless scalar field with nonzero cosmological constant are associated with shrinking or expanding Ricci solitons [1].

From the work by Perelman, we know that closed Riemannian Ricci solitons are necessarily gradient and, moreover, closed expanding or steady Ricci solitons are necessarily Einstein [6], which is derived from maximum principles for the Laplace operator that is an elliptic operator on Riemannian manifolds. In [10], we show the existence of non-trivial and non-gradient steady Ricci solitons on a special group of compact indecomposable Lorentzian 3-manifolds admitting a parallel light-like vector field with closed orbits. The aim of this paper is to study the Ricci soliton equation on compact indecomposable Lorentzian 3-manifolds with a parallel light-like vector field with closed orbits, that were classified recently in [2], in order to construct more examples of closed Lorentzian steady Ricci solitons with zero scalar curvature. In each case, the associated potential vector field with the Ricci soliton structure is space-like. Furthermore, these Ricci solitons are geodesically complete because compact indecomposable Lorentzian 3-manifolds with a parallel light-like vector field are pp-waves, a special class of Lorentzian manifolds admitting a parallel null vector field, which occur whenever the Ricci tensor is completely determined by the parallel null vector. It was proved in [11] that every compact pp-wave is geodesically complete. Also, we construct the examples of closed pseudo-Riemannian steady Ricci solitons in dimension 4 with the neutral signature $(2, 2)$ that admit a parallel light-like vector field. They are geodesically complete and do not admit a conformal-Killing vector field. But they can admit a time-like Killing vector field. The potential vector field associated with a Ricci soliton structure, in this case, can be space-like, time-like or null. Whereas for a non-trivial gradient close pseudo-Riemannian steady Ricci soliton with constant scalar curvature, if there exists, the gradient vector field is necessarily a null vector field. Furthermore, the vector field X associated with a Ricci soliton structure is a harmonic vector field [15], and in our examples of closed pseudo-Riemannian Ricci solitons, X is divergence free and $\|L_X g\|^2 = 0$.

This paper is organized as follows. In Section 2, we give some preliminaries about Ricci soliton structures. Section 3 is devoted to studying the Ricci soliton structure on orientable compact indecomposable Lorentzian manifolds admitting a parallel light-like vector field with closed orbits. Finally, in Section 4, we construct the examples of non-trivial and non-gradient closed pseudo-Riemannian Ricci solitons with the neutral signature $(2, 2)$ in the steady case with zero scalar curvature.

2. Preliminaries

Let (M, g, X) be an n-dimensional Ricci soliton. Then, by tracing equation (1.1), we get

$$\mathrm{Div}(X) + \tau = n\lambda, \tag{2.1}$$

where τ is the scalar curvature and $\mathrm{Div}(X)$ is the divergence of the vector field X. If the manifold M is closed and orientable, then by the Divergence Theorem, we have

$$\int_M \tau \, d\mu_g = n\lambda \, \mathrm{vol}(M). \tag{2.2}$$

So $\lambda = n^{-1} r$, where $r = \mathrm{vol}(M)^{-1} \int_M \tau \, d\mu_g$ is the mean scalar curvature.

Proposition 2.1. *Let (M, g, X) be a closed pseudo-Riemannian Ricci soliton with constant scalar curvature. Then $\tau = n\lambda$ and $\mathrm{Div}(X) = 0$. In particular, in the steady case the scalar curvature is zero.*

Proof. If (M, g, X) is a closed pseudo-Riemannian Ricci soliton with constant scalar curvature, then by (2.2), we have $\tau \ \mathrm{vol}(M) = n\lambda \, \mathrm{vol}(M)$. Therefore, $\tau = n\lambda$. Thus, equation (2.1) implies that $\mathrm{Div}(X) = 0$. □

Remark 2.2. Let (M, g, f) be a gradient Ricci soliton. Then, by tracing equation (1.2), we have

$$\Delta_g f = n\lambda - \tau, \tag{2.3}$$

where Δ_g is the Laplace-Beltrami operator with respect to the pseudo-Riemannian metric g that is an ultrahyperbolic operator in pseudo-Riemannian cases or normally hyperbolic operator when the metric is given in the Lorentzian signature. Since the solutions of the Laplace equation Δ_g on a closed Riemannian manifold are necessarily trivial, i.e., constant functions, then we have no non-trivial closed Riemannian Ricci solitons with constant scalar curvature. Whereas, in pseudo-Riemannian cases, the Laplace equation may have non-trivial solutions.

Therefore, we can find non-trivial examples of pseudo-Riemannian gradient Ricci solitons with constant scalar curvature.

Proposition 2.3. *The gradient vector field associated with a closed pseudo-Riemannian steady gradient Ricci soliton with constant scalar curvature is a null vector field.*

Proof. Let (M, g, f) be a gradient Ricci soliton with constant scalar curvature. Then $\|\nabla f\|^2 - 2\lambda f = \text{const}$, see [3, Lemma 11.14]. Therefore, in the steady case, $\|\nabla f\|^2 = \text{const}$. On the other hand, for an arbitrary function f on a pseudo-Riemannian manifold (M, g), we have $\Delta_g f^2 = 2f\Delta_g f + \|\nabla f\|^2$, see [13, p. 94]. Hence, if (M, g) is closed and orientable, then by the divergence theorem,

$$\int_M \|\nabla f\|^2 \, d\mu_g = -\int_M 2f\Delta_g f \, d\mu_g. \tag{2.4}$$

Now we suppose that (M, g, f) is a closed gradient Ricci soliton with constant scalar curvature. Then, by Proposition 2.1, we have $\Delta_g f = 0$. Therefore, equation (2.4) implies that $\int_M \|\nabla f\ \|^2 \, d\mu_g = 0$. But, since $\|\nabla f\|^2$ is constant, then $\|\nabla f\|^2$ is necessarily zero. □

3. Ricci solitons on orientable, compact indecomposable Lorentzian 3-manifolds admitting a parallel light-like vector field with closed orbits

For any $n \in \mathbb{N}$, we let Γ_n be a group of diffeomorphisms of $\mathbb{R}^3$ generated by the maps

$$\begin{aligned} \tau_x(x, y, z) &= (x + 1, y, z), \\ \tau_y(x, y, z) &= (x, y + 1, z), \\ \tau_{z,n}(x, y, z) &= (x + ny, y, z + 1) \end{aligned}$$

that preserve the moving frame $(\partial_x, \partial_y + nz\partial_x, \partial_z)$ on $\mathbb{R}^3$. We call $\mathbb{R}^3/\Gamma_n$ a *parabolic torus*, as a suspension of the parabolic automorphism $\tau_{z,n}(x, y, z)$ of $\mathbb{R}^2/\mathbb{Z}^2$ over $\mathbb{R}/\mathbb{Z}$. See [2] for more details.

Proposition 3.1. *If (M, g) is an orientable, compact indecomposable Lorentzian 3-manifold endowed with a parallel light-like vector field with closed orbits, then it is isometric to $(\mathbb{R}^3/\Gamma_n, g)$, where g is the metric induced by a metric $\tilde{g}$ on $\mathbb{R}^3$ whose matrix in the Γ_n-invariant moving frame $(\partial_x, \partial_y + nz\partial_x + \theta\partial_z, \partial_z)$, $\theta \in \mathbb{R}$, is*

$$\begin{pmatrix} 0 & 0 & \Lambda \\ 0 & L^2(y, z) & \nu(y, z) \\ \Lambda & \nu(y, z) & \mu(y, z) \end{pmatrix}, \tag{3.1}$$

where L, μ and ν are $(1, 1)$-biperiodic functions and Λ is a non-zero real number. Also, L is a non-vanishing function because of the non-degeneracy of the metric g at any point on the manifold.

The proof can be found in [2, Section 4].

Proposition 3.2. *Let $X = (X_1, X_2, X_3)$ be a vector field on $\mathbb{R}^3/\Gamma_n$, where $X_i = X_i(x, y, z)$, $0 \leq i \leq 3$, are real-valued functions that are Γ_n-invariant. Then X defines a Ricci soliton in the steady case with respect to the metric g if $X_1 = X_1(y, z)$ and $X_2 = X_2(y, z)$ are $(1, 1)$-biperiodic functions and $X_3 = c$ is constant that satisfy the following partial differential equations:*

$$\partial_y(LX_2) + \theta\partial_z(LX_2) + c\partial_z L = 0, \tag{3.2}$$

$$\Lambda(\partial_y X_1 + \theta\partial_z X_1 - nc) + (\partial_y X_2 + \theta\partial_z X_2)\nu + (\partial_y \nu + \theta\partial_z \nu)X_2 + L^2\partial_z X_2 + c\partial_z \nu = 0, \tag{3.3}$$

$$L[2\Lambda\partial_z X_1 + 2\nu\partial_z X_2 + (\partial_y \mu + \theta\partial_z \mu + 2n\Lambda)X_2 + c\partial_z \mu] = H_y + \theta H_z + L_{zz}, \tag{3.4}$$

where

$$H(y, z) := \frac{1}{2L}\Big(2n\Lambda + \partial_z \mu + \partial_y \mu - 2\partial_z \nu\Big). \tag{3.5}$$

Furthermore,

$$\|X\|^2 = L^2 X_2^2 + 2c\Lambda X_1 + 2c\nu X_2 + c^2\mu. \tag{3.6}$$

Proof. Since the scalar curvature of $(\mathbb{R}^3/\Gamma_n, g)$ is zero, then by Proposition 2.1, we can only have steady Ricci solitons on these manifolds. By considering the Ricci soliton equation in the steady case with respect to the metric g, we get a system of partial differential equations. In particular, we have $\partial_x X_3 = 0$ that implies $X_3 = X_3(y, z)$ to be a $(1, 1)$-biperiodic function. Thus, we get the following system of partial differential equations:

$$\Lambda(\partial_y X_3 + \theta\partial_z X_3) + L^2\partial_x X_2 = 0, \tag{3.7}$$

$$\Lambda(\partial_x X_1 + \partial_z X_3) + \nu\partial_x X_2 = 0, \tag{3.8}$$

$$2\nu(\partial_y X_3 + \theta\partial_z X_3) + 2L(nzL\partial_x X_2 + \partial_y(LX_2) + \theta\partial_z(LX_2) + X_3\partial_z L) = 0, \tag{3.9}$$

$$\begin{aligned}\nu\partial_z X_3 + (\partial_y X_3 + \theta\partial_z X_3)\mu + L^2\partial_z X_2 &+ (\partial_y X_2 + \theta\partial_z X_2 + nz\partial_x X_2)\nu \\ &+ \Lambda(\partial_y X_1 + \theta\partial_z X_1 + nz\partial_x X_1 - nX_3) \\ &+ X_2(\partial_y \nu + \theta\partial_z \nu) + X_3\partial_z \nu = 0,\end{aligned} \tag{3.10}$$

$$2\mu\partial_z X_3 + 2\nu\partial_z X_2 + X_2(2n\Lambda + \partial_y \mu + \theta\partial_z \mu) + 2\Lambda\partial_z X_1 + X_3\partial_z \mu = L^{-1}\big(\partial_y H + \theta\partial_z H + \partial_{zz}\big). \tag{3.11}$$

If we take the partial derivative of equation (3.7) with respect to x, then $\partial_{xx} X_2 = 0$. Since X_2 is Γ_n-invariant, then $X_2 = X_2(y, z)$ is a $(1, 1)$-biperiodic function. Also, by taking the partial derivative of equation (3.8) with respect to x, we have $\partial_{xx} X_1 = 0$ that implies $X_1 = X_1(y, z)$ to be a $(1, 1)$-biperiodic function, because X_1 is Γ_n-invariant. Therefore, equation (3.8) leads to $\partial_z X_3 = 0$. Hence, $X_3 = X_3(y)$ is a periodic function. But, by equation (3.7), $X_3'(y) = 0$. Therefore, $X_3 = c$ is a constant function. Thus, equations (3.9), (3.10) and (3.11) lead to (3.2), (3.3) and (3.4), respectively. □

In the following, we identify the behavior of Ricci solitons when $\theta = 0$. We show that when L is a non-zero constant function or when L just depends on y, there are not any non-trivial Ricci solitons. But if L just depends on z, then we can construct non-trivial Ricci solitons in the steady case that admit a parallel light-like vector field. The potential vector fields in all these cases are space-like that cannot be gradient vector fields.

Theorem 3.3. *If we consider the metric g defined in Proposition 3.1 on the parabolic torus $\mathbb{R}^3/\Gamma_n$ when L is a non-zero constant function, $\nu(y,z)$ and $\mu(y,z)$ are arbitrary $(1,1)$-biperiodic functions and $\theta = 0$, then there are not any non-trivial Ricci solitons on $(\mathbb{R}^3/\Gamma_n, g)$.*

Proof. Suppose that $L \equiv 1$ but $\nu = \nu(y,z)$ and $\mu = \mu(y,z)$ are arbitrary $(1,1)$-biperiodic functions. If $X = (X_1, X_2, X_3)$ is a vector field that defines a Ricci soliton structure on $\mathbb{R}^3/\Gamma_n$ with respect to the metric g, then by Proposition 3.2, $X_3 = c$ is a constant function and due to the assumptions, we get the system of partial differential equations:

$$\partial_y X_2 = 0, \tag{3.12}$$
$$\Lambda(\partial_y X_1 - nc) + \partial_y(\nu X_2) + \partial_z X_2 + c\partial_z \nu = 0, \tag{3.13}$$
$$2\Lambda\partial_z X_1 + 2\nu\partial_z X_2 + (\partial_y \mu + 2n\Lambda)X_2 + c\partial_z \mu = \frac{1}{2}\partial_{yy}\mu - \partial_{yz}\nu. \tag{3.14}$$

By equation (3.12), $X_2 = X_2(z)$ is a periodic function. Now, by integrating equation (3.13) with respect to y, we have

$$\Lambda X_1 + \nu X_2(z) = nc\Lambda y - X_2'(z)y + c\int \partial_z \nu \, dy + G(z). \tag{3.15}$$

Since the left-hand side of equation(3.15) is a $(1,1)$-biperiodic function, then $G(z)$ is periodic and

$$nc\Lambda - X_2'(z) + c\int_0^1 \partial_z \nu \, dy = 0. \tag{3.16}$$

See [5] for the integral of a periodic function. Now, if equation (3.16) is satisfied, then

$$\int_0^1 \left[nc\Lambda - X_2'(z) + c\int_0^1 \partial_z \nu \, dy \right] dz = 0. \tag{3.17}$$

But

$$\int_0^1 X_2'(z)\, dz = X_2(1) - X_2(0) = 0,$$

because X_2 is a periodic function with period 1. Also, since $\nu(y,z)$ is a $(1,1)$-biperiodic function, then we have

$$\int_0^1\int_0^1 \partial_z \nu \, dy\, dz = \int_0^1\int_0^1 \partial_z \nu \, dz\, dy = \int_0^1 [\nu(y,1) - \nu(y,0)]\, dy = 0.$$

Hence, $nc\Lambda = 0$ which implies $c = 0$. Thus, X_3 is the zero function. Therefore, equation (3.16) implies that $X_2'(z) = 0$. Since X_2 is periodic, then X_2 is a constant function. Let $X_2 = b$. Hence we have the following system of partial differential equations:

$$\Lambda\partial_y X_1 + b\partial_y \nu = 0, \tag{3.18}$$

$$2\Lambda\partial_z X_1 + (\partial_y \mu + 2n\Lambda)b = \frac{1}{2}\partial_{yy}\mu - \partial_{yz}\nu. \tag{3.19}$$

By integrating equation (3.19) with respect to y, we get

$$\frac{1}{2}\partial_y \mu - \partial_z \nu - b\mu = 2nb\Lambda y + 2\Lambda \int \partial_z X_1 \, dy + G(z). \tag{3.20}$$

Since the left-hand side of equation (3.20) is a $(1,1)$-biperiodic function, then $G(z)$ is periodic and

$$nb + \int_0^1 \partial_z X_1 \, dy = 0$$

that implies

$$\int_0^1 [nb + \int_0^1 \partial_z X_1 \, dy] \, dz = 0.$$

Since X_1 is a $(1,1)$-biperiodic function, then

$$\int_0^1 \int_0^1 \partial_z X_1 \, dy \, dz = 0.$$

Therefore, $nb = 0$. Thus, $b = 0$, and X_2 is the zero function. Now equation (3.18) implies that $\partial_y X_1 = 0$. Hence, $X_1 = X_1(z)$ is a periodic function of z. Now, by equation (3.20), we have

$$\partial_y \mu - 2\partial_z \nu = 4\Lambda X_1'(z) y + G(z).$$

Since the left-hand side of this equation is biperiodic, then $X_1'(z) = 0$ that implies X_1 to be a constant function. Therefore, $L_X g = 0$, and X is a Killing vector field. □

Theorem 3.4. *Let g be the metric defined in Proposition 3.1 on the parabolic torus $\mathbb{R}^3/\Gamma_n$ when $L = L(y)$ is a non-vanishing periodic function, $\nu(y,z)$ and $\mu(y,z)$ are arbitrary $(1,1)$-biperiodic functions and $\theta = 0$. Then there are not any non-trivial Ricci solitons on $(\mathbb{R}^3/\Gamma_n, g)$.*

Proof. Let $X = (X_1, X_2, X_3)$ be a vector field on $(\mathbb{R}^3/\Gamma_n, g)$ that defines a Ricci soliton with respect to the prescribed metric g when $L = L(y)$ is a periodic function and $\theta = 0$. Then, by Proposition 3.2, $X_3 = c$ is a constant function, as well as $X_1 = X_1(y,z)$ and $X_2 = X_2(y,z)$ are $(1,1)$-biperiodic functions that satisfy the following system of partial differential equations

$$\partial_y (L(y) X_2) = 0, \tag{3.21}$$

$$\Lambda(\partial_y X_1 - nc) + \partial_y(\nu X_2) + L^2(y)\partial_z X_2 + c\partial_z \nu = 0, \tag{3.22}$$

$$\begin{aligned} &2\Lambda\partial_z X_1 + 2\nu\partial_z X_2 + (\partial_y\mu + 2n\Lambda)X_2 + c\partial_z\mu \\ &= \frac{-n\Lambda L'(y)}{L^3(y)} + \frac{1}{2L(y)}\partial_y(\frac{\partial_y\mu}{L(y)}) - \frac{1}{L(y)}\partial_y(\frac{\partial_z\nu}{L(y)}). \end{aligned} \tag{3.23}$$

By integrating equation (3.22) with respect to z, we have

$$L^2 X_2 + c\nu = -\Lambda \int (\partial_y X_1 - nc)\, dz - \int \partial_y(\nu X_2)\, dz + F(y).$$

Since the left-hand side of this equation is $(1,1)$-biperiodic, then $F(y)$ is periodic and

$$\Lambda \int_0^1 (\partial_y X_1 - nc)\, dz - \int_0^1 \partial_y(\nu X_2)\, dz = 0$$

that implies

$$\int_0^1 [\Lambda \int_0^1 (\partial_y X_1 - nc)\, dz - \int_0^1 \partial_y(\nu X_2)\, dz]\, dy = 0.$$

But X_1 and νX_2 are $(1,1)$-biperiodic functions. Hence,

$$\int_0^1 \int_0^1 \partial_y X_1\, dzdy = \int_0^1 \int_0^1 \partial_y(\nu X_2)\, dzdy = 0.$$

Therefore, $nc\Lambda = 0$ that implies $c = 0$. Thus, X_3 is the zero function. But, by equation (3.21), $X_2 = L^{-1}(y)F(z)$, where $F(z)$ is a periodic function. By substituting $L^{-1}(y)F(z)$ instead of X_2 in equation (3.22) and integrating this equation with respect to y, we have

$$\Lambda X_1 + \nu L^{-1}(y)F(z) = -F'(z) \int L(y)\, dy + G(z).$$

Since the left-hand side of this equation is $(1,1)$-biperiodic, then $G(z)$ is periodic and

$$F'(z) \int_0^1 L(y)\, dy = 0$$

that implies $F'(z) = 0$ because $\int_0^1 L(y)\, dy$ is a non-zero real number. Since F is periodic, then F is a constant function. Let $F(z) \equiv a$, where a is a real number. Thus $X2 = aL^{-1}(y)$. Then equation (3.22) leads to

$$\Lambda\partial_y X_1 + a\partial_y(\nu L^{-1}) = 0$$

that implies $X_1 = -a(\Lambda L)^{-1}\nu + G(z)$. But, by equation (3.23), we have

$$\begin{aligned} &-2a\partial_z\nu + 2\Lambda G'(z) + a\partial_y\mu + 2na\Lambda \\ &\qquad = \frac{-n\Lambda L'(y)}{L^2(y)} + \frac{1}{2}\partial_y\left(\frac{\partial_y\mu}{L(y)}\right) - \partial_y\left(\frac{\partial_z\nu}{L(y)}\right). \end{aligned} \tag{3.24}$$

By integrating equation (3.24) with respect to y, we get

$$\frac{n\Lambda}{L} + \frac{\partial_y \mu}{2L} - \frac{\partial_z \nu}{L} - a\mu = 2\Lambda(na + G'(z))y - 2a \int \partial_z \nu \, dy + H(z). \tag{3.25}$$

Since the left-hand side of equation (3.25) is $(1,1)$- biperiodic, we obtain

$$2\Lambda(na - G'(z)) + 2a \int_0^1 \partial_z \nu \, dy = 0$$

that implies

$$\int_0^1 [2\Lambda(na + G'(z)) + 2a \int_0^1 \partial_z \nu \, dy] \, dz = 0.$$

Since $G(z)$ is periodic and ν is biperiodic, then

$$\int_0^1 G'(z) \, dz = \int_0^1 \partial_z \nu \, dz \, dy = 0.$$

Therefore, $2na\Lambda = 0$. Thus, $a = 0$, and X_2 is the zero function, as well as $X_1 = X_1(z)$ is a periodic function of z. Now equation (3.23) leads to

$$2\Lambda L(y) X_1'(z) = \frac{-n\Lambda L'(y)}{L^2(y)} + \frac{1}{2}\partial_y \left(\frac{\partial_y \mu}{L(y)} \right) - \partial_y \left(\frac{\partial_z \nu}{L(y)} \right). \tag{3.26}$$

By integrating equation (3.26) with respect to y, we get

$$\frac{n\Lambda}{L(y)} + \frac{\partial_y \mu}{2L(y)} - \frac{\partial_z \nu}{L(y)} = 2\Lambda X_1'(z) \int L(y) \, dy + H(z). \tag{3.27}$$

Since the left-hand side of equation (3.27) is $(1,1)$-biperiodic, then

$$\Lambda X_1' \int_0^1 L \, dy = 0$$

that implies $X_1'(z) = 0$ because $\int_0^1 L \, dy$ is a non-zero real number. Hence, X_1 is a constant funtion. Thus, $L_X(g) = 0$, and X is a Killing vector field. □

Theorem 3.5. *Let g be the metric defined in Proposition 3.1 on the parabolic torus $\mathbb{R}^3/\Gamma_n$, where $L = L(z)$ is a non-vanishing periodic function, $\nu(y,z)$ and $\mu(y,z)$ are arbitrary $(1,1)$-biperiodic functions and $\theta = 0$. Then for a vector field X on $(\mathbb{R}^3/\Gamma_n, g)$, if X defines a Ricci soliton in the steady case, then $X = (X_1(y,z), b, 0)$, where $b = \int_0^1 (2n\Lambda L)^{-1} L'' \, dz$ is a non-zero real number and $X_1(y,z)$ is a $(1,1)$-biperiodic function that satisfies the system of partial differential equations:*

$$\Lambda \partial_y X_1 + b \partial_y \nu = 0, \tag{3.28}$$

$$2\Lambda \partial_z X_1 + (\partial_y \mu + 2n\Lambda) b = \frac{\partial_{yy} \mu}{2L^2(z)} - \frac{\partial_{yz} \nu}{L^2(z)} + \frac{L''(z)}{L(z)}. \tag{3.29}$$

Specially, if $\partial_{yz}\nu = 0$*, then* μ *is constant or* $\mu = \mu(z)$ *is a periodic function, as well as for some periodic functions* $\kappa(y)$ *and* $\eta(z)$*,* $\nu(y,z) = \kappa(y) + \eta(z)$*. In this case,* $X_1(y,z) = F(y) + G(z)$*, where* $F(y) = -b\Lambda^{-1}\kappa(y)$ *and* $G(z) = \int (2\Lambda L)^{-1} L''(z)\, dz - nbz$ *are periodic functions. But, if* $\partial_{yz}\nu \not\equiv 0$*, then* $\partial_{yy}\mu - 2\partial_{yz}\nu = 0$ *and* $X_1 = X_1(y,z)$ *is a biperiodic function that satisfies the following partial differential equations:*

$$\Lambda \partial_y X_1 + b\partial_y \nu = 0, \tag{3.30}$$

$$2\Lambda \partial_z X_1 + (\partial_y \mu + 2n\Lambda)b = \frac{L''(z)}{L(z)}. \tag{3.31}$$

Also, $\| X \|^2 = b^2 L^2(z)$ *and* X *is space-like.*

Proof. Let $X = (X_1, X_2, X_3)$ be a vector field on $(\mathbb{R}^3/\Gamma_n, g)$ that defines a Ricci soliton with respect to the prescribed metric g when $L = L(z)$ is a nonvanishing periodic function and $\theta = 0$. Then by Proposition 3.2, $X_3 = c$ is a constant function, as well as $X_1 = X_1(y,z)$ and $X_2 = X_2(y,z)$ are $(1,1)$-biperiodic functions that satisfy the system of partial differential equations:

$$L(z)\partial_y X_2 + cL'(z) = 0, \tag{3.32}$$

$$\Lambda(\partial_y X_1 - nc) + \partial_y(\nu X_2) + L^2(z)\partial_z X_2 + c\partial_z \nu = 0, \tag{3.33}$$

$$2\Lambda \partial_z X_1 + 2\nu \partial_z X_2 + (\partial_y \mu + 2n\Lambda)X_2 + c\partial_z \mu = \frac{\partial_{yy}\mu}{2L^2} - \frac{\partial_{yz}\nu}{L^2} + \frac{L''}{L}. \tag{3.34}$$

By integrating equation (3.32) with respect to y, we have

$$L(z)X_2 = -cL'(z)y + G(z).$$

Since LX_2 is a $(1,1)$-biperiodic function, then $G(z)$ is periodic and $c = 0$. Hence, X_3 is the zero function. Now equation (3.32) implies that $\partial_y X_2 = 0$. Therefore, $X_2 = X_2(z)$ is a periodic function. On the other hand, by integrating equation (3.33) with respect to y, we have

$$\Lambda X_1 + X_2(z)\nu = -L^2(z)X_2'(z)y + G(z).$$

Since the left-hand side of this equation is biperiodic, then $G(z)$ is periodic and $X_2'(z) = 0$ that implies X_2 to be a constant function. Let $X_2 = b$. Then we get the system of partial differential equations:

$$\Lambda \partial_y X_1 + b\partial_y \nu = 0, \tag{3.35}$$

$$2\Lambda \partial_z X_1 + (\partial_y \mu + 2n\Lambda)b = \frac{\partial_{yy}\mu}{2L^2(z)} - \frac{\partial_{yz}\nu}{L^2(z)} + \frac{L''(z)}{L(z)}. \tag{3.36}$$

By integrating equation (3.36) with respect to y, we have

$$\frac{\partial_y \mu}{2L^2(z)} - \frac{\partial_z \nu}{L^2(z)} - b\mu = 2\Lambda \int \partial_z X_1\, dy + \Big(2nb\Lambda - \frac{L''(z)}{L(z)}\Big)y + F(z). \tag{3.37}$$

Since the left-hand side of equation (3.37) is $(1,1)$-biperiodic, then $F(z)$ is periodic and

$$2\Lambda \int_0^1 \partial_z X_1 \, dy + 2nb\Lambda - \frac{L''(z)}{L(z)} = 0, \tag{3.38}$$

see [5] for the integral of a periodic function that implies

$$\int_0^1 [2\Lambda \int_0^1 \partial_z X_1 \, dy + 2nb\Lambda - \frac{L''(z)}{L(z)}] \, dz = 0. \tag{3.39}$$

But

$$\int_0^1 \int_0^1 \partial_z X_1 \, dy \, dz = 0$$

because X_1 is a $(1,1)$-biperiodic function. Hence,

$$2nb\Lambda - \int_0^1 L^{-1} L'' \, dz = 0.$$

Then,

$$b = \frac{1}{2n\Lambda} \int_0^1 \frac{L''}{L} \, dz. \tag{3.40}$$

But, by integration by parts, we have

$$\int_0^1 \frac{L''}{L} \, dz = \int_0^1 \frac{L'^2}{L^2} \, dz. \tag{3.41}$$

Hence, $\int_0^1 L^{-1} L'' \, dz$ is a non-negative number. Therefore, b is a non-zero real number unless L is a constant function. If L is constant, then by Theorem 3.3, there is not any non-trivial Ricci soliton on $(\mathrm{R}^3/\Gamma_n, g)$. Hence, b is necessarily a non-zero real number. Since $X_2 = b$ is a non-zero constant function and $c = 0$, then by equation (3.6), $\|X\|^2 = b^2 L^2(z)$. Therefore, $X = (X_1(y,z), b, 0)$ is a space-like vector field. In the following, we determine some solutions of equations (3.28) and (3.29) when $\partial_{yz}\nu = 0$.

First, we suppose that $\partial_y \nu = 0$ that implies $\partial_{yz}\nu = 0$. Then ν is constant or $\nu = \nu(z)$ is a periodic function. Thus, by equation (3.35), $X_1 = X_1(z)$ is a periodic function. But, by equation (3.36), we have

$$2\Lambda X_1'(z) + (\partial_y \mu + 2n\Lambda) b = \frac{\partial_{yy}\mu}{2L^2(z)} + \frac{L''(z)}{L(z)}. \tag{3.42}$$

If $\partial_{yy}\mu = 0$, then $\mu = F(z)y + G(z)$. Since μ is a biperiodic function, then μ is a constant or a periodic function of z. Hence, by equation (3.42), we have

$$2\Lambda X_1'(z) + 2nb\Lambda = \frac{L''(z)}{L(z)}. \tag{3.43}$$

Therefore,

$$X_1(z) = \int \frac{L''(z)}{2\Lambda L(z)} \, dz - nbz \tag{3.44}$$

is a periodic function because $b = \int_0^1 (2n\Lambda L)^{-1} L''\, dz$. See [5] for the integral of a periodic function. But, if $\partial_{yy}\mu \not\equiv 0$, then by equation (3.42), we get a first-order ordinary differential equation with respect to the y variable for the function $\partial_y\mu$. Therefore,

$$\begin{aligned}\mu_y &= e^{2bL^2(z)y} \int e^{-2bL^2(z)y}[2\Lambda X_1'(z) + 4nb\Lambda L^2(z) - 2LL''(z)]\, dy + G(z)\\ &= \frac{2\Lambda X_1'(z) + 4nb\Lambda L^2(z) - 2LL''(z)}{-2bL^2(z)} + G(z).\end{aligned} \tag{3.45}$$

Hence, $\partial_y\mu$ is a periodic function of z which implies that μ cannot be a biperiodic function. Thus, $\partial_{yy}\mu = 0$ and μ is a constant or a periodic function of z. Now we suppose that $\partial_z\nu = 0$ that implies $\partial_{yz}\nu = 0$. Then ν is a constant or a periodic function of y. Thus, by equations (3.35) and (3.36),

$$X_1 = G(z) - \Lambda^{-1} b\nu(y),$$

where

$$2\Lambda G'(z) + (\partial_y\mu + 2n\Lambda)b = 2^{-1}L^{-2}(z)\partial_{yy}\mu + L^{-1}(z)L''(z). \tag{3.46}$$

But, similar to what we saw in equation (3.42), equation (3.46) is satisfied if and only if μ is a constant or $\mu = \mu(z)$ a periodic function of z. Thus,

$$G(z) = \int \frac{L''(z)}{2\Lambda L(z)}\, dz - nbz, \tag{3.47}$$

is a periodic function. Finally, we suppose that $\partial_{yz}\nu = 0$, but neither $\partial_y\nu = 0$ nor $\partial_z\nu = 0$. Hence, $\nu(y, z) = \kappa(y) + \eta(z)$ for some periodic functions $\kappa(y)$ and $\eta(z)$. Therefore, by equations (3.35) and (3.36), $X_1(y, z) = G(z) - b\Lambda^{-1}\kappa(y)$, where $G(z)$ is a periodic function that satisfies equation (3.46). Hence, μ is a constant or a periodic function of z. Thus we get the periodic function $G(z)$ by (3.47) for a given nonvanishing periodic function $L(z)$.

Now we suppose that $\partial_{yz}\nu \not\equiv 0$. By integrating equation (3.35) with respect to y, $\Lambda X_1 = E(z) - b\nu$, where $E(z)$ is an arbitrary periodic function. If we take the partial derivative of this equation with respect to z, then $2\Lambda\partial_z X_1 = E'(z) - 2b\partial_z\nu$. Thus, by substituting $E'(z) - 2b\partial_z\nu$ instead of $2\Lambda\partial_z X_1$ in equation (3.36), we have

$$\partial_{yz}\nu - 2bL^2\partial_z\nu = -E'L^2 - bL^2\partial_y\mu - 2nb\Lambda L^2 + \frac{1}{2}\partial_{yy}\mu + LL'', \tag{3.48}$$

which is a first-order ordinary differential equation with respect to the y variable for the function $\partial_z\nu$. Therefore,

$$\begin{aligned}\partial_z\nu &= e^{2bL^2y} \int L^2 e^{-2bL^2y}[-E' - bL^2\partial_y\mu - 2nb\Lambda + \frac{1}{2L^2}\partial_{yy}\mu + \frac{L''}{L}]\, dy + G(z)\\ &= \frac{1}{2b}E'(z) + \frac{1}{2}\partial_y\mu - \frac{L''}{2bL} + n\Lambda + G(z).\end{aligned} \tag{3.49}$$

Thus, $\partial_{yy}\mu - 2\partial_{yz}\nu = 0$ and we take equations (3.30) and (3.31) from (3.35) and (3.36), respectively. □

Remark 3.6. By Theorem 3.5, there is not any non-trivial Ricci soliton on the parabolic torus $(\mathbb{R}^3/\Gamma_n, g)$, prescribed in Proposition 3.1 when $\theta = 0$, $L = L(z)$ and $\partial_{yz}\nu \not\equiv 0$, whereas μ is a constant or a periodic function of z.

In the following, we consider the gradient Ricci soliton equation on the parabolic torus $(\mathbb{R}^3/\Gamma_n, g)$. We see that the potential function f is a periodic function of $\xi = z - \theta y$.

Proposition 3.7. *If $(\mathbb{R}^3/\Gamma_n, g, f)$ is a gradient Ricci soliton, where $(\mathbb{R}^3/\Gamma_n, g)$ is defined in Proposition 3.1, then $f = f(z - \theta y)$ is a periodic function with period 1 and θ is an integer number that satisfies the partial differential equation*

$$Lf_{zz} = \partial_y H + \theta\partial_z H + L_{zz}, \tag{3.50}$$

where

$$H(y, z) := \frac{n\Lambda}{L} + \frac{\theta}{2}\frac{\partial_z\mu}{L} + \frac{1}{2}\frac{\partial_y\mu}{L} - \frac{\partial_z\nu}{L}. \tag{3.51}$$

Proof. Let f be a smooth real-valued function on $\mathbb{R}^3/\Gamma_n$ that defines a gradient Ricci soliton in the steady case with respect to the metric g defined by Proposition 3.1. By considering the gradient Ricci soliton equation, we get $\partial_{xx}f = 0$ that implies $f = f_1(y, z)x + f_2(y, z)$. Since f is Γ_n-invariant, then $f = f(y, z)$ is a $(1, 1)$-biperiodic function. On the other hand, by Proposition 2.3, ∇f is a null vector field. Thus, $\|\nabla f\|^2 = (\theta\partial_z f + \partial_y f)/L^2 = 0$ that implies $f = f(z - \theta y)$ to be a $(1, 1)$-biperiodic function. Hence, $f = f(z - \theta y)$ is a periodic function with period 1 and θ is an integer number. Finally, we get equation (3.50) by the gradient Ricci soliton equation. □

4. Non-gradient closed pseudo-Riemannian steady Ricci solitons with time-like or null potential vector fields

In this section, we construct the examples of non-trivial and non-gradient closed pseudo-Riemannian steady Ricci solitons with zero scalar curvature in the neutral signature $(2, 2)$ such that the associated potential vector fields can be time-like or null.

For any $n \in \mathbb{N}$, define the quotient manifold $\mathbb{R}^4/\Gamma_n$, where Γ_n is the group of diffeomorphisms generated by the maps

$$\begin{aligned}
\tau_t(t, x, y, z) &= (t + 1, x, y, z),\\
\tau_x(t, x, y, z) &= (t, x + 1, y, z),\\
\tau_y(t, x, y, z) &= (t, x, y + 1, z),\\
\tau_{z,n}(t, x, y, z) &= (t, x + ny, y, z + 1),
\end{aligned}$$

and consider the metric g that is the metric induced by the metric $\tilde{g}$ on $\mathbb{R}^4$ whose matrix in the Γ_n-invariant frame field $(\partial_t, \partial_x, \partial_y + nz\partial_x, \partial_z)$ is

$$\begin{pmatrix} -K^2(z) & 0 & 0 & 0\\ 0 & 0 & 0 & \Lambda\\ 0 & 0 & L^2(z) & 0\\ 0 & \Lambda & 0 & 0 \end{pmatrix}, \tag{4.1}$$

where $L^2(z)$ and $K^2(z)$ are positive periodic functions and Λ is a non-zero real number. Then $(\mathbb{R}^4/\Gamma_n, g)$ is an orientable closed pseudo-Riemannian manifold in the neutral signature $(2,2)$ with zero scalar curvature. In this case, the image of ∂_x is a well-defined light-like and parallel vector field. Furthermore,

$$\operatorname{Ric}(g) = -\left(\frac{L''(z)}{L(z)} + \frac{K''(z)}{K(z)}\right)\omega_4 \otimes \omega_4, \tag{4.2}$$

where ω_4 is the 1-form associated with the image of the null parallel vector field ∂_x. A curve $\gamma(s) = (t(s), x(s), y(s), z(s))$ is a geodesic curve on $(\mathbb{R}^4, \tilde{g})$ if its components satisfy the second-order system of differential equations:

$$t''(s) = 2t'(s)z'(s)K'(z(s))K^{-1}(z(s)), \tag{4.3}$$

$$x''(s) = \Lambda^{-1}\left(y'^2(s)L(z(s))L'(z(s)) - t'^2(s)K(z(s))K'(z(s))\right) - ny'(s)z'(s), \tag{4.4}$$

$$y''(s) = n\Lambda z'^2(s)L^{-2}(z(s)) - 2y'(s)z'(s)L^{-1}(z(s))L'(z(s)), \tag{4.5}$$

$$z''(s) = 0. \tag{4.6}$$

By equation (4.6), we have $z(s) = as + b$. If $a \neq 0$, then equation (4.5) leads to

$$y''(s) + \frac{2aL(as+b)L'(as+b)}{L(as+b)}y'(s) - \frac{na^2\Lambda}{L^2(as+b)} = 0. \tag{4.7}$$

Hence,

$$y'(s) = \frac{na^2\Lambda s + c_3}{L^2(as+b)} \tag{4.8}$$

that implies

$$y(s) = \int \frac{na^2\Lambda s + c_3}{L^2(as+b)}\, ds + e_3. \tag{4.9}$$

Also, by equation (4.3), we have

$$t'(s) = \frac{c_1}{K^2(as+b)}, \tag{4.10}$$

and thus

$$t(s) = \int \frac{c_1}{K^2(as+b)}\, ds + e_1. \tag{4.11}$$

Now, by substituting the functions $z'(s)$, $y'(s)$ and $t'(s)$ in equation (4.4), we have

$$x''(s) = \frac{-na(na^2\Lambda s + c_3)}{L^2(as+b)} + \frac{(na^2\Lambda s + c_3)^2 L'(as+b)}{\Lambda L^3(as+b)} - \frac{c_1^2 K'(as+b)}{\Lambda L^3(as+b)}. \tag{4.12}$$

By integrating equation (4.12) and using the integration by parts formula, we get

$$x'(s) = \frac{-(na^2\Lambda s + c_3)^2}{2a\Lambda L^2(as+b)} + \frac{c_1^2}{2a\Lambda K^2(as+b)} + c_2. \tag{4.13}$$

Then,

$$x(s) = \int \frac{-(na^2\Lambda s + c_3)^2}{2a\Lambda L^2(as+b)}\,ds + \int \frac{c_1^2}{2a\Lambda K^2(as+b)}\,ds + c_2 s + e_2. \tag{4.14}$$

Thus, if $a \neq 0$ and $z(s) = as + b$, then equations (4.9), (4.11), and (4.14) show that every maximal geodesic is defined on the entire real line. Furthermore, by equations (4.8), (4.10) and (4.13), $\|\gamma'(s)\|^2 = 2ac_2\Lambda$. Therefore, geodesics can be space-like, time-like, or null. Also, when $a = 0$ and $z(s)$ is a constant function, every maximal geodesic is defined on the entire real line. Thus, $(\mathbb{R}^4, \tilde{g})$ is geodesically complete that implies the quotient manifold $(\mathbb{R}^4/\Gamma_n, g)$ to be a geodesically complete compact manifold.

Let $X = (X_1, X_2, X_3, X_4)$ be a vector field on $\mathbb{R}^4/\Gamma_n$. Then X defines a non-trivial steady pseudo-Riemannian Ricci soliton with respect to the metric g if and only if X_4 is the zero function, $X_1 \equiv a$ and $X_3 \equiv c$ are constant functions, and $X_2 = X_2(z)$ is a periodic function that satisfies the equation

$$2\Lambda X_2'(z) + 2nc\Lambda - \frac{L''(z)}{L(z)} - \frac{K''(z)}{K(z)} = 0. \tag{4.15}$$

Then,

$$X_2(z) = \frac{1}{2\Lambda}\int \left(\frac{L''}{L} + \frac{K''}{K}\right) dz - ncz, \tag{4.16}$$

where

$$c = \frac{1}{2n\Lambda}\int_0^1 \left(\frac{L''}{L} + \frac{K''}{K}\right) dz \tag{4.17}$$

is a non-zero real number that implies $X_2(z)$ to be a periodic function. Thus, we see that for the given non-vanishing periodic functions $L(z)$ and $K(z)$, and a non-zero real number Λ, there are many vector fields that differ only in their first component, which is a constant function, that define a Ricci soliton on $(\mathbb{R}^4/\Gamma_n, g)$. But $\|X\|^2 = c^2L^2 - a^2K^2$. Thus, as we see in the following, with a suitable choice of the first component of the vector field X, we have Ricci solitons with space-like, time-like, or null potential vector fields. If $a = 0$, then X is a space-like vector field. Now we suppose that $a \neq 0$. Since $L^2(z)$ and $K^2(z)$ are smooth positive periodic functions and thus bounded, then there are positive real numbers m and M such that $c^2L^2(z) < M$ and $ma^2 < a^2K^2(z)$. Hence, if we choose $\boldsymbol{a}$ such that $m^{-1}M < a^2$, then $\|X\|^2 < 0$ and X is a time-like vector field. Therefore, for the given non-vanishing periodic functions $L(z)$ and $K(z)$, and a non-zero real number Λ, with a suitable choice of $\boldsymbol{a}$ we can construct non-trivial steady pseudo-Riemannian Ricci solitons with time-like potential vector field which is not a gradient vector field.

If $K(z) = bL(z)$, where $\boldsymbol{b}$ is a non-zero real number, then equation (4.15) leads to

$$\Lambda X_2' + nc\Lambda - L^{-1}L'' = 0.$$

Thus,

$$X_2 = \int (\Lambda L)^{-1} L''\,dz - ncz,$$

where $c = (n\Lambda)^{-1} \int_0^1 L^{-1}L''\,dz$ is a non-zero real number that implies $X_2(z)$ to be a periodic function. But $\|X\|^2 = L^2(c^2 - a^2b^2)$. Let $a^2 = b^{-2}c^2$, then we have a non-trivial steady pseudo-Riemannian Ricci soliton with null potential vector field.

Remark 4.1. It was proved in [14] that a compact pseudo-Riemannian manifold that admits a time-like conformal-Killing vector field is geodesically complete. But the quotient manifold $(\mathbb{R}^4/\Gamma_n, g)$ is an example of a geodesically complete compact pseudo-Riemannian manifold that cannot admit a (time-like) conformal-Killing vector field. However, if we let $X = (a, b, 0, 0)$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero real numbers, then X defines a time-like Killing vector field on $(\mathbb{R}^4/\Gamma_n, g)$, and $\|X\|^2 = -a^2K^2(z)$.

Remark 4.2. If we consider $K^2(z)$ instead of $-K^2(z)$ in the matrix representation of the metric $\tilde{g}$, (4.1), then $(\mathbb{R}^4/\Gamma_n, g)$ is a compact Lorentzian pp-wave manifold in dimension 4. In this case, the associated potential vector field with the structure of Ricci soliton is necessarily a space-like vector field.

References

[1] M.M. Akbar and E. Woolgar, *Ricci solitons and Einstein-scalar field theory*, Classical Quantum Gravity **26** (2009), No. 5, 055015.

[2] C. Boubel and P. Mounoud, *Affine transformations and parallel lightlike vector fields on compact Lorentzian* 3*-manifolds*, Trans. Amer. Math. Soc. **368** (2016), 2223–2262.

[3] E. Calviño-Louzao, E. García-Río, P. Gilkey, J.H. Park, and R. Vázquez-Lorenzo, *Aspects of Differential Geometry*, **III**, Synthesis Lectures on Mathematics and Statistics, Morgan and Claypool Publishers, 2017.

[4] H.D. Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), **11**, Int. Press, Somerville, MA, 2010, 1–38.

[5] S.F. Ellermeyer and D.G. Robinson, *Integrals of periodic functions*, Math. Mag. **74** (2001), 393–396.

[6] M. Eminenti, G. La Nave, and C. Mantegazza, *Ricci solitons: the equation point of view*, Manuscr. Math. **127** (2008), 345–367 .

[7] S. Gavino-Fernández, *The geometry of Lorentzian Ricci solitons*, Ph.D. thesis, Universidade de Santiago de Compostela, 2012, http://www.tesisenred.net/handle/10803/84770.

[8] R.S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math. **71** (1988), 237–262.

[9] R.S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differ. Geom. **17** (1982), 255–306.

[10] M. Jamreh and M. Nadjafikhah, *Closed pseudo-Riemannian Ricci solitons*, J. Math. Phys. **58** (2017), no. 10, 101505.

[11] T. Leistner and D. Schliebner, *Completeness of compact Lorentzian manifolds with special holonomy*, Math. Ann. **364** (2016), 1469–1503.

[12] J. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Mathematics Monographs, **3**, Ame. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[13] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics, **103**, Academic Press, New York, 1983.

[14] M. Sánchez, *An Introduction to the Completeness of Compact Semi-Riemannian Manifolds,* Sémin. Théor. Spectr. Géom., 13, Univ. Grenoble I, Saint-Martin-d'Hères, 1995, 37–53.

[15] S.E. Stepanov and V.N. Shelepova, *A remark on Ricci solitons,* Mat. Zametki **86** (2009), 474–477 (Russian); Engl. transl.: Math. Notes **86** (2009), 447–450.

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Деякі нетривіальні та неградієнтні замкнуті псевдо-риманові стійкі солітони Річчі

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У статті вивчається рівняння солітона Річчі на компактних 3-вимірних незвідних лоренцевих многовидах, що допускають паралельне світлоподібне векторне поле із замкнутими орбітами. Ці компактні структури, які є геодезично повними, допускають нетривіальні, тобто не ейнштейнові та неградієнтні, стаціонарні солітони Річчі з нульовою скалярною кривиною, які показують різницю між замкнутими рімановими та псевдо-рімановими солітонами Річчі. Асоційоване потенційне векторне поле солітонної структури Річчі у всіх випадках, які ми будуємо на цих многовидах, є простороподібним векторним полем. Однак ми показуємо, що є приклади замкнутих псевдо-ріманових стійких солітонів Річчі в нейтральній сигнатурі $(2, 2)$ з нульовою скалярною кривиною, такі, що асоційоване потенційне векторне поле може бути часоподібним чи нульовим. Ці компактні многовиди також геодезично повні і не допускають конформне кілінгове векторне поле.

Ключові слова: солітони Річчі, замкнуті псевдоріманові многовиди, паралельне світлоподібне векторне поле.